HOPF SUBALGEBRAS AND TENSOR POWERS OF GENERALIZED PERMUTATION MODULES

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Abstract. By means of a certain module V and its tensor powers in a finite tensor category, we study a question of whether the depth of a Hopf subalgebra R of a finite-dimensional Hopf algebra H is finite. The module V is the counit representation induced from R to H, which is then a generalized permutation module, as well as a module coalgebra. We show that if in the subalgebra pair either Hopf algebra has finite representation type, or V is either semisimple with H pointed, projective, or its tensor powers satisfy a Burnside ring formula over a finite set of Hopf subalgebras including R, then the depth of R in H is finite. One assigns an non-negative integer depth to V, or any other H-module, by comparing the truncated tensor algebras of V in a finite tensor category and so obtains an upper bound for depth of a Hopf subalgebra. For example, relative Hopf restricted modules have depth 1.

1. Introduction and Preliminaries

Two modules are said to be similar if each module is isomorphic to a direct summand of a multiple of the other: briefly formulated, each module divides a multiple of the other. If the modules are finitely-generated over a finite-dimensional algebra, similarity is equivalent by the Krull-Schmidt theorem to their having the same constituent indecomposables. A subalgebra B in an algebra A is said to have finite depth if $A^{\otimes_B (n+1)}$ is similar to $A^{\otimes_B n}$ for some $n \geq 0$ as X-Y-bimodules for any four choices of $X, Y \in \{B, A\}$ [3, 22]: see below in this section for the precise definition of depth and h-depth $d(B, A)$ and $d_h(B, A)$. It is rather easy to see that $A^{\otimes_B n}$ divides $A^{\otimes_B (n+1)}$ for all n, whence if A is finite dimensional and any one of A^e, B^e, $A \otimes B^{\rm op}$ or $B \otimes A^{\rm op}$ has finite representation type, the depth is finite, since the constituent indecomposables of the tensor powers of A over B are increasing subsets within a finite set of representative indecomposable modules from isomorphism classes.

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The analogy for this in number theory is to ask if a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers is affinely generated over the primes. For example, this is the case if $a_n = a^n$ for some integer a, but is not the case if $a_n = p_n$, a sequence of increasing primes. As a type of toy model for what we do, say that the sequence $\{a_n\}$ has depth m if a_{m+1+j} divides a power of $a_1 \cdots a_{m+j}$ for each $j = 0, 1, 2, \ldots$. Then the sequence $a_n = a^n$ has minimum depth one, and the sequence $a_n = p_n$ has infinite depth. As another example, the Fibonacci sequence $\{u_n\}$ has infinite depth, since u_{p_n} is a subsequence of increasing primes [17, Theorem 179 (iv)]. A sequence $\{a_n\}$ of minimum depth m for each $m \geq 1$ may be obtained from m natural numbers u_i with $\gcd(u_1, \ldots, u_m) = 1$ by letting $a_n = u_r^{q+1}$ where $n = mq+r$ for $q \geq 0, 1 \leq r \leq m$. Of course the analogy is far from perfect if we replace $a \in \mathbb{N}$ with a module in a finite tensor category $\mathcal{C}$ [13] where it rarely has depth 1 (although relative Hopf restricted modules are an exception, see below) and it is unclear if it has finite depth in a category $\mathcal{C}$ of tame or wild representation type.

In [3, Boltje-Danz-Külshammer] the depth of a finite group algebra extensions $K[H] \subseteq K[G]$, where K is a commutative ring, is shown to be finite. The authors propose the problem of determining whether depth of a Hopf subalgebra R in a finite dimensional Hopf algebra H is finite [3, p. 259]. We take up this problem in this paper by showing that any tensor power of the bimodule ${}_RH_R$ is a functorial image of a smaller tensor power of $V := H/R^+H$ where R^+ is the augmentation ideal of R; more precisely, $H^{\otimes_R n} \cong H \otimes V^{\otimes n-1}$ as H-H-bimodules where the unadorned tensor is the tensor in the finite tensor category $\mathcal{M}_H$, the category of finite-dimensional right H-modules. We then define depth of a module W in a finite tensor category $\mathcal{C}$ to be n if the truncated tensor algebra $T_{n+1}(W)$ divides a multiple of $T_n(W)$. It follows that $T_{n+m}(W)$ and $T_{n+m+1}(W)$ are similar in $\mathcal{C}$ for each integer $m \geq 0$; let $d(W, \mathcal{C})$ denote the least such n. It is then shown that the depth of the Hopf subalgebra $d(R, H) \leq 2 + 2d(V, \mathcal{M}_R)$. We note from this that finite depth follows if V is projective, semisimple and H pointed, or V is in a tensor category of finite representation type. In addition, the result in [3] that finite group algebra extensions have finite depth is recovered in Section 4, in the case of an arbitrary ground field. Along the way, depth of relative Hopf restricted modules are shown to be 1 in Section 3, we extend a normality result of Masuoka in Section 2, and depth of V as a $\mathcal{G}$-module for a complex group algebra pair of a corefree subgroup $\mathcal{H} \subseteq \mathcal{G}$ is noted in Section 2 to be less than $2m + 1$ where the permutation character takes m values in $\{0, 1, \ldots, |\mathcal{G} : \mathcal{H}|\}$.

1.1. **Sketch of subalgebra depth.** Let A be a unital associative algebra over a field k. In this paper we will assume all algebras and modules to be finite-dimensional vector spaces for the sake of staying focused on the problem of Boltje *et al*, although much below remains true without this assumption [22]. Two modules M_A and N_A are *similar* (or H-equivalent) if $M \oplus * \cong qN = N \oplus \cdots \oplus N$ (q times) and $N \oplus * \cong rM$ for some $r, q \in \mathbb{N}$. This is briefly denoted by $M \,|\, qN$ and $N \,|\, rM \Leftrightarrow M \sim N$.

Let B be a subalgebra of A (always supposing $1_B = 1_A$). Consider the natural bimodules ${}_AA_A$, ${}_BA_A$, ${}_AA_B$ and ${}_BA_B$ where the last is a restriction of the preceding, and so forth. Denote the tensor powers of ${}_BA_B$ by $A^{\otimes_B n} = A \otimes_B \cdots \otimes_B A$ for $n = 1, 2, \ldots$ which again is a natural bimodule over A and B in any one of four ways; set $A^{\otimes_B 0} = B$ which is only a natural B-B-bimodule.

If $A^{\otimes_B (n+1)}$ is similar to $A^{\otimes_B n}$ as X-Y-bimodules, one says $B \subseteq A$ has

- depth $2n + 1$ if $X = B = Y$;
- left depth $2n$ if $X = B$ and $Y = A$;
- right depth $2n$ if $X = A$ and $Y = B$;
- h-depth $2n - 1$ if $X = A = Y$.

valid for even depth and h-depth if $n \geq 1$ and for odd depth if $n \geq 0$.

For example, $B \subseteq A$ has depth 1 iff ${}_BA_B$ and ${}_BB_B$ are similar [5, 22]. In this case, it is easy to show that A is algebra isomorphic to $B \otimes_{Z(B)} A^B$ where $Z(B), A^B$ denote the center of B and centralizer of B in A. Another example, $B \subset A$ has right depth 2 iff ${}_AA_B$ and ${}_AA \otimes_B A_B$ are similar. If $A = \mathbb{C}\, G$ is a group algebra over finite group G and $B = \mathbb{C}\, H$ a group algebra over subgroup H of G, then $B \subseteq A$ has right depth 2 iff H is a normal subgroup of G iff $B \subseteq A$ has left depth 2 [20]; a similar statement is true for a Hopf subalgebra $R \subseteq H$ of finite index [4].

Note that $A^{\otimes_B n} \,|\, A^{\otimes_B (n+1)}$ for all $n \geq 2$ and in any of the four natural bimodule structures: one applies 1 and multiplication to obtain a split monic, or split epi oppositely. For three of the bimodule structures, it is true for $n = 1$; as A-A-bimodules, equivalently $A \,|\, A \otimes_B A$ as A^e-modules, this is the separable extension condition on $B \subseteq A$. But $A \otimes_B A \,|\, qA$ as A-A-bimodules for some $q \in \mathbb{N}$ is the H-separability condition and implies A is a separable extension of B [19]. Somewhat similarly, ${}_BA_B \,|\, q({}_BB_B)$ implies ${}_BB_B \,|\, {}_BA_B$ [22]. It follows that subalgebra depth and h-depth may be equivalently defined by replacing similarity above with $A^{\otimes_B (n+1)} \,|\, qA^{\otimes_B n}$ for some positive integer q [3, 21, 22].

Note that if $B \subseteq A$ has h-depth $2n-1$, the subalgebra has (left or right) depth $2n$ by restriction of modules. Similarly, if $B \subseteq A$ has depth $2n$, it has depth $2n+1$. If $B \subseteq A$ has depth $2n+1$, it has depth $2n+2$ by tensoring either $- \otimes_B A$ or $A \otimes_B -$ to $A^{\otimes_B (n+1)} \sim A^{\otimes_B n}$. Similarly, if $B \subseteq A$ has left or right depth $2n$, it has h-depth $2n+1$. Denote the minimum depth of $B \subseteq A$ (if it exists) by $d(B, A)$ [3]. Denote the minimum h-depth of $B \subseteq A$ by $d_h(B, A)$. Note that $d(B, A) < \infty$ if and only if $d_h(B, A) < \infty$; in fact, $|d(B, A) - d_h(B, A)| \leq 2$ if either is finite.

For example, the permutation groups $\Sigma_n < \Sigma_{n+1}$ and their corresponding group algebras $B \subseteq A$ over any commutative ring K has depth $d(B, A) = 2n - 1$ [7, 3]. Depths of subgroups in $PGL(2, q)$, twisted group algebras and Young subgroups of Σ_n are computed in [15, 11, 16]. If B and A are semisimple complex algebras, the minimum odd depth is computed from powers of an order r symmetric matrix of nonnegative entries $S := MM^t$ where M is the inclusion matrix $K_0(B) \to K_0(A)$ and r is the number of irreducible representations of B in a basic set of $K_0(B)$; the depth is $2n+1$ if S^n and S^{n+1} has an equal number of zero entries [7]. Similarly, the minimum h-depth of $B \subseteq A$ is computed from powers of an order s symmetric matrix $T = M^t M$, where s is the rank of $K_0(A)$, and the power n at which the number of zero entries of T^n stabilizes [21]. It follows that the subalgebra pair of semisimple complex algebras $B \subseteq A$ always has finite depth. In particular, a Hopf subalgebra of a semisimple complex Hopf algebra has finite depth, since it is semisimple [26].

2. Tensor powers of bimodules and modules in tensor categories

Let H be a finite-dimensional Hopf algebra over a field k with coproduct Δ, counit ε and antipode S. Suppose R is a Hopf subalgebra of H. In this case $\Delta(R) \subseteq R \otimes R$ and $S(R) \subseteq R$. Let R^+ denote $\ker \varepsilon$, the counit restricted to R; e.g., $r - \varepsilon(r)1 \in R^+$ for each $r \in R$. Note that R^+H is a coideal in H as well as right ideal. Form the module coalgebra (and generalized quotient of a Hopf subalgebra) $V := H/R^+H$ where $\overline{h} := h + R^+H$ and Δ induces the coproduct $\Delta(\overline{h}a) = \overline{h_{(1)}}a_{(1)} \otimes \overline{h_{(2)}}a_{(2)}$ for every $h, a \in H$, an H-module morphism, as is the counit: V is a coalgebra in the category $\mathcal{M}_H = \mathrm{Mod}{-}H$. The next lemma is concerned only with the right H-module structure on V (although the given isomorphism is also a right comodule morphism).

Lemma 2.1. *Let A be an algebra and M an A-H-bimodule. Then $M \otimes_R H \cong M \otimes V$ as A-H-bimodules via $m \otimes_R h \mapsto mh_{(1)} \otimes \overline{h_{(2)}}$.*

Proof. Since $\overline{rh} = \varepsilon(r)\overline{h}$ for every $r \in R$, this mapping is well-defined. Of course the mapping is a left A-module mapping. Recall that in the category of H-modules the tensor product of two modules U, W has H-module structure given by $(u \otimes w)h = uh_{(1)} \otimes wh_{(2)}$ for every $u \in U, w \in W, h \in H$; the mapping in the lemma is a right H-module morphism where this is the H-module structure on $M \otimes V$.

Finally the mapping above has inverse mapping given by $m \otimes \overline{h} \mapsto mS(h_{(1)}) \otimes_R h_{(2)}$, which is well-defined since $S(r_{(1)}h_{(1)}) \otimes_R r_{(2)}h_{(2)} = S(h_{(1)}) \otimes_R \varepsilon(r)h_{(2)}$ for all $r \in R, h \in H$. □

A lemma may be similarly established for a bimodule ${}_HN_A$ where $H \otimes_R N \cong H/HR^+ \otimes N$ via a mapping $h \otimes_R n \mapsto \overline{h_{(1)}} \otimes h_{(2)}n$.

Note that $\dim V = \frac{\dim H}{\dim R}$ by an application of the Nichols-Zoeller freeness theorem; in fact, $H \cong R \otimes V$ as left R-modules (for more structure, see [26, Chap. 8]).

Lemma 2.2. *The right H-module $V = H/R^+H \cong t_RH$ where t_R is a nonzero right integral in R.*

Proof. Note the epimorphism $V \to t_RH$, $\overline{h} \mapsto t_Rh$. Let $q = \dim V$ and $h_1, \ldots, h_q \in H$ be a basis for the free module ${}_RH$. Suppose $h = \sum_{i=1}^q r_ih_i$ and $t_Rh = 0$, equivalently $\sum_i \varepsilon(r_i)t_Rh_i = 0$, then $\varepsilon(r_i)t_R = 0$, so $\varepsilon(r_i) = 0$ for each $i = 1, \ldots, q$. Then $h \in R^+H$. It follows that $V \to t_RH$ is also injective. □

From this we (C. Lomp and I) note that if t_R is a normal element in H, i.e., $t_RH = Ht_R$, then V is a trivial R-module, and our results below, Def. 3.1 and Theorem 4.1, together with the characterization of depth two Hopf subalgebras as ad-stable [4], shows that R is ad-stable in H. As a corollary then is one direction of Masuoka's theorem that t_R is central in a semisimple Hopf algebra H iff R is ad-stable in H.

At this point, it is informative to extend [22, Prop. 2.6] from its hypothesis of unimodularity on H, with thanks to C. Young for the ι-method.

Corollary 2.3. *Suppose $R \subseteq H$ is a Hopf subalgebra of h-depth 1. Then $R = H$.*

Proof. Assume that $\iota : H \otimes V \hookrightarrow qH$ is an H-H-bimodule monomorphism, which must exist since $H \otimes_R H \,|\, qH$ for some $q \in \mathbb{N}$. Let t_H be a nonzero right integral in the space of integrals $\int_H^r$; let $\alpha \in H^*$ be its modular function, an augmentation of H defined by $xt_H = \alpha(x)t_H$. Then for each $v \in V$ and $h \in H$, $h\iota(t_H \otimes v) = \iota(ht_H \otimes v) = \alpha(h)\iota(t_H \otimes v)$, which implies that $\iota(t_H \otimes V) \subseteq q\int_H^r$, since each of the q component is an augmented Frobenius algebra (H, α) with 1-dimensional space of left integrals kt_H [19]; it follows that $q \geq \frac{\dim H}{\dim R}$. Also $\iota(t_H \otimes v)h = \iota(t_H\varepsilon(h_{(1)}) \otimes vh_{(2)}) = \iota(t_H \otimes vh)$, then since $\iota(t_H \otimes V) \subseteq q\int_H^r$, it follows that $\iota(t_H \otimes vh) = \iota(t_H \otimes v\varepsilon(h))$, whence $vh = v\varepsilon(h)$ for every $v \in V, h \in H$. In particular, $\overline{1_H}h = \overline{h} = \varepsilon(h)\overline{1_H}$ for each $h \in H$, so that $\dim V = 1$. Hence $\dim R = \dim H$. □

In the two extreme cases of Hopf subalgebra, when $R = H$ and $R = k1_H$, we have in the first case $V = k_\varepsilon$, which is simple (but not projective unless H is semisimple), and in the second case, $V = H$, which is free over H or R (and therefore not semisimple unless H or R is so).

If R is normal, or ad-stable in H, then $R^+H = HR^+$ so $V = H/HR^+$ is a trivial right R-module (given by the counit). Then taking $M = H$ in the lemma, $H \otimes_R H \cong H \otimes V \cong H^{\dim V}$ as H-R-bimodules, the right depth two condition. The left depth two condition is similarly obtained from the variant of the lemma just mentioned; the converse that a left (right) depth two Hopf subalgebra is right (left) ad-stable is shown in [4].

Note that $V \cong k \otimes_R H$, since the annihilator ideal of the R-module k is R^+. It follows that the H-module V is the induced module of the one-dimensional trivial R-module; thus, V_H is R-relative projective.

Example 2.4. Consider the group algebras $R = k[\mathcal{H}]$ and $H = k[\mathcal{G}]$ where $\mathcal{H} \subseteq \mathcal{G}$ is a subgroup of a finite group. Then $V \cong k[\mathcal{G}/\mathcal{H}]$ the permutation module of right cosets (via $\overline{g} \mapsto \mathcal{H}g$). Suppose for a moment that $k = \mathbb{C}$; the character of V is the induced principal character $\eta := 1_{\mathcal{H}}{}^{\mathcal{G}}$. This character is faithful if $\mathcal{H}$ is corefree in $\mathcal{G}$, i.e., there is a trivial intersection of conjugates of $\mathcal{H}$ [18]. In this case, the Brauer-Burnside theorem [18, p. 49] guarantees that each irreducible character ψ of $\mathcal{G}$ is a constituent of a power η^n where $0 \leq n < m \leq |\mathcal{G} : \mathcal{H}|$ and m is the number of distinct values taken by $\eta(g)$ as $g \in \mathcal{G}$. Putting Prop. 2.7 together with Def. 3.1, one obtains that η^{m-1} contains each irreducible character as a constituent, so the depth $d(V, \mathrm{Mod}{-}\mathbb{C}G) \leq m - 1$. Applying the main theorem 4.1, we then see that the depth of the corefree subgroup is $d_{\mathbb{C}}(\mathcal{H}, \mathcal{G}) \leq 2m \leq 2|\mathcal{G} : \mathcal{H}|$.

Because of the example, we suggest referring to the right H-module V as the *generalized permutation module* of the Hopf subalgebra pair $R \subseteq H$.

Theorem 2.5. *The right H-module V is projective if and only if R is semisimple if and only if V_R is projective.*

Proof. ($\Leftarrow$) If R is semisimple, then k is a projective R-module by Maschke's theorem. Since $V = \mathrm{Ind}_R^H k$, it follows that V_H is projective. (Alternatively, choose t_R in the lemma above to be an idempotent.)

($\Rightarrow$) If V is projective, then the short exact sequence

$$0 \to R^+H \to H \stackrel{\pi}{\to} V \to 0 \tag{1}$$

splits, where $\pi(h) = \overline{h}$; let $\sigma : V_H \to H_H$ satisfy $\pi \circ \sigma = \mathrm{id}_V$. Consider $e := \sigma(\overline{1})$. Then $e^2 = e$ since $\overline{h} = \overline{h'}$ iff $h - h' \in R^+H$, $\overline{e} = \overline{1}$ and $\sigma(\overline{1})e = \sigma(\overline{e}) = \sigma(\overline{1})$. It follows that $V \cong eH$ via $\overline{h} \mapsto eh$. Note that for every $r \in R$ we have $er = e\varepsilon(r)$, since $r - \varepsilon(r)1_H \in R^+H$. Also note that from $e - 1 \in R^+H$ it follows that $\varepsilon(e) = 1$.

Let $\{h_i\}$, $\{f_i : H_R \to R_R\}$ be dual bases for the free module H_R. Then for each i, $f_i(e)$ is a right integral of R, thus $f_i(e) = c_i t_R$ for some scalar c_i and a fixed nonzero right integral t_R in R. Then $e = \sum_i h_i f_i(e) = (\sum_i c_i h_i)t_R$, so that $\varepsilon(t_R) \neq 0 \stackrel{\text{Maschke}}{\Rightarrow} R$ is semisimple.

Finally, if V_H is projective, so is V_R since H_R is free. If V_R is projective, then $V \otimes_R H$ is a projective H-module. But the natural epimorphism $V \otimes_R H \to V$ is R-split, so H-split since V is R-relative projective. Then V_H is projective. □

For example, Lorenz has proven that if H is involutory and non-semisimple Hopf algebra where $\mathrm{char}\, k = p$, then p divides the dimension of any projective H-module [25]: thus if H is a finite group algebra $k[\mathcal{G}]$, then $p||G|$, and if a Hopf subalgebra $k[\mathcal{H}]$, where $\mathcal{H} \leq \mathcal{G}$, has projective permutation $\mathcal{G}$-module V, then $p| \dim V = |\mathcal{G} : \mathcal{H}|$; indeed consistent with the proposition since $k[\mathcal{H}]$ is semisimple iff p does not divide the order $|\mathcal{H}|$.

Note that the short exact sequence (1) may be derived from the induction functor, $- \otimes_R H$ (where ${}_RH$ is faithfully flat) applied to the R-module sequence:

$$0 \to R^+ \to R \stackrel{\varepsilon}{\to} k \to 0. \tag{2}$$

Let $H \otimes_R \cdots \otimes_R H$ (n times H) be denoted by $H^{\otimes_R n}$, a natural H-H-bimodule for each $n \geq 1$.

Proposition 2.6. *For each integer $n \geq 2$, $H^{\otimes_R n} \cong H \otimes V^{\otimes(n-1)}$ as H-H-bimodules.*

Proof. The statement is true for $n = 2$ by the lemma with $M = H$ and $A = H$. Then by induction with $n > 2$,

$$H^{\otimes_R n} \cong H^{\otimes_R (n-1)} \otimes_R H \cong (H \otimes V^{\otimes(n-2)}) \otimes_R H \cong H \otimes V^{\otimes(n-1)}$$

where we apply the lemma with the H-H-bimodule $M = H \otimes V^{\otimes(n-2)}$ in the last isomorphism. □

The isomorphism in the proposition is given by

$$x \otimes y \otimes \cdots \otimes z \mapsto xy_{(1)} \cdots z_{(1)} \otimes \overline{y_{(2)} \cdots z_{(2)}} \otimes \cdots \otimes \overline{z_{(n)}},$$

with inverse mapping given by

$$u \otimes \overline{v} \otimes \overline{w} \otimes \cdots \longmapsto uS(v_{(1)}) \otimes_R v_{(2)} S(w_{(1)}) \otimes_R w_{(2)} \cdots .$$

Proposition 2.7. *If W is a finite-dimensional right H-module coalgebra, or dually a left H-module algebra, then $W^{\otimes n} \mid W^{\otimes (n+1)}$ for each $n \geq 1$ as H-modules (if H is semisimple, $n \geq 0$). In particular, this applies to V, which additionally satisfies $k_R \mid V$.*

Proof. Notice that the coproduct $\Delta_W : W \to W \otimes W$ is a split monomorphism of H-modules with respect to the counit ε_W; hence, $\Delta_W \otimes \mathrm{id}^{\otimes(n-1)} : W^{\otimes n} \to W^{\otimes(n+1)}$ is a split monic for each n. Note that $\varepsilon_W : W \to k$ is an epi of H-modules, which is split if H is semisimple (equivalently, k_H is projective).

If W is a right H-module coalgebra, then observe that W^* is a left H-module algebra via the dual algebra structure and the left H-module structure $\langle hw^*, w \rangle = \langle w^*, wh \rangle$ for all $w^* \in W$, $w \in W$ and $h \in H$. It is not hard to show that $(W^*)^{\otimes n} \cong (W^{\otimes n})^*$ from which is follows from $W^{\otimes n} \mid W^{\otimes(n+1)}$ that $(W^*)^{\otimes n} \mid (W^*)^{\otimes(n+1)}$. Alternatively, the face and degeneracy maps $A^{\otimes n} \to A^{\otimes(n \pm 1)}$ of a left H-module algebra A are left H-module arrows; in particular, $A^{\otimes n} \mid A^{\otimes(n+1)}$.

Finally note that V is a right H-module (and right R-module) coalgebra, and the mapping $k \to V$ given by $\lambda \mapsto \lambda \overline{1}$ splits ε_V as an R-epi, since $\overline{r} = \varepsilon(r)\overline{1}$ for each $r \in R$. □

If H is a right semisimple extension of R (i.e. all H-modules are (H,R)-relative projective), then $k_H \mid V_H$, since k_H is relative projective and the image of the R-split epi $\varepsilon_V : V_H \to k_H$. The next proposition finds a projective cover and injective hull for the cyclic H-module $V = H/R^+H$.

Proposition 2.8. *The projective cover of V is $eH \stackrel{\pi}{\longrightarrow} V$ for some idempotent $e \in H$ with $\varepsilon(e) = 1$ and $\ker \pi := C \subseteq e\mathrm{Rad}\, H$; there is equality of subsets if and only if V_H is semisimple. There is an idempotent $f \in H$ such that the H-module fH is the injective hull of V and contains $t_R H$.*

Proof. It is a result of a well-known theorem for projective covers (e.g. [23]) which applied to π, results in a direct decomposition of $H = eH \oplus (1-e)H$ for some idempotent $e \in H$, where $(1-e)H \subseteq \ker \pi = R^+H$ ($\subseteq H^+$, so $\varepsilon(e) = 1$) and C, the kernel of the restriction of π to eH, a small submodule. Then C is contained in $\mathrm{Rad}\, eH = e\mathrm{Rad}\, H$. Then

$\pi : eH \to V$ is the projective cover. Since $\mathrm{Rad}\, V = \mathrm{Rad}\, eH/C = (e\mathrm{Rad}\, H)/C = 0$ iff $C = e\mathrm{Rad}\, H$.

By Lemma 2.2, $V \cong t_R H$ as right H-modules; but $t_R H \hookrightarrow H_H$, which is projective and injective. By standard injective hull theory (for example, large submodules and essential extensions [23]) there is an injective submodule $fH \subseteq H$, where $f^2 = f$ since it is also projective, such that $fH \supseteq t_R H$ is an essential extension. □

3. A module's depth in a tensor category

We define an object W's depth in a finite tensor category $\mathcal{C}$ that is naturally isomorphic to a category $\mathcal{M}_H$ of finite-dimensional right modules over a finite-dimensional Hopf algebra H over any field k. (A tensor category is an abelian category with a tensor that is distributive over direct sums, and has a unit module, in this case k_ε. Following [13] a tensor category $\mathcal{D}$ is a *finite tensor category* over an algebraically closed field $\tilde{k}$ if every object has finite length, $\mathcal{D}$ has finitely many simple objects, and each simple X has projective cover $P(X)$. This is equivalent to $\mathcal{D} \cong \mathrm{Mod}{-}A$ for some finite-dimensional algebra A over $\tilde{k}$. If $\mathcal{C}$ has a fiber functor $F : \mathcal{D} \to \tilde{k}{-}\mathrm{Vect}$, a Tannakian reconstruction shows that $\mathcal{D} \cong \mathrm{Mod}{-}A$ where A is a finite-dimensional Hopf algebra.) Proposition 2.6 provides a clue that defining a depth of W in $\mathcal{M}_R$ is useful for obtaining an upper bound on depth $d(R, H)$ of a Hopf subalgebra $R \subseteq H$. Note the unit object $\mathbf{1} = k_\varepsilon$ in $\mathcal{M}_H$. We make use of the truncated tensor algebra of W, $T_n(W) := \mathbf{1} \otimes W \oplus (W \otimes W) \oplus \cdots \oplus W^{\otimes n}$, since $T_n(W)$ clearly divides $T_{n+1}(W)$ in the abelian category $\mathcal{C}$ for each $n \geq 0$.

Definition 3.1. Say that the object W has depth $n \geq 0$ in $\mathcal{C}$ if $T_{n+1}(W)$, equivalently $W^{\otimes(n+1)}$, divides a multiple of $T_n(W)$; briefly, $W^{\otimes(n+1)} \,|\, qT_n(W)$. Note that tensoring this by $- \otimes W$ shows that if W has depth n in $\mathcal{C}$, then it has depth $n+1$. Denote the minimum depth of W, if it exists, by $d(W, \mathcal{C})$. Write $d(W, \mathcal{C}) = \infty$ if W has no finite depth.

Note that for coalgebras and algebras in the category $\mathcal{C}$ we may simplify the definition of depth n object with the equivalent condition $W^{\otimes(n+1)} \,|\, qW^{\otimes n}$ by making use of Proposition 2.7. Next are a series of lemmas for computing depth of modules in finite tensor categories; we switch back to our point-of-view in the module category $\mathcal{M}_H$ of a finite-dimensional Hopf algebra H. The following lemma may be applied to either inclusions of Hopf subalgebras or epis to Hopf algebra quotients.

Lemma 3.2. *Given a Hopf algebra homomorphism $f : R \to H$, if depth $d(W, \mathcal{M}_H) \leq n$, then depth of its pullback or restriction along f, $d((W_f, \mathcal{M}_R) \leq n$.*

Proof. We first note that the functor $U \mapsto U_f$ is a tensor functor from $\mathcal{M}_H \to \mathcal{M}_R$, since $(U \otimes W)_f = U_f \otimes W_f$ as R-modules follows from the coalgebra morphism property of f, and $\mathbf{1}_f = 1_R$ from $\varepsilon_H \circ f = \varepsilon_R$. If $W^{\otimes(n+1)} \mid T_n(W)$ for integer $n \geq 0$, then ${W_f}^{\otimes(n+1)} \mid T_n(W_f)$ follows readily. □

For example, if R is an ad-stable Hopf subalgebra of H, and V its generalized permutation module, then $V = H/R^+H = \overline{H}$, the quotient Hopf algebra, which is a relative Hopf module over itself: these have depth $d(V, \mathcal{M}_{\overline{H}}) \leq 1$ as noted below in Corollary 3.9. Then by lemma applied to $H \to \overline{H}$, $d(V, \mathcal{M}_H) \leq 1$, then applied to $R \hookrightarrow H$, $d(V, \mathcal{M}_R) \leq 1$. On the other hand, $R^+H = HR^+$, so that $V = H/HR^+$ is a trivial right R-module with depth 0.

Lemma 3.3. *Given a module W in $\mathcal{M}_H$ let $\mathcal{P}_n(W)$ be the full set of pairwise nonisomorphic constituent indecomposables of $T_n(W)$. Then*

1. *$\mathcal{P}_n(W) \subseteq \mathcal{P}_{n+1}(W)$, with equality iff W has depth n in $\mathcal{C}$;*
2. *If $U \mid W$ in $\mathcal{C}$, then $d(W, \mathcal{C}) \leq n$ implies $d(U, \mathcal{C}) \leq n + |\mathcal{P}_n(W)|$.*

Proof. The inclusion follows from Krull-Schmidt applied to $T_n(W) \oplus W^{\otimes(n+1)} \cong T_{n+1}(W)$. The opposite inclusion $\mathcal{P}_n(W) \supseteq \mathcal{P}_{n+1}(W)$ follows in the same way from $T_{n+1}(W) \oplus * \cong qT_n(W)$ (indeed, equivalently).

If $U \mid W$, then using distributive law, one obtains $U^{\otimes n} \mid W^{\otimes n}$ for each $n \in \mathbb{N}$. It follows that $\mathcal{P}_n(U) \subseteq \mathcal{P}_n(W) = \mathcal{P}_{n+k}(W)$ for all $k \geq 0$. Then $\mathcal{P}_{n+s}(U) = \mathcal{P}_{n+s+1}(U)$ for $s = |\mathcal{P}_n(W)|$. □

Suppose $s = |\{$ isomorphism classes of simples in $\mathcal{M}_H\}| = |\{$ isomorphism classes of projective indecomposables in $\mathcal{M}_H\}| = |\{$ isomorphism classes of injective indecomposables in $\mathcal{M}_H\}|$ (via bijection $X \mapsto P(X)$) [12].

Proposition 3.4. *If W is a projective module in $\mathcal{M}_H$, then its depth $d(W, \mathcal{M}_H) \leq s$.*

Proof. The nonzero tensor powers of W are projective modules in $\mathcal{M}_H$ ([13, Prop. 2.1], also noted below for k not necessarily algebraically closed). Therefore the indecomposable summands of $W^{\otimes n}$ are projective indecomposables of which there are N different modules. Since $\mathcal{P}_n(W) \subseteq \mathcal{P}_{n+1}(W)$ for all $n > 0$, and these are bounded above by a finite set of cardinality $s + 1$, it follows that $\mathcal{P}_s(W) = \mathcal{P}_{s+1}(W)$, so W has depth s in $\mathcal{C}$. □

A Hopf algebra H has the Chevalley property, i.e., the tensor product of two simple H-modules is semisimple iff the radical ideal of H is a Hopf ideal [25]. If this is not the case, we must distinguish a proper subset, the maximal nilpotent Hopf ideal $J_\omega(H) \subset \operatorname{rad} H$ [8]. As a note of caution for the next proposition, consider the special case where k has characteristic p and H is a group algebra $k[G]$. It is noted in [8] that $J_\omega(H) = R^+H$ for the Hopf subalgebra $R = k[O_p(G)]$, where $O_p(G)$ is the largest normal p-subgroup of G, based on the result in [27] that (nilpotent) Hopf ideals in H are of the form $Hk[N]^+$ for normal (p-) subgroups $N \leq G$. Thus for this Hopf subalgebra pair $V \cong H/J_\omega(H)$ is the Hopf algebra $k[G/O_p(G)]$ a semisimple H-module iff $O_p(G)$ is the Sylow p-subgroup iff $k[G]$ has the Chevalley property.

Proposition 3.5. *Suppose the radical ideal J of a Hopf algebra H is a Hopf ideal (e.g., H is a pointed Hopf algebra* [26, 5.2.8]*). If W is a semisimple H-module, then $d(W, \mathcal{M}_H) \leq s$.*

Proof. If the radical is just a coideal, it is in fact a Hopf ideal, in which case the set of semisimple A-modules is closed under tensor product [27, cor. 8]. Then W and its powers are semisimple modules made up of s different simples. The rest of the proof proceeds as the proof of the previous proposition. □

The paper [27] proves that if an H-module W has annihilator ideal that contains no nonzero Hopf ideal of H, then the tensor H-module algebra $T(V)$ is faithful as an H-module. The paper [24] classifies pointed Hopf algebras H of finite corepresentation type over an algebraically closed field,; equivalently, classifies basic Hopf algebras H^* of finite representation type. The paper [1] classifies triangular Hopf algebras over $\mathbb{C}$ with the Chevalley property.

The generalized permutation module $V = H/R^+H$ is a semisimple R-module in the following three circumstances:

- if R is an ad-stable Hopf subalgebra (as noted above);
- if the radical ideal J of R is left ad-stable in H (for a short computation shows that $HJ \subseteq JH$, then $VJ = 0$);
- if $\operatorname{rad} R \subseteq \operatorname{rad} H$ and V_H semisimple.

Finally suppose W is a module in the finite tensor category $\mathcal{M}_H$ having only a finite number of indecomposables, say t of these; i.e., the Hopf algebra H has finite representation type. In this case we have the proposition below by an argument similar to the proofs in the propositions directly above.

Proposition 3.6. *If H has finite representation type, then a module W has depth $d(W, \mathcal{M}_H) \leq t$.*

Example 3.7. Let $H = \overline{U_q}(sl_2(\mathbb{C}))$ at the 4'th root of unity $q = i$, which is generated as an algebra by K, E, F where $K^2 = 1$, $E^2 = 0 = F^2$, $EF = FE$, $KE = -EK$, and $KF = -KF$. This is an 8 dimensional algebra with coalgebra structure given by $\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes 1 + K \otimes E$ and $\Delta(F) = 1 \otimes F + F \otimes K$, from which the rest of the Hopf algebra structure follows. Let R be the Hopf subalgebra of dimension 4 generated by K, F (isomorphic to the Sweedler algebra considered below in Example 4.3). Note that neither R nor $J := \mathrm{rad}\, R$ are ad-stable, and that H is a basic algebra, self-dual and pointed. We denote the simples from $H \to H/J \cong \mathbb{C}^2$ by S_1, S_2.

Consider $V := H/R^+H$ as an R-module, which is spanned by $\overline{1}$ and $\overline{E}$, where $\overline{1}K = \overline{1}$, $\overline{E}K = -\overline{E}$, and F annihilates. From Theorem 2.5, one notes that V_R is not projective, also obtainable from Doi's observation: the module coalgebra V is projective iff there is a right R-module map $\psi : C \to R$ such that $\varepsilon_V \psi = \varepsilon_R$ [11]. Supposing $\psi(\overline{1}) = a1 + bK + cF + dFK$, we conclude from $\psi(\overline{1})K = \psi(\overline{1})$ that $a = b$, $c = d$, from $\psi(\overline{1})F = 0$ that $a = 0 = b$ and from $\varepsilon(\psi(\overline{1})) = 1$ that $2a = 1$, a contradiction.

The projective cover is eH where $e = \frac{1+K}{2}$ and $\pi : eH \to V$ has kernel C spanned by eF and eEF. Note that V_H is not semisimple, since $C \neq e\mathrm{Rad}\, H$. However, V_R is a semisimple module, since the radical ideal J in R (spanned by F and FK in R^+) satisfies $HJ = JH$. The number of R-simples is $N = 2$.

The injective hull of $V \cong t_R H$ where $t_R = F(1+K)$ is $W = (\frac{1-K}{2})H$ (so the idempotent $f = \frac{1-K}{2}$ in this application of Prop. 2.8). This is because $EF(1-K) \in t_R H$ spans a space that nontrivially intersects nonzero submodules of W.

The composition series of eH and fH have length 4 and are non-unique, the radical length is 3: $eA \supset eJ \supset eJ^2 \supset \{0\}$. The Cartan map $K_0(H) \to G_0(H)$ is given by $[xH] \mapsto 2[S_1] + 2[S_2]$ for both $x = e, f$. By [24, 3.1], a Hopf algebra that is basic has finite representation type (f.r.t.) if and only if it is a Nakayama algebra (i.e. each projective indecomposable has a unique composition series [2]). It follows that H does not have f.r.t.

3.1. **Relative Hopf modules.**

Next we show that finite-dimensional relative Hopf restricted modules have depth 1. Again let $R \subseteq H$ be a Hopf subalgebra pair. Recall that a vector space N is a right (H, R)-Hopf module if N is a right R-module, and N is a right H-comodule such that $(nr)_{(0)} \otimes (nr)_{(1)} = n_{(0)} r_{(1)} \otimes n_{(1)} r_{(2)}$ for all $r \in R, n \in N$; if N is moreover the restriction of an H-module, we refer to it as a relative

Hopf restricted module. The next proposition extends [26, Lemma 3.1.4] for our purposes.

Proposition 3.8. *Given a relative Hopf module N and right H-module M, the following is an isomorphism of right R-modules:*

$$N_{\cdot} \otimes M \cong N_{\cdot} \otimes M_{\cdot} \tag{3}$$

where the right-side is the tensor product in $\mathcal{M}_R$.

Proof. The forward mapping is given by $n \otimes m \mapsto n_{(0)} \otimes m n_{(1)}$. The inverse mapping is given by $n' \otimes m' \mapsto n'_{(0)} \otimes m' \overline{S}(n'_{(1)})$, where $\overline{S}$ is the composition-inverse of the antipode. □

Since H and H^n, for each free module over a Hopf algebra H, are Hopf modules and relative Hopf modules (with respect to any Hopf subalgebra), it follows from the Proposition that $P \otimes M$ is projective in $\mathcal{M}_H$ for any projective $P \in \mathcal{M}_H$ for any ground field k.

Corollary 3.9. *Suppose N is a right (H, R)-Hopf restricted module. Then depth $d(N, \mathcal{M}_R) \leq 1$.*

Proof. Let $M = N$ in the proposition, so that $N \otimes N \cong (\dim N)N$ in additive notation, a depth 1 condition in Mod-R. □

If one asks when the module V, formed from a Hopf subalgebra $R \subseteq H$, is a relative Hopf restricted module, a necessary condition for this is that V_R is free, since relative Hopf modules are free by a result of Nichols-Zoeller [26]. So a necessary condition for V to be relative Hopf module is that $\dim H = (\dim R)^2 r$ for some integer $r \geq 1$, and that R is semisimple by Proposition 2.5. If V is a relative Hopf module, it follows from Theorem 4.1 below that the depth $d(R, H) < 5$. The next proposition provides a sufficient condition for V to be a relative Hopf module; the proof is straightforward and left to the reader.

Proposition 3.10. *Given $R \subseteq H$ a finite Hopf subalgebra pair, suppose there is a module morphism $\phi : V_R \to H_R$ of modules that is simultaneously a coalgebra morphism. Then $\rho := (V \otimes \phi) \circ \Delta_V$ defines a right H-comodule structure on V making V an (H, R)-Hopf module.*

It follows from the injectivity of ρ and Δ_V that ϕ must be injective.

Remark 3.11. It would be interesting to pursue the depth of the left H- or R-module algebra V^* where $V = H/R^+H$ is a right H-module coalgebra studied briefly in Section 2. As noted there, V^* has a left H-module algebra structure; there is also an augmentation $V^* \to k$ defined by $v^* \mapsto v^*(\overline{1_H})$. Since one shows that $(V^{\otimes n})^* \cong (V^*)^{\otimes n}$ as left H-modules, it follows that the depths are equal, $d(V, \mathcal{M}_H) =$

$d(V^*, {}_H\mathcal{M})$ (via usual duality for finite-dimensional algebra). It is also clear that $V^* \hookrightarrow H^*$ as the dual of the epi $H \stackrel{\pi}{\to} V$, so that the smash product $V^*\#H \hookrightarrow H^*\#H \cong M_{\dim H}(k)$, embeds in the Heisenberg double of H [26, Ch. 9].

4. Upper bounds for depth

Again suppose R is a Hopf subalgebra of a Hopf algebra H with V denoting the right R-module H/R^+H.

Theorem 4.1. *The depths of a Hopf subalgebra and its module V are related by $d(R,H) \leq 2d(V, \mathcal{M}_R)+2$. If H is a semisimple Hopf algebra, then $d_h(R,H) \leq 2d(V, \mathcal{M}_H) + 1$.*

Proof. Given a bimodule ${}_HM_R$ let $\hat{T}_n(M) := M \oplus \cdots \oplus M^{\otimes_R n}$ denote the truncated tensor algebra from degree 1 up to n. Suppose $d(V, \mathcal{M}_R) = n$. Then $T_{n+1}(V) \,|\, qT_n(V)$ as right R-modules for some $q \in \mathbb{N}$. It follows from tensoring by ${}_HH_{.} \otimes -$ and Prop. 2.6 that $\hat{T}_{n+2}({}_HH_R) \,|\, q\hat{T}_{n+1}({}_HH_R)$. Note then that $H^{\otimes_R (n+2)} \,|\, q\hat{T}_{n+1}({}_HH_R)$. But $H \,|\, H^{\otimes_R 2} \,|\, \cdots \,|\, H^{\otimes_R (n+1)}$, so that $H^{\otimes_R (n+2)} \,|\, q(n+1)H^{\otimes_R (n+1)}$ as H-R-bimodules; this is the right depth $2n+2$ condition.

If H is semisimple, it is a separable algebra, whence $H \,|\, H^{\otimes 2}$ as H-H-bimodules. Suppose $d(V, \mathcal{M}_H) = n$. Then arguing as in the first paragraph, one arrives at $H^{\otimes_R (n+2)} \,|\, q(n+1)H^{\otimes_R (n+1)}$ as H-H-bimodules, which is the h-depth $2n+1$ condition. □

For example, if R is ad-stable in a finite-dimensional H, then V has $d(V, \mathcal{M}_R) = 0$, whence $d(R,H) \leq 2$ [20]. Another example: if $\mathrm{rad}\, R$ is ad-stable in a pointed Hopf algebra H, then $d(R,H) < \infty$, since V is a semisimple R-module (see Section 3).

Example 4.2. The 4- and 8-dimensional Hopf subalgebra pair in Example 3.7 has $d(R,H) \leq 6$ since $d(V, \text{Mod-}R) \leq 2$.

Example 4.3. Consider the Taft Hopf algebras $H = H_n = \langle g, x \mid g^n = 1, x^n = 0, gxg^{-1} = qx\rangle$ where q is a primitive n'th root of unity in the ground field k. These are Hopf algebras, where g, x, is a grouplike, skew primitive element, of dimension n^2 (for each n the algebras are examples of basic algebras, Nakayama algebras, pointed Hopf algebras, algebras having finite representation and corepresentation type, as well as monomial algebras [24]). Consider the cyclic group subalgebra $R \cong k\mathbb{Z}_n$ affinely generated by g in H. The category $\text{Mod} - R$ is semisimple with n simples; thus the theorem computes the depth $d(R,H) \leq 2n+2$ by an application of Prop. 3.5. We improve this by computing V and its depth. Since $V = H/R^+H$ is n-dimensional, and R^+ is spanned by

$1-g^j$ for $j = 1,\dots,n-1$, V is spanned by $\overline{x^i}$ where $i = 0,1,\dots,n-1$. The R-module action on V is given by $\overline{1}g^i = \overline{1}$, $\overline{x^j}g^i = q^{-ij}\overline{x^j}$. (Thus V is a free R-module via $V_R \to R_R$, $\overline{x^{n-i}} \mapsto e_i$ where $e_i = \frac{1}{n}\sum_{j=0}^{n-1}(q^{-i}g)^j$.) We compute that $V \otimes V \cong nV$, since

$$\overline{x^j} \otimes \overline{x^k} \mapsto (0,\dots,\overline{x^{[j+k]}},0,\dots,0)$$

where the nonzero term occurs in the $k+1$'st coordinate and $[j+k]$ is congruent to $j+k \pmod n$ and in the interval $0 \leq [j+k] < n$: this follows from noting $(\overline{x^j}\otimes\overline{x^k})g^i = q^{-i(j+k)}\overline{x^j}\otimes\overline{x^k}$. For example, if $n=2$

$$\overline{1}\otimes\overline{1} \mapsto (\overline{1},0) \qquad \overline{1}\otimes\overline{x} \mapsto (0,\overline{x})$$
$$\overline{x}\otimes\overline{1} \mapsto (\overline{x},0) \qquad \overline{x}\otimes\overline{x} \mapsto (0,\overline{1})$$

It follows that $T_2(V) \,|\, (n+1)T_1(V)$, thus $d(V,\mathcal{M}_R) = 1$. Then by theorem $d(R,H) \leq 4$. It is computed by other means in [29] that $d(R,H) = 3$. (As an H-module $V \cong e_0H$ where $e_0 = \frac{1+g+\cdots+g^{n-1}}{n}$ is projective (indecomposable) but not semisimple; cf. Prop. 2.5.)

Again let $R \subseteq H$ denote a Hopf subalgebra. The ground field k is of arbitrary characteristic.

Corollary 4.4. *If either R or H has finite representation type (e.g., either is semisimple or Nakayama), then depth $d(R,H)$ is finite.*

Proof. If R has f.r.t. then the right R-module $V = H/R^+H$ has finite depth by an application of Proposition 3.6. Then $d(R,H) < \infty$ from the theorem above. If H has f.r.t. with n nonisomorphic indecomposables in all, then the right H-module V has depth $d(V,\mathcal{M}_H) = n$, and its restriction along $R \hookrightarrow H$ has depth $d(V,\mathcal{M}_R) \leq n$ by Lemma 3.2. Thus $d(R,H) \leq 2n+2$ by theorem. □

An example of a Hopf subalgebra $R \subseteq H$ where neither R nor H has f.r.t. and the generalized permutation module V_H is neither semisimple nor projective is $\operatorname{char} k = p$, $\mathcal{G}$ a finite group with noncyclic $O_p(\mathcal{G}) \neq$ the Sylow p-subgroup of $\mathcal{G}$, and $R = k[O_p(\mathcal{G})] \subset H = k[\mathcal{G}]$; see the discussion above Prop. 3.5, Prop. 2.5 and [28]. However, this example has finite depth for another (Burnside-Mackey-theoretic) reason, as we see next.

Suppose a finite set of Hopf subalgebras $R_i \subseteq H$ have induced modules $V_i = H/R_i^+H$ for $i = 1,\dots,n$, where the modules V_i enjoy a tensor product theorem, or Burnside ring formula; then each depth $d(R_i,H)$ is finite, as we record below, noting the classical case where H is a group algebra.

Corollary 4.5. *Suppose* $V_i \otimes V_j \cong \sum_{k=1}^n \oplus\, a_{ij}^k V_k$ *for* n^3 *nonnegative integers* a_{ij}^k. *Then* $d(R_i, H) \leq 2n+2$ *for each Hopf subalgebra* $R_i \subseteq H$. *In particular, finite group algebra extensions have finite depth* [3].

Proof. Since each V_i and its tensor powers $V_i^{\otimes n}$ are direct sums of multiples (of the form $a_{ii}^k a_{ki}^r a_{ri}^s \ldots$) of $\mathcal{V} := \{V_1, \ldots, V_n\}$, the chain of $\mathcal{V}$-constituents

$$\cdots \subseteq \{V_j \in \mathcal{V} \,:\, V_j \,|\, T_r(V_i)\} \subseteq \{V_j \in \mathcal{V} \,:\, V_j \,|\, T_{r+1}(V_i)\} \subseteq \cdots$$

stops strictly increasing after at most n steps $r = 1, \ldots, n$. Thus $T_{n+1}(V_i) \,|\, qT_n(V_i)$ for some large enough q. Apply the theorem to conclude that $d(R_i, H) \leq 2n + 2$.

Let $\mathcal{H}_1, \mathcal{H}_2 < \mathcal{G}$ be two subgroups of a finite group. Consider the group algebras $k[\mathcal{H}_i] \subseteq k[\mathcal{G}]$ a Hopf subalgebra pair where the permutation module $V_i = \mathrm{Ind}_{\mathcal{H}_i}^{\mathcal{G}} k := (k|_{\mathcal{H}_i})^{\mathcal{G}}$ has dimension $|\mathcal{G} : \mathcal{H}_i|$ $(i = 1, 2)$. The tensor product theorem (valid as well for a commutative ground ring) [9, 10.18] applied to V_i gives

$$V_1 \otimes V_2 \cong \sum_{x^{-1}y \in D} \oplus (k|_{{}^x\mathcal{H}_1 \cap {}^y\mathcal{H}_2})^{\mathcal{G}} \tag{4}$$

where D is a set of representatives of $(\mathcal{H}_1, \mathcal{H}_2)$-double cosets in $\mathcal{G}$ and ${}^y\mathcal{H}_i$ denotes the conjugate subgroup $y\mathcal{H}_i y^{-1}$. Thus given a subgroup $\mathcal{H}$ and its associated permutation module V, let I be the finite index set of conjugate subgroups of $\mathcal{H}$ and their intersecting subgroups, i.e. $I = \{{}^x\mathcal{H} \cap \cdots \cap {}^z\mathcal{H} \,:\, x, \ldots, z \in \mathcal{G}\}$, $n = |I|$ and V_i the permutation modules associated with each these, beginning with $V_1 = V$. Then the formula (4) for $V^{\otimes 2}$ and its extension to $V_i \otimes V_j$ are indeed formulas as in the hypothesis of the corollary. Thus, $d(V, \mathrm{Mod}{-}k[\mathcal{G}]) \leq n$, then apply Lemma 3.2 and Theorem 4.1. □

For example, if $\mathcal{H} \lhd \mathcal{G}$, then $|I| = 1$; if $\mathcal{G}$ is a Frobenius group and $\mathcal{H}$ is the complementary subgroup, then $|I| = 2$. Based on [7, Section 6], one might show the depth of $V \leq$ the minimum number of conjugates of $\mathcal{H}$ intersecting in the core of $\mathcal{H}$.

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