ABSENCE OF SPECIFICATION FOR PARTIALLY HYPERBOLIC FLOWS

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Abstract. In this article we prove that if a flow exhibits a partially hyperbolic attractor Λ with splitting $T_\Lambda M = E^s \oplus E^c$ and two periodic saddles with different indices and the stable index of one of them coincides with the dimension of E^s then it does not satisfy the specification property. As an application, we prove that all Lorenz attractors do not satisfy the specification property.

1. Introduction and statement of the main results

Throughout, let M be a closed manifold with $\dim M \geq 3$, where $\dim E$ denotes the dimension of E, and let $\mathfrak{X}^1(M)$ be the space of C^1-vector fields of a closed C^∞ manifold M endowed with the C^1-topology. Given $X \in \mathfrak{X}^1(M)$ a C^1-vector field it generates a flow $(X_t)_{t\in\mathbb{R}}$ on M. We say that a compact X_t-invariant subset $\Lambda \subset M$ is a *hyperbolic set* for $(X_t)_t$ if there exists a dominated decomposition $T_\Lambda M = E^s \oplus E^c \oplus E^u$ such that E^c_x is the one dimensional subspace generated by $X(x)$ and there are constants $C > 0$ and $\lambda \in (0, 1)$ so that $\|DX_t(x) \mid E^s_x\| \leq C\lambda^t$ and $\|(DX_t(x) \mid E^u_x)^{-1}\| \leq C\lambda^t$ for every $t \geq 0$ and $x \in \Lambda$. If $\Lambda = M$ we say that $(X_t)_t$ is an *Anosov flow*. We say that a X_t-invariant compact set Λ is *sectional-hyperbolic* if every singularity in Λ is hyperbolic and there exist a continuous non-trivial invariant splitting $T_\Lambda M = E^s \oplus E^c$ over Λ and constants $C > 0$ and $\lambda \in (0, 1)$ such that for every $x \in \Lambda$ and $t \geq 0$

(i) $\|DX_t \mid E^s_x\| \, \|DX_{-t} \mid E^c_{X_t(x)}\| < C\lambda^t$;
(ii) $\|DX_t(x) \mid E^s_x\| \leq C\lambda^t$;
(iii) $|\det(DX_t(x) \mid_{L_x})| > C\lambda^t$ for every plane $L_x \subset F_x$.

More generally, an invariant splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ is dominated if for any $1 \leq l \leq k-1$, $(E_1 \oplus \cdots \oplus E_l) \oplus (E_{l+1} \oplus \cdots \oplus E_k)$ is dominated, that is, satisfies condition (i) above. Any subbundle F satisfying condition (iii) above, expressing that the flow expands volume in all two-planes contained in F, is said to be sectionally expanding. Moreover, it is well known that the sub-bundle E^s is uniquely integrable and hence there is a foliation $\mathcal{F}^{ss}$ which is tangent to E^s (see e.g. [11]). We refer to $\mathcal{F}^{ss}$ as the strong stable foliation. Let us also mention that sectional hyperbolic flows in three-dimensional manifolds coincide with the notion of singular-hyperbolicity for flows that arised in the characterization of robustly transitive attractors in dimension three. We observe that if a sectional hyperbolic flow does not have singularities then it is necessarily hyperbolic (see e.g. [15] for

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more details). Finally, an orbit of a point x by the flow is called a *critical element* if it is either periodic or x it is constant (that is, x a singularity for the vector field).

The specification property for maps was introduced by Bowen in [7, 6] and roughly means that an arbitrary number of pieces of orbits can be "glued" to obtain a real orbit that shadows the previous ones. Dynamical systems satisfying the specification property are intensively studied from an ergodic viewpoint [5, 21, 26, 18] and an algebraic viewpoint [1, 13]. We shall recall this notion for flows.

A X_t-invariant compact set Λ is an *attractor* if it is transitive and admits a compact neighborhood U such that $\Lambda = \bigcap_{t\in\mathbb{R}_0^+} X_t(U)$. We say that an invariant compact subset $\Lambda \subset M$ has the *specification property* if for any $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that the following property holds: given any finite colection of intervals $I_i = [a_i, b_i]$ $(i = 1 \dots m)$ of the real line satisfying $a_{i+1} - b_i \geq T(\epsilon)$ for every i and every map $P : \bigcup_{I_i\in\tau} I_i \to \Lambda$ such that $X_{t_2}(P(t_1)) = X_{t_1}(P(t_2))$ for any $t_1, t_2 \in I_i$ there exists $x \in \Lambda$ so that $d(X_t(x), P(t)) < \epsilon$ for all $t \in \bigcup_i I_i$. When the previous shadowing property is required only to specifications made by two pieces of orbits ($m = 2$ above) we shall refer to this as the *weak specification property.* These properties clearly implies the flow to be transitive on Λ and not to admit sources nor sinks.

Recently, several authors studied the specification property from a viewpoint of geometric theory of dynamical systems. In [22], Sakai and the first and third authors proved that the C^1-interior of the set of all diffeomorphisms satisfying the specification property coincides with the set of all transitive Anosov diffeomorphisms. Moriyasu, Sakai and the third author extended the above results to regular maps, and proved that C^1-generically, regular maps satisfy the specification property if and only if they are transitive Anosov ([17]). Owing to these results, the relation to hyperbolicity turns out to be clear. Following the ideas in [22], Arbieto, Senos and Todero [3] proved that any isolated set for a flow $(X_t)_t$ that satisfies the (weak) specification property robustly is a topologically mixing hyperbolic set and, consequently, if X is a vector field which has the weak specification property robustly then it generates a topologically mixing Anosov flow. In particular the characterization was to prove first that robust specification would lead to sectional-hyperbolicity and then, by robustness, to use perturbative techniques and rule out singularities. Hence, a natural question left in [3] was to characterize the set of Lorenz attractors that do satisfy the specification property. Recall that Lorenz attractors do not satisfy the shadowing property with rare exceptions (c.f. [12]). Here we provide an answer to this question.

Theorem A. *Every transitive sectional-hyperbolic attractor is either hyperbolic or does not satisfy the weak specification property.*

Observe that, by the definition of sectional hyperbolicity, all singularities are hyperbolic and all periodic orbits have stable index equal to $\dim E^s$. Recently, periodic orbits for sectional-hyperbolic attractors were constructed by Lopez [14], and in [2, Proposition 10] Arbieto and Morales showed that the stable indices of sigularities for every nontrivial transitive sectional-hyperbolic set are equal to $\dim E^s + 1$. Moreover, every sectional-hyperbolic flow without singularities is actually hyperbolic. Hence, Theorem A is actually a consequence of the more general result:

Theorem B. *Let $X \in \mathfrak{X}^1(M)$ be a vector field and let Λ be an attractor so that the flow $(X_t)_{t\in\mathbb{R}}$ admits a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c$. Assume there* *are two hyperbolic critical elements p and q such that* $\dim\, E^s = \dim\, W^{ss}(p) < \dim\, W^{ss}(q)$. *Then $X \mid_\Lambda$ does not satisfy the weak specification property.*

In the case of three-dimensional manifolds the singularity of the Lorenz attractors have stable index larger than the one of the strong stable direction. Hence, we obtain the following immediate consequence:

Corollary 1. *Assume that $\dim M = 3$ and that $(X_t)_t$ is a flow that admits a geometric Lorenz attractor Λ. Then $(X_t)_t$ does not satisfy the weak specification property on Λ.*

We notice that if $\dim M = 3$ then every C^1-robustly transitive set with singularities Λ is a singular-hyperbolic set up to flow-reversing [16] and consequently, $X_t \mid_\Lambda$ does not satisfy the weak specification property. Observe that the previous theorem also applies for partially hyperbolic sets Λ with a decomposition $E^u \oplus E^c$ just by considering the vector field $-X$. Moreover, even in the case of an Anosov flow $(X_t)_t$ the time-1 map $f = X_1 : M \to M$ of an Anosov flow is a strongly partially hyperbolic diffeomorphism that admits no hyperbolic periodic points. In particular an analogous theorem as the previous one for flows does not follow from the ones obtained for partially hyperbolic diffeomorphisms in [24]. Nevertheless some corollaries of the main result in [24] for strongly partially hyperbolic diffeomorphisms on three-manifolds can be expected to hold for strongly partially hyperbolic flows on four-manifolds due to the neutral direction of the vector field. We shall discuss now such extensions.

We say that a flow is strongly partially hyperbolic with d-dimensional central direction ($d \geq 1$) if it admits a decomposition $TM = E^s \oplus E^c \oplus E^u$ with d-dimensional central direction. Denote by $\mathcal{SPHF}_d(M)$ the set of such flows and note that it is an open subset of $\mathfrak{X}^1(M)$. We say a flow $(X_t)_t$ generated by a vector field X is *robustly transitive* if all flows generated by vector fields in a C^1-open neighborhood of X are transitive, that is, have a dense orbit. If the vector field X has an attractor $\Lambda_X := \bigcap_{t\geq 0} X_t(U)$ we say that Λ is a *robustly transitive attractor* if for any vector field Y in a C^1-open neighborhood of X the attractor $\Lambda_Y := \bigcap_{t\geq 0} Y_t(U)$ is transitive. Finally, we denote by $\mathcal{RNTF}$ the set of robustly non-hyperbolic transitive flows (that is, flows generated by vector fields X so that every C^1-vector field Y in a C^1-neighborhood of X generates a non-hyperbolic and transitive flow) endowed with the C^1-topology in the space of vector fields.

Given a vector field $X \in \mathfrak{X}^1(M)$ a point $p \in M$ is a singularity if $X(p) = 0$, otherwise it is referred as a regular point. A point $p \in M$ is *periodic* if there exists a minimum period $T > 0$ so that $X_T(p) = p$ and we say that p is a *periodic hyperbolic point* if the orbit $\mathcal{O}(p) = \cup_{t\in[0,T]} X_t(p)$ is a hyperbolic set for X. If this is the case, the strong-stable set

$$W^{ss}(p) = \left\{ x \in M : \lim_{t\to+\infty} d(X_t(x), X_t(p)) = 0 \right\}$$

is indeed a C^1-submanifold tangent to E^s.Let d^{ss} be the distance in $W^{ss}(p)$ induced in the Riemannian metric. The *local stable manifold* at p is defined by $W^s_\varepsilon(p) = \bigcup_{|t|\leq\varepsilon} X_t(W^{ss}_\varepsilon(p))$ where

$$W^{ss}_\varepsilon(p) = \{x \in W^{ss}(p) : d^{ss}(x,p) \leq \varepsilon\}$$

for $\varepsilon > 0$. Local strong-unstable and unstable manifolds are defined analogously. Moreover, observe that for $\varepsilon > 0$ there exists $\varepsilon_0 > 0$ such that

$$\bigcap_{T \geq 0} B_T(p, \varepsilon_0) \subset W^s_\varepsilon(p)$$

where $B_T(p, \varepsilon_0) = \{x \in M \colon d(X_t(x), X_t(p)) \leq \varepsilon_0, \forall 0 \leq t \leq T\}$ and consequently, $W^s_\varepsilon(p)$ contains the intersection of dynamical balls computed only for future iterates (see Lemma 2.1). In the case that the central direction E^c is two dimensional, any two hyperbolic periodic points with different indices verify the assumptions of Theorem B. Thus we obtain the following direct consequence.

Corollary 2. *Let $X \in \mathcal{SPHF}_2(M)$ and suppose that there exist two hyperbolic critical elements with different indices. Then X does not satisfy the weak specification property.*

Proof. Let p, q be hyperbolic critical elements so that $ind^s(p) \neq ind^s(q)$. Observe that due to transitivity p and q are not attractors nor repellers. We shall prove that either X or $-X$ satisfies the conditions of Theorem B and, consequently, X does not satisfy the weak specification property. For simplicity, we assume that $\dim M = 4$.

(i) If p, q are both periodic points then necessarily $ind^s(p) \in \{1, 3\}$ or $ind^s(q) \in \{1, 3\}$. Assume without loss of generality that $ind^s(p) \in \{1, 3\}$. If $ind^s(p) = 1$ then the vector field X satisfies the conditions of Theorem B. If $ind^s(p) = 3$ then $-X$ satisfies the conditions of Theorem B.

(ii) If p, q are both singularities then $ind^s(p)$ and $ind^s(q)$ cannot be simultaneously 2. The argument is completely analogous to the previous case.

(iii) If p is a periodic point and q is a singularity. If $ind^s(p) = 1 = \dim E^s$, since $ind^s(q) \neq 1$ then X satisfies the assumptions of Theorem B. If $ind^s(p) = 2$ then $ind^u(p) = 1$ and just consider $-X$. This completes the proof of the corollary. □

Since $X(x)$ is in the central direction for a nonsingular partially hyperbolic flow $(X_t)_t$, we can obtain the following corollary in a similar way as above.

Corollary 3. *Let $X \in \mathcal{SPHF}_3(M)$. If X is nonsingular and if there exist two hyperbolic critical elements with different indices, then X does not satisfy the weak specification property.*

Using C^1-perturbative techniques it follows that hyperbolic flows coincide with the class star-flows $\mathcal{G}^1(M)$ (i.e. flows such that all critical elements are hyperbolic C^1-robustly) (see e.g. [3] for a more precise description). We deduce that most robustly non-hyperbolic and transitive partially hyperbolic flows with two dimensional central direction do not have the specification property.

Corollary 4. *There is a C^1-open and dense subset O in $\mathcal{RNTF} \cap \mathcal{SPHF}_3(M)$, such that every $X \in O$ does not satisfy the weak specification property.*

Proof. Put $\mathcal{U} = \mathcal{RNTF} \cap \mathcal{SPHF}_3(M)$. Since $X \in \mathcal{U}$ is robustly transitive, X has no singularity (see [27]). We note that $\mathcal{U} \cap \mathcal{G}^1(M) = \emptyset$. Indeed, we assume by contradiction that there exists $X \in \mathcal{U} \cap \mathcal{G}^1(M)$. In [10] Gan and Wen showed that if $X \in \mathcal{G}^1(M)$ has no singularity, then the nonwandering set of X is hyperbolic, which means that X is Anosov. This contradicts the fact that X is not hyperbolic.

Let $X \in \mathcal{U}$. Since $X \notin \mathcal{G}^1(M)$, X can be approximated by a flow $Y \in \mathcal{U}$ having a non-hyperbolic periodic orbit. By the proof of Theorem 4.3 in [3], we can find

$Z \in \mathcal{U}$ arbitrarily close to Y and having two hyperbolic periodic orbits with different indices, which is a C^1-open condition. Thus Corollary 3 implies that Z does not satisfy the weak specification property C^1-robustly. □

We can expect to extend the previous result by removing the partial hyperbolicity assumption in a lower dimensional setting. In the case that $\dim M = 3$, Doering [9] proved that every C^1-robustly transitive flow on a three-dimensional manifolds is Anosov and consequently satisfies the specification property. If $\dim M = 4$ we can remove the assumption of partial hyperbolicity from the previous corollary.

Corollary 5. *Suppose that* $\dim M = 4$. *Then there is a* C^1*-open and dense subset* O *in* $\mathcal{RNTF}$ *so that every* $X \in O$ *does not satisfy the weak specification property.*

Proof. Following [27], given $X \in \mathcal{RNTF}$ it follows that X has no singularity and the linear Poincaré flow $P^t = \pi_{\mathcal{N}_{X_t(x)}} \circ DX_t(x) : \mathcal{N}_x \to \mathcal{N}_{X_t(x)}$ admits a dominated splitting: for every $x \in M$ there exists a DP^t-invariant and continuous decomposition of the normal space $\mathcal{N}_x = E_x \oplus F_x$ and constants $C > 0$ and $0 < \lambda < 1$ so that

$$\|DP^t E_x\| \, \|(DP^t F_{X_t(x)})^{-1}\| \leq C\lambda^t$$

for every $t \geq 0$.

We now proceed to prove that the one-dimensional subbunddle is hyperbolic. Assume for simplicity that $\dim E = 1$ and $\dim F = 2$. Since a robustly transitive flow does not present repelling periodic orbits we claim that there exists exists $\delta > 0$ such that $|\lambda_E(p)| \leq (1-\delta)^T < 1$ for every periodic point p of period T, where $\lambda_E(p)$ denotes the eigenvalues of $DP^T(p) \mid_{E_p}$ (since otherwise one could use the Frank's lemma for flows as in the proof of [8, Lemma 4.5] to create a repelling periodic orbit). The proof that E is uniformly contracting follows by the well known strategy in the proof of the stability conjecture using the ergodic closing lemma proved by Wen (c.f. Step 3 in [28, page 347]).

We put $E^c_x = \langle X(x) \rangle \oplus F_x (\subset T_x M)$ for $x \in M$. Here $\langle X(x) \rangle$ denotes the one dimensional subspace generated by $X(x)$. Then E^c is a DX_t-invariant subbundle. Since E is uniformly contracting, as in the proof of [23, Theorem 1.5], we can define a DX_t-invariant continuous one-dimentional subbundle $E^s \subset \langle X \rangle \oplus E$ such that the splitting $E^s \oplus E^c$ is partially hyperbolic.

By [10, Theorem A] every non-singular star-flow is Axiom A without cycles. Since X generates a non-hyperbolic robustly transitive flow without singularities then $X \notin \mathcal{G}^1(M)$ and, consequently, X can be C^1-approximated by a flow $Z \in \mathcal{U}$ arbitrarily close to X two hyperbolic periodic orbits with different indices, which is a C^1-open condition. This finishes the proof of the corollary. □

In conclusion, together with the results by Komuro [12] we obtain that the Lorenz attractors do not satisfy both the specification and shadowing properties. On the other hand, several authors considered more recently either measure theoretical non-uniform specification properties (see e.g. [18, 26]) or almost specification properties (see e.g. [20, 25]) to the study of the ergodic properties of a given dynamical system. One remaining interesting question is to understand which partially hyperbolic maps admit weaker specification properties. A global picture that includes the characterization of dynamical systems satisfying these weak kinds of specification is still incomplete.

2. Preliminaries

In this section we provide necessary definitions and prove some auxiliary results used in the proofs of the main results.

Lemma 2.1. *For every hyperbolic periodic point p and $\varepsilon > 0$, we can choose $\epsilon_0 \in (0, \epsilon)$ such that for $x \in M$, if $d(X_t(x), X_t(p)) \leq \epsilon_0$ for every $t \geq 0$ then*

$$x \in W^s_\varepsilon(p) = \bigcup_{|t|\leq\varepsilon} X_t(W^{ss}_\varepsilon(p)).$$

Proof. Let $T_\Gamma M = E^s \oplus E^c \oplus E^u$ be the decomposition as in the definition of hyperbolicity where Γ is the orbit of p. We fix a continuous extension $\tilde{E}^s \oplus \tilde{E}^c \oplus \tilde{E}^u$ of $E^s \oplus E^c \oplus E^u$ to some small neighborhood U of Γ and define $C^u_\kappa(x) = \{v = v_1 + v_2 \in (\tilde{E}^s_x \oplus \tilde{E}^c_x) \oplus \tilde{E}^u_x : \|v_1\| \leq \kappa\|v_2\|\}$ ($x \in U$, $\kappa > 0$). By the definition of hyperbolicity, there are $\kappa > 0$, $0 < \lambda < 1$ and $T > 0$ with $X_T(p) = p$ such that if $X_s(x) \in U$ for $0 \leq s \leq T$, then

$$\begin{aligned} D_x X_T(C^u_\kappa(x)) &\subset C^u_\kappa(X_T(x)), \\ \|D_x X_T(v)\| &\geq \lambda^{-1}\|v\| \ (v \in C^u_\kappa(x)). \end{aligned} \tag{2.1}$$

Increasing T if necessary we may assume that $X_T(W^s_\varepsilon(p)) \subset W^s_\varepsilon(p)$. Choose $\delta_0 > 0$ (depending on T) such that if $d(x, p) \leq \delta_0$, then $X_t(x) \in U$ for $0 \leq t \leq T$. Since $W^{ss}_\varepsilon(p)$ is a C^1 disk with $T_p W^{ss}_\varepsilon(p) = E^s_p$, we have that $W^s_\varepsilon(p)$ is a C^1 disk with $T_p W^s_\varepsilon(p) = E^s_p \oplus E^c_p$. So we can take $0 < \varepsilon_0 < \theta(< \delta_0/2)$ such that if $d(x, p) \leq \varepsilon_0$, then the following hold:

(1) There is a C^1 disk $D \subset U$ centered at x of radius θ such that

$$\dim D = \dim E^u_p \text{ and } T_y D \subset C^u_\kappa(y) \ (y \in D). \tag{2.2}$$

(2) Any disk centered at x of radius θ satisfying (2.2) intersects $W^s_\varepsilon(p)$ at a unique point transversely. (Such an intersection y satisfies

$$d(y, p) \leq d(y, x) + d(x, p) \leq \theta + \varepsilon_0 < \delta_0.) \tag{2.3}$$

Assume that $x \in M$ satisfies $d(X_t(x), X_t(p)) \leq \epsilon_0$ for $t \geq 0$. Let D_0 be a C^1 disk centered at x of radius θ satisfying (2.2) and y be an intersection between D_0 and $W^s_\varepsilon(p)$ (see (2.3)). Since D_0 is contained in a ball centered at p with radius δ_0, we have $X_t(D_0) \subset U$ for $0 \leq t \leq T$. By (2.1) and (2.2), $X_t(D_0)$ contains a C^1 disk centered at $X_T(x)$ of radius $\lambda^{-1}\theta$ satisfying (2.2). Denote as $D_1 \subset X_T(D_0)$ a C^1 disk centered at $X_T(x)$ of radius θ. Since $X_T(y) \in X_T(W^s_\varepsilon(p)) \subset W^s_\varepsilon(p)$ and since both D_1 and $X_T(D_0)$ intersect $W^s_\varepsilon(p)$ at a unique point respectively, we have

$$\{X_T(y)\} = X_T(D_0) \cap W^s_\varepsilon(p) = D_1 \cap W^s_\varepsilon(p).$$

Moreover, since $X_T(x), X_T(y) \in D_1$, we have

$$\begin{aligned} d(x, y) &= d(X_{-T}(X_T(x)), X_{-T}(X_T(y))) \\ &\leq \lambda d(X_T(x), X_T(y)) \leq \lambda\theta. \end{aligned}$$

Repeating this procedure, we find C^1 disks D_n ($n \geq 0$) centered at $X_{nT}(x)$ of radius θ satisfying (2.2) such that

$$D_{n+1} \subset X_T(D_n) \text{ and } X_{nT}(x), X_{nT}(y) \in D_n$$

for $n \geq 0$. So

$$\begin{aligned} d(x,y) &= d(X_{-nT}(X_{nT}(x)), X_{-nT}(X_{nT}(y))) \\ &\leq \lambda^n \theta \ (n \geq 0), \end{aligned}$$

which means $d(x,y) = 0$, and so $x \in W^s_\varepsilon(p)$. This finishes the proof of the lemma. □

Remark 2.1. An analogous result holds for the local unstable manifold as follows: for every hyperbolic periodic point p and $\varepsilon > 0$, we can choose $\epsilon_0 \in (0, \epsilon)$ such that for $x \in M$, if $d(X_t(x), X_t(p)) \leq \epsilon_0$ for $t \leq 0$, then $x \in W^u_\varepsilon(p)$.

Lemma 2.2. *Suppose that the flow $(X_t)_t$ restricted to the attractor Λ satisfies the weak specification property. Then for every hyperbolic critical element p, the strong stable manifold $W^{ss}(p)$ is dense in Λ.*

Proof. We prove only the case when p is periodic since the singularity case can be shown similarly. Let $\varepsilon > 0$ and $z \in \Lambda$ be fixed arbitrarily. Since $(X_t)_t$ is the flow generated by the vector field X we can take $0 < t_0 < \epsilon$ so that $d(x, X_t(x)) \leq \epsilon$ for any $x \in \Lambda$ and $|t| \leq t_0$. By Lemma 2.1 we can choose $\epsilon_0 \in (0, t_0)$ such that if $d(X_t(x), X_t(p)) \leq \epsilon_0$ for every $t > 0$ then

$$x \in W^s_{t_0}(p) = \bigcup_{|t| \leq t_0} X_t(W^{ss}_{t_0}(p)). \tag{2.4}$$

Let $T(\epsilon_0) > 0$ be as in the definition of the specification property and choose $T \geq T(\epsilon_0)$ so that $X_T(p) = p$. By the weak specification property, there are $x_n \in \Lambda$ so that $d(x_n, z) \leq \epsilon_0$ and $d(X_t(X_T(x_n)), X_t(p)) \leq \epsilon_0$ for every $t \in [0, n]$. By compactness of Λ we may assume, without loss of generality, that $(x_n)_n$ is convergent to some point $x \in \Lambda$ satisfying $d(x, z) \leq \epsilon_0$ and $d(X_t(X_T(x)), X_t(p)) \leq \epsilon_0$ for every $t > 0$. Using (2.4), we have

$$X_T(x) \in \bigcup_{|t| \leq t_0} X_t(W^{ss}_{t_0}(p))$$

and so we can find $t_1 \in [-t_0, t_0]$ such that $X_T(x) \in X_{t_1}(W^{ss}_{t_0}(p))$. Since T is the period of p, we have $x \in X_{t_1}(W^{ss}(p))$. Thus, there exists a point $y \in W^{ss}(p)$ such that $x = X_{t_1}(y)$ and consequently

$$d(y, z) \leq d(y, x) + d(x, z) \leq \epsilon + \epsilon_0 \leq 2\epsilon,$$

which implies that $W^{ss}(p)$ is dense in Λ. □

In the next proposition, the time-continuous version of [4, Proposition 3], we recall some necessary results relating some shadowing properties with the location of the shadowing point in unstable disks. First we introduce a notation. Set $W^s(p) = \bigcup_{t \in \mathbb{R}} X_t(W^{ss}(p))$. For $x \in W^s(p)$ and $\eta > 0$ we will consider the local unstable disk around x in $W^s(p)$ given by $\gamma^s_\eta(x) := \{z \in W^s(p) : d^s(x, z) \leq \eta\}$ where d^s is the distance in $W^s(p)$ induced in the Riemannian metric.

Proposition 2.3. *Let p be a hyperbolic critical element for the flow. There are $\varepsilon_1 > 0$ and $L > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ the following holds: if $x \in W^{ss}(p)$ and $d(X_t(z), X_t(x)) \leq \varepsilon$ for any $t > 0$ then $z \in \gamma^s_{L\varepsilon}(x)$.*

Proof. We prove only the case when p is periodic since the singularity case can be shown similarly. Put $\kappa = \min\{\|X(X_t(p))\| : t \in \mathbb{R}\}(> 0)$. Then we can take $t_0 > 0$ such that

$$d(X_t(p), X_s(p)) \geq \kappa|t-s|/2 \tag{2.5}$$

for $|t-s| \leq t_0$. Since Λ is partially hyperbolic, there exists $0 < \mu \leq t_0$ such that if $x \in \Lambda$ and $y \in W^{ss}_{\mu}(x)$, then

$$d^{ss}(x, y) \leq 2d(x, y). \tag{2.6}$$

Recall that d^{ss} is the distance in $W^{ss}(x)$ induced in the Riemannian metric. By Lemma 2.1 we can choose $0 < 2\varepsilon_0 < \mu/4$ such that for $x \in M$ and $s \in \mathbb{R}$, if $d(X_t(x), X_{t+s}(p)) \leq 2\varepsilon_0$ for every $t \geq 0$, then

$$x \in W^s_{\mu/4}(X_s(p)) = \bigcup_{|t|\leq\mu/4} X_t(W^{ss}_{\mu/4}(X_s(p))). \tag{2.7}$$

Put $K = \max\{\|X(x)\| : x \in M\}$ and $L_0 = 1 + 2K/\kappa$. Let $0 < \varepsilon < \varepsilon_1 = \min\{\varepsilon_0, \mu/4L\}$. Since $x \in W^{ss}(p)$, there is a sufficiently large $T > 0$ such that

$$d(X_{t+T}(x), X_{t+T}(p)) \leq \varepsilon_0$$

for all $t \geq 0$. By the definition of $W^{ss}(p)$ and (2.7) we have

$$X_T(x) \in W^{ss}_{\mu/4}(X_T(p)). \tag{2.8}$$

By the assumption of z, we have

$$d(X_{t+T}(z), X_{t+T}(p)) \leq d(X_{t+T}(z), X_{t+T}(x)) + d(X_{t+T}(x), X_{t+T}(p)) \leq 2\varepsilon_0$$

for $t \geq 0$. By (2.7) we can find t_1 with $|t_1| \leq \mu/4$ such that

$$X_T(z) \in X_{t_1}(W^{ss}_{\mu/4}(X_T(p))). \tag{2.9}$$

Combining (2.8) and (2.9) we have

$$X_{T-t_1}(z) \in W^{ss}_{\mu/2}(X_T(x)). \tag{2.10}$$

Since $x \in W^{ss}(p)$, we have $d(X_t(x), X_t(p)) \to 0$ $(t \to \infty)$. Put $K_0 = \max\{\|D_x X_{t_1}\| : x \in M\}$. By (2.9) we have

$$d(X_t(z), X_{t+t_1}(p)) \quad \leq \quad K_0 d(X_{t-t_1}(z), X_t(p)) \to 0 \ (t \to \infty).$$

Thus it follows from (2.5) that

$$\begin{aligned} \varepsilon &\geq d(X_t(x), X_t(z)) \\ &\geq d(X_t(p), X_{t+t_1}(p)) - d(X_t(p), X_t(x)) - d(X_{t+t_1}(p), X_t(z)) \\ &\geq \kappa|t_1|/2 - d(X_t(p), X_t(x)) - d(X_{t+t_1}(p), X_t(z)) \\ &\to \kappa|t_1|/2 \ (t \to \infty), \end{aligned}$$

which means that $|t_1| \leq 2\varepsilon/\kappa$. Recall $L_0 = 1 + 2K/\kappa$. then we have

$$\begin{aligned} d(X_t(x), X_{t-t_1}(z)) &\leq d(X_t(x), X_t(z)) + d(X_t(z), X_{t-t_1}(z)) \\ &\leq \varepsilon + K|t_1| \\ &\leq (1 + 2K/\kappa)\varepsilon = L_0\varepsilon \end{aligned} \tag{2.11}$$

for $t \geq 0$. We take a small $t_2 > 0$ such that

$$K_1 = \max\{\|D_x X_{-t}\| : x \in M, 0 \leq t \leq t_2\} \leq 2.$$

Put $I = \{t \in [0,\infty) : d^{ss}(X_t(x), X_{t-t_1}(z)) \le 2L_0\varepsilon\}$ and $t_0 = \inf I$. By (2.6), (2.10) and (2.11) we have $T \in I$. Assume by contradiction that $t_0 > 0$. Since

$$d^{ss}(X_{t_0-t_2}(x), X_{t_0-t_1-t_2}(z)) \quad \le \quad K_1 d^{ss}(X_{t_0}(x), X_{t_0-t_1}(z)) \le 4L_0\varepsilon \le \mu,$$

by (2.6) and (2.11) we have $t - t_2 \in I$, which is a contradiction. Thus $0 = \inf I$. Therefore

$$\begin{aligned} d^s(x,z) &\le d^{ss}(x, X_{-t_1}(z)) + d^s(X_{-t_1}(z), z) \\ &\le 2L_0\varepsilon + K|t_1| \le (2L_0 + 2K/\kappa)\varepsilon \le 3L_0\varepsilon. \end{aligned}$$

□

Let us now recall the *tubular neighborhood theorem* and refer the reader to e.g. [19] for the proof and more details.

Proposition 2.4. *Let M be a compact Riemannian manifold of dimension d. Given $X \in \mathfrak{X}^1(M)$ and a regular point $x \in M$ there exists $\delta = \delta_x > 0$, an open neighborhood U_x^δ of x, and a C^1-diffeomorphism $\Psi_x : U_x^\delta \to (-\delta,\delta) \times B(x,\delta) \subset \mathbb{R} \times \mathbb{R}^{d-1}$ such that the vector field X on U_x^δ is the pull-back of the vector field $Y := (1,0,\dots,0)$ on $(-\delta,\delta) \times B(x,\delta)$, that is,*

$$Y = (\Psi_x)_* X := D(\Psi_x)_{\Psi_x^{-1}} X(\Psi_x^{-1}).$$

In particular $Y^t(\cdot) = \Psi_x(X^t(\Psi_x^{-1}(\cdot)))$ for every $|t| < \delta$.

In fact, using the previous result finitely many times one can prove the *long tubular neighborhood theorem* for arbitrary long compact pieces of regular non-periodic orbits as follows (see e.g. [19] for more details):

Proposition 2.5. *Let M be a compact Riemannian manifold of dimension d and take $X \in \mathfrak{X}^1(M)$, a regular point $x \in M$ and $T > 0$. Then there exists $\delta_T = \delta_T(x) > 0$, an open neighborhood U_x^T of x, and a C^1-diffeomorphism $\Psi_x : U_x^T \to (-T,T) \times B(x,\delta_T) \subset \mathbb{R} \times \mathbb{R}^{d-1}$ such that $Y^t(\cdot) = \Psi_x(X^t(\Psi_x^{-1}(\cdot)))$ for every small $|t| < T$, where $(Y^t)_t$ is the flow generated by the vector field $Y := (1,0,\dots,0)$ on $(-T,T) \times B(x,\delta_T)$.*

We finish this section with some considerations on the existence of stable foliations and holonomies. Since we assume Λ to be a partially hyperbolic attractor and that the subbundle E^s is non-empty, it is well known that the sub-bundle E^s is uniquely integrable and hence we have a foliation $\mathcal{F}^s$ which are tangent to E^s, called the strong stable foliation (see [11]). This is enough to guarantee the following.

Lemma 2.6. *Let Λ be a partially hyperbolic attractor for the flow $(X_t)_t$ generated by the vector field $X \in \mathfrak{X}^1(M)$. If $p \in \Lambda$ is a hyperbolic critical element for X then there exists an open neighborhood $\mathcal{U}$ of p so that the strong stable leaves foliate $\mathcal{U}$. In particular there is a well defined stable holonomy map $\pi^s : \mathcal{U} \to \pi^s(\mathcal{U}) \subset W^u(p)$.*

Proof. The key observation is that an attractor contains strong unstable manifolds of hyperbolic critical elements. Assume first that p is a singularity and $\mathcal{U}$ is a small neighborhood of p (smaller than the uniform size of local stable leaves through points of the attractor) and, in particular, $W^{uu}(p) = W^u(p)$. Then, we notice that the set $W^u_{\mathcal{U}}(p) := \mathcal{U} \cap W^u(p)$ is contained in Λ and that $\mathcal{U}$ is foliated by the sets $\{\mathcal{F}^s(x) \cap \mathcal{U}\}_{x \in W^u_{\mathcal{U}}(p)}$. The stable holonomy map in the neighborhood $\mathcal{U}$ is defined simply by $\pi^s(x) = \mathcal{F}^s(x) \cap W^u_{\mathcal{U}}(p)$.

Assume now that p is a regular point. In this case, using the invariance of Λ by the flow and that $W^{uu}(p) \subset \Lambda$ we deduce that $W^u(p) \subset \Lambda$. Let $\mathcal{U}$ be a small neighborhood of p given by the tubular neighborhood theorem and set $W^u_{\mathcal{U}}(p) := \mathcal{U} \cap W^u(p)$. Then it is clear that $\mathcal{U}$ is foliated by centre-stable leaves and that each of these are subfoliated by strong stable ones. The stable holonomy map is defined as before. □

3. Proof of Theorem B

The aim of this section is to prove our main result in this paper. Let $X \in \mathfrak{X}^1(M)$ be a C^1 vector field so that the flow $(X_t)_{t\in\mathbb{R}}$ admits a partially hyperbolic attractor Λ with splitting $T_\Lambda M = E^s \oplus E^c$ and assume that there are two hyperbolic critical elements p and q such that $\dim E^s = \dim W^{ss}(p) < \dim W^{ss}(q)$.

Assume by contradiction that $(X_t)_t$ satisfies the weak specification property. Then $(X_t)_t$ is topologically mixing (c.f. [3, Lemma 3.1]) and it admits neither attracting nor repelling critical elements. There are four situations to consider depending on wether the critical elements p and q defined above are singularities or periodic orbits.

First case: *p and q are singularities*

Since p, q are singularities then $W^s_\mu(p) = W^{ss}_\mu(p)$ and $W^u_\mu(q) = W^{uu}_\mu(q)$ for any $\mu > 0$. Take an open disk $D_0 = W^u_\mu(p) \subset \Lambda$ containing p that it is transverse to the local stable foliation through points of D_0. For any open disk U contained in D_0, $\mathcal{A}(U) := \bigcup_{z\in U} \mathcal{F}^s_\mu(z)$ is homeomorphic to $U \times [-\mu, \mu]^{\dim\ E^s}$, where we set

$$\mathcal{F}^s_\mu(z) := \{w \in \mathcal{F}^s(z) : d^s(z, w) \leq \mu\},$$

and d^s is the distance in $\mathcal{F}^s(z)$ induced by the Riemannian metric. Let $\epsilon_1 > 0$ and $L > 0$ be given by Proposition 2.3. We claim the following:

Claim: There are $\mu > 0$, $\epsilon \in (0, \epsilon_1)$ with $\epsilon < \mu/L$ and $x \in W^{ss}(p)$ so that if $T = T(\epsilon)$ is given by the specification property then $X_{-T}(\gamma^s_\mu(x)) \cap W^u_\mu(q) = \emptyset$.

Proof of the claim. Take $\mu > 0$ small. Set $\epsilon := \min\{\mu/5, \epsilon_1/2\}$ and let $T(\epsilon)$ be as above. By the long tubular neighborhood theorem the intersection $X_T(W^u_\mu(q)) \cap \mathcal{A}(\Delta_0)$ is either empty or a submanifold of dimension $\dim E^u_q < \dim E^u_p$. Since $X_T(W^u_\mu(q))$ is tangent to E^c, i.e.

$$T_y(X_T(W^u_\mu(q))) \subset E^c_y,$$

for $y \in X_T(W^u_\mu(q))$, the projection $\pi^s(X_T(W^u_\mu(q)))$ along the stable holonomy consists of a finite set of disks of dimension smaller than or equal to $\dim E^u_q$ on D_0. Since the complementar of $\pi^s(X_T(W^u_\mu(q)))$ is open in D_0, there exists an open disk $U \subset D_0$ so that $\mathcal{A}(U) \cap X_T(W^u_\mu(q)) = \emptyset$. Since $U \subset \Lambda$ (c.f. proof of Lemma 2.6) and the stable manifold $W^s(p)$ is dense in Λ then there exists $x \in D_0 \cap \mathcal{A}(U) \cap W^s(p)$ with $\mathcal{F}^s_\mu(x) \subset \mathcal{A}(U)$ disjoint from $X_T(W^u_\mu(q))$. Hence $X_{-T}(\gamma^s_\mu(x)) \cap W^u_\mu(q) = \emptyset$ □

We proceed to prove Theorem B in this first setting. On the one hand, by the claim there exists $\mu > 0$, $0 < \epsilon < \min\{\mu/L, \epsilon_1\}$ and $x \in W^s(p)$ so that $X_{-T}(\gamma^s_\mu(x))$ does not intersect $W^u_\mu(q)$, where $T = T(\epsilon) > 0$ is given by the specification property.

On the other hand, for the singularity q and $x \in W^{ss}(p)$ given by the previous claim, by compactness of Λ and the specification property there exists $z \in \Lambda$ such that $d(X_t(z), X_t(x)) \leq \epsilon$ and $d(X_{-t}(X_{-T}(z)), X_{-t}(q)) \leq \epsilon$ for all $t \geq 0$. Since

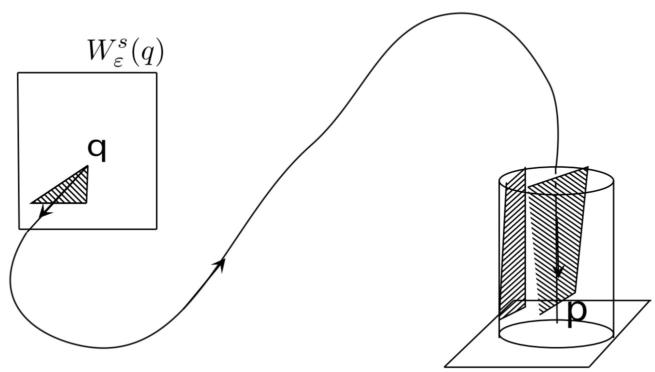

FIGURE 1

$\epsilon \in (0, \epsilon_1)$, Proposition 2.3 guarantees that $z \in X_{-T}(\gamma^s_{L\epsilon}(x)) \cap W^u_{L\epsilon}(q)$, which is a contradiction since $L\epsilon < \mu$. This finishes the proof of Theorem B in this first case.

Second case: *p and q are periodic orbits*

The strategy is again to deduce a contradiction by assuming the specification property. Given μ small consider the disk $D_0 = \cup_{|t|\le\mu} X_t(W^{uu}_\mu(p))$ containing p and the strong stable holonomy π^s defined in $\mathcal{A}(D_0) := \bigcup_{z\in D_0} \mathcal{F}^s_\mu(z)$ (c.f. Lemma 2.6). Moreover we assume that

$$0 < \mu < \min\{l_p, l_q\}/10 \tag{3.1}$$

where l_p and l_q denote the prime periods of p and q, respectively.

Let $\epsilon_1 > 0$ and $L > 0$ be as in Proposition 2.3. Then (up to considering the reverse time flow in Proposition 2.3) we can choose $0 < \epsilon \le \min\{\epsilon_1, \mu/L\}$ such that for $x \in \Lambda$, if $d(X_{-t}(x), X_{-t}(q)) \le \epsilon$ for $t \ge 0$, then

$$x \in W^u_\mu(q) = \bigcup_{|t|\le\mu} X_t(W^{uu}_\mu(q)). \tag{3.2}$$

By definition, $W^u_\mu(q)$ is foliated by pieces of orbit of points in $W^{uu}_\mu(q)$ and so $\dim W^u_\mu(q) = 1 + \dim W^{uu}_\mu(q) = 1 + \dim E^u_q$. Let $T = T(\epsilon) > 0$ be as in the definition of the specification property. On the one hand, since $X_T(W^u_\mu(q))$ is a $(1+\dim E^u_q)$-dimensional submanifold, the intersection of $X_T(W^u_\mu(q))$ with $\mathcal{A}(D_0)$, if nonempty, consists of finitely many compact disks of dimension $1 + \dim E^u_q$. Moreover, $X_T(W^u_\mu(q))$ is tangent to E^c, and so $\pi^s(X_T(W^u_\mu(q)) \cap \mathcal{A}(D_0))$ is a union of finitely many compact disks of dimension $1 + \dim E^u_q$.

Let $\pi : D_0 \to W^{uu}_\mu(p)$ be the projection along the orbit, i.e. if $x = X_t(z) \in D_0$ for some $|t| \le \mu$ and $z \in W^{uu}_\mu(p)$, then $\pi(x) = z$. Since $X_t \circ \pi_s = \pi_s \circ X_t$ in $\mathcal{A}(D_0)$ then $\pi^s(X_T(W^u_\mu(q)) \cap \mathcal{A}(D_0))$ is a union of finitely many compact disks which are foliated by pieces of orbit. Thus the dimension of $(\pi \circ \pi^s)(X_T(W^u_\mu(q)) \cap \mathcal{A}(D_0))$ is exactly (less than) $\dim E^u_q < \dim E^u_p = \dim W^{uu}_\mu(p)$. Since the complementar is dense and open in $W^{uu}_\mu(p)$, there exists an open disk $U \subset W^{uu}_\mu(p)$ so that

$$U \cap (\pi \circ \pi^s)(X_T(W^u_\mu(q)) \cap \mathcal{A}(D_0)) = \emptyset,$$

which means that $\mathcal{A}(\pi^{-1}(U)) \cap X_T(W^u_\mu(q)) = \emptyset$.

On the other hand, since $W^{ss}(p)$ is dense in Λ (Lemma 2.2), we have $\mathcal{A}(\pi^{-1}(U)) \cap W^{ss}(p) \neq \emptyset$. Furthermore, we can choose a point $w \in \pi^{-1}(U) \cap W^{ss}(p) (\subset D_0)$ which is sufficiently close to $W^{ss}_\mu(p)$. By the specification property, there exists $y \in \Lambda$ so that $d(X_{-t}(y), X_{-t}(q)) \le \epsilon$ and $d(X_t(X_T(y)), X_t(w)) \le \epsilon$ for all $t \ge 0$. By (3.2) and Proposition 2.3, we have $y \in W^u_\mu(q)$ and $X_T(y) \in \gamma^s_{L\epsilon}(w)$. Since ϵ is sufficiently small, we may assume that $\gamma^s_{L\epsilon}(w) \subset \mathcal{A}(\pi^{-1}(U))$. Thus we have $\mathcal{A}(\pi^{-1}(U)) \cap X_T(W^u_\mu(q)) \neq \emptyset$, which is a contradiction.

Third case: *p is a singularity and q is a periodic orbit*

The strategy is again to deduce a contradiction by assuming the specification property. Let us observe that in this setting

$$\dim W^u(q) = 1 + \dim E^u_q = n - \dim E^s_q < n - \dim E^s_p = \dim W^u(p).$$

Thus, the argument proving that the complement of the set $\pi^s(X_T(W^u_\mu(q)))$ (here π^s denotes again the strong strong stable holonomy map in a neighborhood of p on a disk $D_0 \subset W^u(p)$) contains open sets $\mathcal{U} \subset D_0$ survives, as well as the proof that this property prevents specification.

Remark 3.1. Let us mention that simpler third case is only relevant in the dimension larger or equal to 4. In fact, if $\dim M = 3$ then necessarily $\dim E^u_q = \dim E^s_q = 1$ and $\dim E^s_p < \dim E^s_q$ leads to a contradiction to the fact that E^s is non-trivial.

Fourth case: *q is a singularity and p is a periodic orbit*

To finish the proof of Theorem B we are left to deal with the case that q is a singularity and p is a periodic orbit, in which case the relations $\dim M = \dim E^s_q + \dim E^u_q$ and also $\dim M = \dim E^s_p + \dim E^u_p + 1$ together with $\dim E^s_p \le \dim E^s_q - 1$ yield that $\dim E^u_q \le \dim E^u_p$. If the strict inequality holds we can proceed as in the third case. Otherwise, the difficulty occurs if $\dim E^u_q = \dim E^u_p$. Nevertheless, π^s is a projection defined in a neighborhood of p onto the local weak unstable manifold $W^u(p)$, and $\dim W^u(p) = 1 + \dim E^u_p > \dim E^u_q$. The argument follows as before: taking $D_0 = W^u_\mu(p)$ it follows that $\pi^s(X_T(W^{uu}_\mu(q)) \cap \mathcal{A}(D_0))$ is a union of finitely many compact disks of dimension $\dim E^u_q < 1 + \dim E^u_p = \dim W^u(p)$.

Since $X_t(x) \in W^{uu}(q) = W^u(q)$ for $x \in W^{uu}_\mu(q)$ and $t \in \mathbb{R}$, $W^{uu}_\mu(q) \setminus \{q\}$ is foliated by pieces of orbit. Thus $\pi^s(X_T(W^{uu}_\mu(q)) \cap \mathcal{A}(D_0))$ is a union of finitely many compact disks which are foliated by pieces of orbit. Let $\pi : D_0 \to W^{uu}_\mu(p)$ be the projection along the orbit. Then the dimension of $(\pi \circ \pi^s)(X_T(W^{uu}_\mu(q)) \cap \mathcal{A}(D_0))$ is less than $\dim E^u_q - 1 < \dim E^u_p = \dim W^{uu}_\mu(p)$. Since the complementar is dense and open in $W^{uu}_\mu(p)$, there exists an open disk $U \subset W^{uu}_\mu(p)$ so that

$$U \cap (\pi \circ \pi^s)(X_T(W^u_\mu(q)) \cap \mathcal{A}(D_0)) = \emptyset,$$

which means that $\mathcal{A}(\pi^{-1}(U)) \cap X_T(W^u_\mu(q)) = \emptyset$. The proof follows the same lines of the previous arguments.

Remark 3.2. In fact, in the case of the geometric Lorenz attractor in dimension 3 necessarily $\dim E^u_p = \dim E^s_p = 1$ and for the singularity $\dim E^u_p = 1$ and consequently $\dim E^u_p = \dim E^u_q$ leading to the fourth situation.

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