On the Yor integral and a system of polynomials related to the Kontorovich-Lebedev transform

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Abstract

In this paper we establish different representations of the so-called Yor integral, which is one of the key ingredient in mathematical finance, in particular, to compute normalized prices of Asian options. We show, that the Yor integral is related with the Kontorovich-Lebedev transform. Also we discuss its relationship with a system of polynomials recently introduced by the author. We derive new important properties of these polynomials, including upper bounds, an exact asymptotic behavior for large values of their degree and explicit formula of coefficients.

Keywords: *Yor integral, Kontorovich-Lebedev transform, Parseval equality, modified Bessel functions, Asian options, Hartman-Watson distribution, Brownian motion*
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1 Introduction and preliminary results

In 1980 Yor [14] (see also in [3], [4], [15]) expressed the density of the Hartman -Watson distribution [2], which is related to the pricing of Asian options in mathematical finance, in the form of elementary integral, involving exponential functions

$$F_t(r) = \frac{e^{\pi^2/2t}}{\sqrt{2\pi^3 t}} \int_0^\infty \exp\left(-\frac{y^2}{2t}\right) \exp\left(-r\cosh y\right) \sinh y \sin\left(\frac{\pi y}{t}\right) dy,\ \ r, t > 0. \tag{1.1}$$

However, despite of the importance of integral (1.1) in applications and a necessity to calculate it in the closed form, this task is quite difficult. Nevertheless, we can perhaps stimulate the corresponding numerical calculations of the integral and finding its asymptotic behavior for small values of t, representing (1.1) in a different form. Precisely, we will show the relation between $F_t(r)$ and the Kontorovich-Lebedev transformation [9], [10], [11]

$$(Gf)(\tau) = \int_0^\infty K_{i\tau}(r)f(r)dr,\ \tau \in \mathbb{R}_+, \tag{1.2}$$

where $K_{i\tau}(r)$ is the modified Bessel function of the pure imaginary index $i\tau$ [1]. As it is known, operator (1.2) is bounded

$$G : L_2\left(\mathbb{R}_+; rdr\right) \leftrightarrow L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$$

and the integral in (1.2) converges with respect to the norm in $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$. Moreover, it forms an isometric isomorphism between these Hilbert spaces and the Parseval identity holds

$$\int_0^\infty \tau \sinh \pi\tau |(Gf)(\tau)|^2 d\tau = \frac{\pi^2}{2} \int_0^\infty |f(r)|^2 rdr. \tag{1.3}$$

Reciprocally, the inversion formula

$$f(r) = \frac{2}{r\pi^2} \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(r)(Gf)(\tau)d\tau, \tag{1.4}$$

takes place, where the convergence of the integral (1.4) is understood with respect to the norm of the space $L_2\left(\mathbb{R}_+; rdr\right)$.

The modified Bessel function $K_{i\tau}(x)$ is an eigenfunction of the following second order differential operator

$$\mathcal{A}_x \equiv x^2 - x\frac{d}{dx}x\frac{d}{dx}, \tag{1.5}$$

i.e. we have

$$\mathcal{A}_x\ K_{i\tau}(x) = \tau^2 K_{i\tau}(x). \tag{1.6}$$

It has the asymptotic behavior (cf. [1] relations (9.6.8), (9.6.9), (9.7.2))

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.7}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\nu|}\right),\ z \to 0, \tag{1.8}$$

$$K_0(z) = -\log z + O(1),\ z \to 0. \tag{1.9}$$

Moreover it can be defined by the following integral representations [9, (6-1-2)], [8], Vol. I, relation (2.4.18.4)

$$K_\nu(x) = \int_0^\infty e^{-x\cosh u} \cosh \nu u du, x > 0, \tag{1.10}$$

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-\nu-1}dt,\ x > 0. \tag{1.11}$$

The convolution operator for the Kontorovich-Lebedev transform is defined as follows [10, 11]

$$(f * h)(x) \equiv (f(x) * g(x)) = \frac{1}{2x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x \frac{u^2+y^2}{uy} + \frac{yu}{x}\right)} f(u)h(y)dudy, \ x > 0. \quad (1.12)$$

It is well defined in the Banach ring $L^\alpha(\mathbb{R}_+) \equiv L_1(\mathbb{R}_+; K_\alpha(x)dx), \alpha \in \mathbb{R}$, i.e. the space of all summable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with respect to the measure $K_\alpha(x)dx$ for which

$$||f||_{L^\alpha(\mathbb{R}_+)} = \int_0^\infty |f(x)| K_\alpha(x)dx \quad (1.13)$$

is finite. The following embeddings take place

$$L^\alpha(\mathbb{R}_+) \equiv L^{-\alpha}(\mathbb{R}_+), \ L^\alpha(\mathbb{R}_+) \subseteq L^\beta(\mathbb{R}_+), \ |\alpha| \geq |\beta| \geq 0, \alpha, \beta \in \mathbb{R},$$

$$L^\alpha(\mathbb{R}) \supset L_p(\mathbb{R}_+; xdx), \ 2 < p \leq \infty, \ |\alpha| < 1 - \frac{2}{p},$$

where $L_p(\mathbb{R}_+; xdx)$ is a weighted Banach space with the norm

$$||f||_{L_p(\mathbb{R}_+;xdx)} = \left(\int_0^\infty |f(x)|^p x dx \right)^{1/p}, \ 1 \leq p < \infty,$$

$$||f||_{L_\infty(\mathbb{R}_+;xdx)} = \text{ess sup}_{x \in \mathbb{R}_+} |f(x)|.$$

The factorization property is true for the convolution (1.12) in terms of the Kontorovich-Lebedev transform (1.2) in the space $L^\alpha(\mathbb{R}_+)$, namely

$$(G[f * h])(\tau) = (Gf)(\tau)(Gh)(\tau), \ \tau \in \mathbb{R}_+. \quad (1.14)$$

This property is based on the Macdonald formula [1]

$$K_\nu(x)K_\nu(y) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}\left(t \frac{x^2+y^2}{xy} + \frac{xy}{t}\right)} K_\nu(t) \frac{dt}{t}. \quad (1.15)$$

2 A system of polynomials

In this section we will provide a useful information about a system of polynomials, which is related to the Kontorovich-Lebedev transformation (1.2) and was studied by the author in [12]. In particular, we will derive new properties of these polynomials, including an upper bound, a series representation by the index of the modified Bessel functions of the third kind and an explicit formula expressing their coefficients. In fact, as it is proved in [12], the following functions

$$p_n(x) = (-1)^n e^x \mathcal{A}_x^n \ e^{-x}, \ n \in \mathbb{N}_0, \tag{2.1}$$

where $\mathcal{A}_x^n$ is the n-th iteration of the differential operator (1.5), are n-th degree polynomials, which have the following integral representation

$$p_n(x) = \frac{2(-1)^n}{\pi} e^x \int_0^\infty \tau^{2n} K_{i\tau}(x) d\tau, \ x > 0, \ n \in \mathbb{N}. \tag{2.2}$$

Hence calling the reciprocal formula (1.2), we obtain the Kontorovich-Lebedev transform of $p_n(x)$

$$\int_0^\infty K_{i\tau}(x) e^{-x} p_n(x) \frac{dx}{x} = (-1)^n \ \frac{\pi \tau^{2n-1}}{\sinh(\pi\tau)}. \tag{2.3}$$

The system p_n satisfies the differential recurrence relation of the form

$$p_{n+1}(x) = x^2 p_n''(x) + x(1-2x) p_n'(x) - x p_n(x), \ n = 0, 1, 2, \dots \ .$$

In particular, we derive

$$p_0(x) = 1, \ p_1(x) = -x, \ p_2(x) = 3x^2 - x, \quad p_3(x) = -15x^3 + 15x^2 - x.$$

The generating function of these polynomials is given by the series

$$e^{-2x \sinh^2(t/2)} = \sum_{n=0}^\infty \frac{p_n(x)}{(2n)!} \ t^{2n}.$$

Letting $x = 0$ in the latter equation, we find

$$p_n(0) = 0, \quad n = 1, 2, \dots .$$

The leading coefficient $a_{n,n}$ of these polynomials can be calculated by the formula

$$a_{n,n} = (-1)^n (2n-1)!! = (-1)^n 1 \cdot 3 \cdot 5 \dots \ \cdot (2n-1), \ n \in \mathbb{N}. \tag{2.4}$$

The following lemma proves an upper bound for the system $p_n(x)$. Indeed, we have

Lemma 1. *Let* $x > 0, \ n \in \mathbb{N}, \ \varepsilon \in \left(1 - \frac{\sqrt{3}}{2}, 1\right]$ *and*

$$\alpha \in \left(0, \quad 2 arccos \ \frac{\sqrt{1 + 4(1-\varepsilon)^2}}{2} \ \right]. \tag{2.5}$$

Then

$$|p_n(x)| \le \sqrt{\frac{(2^{4n} - 1)(4n)! \sin(\alpha/2)}{\pi n \ \alpha^{4n} (2^{4n-2} + 6/\pi^2 - 1)}} \ \frac{e^{\varepsilon x}}{2}. \tag{2.6}$$

Proof. In fact, taking representation (2.2), we apply the Schwarz inequality together with integral formula for Bernoulli numbers $B_{4n}, n = 1, 2, \ldots$ [1]

$$\int_0^\infty \frac{\tau^{4n-1}}{\sinh \tau} d\tau = -B_{4n} \frac{(2^{4n}-1)\pi^{4n}}{4n}, \tag{2.7}$$

and relation (2.16.51.8) in [8], Vol. II. Then we deduce

$$|p_n(x)| \le \frac{2}{\pi} e^x \int_0^\infty |K_{i\tau}(x)| \tau^{2n}\, d\tau \le \frac{2}{\pi} e^x \left(\int_0^\infty \tau \sinh(\alpha\tau)\, K^2_{i\tau}(x)\, d\tau \right)^{1/2}$$

$$\times \left(\int_0^\infty \frac{\tau^{4n-1}}{\sinh(\alpha\tau)} d\tau \right)^{1/2} = e^x \left(\frac{\pi}{\alpha}\right)^{2n} \sqrt{\frac{(1-2^{4n})\sin(\alpha/2) B_{4n}}{2\pi n}}$$

$$\times \left(x K_1(2x\cos(\alpha/2))\right)^{1/2}, \ x > 0, \alpha \in (0, \pi).$$

But in the meantime via (1.10) it is not difficult to find that

$$\left(x K_1(2x\cos(\alpha/2))\right)^{1/2} = \left(x \int_0^\infty e^{-2x\cos(\alpha/2)\cosh u} \cosh u du \right)^{1/2}$$

$$= \left(x \int_0^\infty e^{-2x\cos(\alpha/2)\sqrt{v^2+1}} dv \right)^{1/2} \le e^{\varepsilon x - x} \left(x \int_0^\infty e^{-xv} dv \right)^{1/2} = e^{\varepsilon x - x}$$

when $\varepsilon \in \left(1 - \frac{\sqrt{3}}{2}, 1\right]$ and α satisfies condition (2.5). Indeed, in this case

$$2\cos(\alpha/2)\sqrt{v^2+1} \ge v + 2(1-\varepsilon)$$

for any $v \ge 0$ since $2\cos(\alpha/2) \ge \sqrt{1 + 4(1-\varepsilon)^2}$. Moreover, employing a sharp upper bound for the Bernoulli numbers $-B_{4n}$ (see in [2])

$$-B_{4n} \le \frac{(4n)!}{2\pi^{4n}(2^{4n-2} + 6/\pi^2 - 1)},$$

we combine with (2.7) to arrive at inequality (2.6) and complete the proof of Lemma 1. □

Remark 1. As we observe form the Stirling asymptotic formula for factorials [1], inequality (2.6) guarantees the absolute and uniform convergence of series (2.4) for the generating function on any compact set of $x \in [x_0, X_0] \subset \mathbb{R}_+$ and t from the interval $|t| \le t_0 < 2/\alpha$.

Theorem 1. *The system* $p_n(x), x > 0,\ n \in \mathbb{N}$ *can be expressed in terms of the absolutely convergent series*

$$p_n(x) = 2e^x \sum_{m=1}^{\infty} (-1)^m m^{2n} I_m(x), \tag{2.8}$$

where $I_\nu(z)$ *is the modified Bessel function of the third kind [5]. Moreover, we have an explicit formula for these polynomials*

$$p_n(x) = \sum_{k=1}^{n} a_{k,n} x^k, \tag{2.9}$$

where the coefficients $a_{k,n}$ *are given by*

$$a_{k,n} = \frac{1}{k!} \sum_{r=0}^{k} \frac{(-1)^r}{2^r} \binom{k}{r} \sum_{j=0}^{k-r} \frac{(-1)^j}{2^j} \binom{k-r}{j} (r-j)^{2n}. \tag{2.10}$$

Finally, the following combinatorial identities hold

$$\sum_{r=0}^{k} \frac{(-1)^r}{2^r} \binom{k}{r} \sum_{j=0}^{k-r} \frac{(-1)^j}{2^j} \binom{k-r}{j} (r-j)^{2n} = 0, \quad k = n+1,\ n+2, \dots,\ 2n, \tag{2.11}$$

$$\sum_{r=0}^{n} \frac{(-1)^r}{2^r} \binom{n}{r} \sum_{j=0}^{n-r} \frac{(-1)^j}{2^j} \binom{n-r}{j} (r-j)^{2n} = (-1)^n n!\ (2n-1)!!\ . \tag{2.12}$$

Proof. Taking the integral representation (2.2) of $p_n(x)$ and employing the formula (see [6])

$$K_{i\tau}(x) = \frac{\pi}{2\sin(\pi i\tau)} \left[I_{-i\tau}(x) - I_{i\tau}(x) \right],$$

where $I_\nu(z)$ is the modified Bessel function of the third kind with its series representation

$$I_{-i\tau}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k-i\tau}}{k!\Gamma(k+1-i\tau)}, \tag{2.13}$$

where $\Gamma(z)$ is Euler's gamma-function [1], we get the chain of equalities

$$p_n(x) = \frac{e^x}{2} \int_{-\infty}^{\infty} \frac{(i\tau)^{2n}}{\sin(\pi i\tau)} \left[I_{-i\tau}(x) - I_{i\tau}(x) \right]\ d\tau = e^x \int_{-\infty}^{\infty} \frac{(i\tau)^{2n}}{\sin(\pi i\tau)} I_{-i\tau}(x)\ d\tau$$

$$= \frac{e^x}{\pi i} \int_{-i\infty}^{i\infty} s^{2n-1} \Gamma(1+s)\Gamma(1-s) \sum_{k=0}^{\infty} \frac{(x/2)^{2k-s}}{k!\Gamma(k+1-s)}\ ds.$$

The change of the order of integration and summation in the right-hand side of the latter equality is indeed possible by Fubini's theorem owing to the absolute convergence for each $x > 0$, namely

$$\int_{-i\infty}^{i\infty} \left|s^{2n-1}\Gamma(1+s)\Gamma(1-s)\right| \sum_{k=0}^{\infty} \left| \frac{(x/2)^{2k-s}}{k!\Gamma(k+1-s)}\, ds \right| < \infty.$$

Therefore,

$$p_n(x) = \frac{e^x}{\pi i} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!} \int_{-i\infty}^{i\infty} s^{2n-1} \frac{\Gamma(1+s)\Gamma(1-s)}{\Gamma(k+1-s)} \left(\frac{x}{2}\right)^{-s} ds. \tag{2.14}$$

But the integral in (2.14) can be calculated via the Slater theorem (see, for instance, in [8], Vol. III) after $2n-1$ times differentiation with respect to z under the integral sign. This operation is allowed by virtue of the absolute and uniform convergence. Thus we obtain

$$\int_{-i\infty}^{i\infty} s^{2n-1} \frac{\Gamma(1+s)\Gamma(1-s)}{\Gamma(k+1-s)} z^{-s} ds = -\left(z\frac{d}{dz}\right)^{(2n-1)} \int_{-i\infty}^{i\infty} \frac{\Gamma(1+s)\Gamma(1-s)}{\Gamma(k+1-s)} z^{-s} ds$$

$$= 2\pi i \left(z\frac{d}{dz}\right)^{(2n-1)} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} z^{m+1}\Gamma(2+m)}{m!\; \Gamma(2+k+m)}$$

$$= 2\pi i \left(z\frac{d}{dz}\right)^{(2n-1)} \sum_{m=1}^{\infty} \frac{(-1)^m z^m\; m}{(m+k)!}.$$

Hence differentiating under the series sign due to the absolute and uniform convergence, letting $z = x/2$ and combining with (2.13), (2.14), we derive the representation

$$p_n(x) = 2e^x \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!} \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^m\; m^{2n}}{(m+k)!} = 2e^x \sum_{m=1}^{\infty} (-1)^m m^{2n} I_m(x), \tag{2.15}$$

where the inversion of the summation order is motivated by the absolute convergence of the iterated series. Thus we proved (2.8).

On the other hand, calling relation (5.8.5.3) in [8], Vol. II, it is not difficult to find via differentiation with respect to a parameter owing to the absolute and uniform convergence that

$$2\sum_{m=1}^{\infty} m^{2n} I_m(x) = \lim_{a\to 0} \frac{d^{2n}}{da^{2n}}\; e^{x\cosh a}.$$

We calculate the $2n$-th derivative in the right-hand side of the latter equality employing the Hoppe formula [5]. Precisely it gives,

$$\frac{d^{2n}}{da^{2n}}e^{x\cosh a} = e^{x\cosh a}\sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \frac{(-\cosh a)^{k-j}}{j!(k-j)!}\frac{d^{2n}}{da^{2n}}\cosh^j a$$

$$= e^{x\cosh a}\sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \frac{(-\cosh a)^{k-j}}{2^j j!(k-j)!}\frac{d^{2n}}{da^{2n}}\sum_{r=0}^{j}\binom{j}{r}e^{(2r-j)a}$$

$$= e^{x\cosh a}\sum_{k=0}^{2n} x^k \sum_{j=0}^{k} \frac{(-\cosh a)^{k-j}}{2^j j!(k-j)!}\sum_{r=0}^{j}\binom{j}{r}(2r)^{2n-m}(2r-j)^{2n}e^{(2r-j)a}.$$

Therefore, passing to the limit when $a \to 0$, we find the identity

$$2\sum_{m=1}^{\infty} m^{2n} I_m(x) == e^x \sum_{k=0}^{2n}\frac{(-x)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{2^j}\sum_{r=0}^{j}\binom{j}{r}(2r-j)^{2n}$$

$$= e^x \sum_{k=0}^{2n}\frac{(-x)^k}{k!}\sum_{r=0}^{k}\frac{(-1)^r}{2^r}\binom{k}{r}\sum_{j=0}^{k-r}\frac{(-1)^j}{2^j}\binom{k-r}{j}(r-j)^{2n}.$$

On the other hand, since $(-1)^m I_m(x) = I_m(-x)$, we return to (2.15) and appealing to the right-hand side of the latter equality, we derive

$$p_n(x) = \sum_{k=0}^{2n}\frac{(-x)^k}{k!}\sum_{r=0}^{k}\frac{(-1)^r}{2^r}\binom{k}{r}\sum_{j=0}^{k-r}\frac{(-1)^j}{2^j}\binom{k-r}{j}(r-j)^{2n}.$$

However, as it is proved in [11], $p_n(x)$ is a polynomial of degree n. Therefore, all coefficients in front of powers x^k, $k = n+1,\ n+2,\ \ldots,\ 2n$ are surely equal to zero. Thus we establish the explicit formula (2.9) with coefficients (2.10) and combinatorial equality (2.11). Taking into account the value (2.4) of the leading coefficient $a_{n,n}$, we get identity (2.12) and complete the proof of Theorem 1. □

Finally, in this section we will established an exact asymptotic behavior of $p_n(x)$, when $n \to \infty$ and $x > 0$ is a fixed number. Precisely, we have

Theorem 2. *Let $x > 0$ and $\beta \in (0, \pi/2)$ be fixed numbers. Then*

$$p_n(x) = \frac{6x(-1)^n \sin\beta\ (2n)!}{\pi\beta^{2n}(2n+1)^3}e^x\left(1 + O\left(\frac{1}{n}\right)\right),\ n \to \infty. \tag{2.16}$$

Proof. Indeed, employing representation (2.2) and choosing a fixed parameter $\beta \in (0, \pi/2)$, we write

$$p_n(x) = \frac{2(-1)^n}{\pi} e^x \int_0^\infty \tau^{2n} e^{-\beta\tau} \left[\cosh(\beta\tau) + \sinh(\beta\tau)\right] K_{i\tau}(x)\, d\tau.$$

Hence the Parseval identities for the cosine and sine Fourier transforms [9] together with relations (2.5.31.4) in [8], Vol. I and (2.16.48.20) in [8], Vol. II drive us to the equalities

$$p_n(x) = \frac{2(-1)^n (2n)!}{\pi} e^x \int_0^\infty \frac{e^{-x\cos(\beta)\cosh y}}{(\beta^2+y^2)^{n+1/2}} \left[\cos\left((2n+1)\text{arctg}(y/\beta)\right)\cos(x\sin(\beta)\sinh y)\right.$$
$$\left. + \sin\left((2n+1)\text{arctg}(y/\beta)\right)\sin(x\sin(\beta)\sinh y)\right] dy$$
$$= \frac{2(-1)^n (2n)!}{\pi} e^x \int_0^\infty \frac{e^{-x\cos(\beta)\cosh y}}{(\beta^2+y^2)^{n+1/2}} \cos\left[(2n+1)\text{arctg}(y/\beta) - x\sin(\beta)\sinh y\right] dy$$
$$= \frac{2(-1)^n (2n)!}{\pi} e^x \,\text{Re}\left[\int_0^\infty e^{-x\cos(\beta+iy)} \exp\left[-(2n+1)\left(\log\left(\sqrt{\beta^2+y^2}\right)\right.\right.\right.$$
$$\left.\left.\left. + i\text{arctg}(y/\beta)\right)\right] dy\right]. \tag{2.17}$$

The asymptotic behavior for large n of the latter integral in brackets can be treated by the Laplace method [7], Ch. 4. Indeed, it has the form

$$\int_0^\infty q(y) e^{-(2n+1)p(y)} dy,$$

where $p(y) = \log(\beta + iy),\ q(y) = e^{-x\cos(\beta+iy)}$. Thus the integral

$$\int_0^\infty \frac{e^{-x\cos(\beta+iy)}}{(\beta+iy)^{2n+1}} dy \sim \frac{1}{\beta^{2n+1}} \sum_{s=0}^\infty \frac{a_s(x,\beta)\ s!}{(2n+1)^{s+1}},\ n \to \infty,$$

where the calculation of three first coefficients a_s will be enough to determine the main term of the expansion of its real part. Namely, using formulas for coefficients in [7], we have

$$a_0 = -i\beta e^{-x\cos\beta}, \quad a_1 = -i\beta(1 + x\beta\sin\beta)\ e^{-x\cos\beta},$$
$$a_2 = \frac{3}{2}x\beta\sin\beta - \frac{i\beta}{2} e^{-x\cos\beta}\left(\beta^2 x(\cos\beta + x\sin^2\beta) + 1\right).$$

Therefore, returning to (2.17), we find

$$p_n(x) = \frac{6x(-1)^n \sin\beta\ (2n)!}{\beta^{2n}(2n+1)^3\pi} e^x \left(1 + O\left(\frac{1}{n}\right)\right),\ n \to \infty,$$

which proves (2.17).

□

3 Representations of the Yor integral

This section will complete our goal to represent integral (1.1) in a different form. Namely, we will start relating $F_t(r)$ with the inverse Kontorovich-Lebedev transform (1.4). Indeed, taking (1.4) with $\nu = i\tau$ and integrating by parts we come out with the representation of the modified Bessel function

$$\tau K_{i\tau}(r) = r\int_0^\infty e^{-r\cosh u}\sinh u \sin\tau u du,\ \ r>0. \tag{3.1}$$

Hence applying the Parseval equality for the sine Fourier transform [9] with the relation (2.5.36.1) in [8], Vol. 1, integral (1.1) becomes $(r, t > 0)$

$$F_t(r) = \frac{e^{\pi^2/2t}}{r\pi^2}\int_0^\infty \tau K_{i\tau}(r)\left[\exp\left(-\frac{t(\tau-\pi/t)^2}{2}\right) - \exp\left(-\frac{t(\tau+\pi/t)^2}{2}\right)\right] d\tau$$

$$= \frac{2}{r\pi^2}\int_0^\infty e^{-\frac{t}{2}\tau^2}\tau\sinh\pi\tau K_{i\tau}(r)d\tau. \tag{3.2}$$

Hence form (1.2) we obtain reciprocally for each $t > 0$

$$\int_0^\infty K_{i\tau}(r)F_t(r)dr = e^{-\frac{t}{2}\tau^2} \tag{3.3}$$

and via Parseval equality (1.3) we derive the value of the integral

$$\int_0^\infty |F_t(r)|^2 r dr = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)\ e^{-t\tau^2}d\tau$$

$$= \frac{2e^{\pi^2/4t}}{\pi^2}\int_{-\infty}^\infty \tau e^{-t(\tau-\pi/(2t))^2}d\tau = \frac{e^{\pi^2/4t}}{t\sqrt{\pi t}}.$$

Recently (see [12]), the author introduced the following heat kernel for the Kontorovich-Lebedev transform

$$h_t(x,y) = \frac{2}{x\pi^2}\int_0^\infty e^{-t\tau^2/2}\tau\sinh\pi\tau\ K_{i\tau}(x)K_{i\tau}(y)d\tau. \tag{3.4}$$

Hence appealing to the Macdonald formula (1.15) and Fubini's theorem to interchange the order of integration, we come out with the representation of (3.4) as a translation operator of the Yor integral for convolution (1.12). Precisely, minding (3.2) it gives

$$h_t(x,y) = \frac{1}{2x}\int_0^\infty e^{-\frac{1}{2}\left(r\frac{x^2+y^2}{xy}+\frac{xy}{r}\right)} F_t(r)dr,\ \ x,y>0. \tag{3.5}$$

Meanwhile differential and convolution properties of the Yor integral (1.1) are given by the following

Theorem 3. *The function $F_t(r)$ is infinitely differentiable of variables $(r,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, satisfying the estimate*

$$\left|\frac{\partial^m F_t(r)}{\partial t^m}\right| \leq \frac{2^{1/4-m}e^{\pi^2/t}}{t^{m+3/4}\pi^{11/8}}\ K_0^{1/2}(2r)\Gamma^{1/4}(4m+5/2), \quad m \in \mathbb{N}_0. \tag{3.6}$$

Moreover, $F_t(r)$ is a solution of the generalized diffusion equation $(u = u(t,r))$

$$2\frac{\partial u}{\partial t} = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} - r^2 u \tag{3.7}$$

and satisfies the index law in terms of convolution (1.12)

$$F_t(r) = \left(F_{t_1} * F_{t_2}\right)(r), \quad t_1 + t_2 = t. \tag{3.8}$$

Finally, for the Yor integral the following integral equation takes place

$$\int_0^\infty h_{t_1}(r,y)F_{t_2}(y)dy = F_{t_1+t_2}(r). \tag{3.9}$$

Proof. In fact, employing the following inequality (see [11], [13]) for derivatives of the modified Bessel function with respect to x

$$\left|\frac{\partial^m K_{i\tau}(r)}{\partial x^m}\right| \leq\ e^{-\delta\tau}K_m(r\cos\delta),\ x>0,\ \tau>0,\ \delta \in \left[0;\frac{\pi}{2}\right),\ m=0,1,\ldots \tag{3.10}$$

it is not difficult to verify that all positive t the integral in the right-hand side of the latter equality in (3.2) together with derivatives of any order with respect to r converge absolutely and uniformly by $r \geq r_0 > 0$. Therefore $F_t(r)$ is infinitely differentiable with respect to r. Similar motivation can be done for the derivatives of the order $m \in \mathbb{N}_0$ with respect to $t > 0$ and it gives the expression

$$\frac{\partial^m F_t(r)}{\partial t^m} = \frac{2^{1-m}(-1)^m}{\pi^2}\int_0^\infty e^{-t\tau^2/2}\tau^{2m+1}\sinh\pi\tau\ K_{i\tau}(r)d\tau,\ m=0,1,\ldots\ . \tag{3.11}$$

Hence the Schwarz inequality and relation (2.16.52.6) in [8], Vol. II yield

$$\left|\frac{\partial^m F_t(r)}{\partial t^m}\right| \leq \frac{2^{1-m}}{\pi^2}\left(\int_0^\infty e^{-t\tau^2+2\pi\tau}\tau^{2(2m+1)}d\tau\right)^{1/2}\left(\int_0^\infty K_{i\tau}^2(r)d\tau\right)^{1/2}$$

$$\leq \frac{2^{1/2-m}}{\pi\sqrt{\pi}}K_0^{1/2}(2r)\left(\int_{-\infty}^\infty e^{-t\tau^2+4\pi\tau}d\tau\right)^{1/4}\left(\int_0^\infty e^{-t\tau^2}\tau^{4(2m+1)}d\tau\right)^{1/4}$$

$$= \frac{2^{1/4-m}}{t^{m+3/4}\pi^{11/8}} \, e^{\pi^2/t} K_0^{1/2}(2r)\Gamma^{1/4}(4m+5/2).$$

Thus we proved (3.6).

Further, as it follows from (1.5), (1.6) and absolute and uniform convergence of the corresponding integrals, formula (3.11) can be written as the following partial differential equation

$$\frac{\partial^m F_t(r)}{\partial t^m} = (-1)^m 2^{-m} \mathcal{A}_r^m F_t(r), \tag{3.12}$$

where $m = 0, 1, \ldots$ and A_r^m is m-th iterates of the operator (1.5). In particular, letting $m = 1$ we obtain that the Yor integral (1.1) satisfies the generalized diffusion equation (3.7).

Next, equality (3.8) is a direct consequence of (3.3), factorization equality (1.14) and the uniqueness property of the Kontorovich-Lebedev transform [11]. In order to prove (3.9), we multiply $h_{t_1}(x, y)$ by $F_{t_2}(y)$ and integrate by y over $\mathbb{R}_+$. Then using (3.3), (3.4) and the Fubini theorem, which is applicable due to the absolute convergence of the iterated integral (it can be verified, employing inequalities (3.6) with $m = 0$ and (3.10)), we get the result. □

Finally, we will establish a representation of the Yor integral (1.1) in terms of polynomials (2.1) and the heat kernel (3.4).

We have

Theorem 4. *Let $r, t > 0$. Then the Yor integral satisfies the following equation*

$$F_t(r) = \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2(k-1)}}{(2k-1)!} \left(e^{-r} \frac{p_k(r)}{r} * F_t(r) \right), \tag{3.14}$$

where $*$ denotes convolution (1.12). *Moreover, it can be rewritten in the form*

$$F_t(r) = \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2(k-1)} a_k(r,t)}{(2k-1)!}, \tag{3.15}$$

where

$$a_k(r,t) = \int_0^{\infty} e^{-u} h_t(r,u) p_k(u) \frac{du}{u}$$

and h_t is the heat kernel (3.4).

Proof. Calling again integral (3.2), we expand the hyperbolic sine in Taylor's series. Then changing the order of integration and summation due to the estimate

$$\int_0^{\infty} e^{-t\tau^2/2} \tau |K_{i\tau}(r)| \sum_{k=1}^{\infty} \frac{(\pi\tau)^{2k-1}}{(2k-1)!} d\tau \le K_0(r) \sum_{k=1}^{\infty} \frac{\pi^{2k-1}}{(2k-1)!} \int_0^{\infty} e^{-t\tau^2/2} \tau^{2k} d\tau$$

$$= \frac{K_0(r)}{\pi\sqrt{2t}} \sum_{k=1}^{\infty} \frac{\left(\pi\sqrt{2/t}\right)^{2k} \Gamma(k+1/2)}{(2k-1)!} < \infty,$$

we come out with the representation

$$F_t(r) = \frac{2}{r\pi^2} \sum_{k=1}^{\infty} \frac{\pi^{2k-1}}{(2k-1)!} \int_0^{\infty} e^{-t\tau^2/2}\ \tau^{2k} K_{i\tau}(r) d\tau. \tag{3.16}$$

But the integral in (3.16) can be treated with the convolution for the Kontorovich-Lebedev transform (1.12), its factorization property (1.14) and inversion formula (1.4). Thus minding formulas (2.2), (3.2) and (3.3), we derive

$$\frac{2}{r\pi^2} \int_0^{\infty} e^{-t\tau^2/2}\ \tau^{2k} K_{i\tau}(r) d\tau = \frac{(-1)^k}{\pi} \left(e^{-r} \frac{p_k(r)}{r} * F_t(r) \right).$$

Substituting in (3.16), it gives

$$F_t(r) = \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2(k-1)}}{(2k-1)!} \left(e^{-r} \frac{p_k(r)}{r} * F_t(r) \right),$$

and equality (3.14) is proved. On the other hand, employing (3.5), one can express the latter convolution in terms of the heat kernel (3.4). Precisely, we find for each $k \in \mathbb{N}$ (see (1.12))

$$\left(e^{-r} p_k(r) * F_t(r)\right) = \int_0^{\infty} e^{-u} h_t(r,u) p_k(u) \frac{du}{u} = a_k(t,r),$$

which proves (3.15).

□

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