

Certain identities, connection and explicit formulas for the Bernoulli, Euler numbers and Riemann zeta -values

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Abstract

Various new identities, recurrence relations, integral representations, connection and explicit formulas are established for the Bernoulli, Euler numbers and the values of Riemann's zeta function $\zeta(s)$. To do this, we explore properties of some Sheffer's sequences of polynomials related to the Kontorovich-Lebedev transform.

Keywords: *Bernoulli polynomials, Bernoulli numbers, Euler polynomials, Euler numbers, generalized Euler polynomials, Sheffer sequences, Von Staudt-Clausen theorem, Riemann zeta function, modified Bessel functions, Kontorovich-Lebedev transform*

AMS subject classification: 11B68, 11B73, 11B83, 11M06, 12E10, 33C10, 44A15

1 Introduction and preliminary results

In 2009 the author [1] introduced a family of polynomials of degree n , which belongs to Sheffer's sequences (cf. [2]) and related to the Kontorovich-Lebedev transform. Precisely, it has the form

$$p_n(x) = (-1)^n e^x \mathcal{A}^n e^{-x}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

where

$$\mathcal{A} \equiv x^2 - x \frac{d}{dx} x \frac{d}{dx}, \quad (1.2)$$

is the second order differential operator having as an eigenfunction the modified Bessel function $K_{i\tau}(x)$, $\tau \in \mathbb{R}$, (i is the imaginary unit), i.e.

$$\mathcal{A} K_{i\tau}(x) = \tau^2 K_{i\tau}(x). \quad (1.3)$$

The modified Bessel function $K_{i\tau}(x)$ is, in turn, the kernel of the Kontorovich-Lebedev transform (see in [3], [4], [5])

$$(Gf)(\tau) = \int_0^\infty K_{i\tau}(x) f(x) \frac{dx}{x}, \quad \tau \in \mathbb{R}_+. \quad (1.4)$$

As it is known, operator (1.4) extends to a bounded invertible map $G : L_2(\mathbb{R}_+; x^{-1}dx) \rightarrow L_2(\mathbb{R}_+; \tau \sinh \pi \tau d\tau)$ and this map is isometric, i.e.

$$\int_0^\infty \tau \sinh \pi \tau |(Gf)(\tau)|^2 d\tau = \frac{\pi^2}{2} \int_0^\infty |f(x)|^2 \frac{dx}{x}. \quad (1.5)$$

Reciprocally, the inversion formula holds

$$f(x) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi \tau K_{i\tau}(x) (Gf)(\tau) d\tau, \quad x > 0. \quad (1.6)$$

The modified Bessel function $K_{i\tau}(x)$ has the asymptotic behavior [4]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} [1 + O(1/z)], \quad z \rightarrow \infty, \quad (1.7)$$

and near the origin

$$K_\nu(z) = O(z^{-|\operatorname{Re} \nu|}), \quad z \rightarrow 0, \quad (1.8)$$

$$K_0(z) = -\log z + O(1), \quad z \rightarrow 0. \quad (1.9)$$

Moreover it can be defined by the following integral representations

$$K_\nu(x) = \int_0^\infty e^{-x \cosh u} \cosh \nu u du, \quad x > 0, \quad (1.10)$$

$$K_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-t - \frac{x^2}{4t}} t^{-\nu-1} dt, \quad x > 0. \quad (1.11)$$

Returning to the system of polynomials (1.1), we easily observe that all their coefficients $a_{n,k}$, $k = 0, 1, \dots, n$ are integers. It can be represented by the integral

$$p_n(x) = \frac{2(-1)^n}{\pi} e^x \int_0^\infty \tau^{2n} K_{i\tau}(x) d\tau, \quad (1.12)$$

and satisfies the differential recurrence relation of the form

$$p_{n+1}(x) = x^2 p_n''(x) + x(1 - 2x) p_n'(x) - x p_n(x), \quad n = 0, 1, 2, \dots \quad (1.13)$$

In particular, we derive

$$p_0(x) = 1, \quad p_1(x) = -x, \quad p_2(x) = 3x^2 - x, \quad p_3(x) = -15x^3 + 15x^2 - x.$$

The leading coefficient $a_{n,n}$ of these polynomials can be calculated by the formula

$$a_{n,n} = (-1)^n (2n-1)!! = (-1)^n 1 \cdot 3 \cdot 5 \dots \cdot (2n-1), \quad n \in \mathbb{N}. \quad (1.14)$$

Moreover, recently we found the explicit formula of coefficients $a_{n,k}$ (see [6])

$$a_{n,k} = \frac{1}{k!} \sum_{r=0}^k \frac{(-1)^r}{2^r} \binom{k}{r} \sum_{j=0}^{k-r} \frac{(-1)^j}{2^j} \binom{k-r}{j} (r-j)^{2n}, \quad k = 1, \dots, n \quad (1.15)$$

and as a consequence of the definition (1.1) $a_{n,k} \in \mathbb{Z}$, i.e. the right-hand side of (1.15) is an integer. The generating function $\Phi(x, t)$ for this sequence of polynomials is given by the series

$$\Phi(x, t) = e^{-2x \sinh^2(t/2)} = \sum_{n=0}^{\infty} \frac{p_n(x)}{(2n)!} t^{2n}, \quad |t| < \frac{\pi}{4}. \quad (1.16)$$

Letting $x = 0$ in the latter equation, we find

$$p_n(0) = 0, \quad n = 1, 2, \dots$$

A differentiation with respect to x in (1.16) yields the equality

$$\frac{\partial \Phi}{\partial x} = (1 - \cosh t) e^{-2x \sinh^2(t/2)} = \sum_{n=0}^{\infty} \frac{p'_n(x)}{(2n)!} t^{2n}. \quad (1.17)$$

Decomposing the left-hand side of (1.17) as a product of series and equating coefficients in front of t^{2n} we come up with the following recurrence relation

$$p'_n(x) = - \sum_{k=0}^{n-1} \binom{2n}{2k} p_k(x), \quad n \in \mathbb{N}. \quad (1.18)$$

Putting $x = 0$ in (1.17) and using values $p_0(0) = 1$, $p_k(0) = 0$, $k \in \mathbb{N}$ we obtain $p'_n(0) = -1$, $n \in \mathbb{N}$. Analogously, a differentiation with respect to t in (2.4) leads us to

$$\frac{\partial \Phi}{\partial t} = -x \sinh t e^{-2x \sinh^2(t/2)} = \sum_{n=1}^{\infty} \frac{p_n(x)}{(2n-1)!} t^{2n-1}. \quad (1.19)$$

Similarly we derive the relation

$$p_{n+1}(x) = -x \sum_{k=0}^n \binom{2n+1}{2k} p_k(x), \quad n \in \mathbb{N}_0. \quad (1.20)$$

Moreover, differentiating through in (1.19), we call (1.17) and using simple relations for binomial coefficients, we obtain the identity

$$x \sum_{k=0}^n \binom{2n+1}{2k} p'_k(x) = \sum_{k=1}^n \binom{2n+1}{2k-1} p_k(x). \quad (1.21)$$

Comparing (1.17), (1.19), we find that $\Phi(x, t)$ satisfies the following first order partial differential equations

$$\frac{\partial \Phi}{\partial t} + x \sinh t \Phi = 0, \quad (1.22)$$

$$\frac{\partial \Phi}{\partial x} + 2 \sinh^2 \left(\frac{t}{2} \right) \Phi = 0, \quad (1.23)$$

$$x \frac{\partial \Phi}{\partial x} = \tanh \left(\frac{t}{2} \right) \frac{\partial \Phi}{\partial t}. \quad (1.24)$$

Further, returning to representation (1.12) and employing the inversion formula (1.6) of the Kontorovich-Lebedev transform, we obtain the equality

$$\frac{\tau^{2n-1}}{\sinh \pi \tau} = \frac{(-1)^n}{\pi} \int_0^\infty e^{-x} K_{i\tau}(x) p_n(x) \frac{dx}{x}, \quad n \in \mathbb{N}. \quad (1.25)$$

In particular, it yields

$$\int_0^\infty e^{-x} K_0(x) p_n(x) \frac{dx}{x} = 0, \quad n = 2, 3, \dots$$

Integrating with respect to τ in (1.25), we call the value of the integral (2.4.3.1) in [7], Vol. I

$$\int_0^\infty \frac{\tau^{\alpha-1} d\tau}{\sinh \pi \tau} = \frac{2^\alpha - 1}{\pi^\alpha 2^{\alpha-1}} \Gamma(\alpha) \zeta(\alpha), \quad \operatorname{Re} \alpha > 1, \quad (1.26)$$

where $\Gamma(\alpha)$ is Euler's gamma function and $\zeta(\alpha)$ is Riemann's zeta function (cf. [8], Vol. 1) and relation (2.16.48.1) in [7], Vol. II to obtain the following representations of zeta-values at even and odd integers, respectively,

$$\frac{2^{2n} - 1}{2^{2(n-1)}} (-1)^n (2n-1)! \frac{\zeta(2n)}{\pi^{2n}} = \int_0^\infty e^{-2x} p_n(x) \frac{dx}{x}, \quad n \in \mathbb{N}, \quad (1.27)$$

$$(-1)^n (2n)! (2^{2n+1} - 1) \frac{\zeta(2n+1)}{(2\pi)^{2n}} = \int_0^\infty \int_0^\infty \tau K_{i\tau}(x) e^{-x} p_n(x) \frac{d\tau dx}{x}, \quad n \in \mathbb{N}. \quad (1.28)$$

Finally in this section we note that recently in [9] the family of polynomials (1.1) was generalized on the sequence

$$p_n(x; \alpha) = (-1)^n e^x x^{-\alpha} \mathcal{A}^n e^{-x} x^\alpha, \quad n \in \mathbb{N}_0, \quad (1.29)$$

involving an arbitrary parameter α , $\operatorname{Re} \alpha > -1/2$.

2 Identities for the Bernoulli and Euler numbers

In this section we will derive a number of recurrence relations, finite sum, connection and explicit formulas, series and integral representations, which are related to the Bernoulli and Euler numbers. The Bernoulli numbers B_n , $n = 0, 1, 2, \dots$, can be defined via the generating function (see in [8], Vol. I)

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi \quad (2.1)$$

and Bernoulli polynomials $B_n(x)$ by the equality

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |x| < 2\pi. \quad (2.1)$$

In particular, we find the values, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$ and $B_n = 0$ for all odd $n \geq 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geq 1$. We list some important properties of the Bernoulli numbers and polynomials, which will be employed below. All details and proofs can be found in [8], Vol. I. The basic identity for Bernoulli numbers is

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad n \geq 2. \quad (2.3)$$

Concerning the Bernoulli polynomials, it has the explicit formula,

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}. \quad (2.4)$$

Hence, for instance,

$$\begin{aligned} B_0(x) &= 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}. \end{aligned} \quad (2.5)$$

Evidently, $B_n = B_n(0)$. The Bernoulli polynomials and their derivative satisfy the following important relations

$$B'_n(x) = nB_{n-1}(x), \quad (2.6)$$

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad (2.7)$$

$$B_n(1-x) = (-1)^n B_n(x), \quad (2.8)$$

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \right). \quad (2.9)$$

The remarkable Euler formula relates the Bernoulli numbers B_{2n} and Riemann zeta- values $\zeta(2n)$ (see, for instance, in [10])

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}. \quad (2.10)$$

The Euler numbers E_n , $n = 0, 1, 2, \dots$, can be defined, in turn, by the equality (see in [8], Vol. I)

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \frac{\pi}{2}. \quad (2.11)$$

As we see, $E_{2n+1} = 0$ and, in particular, $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. The basic relation for Euler numbers is

$$\sum_{k=0}^n \binom{2n}{2k} E_{2k} = 0, \quad n \geq 1. \quad (2.12)$$

In order to obtain new properties of the Bernoulli and Euler numbers, we will employ Sheffer's sequences, which are associated with the Kontorovich-Lebedev transform (1.4). Indeed, calling identity (1.27), we immediately derive the integral representation of Bernoulli's numbers B_{2n} in terms of the sequence of polynomials (1.1), namely

$$B_{2n} = \frac{n}{1 - 2^{2n}} \int_0^{\infty} e^{-2x} p_n(x) \frac{dx}{x}. \quad (2.13)$$

For the numbers B_{4n} we have the formula (see in [1])

$$B_{4n} = \frac{2n}{1 - 2^{4n}} \int_0^{\infty} e^{-2x} p_n^2(x) \frac{dx}{x}, \quad (2.14)$$

which is the result of the equality

$$\int_0^{\infty} e^{-2x} p_n(x) \frac{dx}{x} = \int_0^{\infty} e^{-2x} p_n^2(x) \frac{dx}{x}. \quad (2.15)$$

But as it is proved in [1], a more general relation takes place

$$\int_0^{\infty} e^{-2x} p_{n+m}(x) \frac{dx}{x} = \int_0^{\infty} e^{-2x} p_n(x) p_m(x) \frac{dx}{x}, \quad (2.16)$$

which holds for any $n, m \in \mathbb{N}_0$ such that at least one is nonzero. Hence, appealing to (2.13), we derive the identity

$$B_{2(n+m)} = \frac{n+m}{1-2^{2(n+m)}} \int_0^\infty e^{-2x} p_{n-k}(x) p_{m+k}(x) \frac{dx}{x} \quad (2.17)$$

being valid for any $k = 0, 1, \dots, n$.

Another definition of Euler's numbers can be given in terms of Sheffer's sequence of polynomials $q_n(x)$ introduced in [1]

$$q_n(x) = e^x \int_x^\infty e^{-t} p_n(t) dt, \quad n \in \mathbb{N}_0, \quad (2.18)$$

which, in turn, is defined via the generating function $F(x, t) = \Phi(x, t) / \cosh t$ (see (1.16))

$$\frac{1}{\cosh t} e^{-2x \sinh^2(t/2)} = \sum_{n=0}^\infty \frac{q_n(x)}{(2n)!} t^{2n}, \quad |t| < \frac{\pi}{4}. \quad (2.19)$$

Hence with the use of (2.11) and the integral representation of Euler numbers [1]

$$E_{2n} = \int_0^\infty e^{-x} p_n(x) dx \in \mathbb{Z} \quad (2.20)$$

we find

$$E_{2n} = q_n(0), \quad n = 0, 1, \dots$$

Moreover, following [1], the sequence $q_n(x)$ has a relationship with $p_n(x)$. Indeed,

$$q_n(x) = \sum_{k=0}^n p_n^{(k)}(x), \quad n \in \mathbb{N}_0, \quad (2.21)$$

$$q_n(x) = \sum_{k=0}^n E_{2(n-k)} \binom{2n}{2k} p_k(x), \quad (2.22)$$

where $p_n^{(k)}(x)$ is the k -th derivative of $p_n(x)$ and relation (2.22) can be obtained employing (2.18) and the binomial type identity for the sequence $p_n(x)$ (cf. [2])

$$p_n(x+y) = \sum_{k=0}^n \binom{2n}{2k} p_k(x) p_{n-k}(y). \quad (2.23)$$

For instance,

$$q_0(x) = 1, \quad q_1(x) = -(x+1),$$

$$q_2(x) = 3x^2 + 5x + 5, \quad q_3(x) = -15x^3 - 30x^2 - 61x - 61.$$

Differentiating through in (2.21), we find

$$q'_n(x) = \sum_{k=0}^n p_n^{(k+1)}(x) = \sum_{k=1}^n p_n^{(k)}(x) = q_n(x) - p_n(x).$$

Thus, $p_n(x) = q_n(x) - q'_n(x)$ and since $p_n(0) = 0, n \in \mathbb{N}$, we get

$$E_{2n} = q'_n(0), \quad n \in \mathbb{N}.$$

Moreover,

$$p'_n(x) = q'_n(x) - q''_n(x)$$

and since $p'_n(0) = -1, n = 1, 2, \dots$ (see (1.18)), it yields

$$q''_n(0) = E_{2n} + 1, \quad n \geq 1. \quad (2.24)$$

Returning to (2.19), we easily derive analogs of the first order partial differential equations (1.22), (1.23), (1.24) for the generating function $F(x, t)$, namely

$$\frac{\partial F}{\partial t} + [x \sinh t + \tanh t] F = 0, \quad (2.25)$$

$$\frac{\partial F}{\partial x} + 2 \sinh^2 \left(\frac{t}{2} \right) F = 0, \quad (2.26)$$

$$x \frac{\partial F}{\partial x} = \tanh \left(\frac{t}{2} \right) \left[\frac{\partial F}{\partial t} + F \tanh t \right]. \quad (2.27)$$

To find the inverse of relation (2.22), we apply the product of series of $\cosh t$ and (2.19), equating its coefficients in front of t^{2n} with the corresponding terms of series (1.16). This is indeed allowed within the interval of the absolute convergence $|t| < \pi/4$. As a result, we deduce

$$p_n(x) = \sum_{k=0}^n \binom{2n}{2k} q_k(x). \quad (2.28)$$

But $q'_n(x) = q_n(x) - p_n(x)$. So, we have

$$q'_n(x) = - \sum_{k=0}^{n-1} \binom{2n}{2k} q_k(x). \quad (2.29)$$

Another source of identities for Bernoulli numbers is a formula related to $p'_n(x)$. To derive it, we call the partial differential equation (1.24) and the Taylor series for the hyperbolic tangent

$$\tanh \left(\frac{x}{2} \right) = 2 \sum_{k=0}^{\infty} \frac{(2^{2(k+1)} - 1) B_{2(k+1)}}{(2(k+1))!} x^{2k+1}.$$

Hence, substituting it in (1.24), making the product of series and equating the coefficients in front of t^{2n} , we obtain

$$x p'_n(x) = \sum_{k=1}^n \binom{2n}{2k-1} \frac{2^{2k}-1}{k} B_{2k} p_{n+1-k}(x). \quad (2.30)$$

In particular, dividing (2.30) by x and passing x to zero, we take into account the value $p'_n(0) = -1$, $n \geq 1$ to derive the identity

$$\sum_{k=1}^n \binom{2n}{2k-1} \frac{(2^{2k}-1)}{k} B_{2k} = 1. \quad (2.31)$$

Further, employing (1.21), we get from (2.30)

$$\sum_{k=1}^n \binom{2n+1}{2k-1} p_k(x) = \sum_{r=1}^n \binom{2n+1}{2r} \sum_{k=1}^r \binom{2r}{2k-1} \frac{(2^{2k}-1)}{k} B_{2k} p_{r+1-k}(x).$$

Thus, multiplying both sides of the latter equality by e^{-x} and integrating over \mathbb{R}_+ , we use (2.20) to find the identity

$$\sum_{r=1}^n \frac{E_{2r}}{(2r-1)!(2(n-r+1))!} = 2 \sum_{r=1}^n \sum_{k=1}^r \frac{(2^{2k}-1) B_{2k} E_{2(r-k+1)}}{(2k)!(2(n-r)+1)!(2(r-k)+1)!}. \quad (2.32)$$

Returning to (2.17) and letting $m = 0$, we make a summation in the right-hand side of (2.17) by k from zero to n

$$\frac{n}{1-2^{2n}} \sum_{k=0}^n \binom{2n}{2k} \int_0^\infty e^{-2x} p_{n-k}(x) p_k(x) \frac{dx}{x}$$

and employ the binomial type identity (2.23) to deduce

$$\sum_{k=0}^n \binom{2n}{2k} \int_0^\infty e^{-2x} p_{n-k}(x) p_k(x) \frac{dx}{x} = \int_0^\infty e^{-x} p_n(x) \frac{dx}{x}.$$

Therefore, we obtain the identity

$$B_{2n} \sum_{k=0}^n \binom{2n}{2k} = \frac{n}{1-2^{2n}} \int_0^\infty e^{-x} p_n(x) \frac{dx}{x}. \quad (2.33)$$

An interesting question is to express the finite sum in the left-hand side of (2.33) in terms of the values $p''_n(0)$. In fact, differentiating two times in (2.28) with respect to x , we let $x = 0$ and use (2.12), (2.24) to obtain

$$p''_n(0) = \sum_{k=2}^n \binom{2n}{2k} q''_k(0) = \sum_{k=0}^n \binom{2n}{2k} - 2,$$

or,

$$\sum_{k=0}^n \binom{2n}{2k} = p_n''(0) + 2, \quad n \geq 1. \quad (2.34)$$

Moreover, differentiating two times in (2.30) and letting then $x = 0$, we get the following recurrence relation for the values $p_n''(0)$

$$p_n''(0) = \frac{1}{2-n} \sum_{k=2}^{n-1} \binom{2n}{2k-1} \frac{2^{2(n-k+1)} - 1}{n-k+1} B_{2(n-k+1)} p_k''(0), \quad n \geq 1, \quad n \neq 2, \quad (2.35)$$

and $p_2''(0) = 6$.

Nevertheless, we are able to calculate explicitly the left-hand side of (2.34) due to trigonometric and exponential series technique developed in [11] and where one can find a great collection of many such formulas. Precisely, employing relation (3.7) in Vol. 6, formula (3.7), it gives

$$\sum_{k=0}^n \binom{2n}{2k} = 2^{2n-1}, \quad n \in \mathbb{N} \quad (2.36)$$

and therefore, $p_n''(0) = 2(2^{2(n-1)} - 1)$. Consequently, identity (2.33) becomes

$$B_{2n} = \frac{2n}{2^{2n} - 2^{4n}} \int_0^\infty e^{-x} p_n(x) \frac{dx}{x}. \quad (2.37)$$

Substituting the value of $p_n''(0)$ in (2.35), we obtain a possibly new identity

$$\begin{aligned} & \sum_{k=2}^{n-1} \binom{2n}{2k-1} \frac{(2^{2(n-k+1)} - 1)(2^{2(k-1)} - 1)}{n-k+1} B_{2(n-k+1)} \\ &= (2^{2(n-1)} - 1)(2 - n), \quad n \geq 1. \end{aligned} \quad (2.38)$$

Meanwhile,

$$p_n(x) = \sum_{k=1}^n a_{n,k} x^k, \quad n \geq 1,$$

where $a_{n,k}$ is defined by (1.15). Thus, substituting it into (2.13) and (2.36), after calculation of the elementary Euler integral and the use of (2.36), we find the following explicit formulas for the Bernoulli numbers, respectively,

$$B_{2n} = \frac{n}{1 - 2^{2n}} \sum_{k=1}^n \frac{1}{2^k} \frac{1}{k} \sum_{r=0}^k \frac{(-1)^r}{2^r} \binom{k}{r} \sum_{j=0}^{k-r} \frac{(-1)^j}{2^j} \binom{k-r}{j} (r-j)^{2n}, \quad (2.39)$$

$$B_{2n} = \frac{2n}{2^{2n}(1-2^{2n})} \sum_{k=1}^n \frac{1}{k} \sum_{r=0}^k \frac{(-1)^r}{2^r} \binom{k}{r} \sum_{j=0}^{k-r} \frac{(-1)^j}{2^j} \binom{k-r}{j} (r-j)^{2n} \quad (2.40)$$

and the equality of integrals

$$\int_0^\infty e^{-x} p_n(x) \frac{dx}{x} = 2^{2n-1} \int_0^\infty e^{-2x} p_n(x) \frac{dx}{x}, \quad n \geq 1. \quad (2.41)$$

Concerning other identities and explicit formulas for Bernoulli's numbers see, for instance, a survey article [12] and in [13], [14]. Further, substituting the right-hand side of (2.41) into (1.26), it becomes

$$\zeta(2n) = \frac{(-1)^n \pi^{2n}}{2(2^{2n}-1)(2n-1)!} \int_0^\infty e^{-x} p_n(x) \frac{dx}{x}, \quad n \in \mathbb{N}. \quad (2.42)$$

An explicit formula for the Euler numbers can be deduced similarly to (2.39), (2.40) with the use of the integral (2.20). Hence, due to (1.15), we obtain for all $n \in \mathbb{N}$

$$E_{2n} = \sum_{k=1}^n k! \sum_{r=0}^k \frac{(-1)^r}{2^r} \frac{1}{r!} \sum_{j=0}^{k-r} \frac{(-1)^j (r-j)^{2n}}{2^j j! (k-r-j)!}.$$

Other explicit formulas for Euler numbers see, for instance, in [15]. The latter equality can give another characteristic of the Euler numbers. In fact, we have

$$\begin{aligned} E_{2n} &= \frac{d^{2n}}{dz^{2n}} \sum_{k=1}^n \sum_{r=0}^k \frac{(-1)^r}{2^r} \binom{k}{r} \sum_{j=0}^{k-r} \frac{(-1)^j}{2^j} \binom{k-r}{j} e^{z(r-j)} \Big|_{z=0} \\ &= \frac{d^{2n}}{dz^{2n}} \sum_{k=1}^n (1 - \cosh z)^k \Big|_{z=0}. \end{aligned}$$

Therefore, we find the formula

$$E_{2n} = \frac{d^{2n}}{dz^{2n}} \frac{(1 - \cosh z)(1 - (1 - \cosh z)^n)}{\cosh z} \Big|_{z=0}, \quad n \geq 1.$$

Analogously, coefficients (1.15) of the polynomial sequence $p_n(x)$ take the form

$$a_{n,k} = \frac{(-1)^k 2^k}{k!} \frac{d^{2n}}{dz^{2n}} \sinh^{2k} \left(\frac{z}{2} \right) \Big|_{z=0}, \quad k = 1, 2, \dots, n.$$

In the meantime, equality (2.41) is quite important to derive connection formulas for the Bernoulli and Euler numbers. In fact, the integral in the left-hand side of (2.40) is calculated in [1] and we have

$$\int_0^\infty e^{-x} p_n(x) \frac{dx}{x} = - \sum_{k=0}^{n-1} \binom{2n-1}{2k} E_{2k}, \quad n \geq 1.$$

Consequently, combining with (2.13) and (2.37), we established the connection formula between the Bernoulli and Euler numbers.

Theorem 1. *The following identity holds valid*

$$B_{2n} = \frac{2n}{2^{2n}(2^{2n}-1)} \sum_{k=0}^{n-1} \binom{2n-1}{2k} E_{2k}, \quad n \in \mathbb{N}.$$

Calling (2.42), we get an immediate

Corollary 1. *For all $n \in \mathbb{N}$ one has*

$$\zeta(2n) = \frac{(-1)^{n+1} \pi^{2n}}{2(2^{2n}-1)} \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!(2(n-k)-1)!}, \quad n \in \mathbb{N}.$$

Calling again (2.22), we differentiate through two times and let $x = 0$. Hence with the use of (2.12) and (2.24) we derive a curious recurrence relation for the Euler numbers. Indeed, it has

Theorem 2. *The following identity holds*

$$E_{2n} = 1 - \sum_{k=0}^{n-1} 2^{2(n-k)-1} \binom{2n}{2k} E_{2k}, \quad n \in \mathbb{N}.$$

As an application, we announce at the end of this section an interesting result about the structure of the Bernoulli numbers B_{2n} and the rational values $\zeta(2n)/\pi^{2n}$ (see (2.10), (2.42)), which is a immediate consequence of the Von Staudt- Clausen theorem [16] about the fractional part of Bernoulli numbers and Fermat's Little theorem.

Precisely, it has

Theorem 3. *The Bernoulli numbers B_{2n} and Riemann zeta-values $\zeta(2n)$ satisfy the following properties, respectively,*

$$2(2^{2n}-1)B_{2n} \in \mathbb{Z}, \quad n \in \mathbb{N}, \quad (2.43)$$

$$2(2^{2n}-1) \frac{\zeta(2n)(2n-1)!}{\pi^{2n}} \in \mathbb{Z}, \quad n \in \mathbb{N}. \quad (2.44)$$

Meanwhile identity (2.13) leads to

Corollary 2. *For all $n \in \mathbb{N}$*

$$2n \int_0^\infty e^{-2x} p_n(x) \frac{dx}{x} \in \mathbb{Z}.$$

3 Riemann's zeta-values

Our main goal here is to establish certain identities, integral and series representations for the Riemann zeta function of positive argument. Concerning zeta-values at integers, as we could see in the previous section, the Euler formula (2.10) gives a direct relationship of $\zeta(2n)$, $n \in \mathbb{N}$ with the Bernoulli numbers. However, similar formula for the values of zeta function at odd integers is unknown and probably does not exist. Our attempts to find a finite relation between $\zeta(2n+1)$ are still unsuccessful. Nevertheless, we will derive several integral and series representations, related to these numbers and general positive numbers greater than one, involving our Sheffer's sequences of polynomials. Some rapidly convergent series for $\zeta(2n+1)$ see, for instance, in [17].

In fact, returning to (1.28) and substituting the modified Bessel function by its representation (1.10), we employ the definition of the improper integral, integration by parts, the absolute and uniform convergence and the Riemann- Lebesgue lemma to make the change of the order of integration and motivate the following equalities

$$\begin{aligned} (-1)^n (2n)! (2^{2n+1} - 1) \frac{\zeta(2n+1)}{(2\pi)^{2n}} &= \lim_{N \rightarrow \infty} \int_0^N \tau \int_0^\infty \int_0^\infty e^{-2x \cosh^2(u/2)} p_n(x) \cos(\tau u) \frac{dudxd\tau}{x} \\ &= \lim_{N \rightarrow \infty} \int_0^\infty \int_0^\infty e^{-2x \cosh^2(u/2)} \sinh u p_n(x) \frac{1 - \cos(Nu)}{u} dudx \\ &= \int_0^\infty \int_0^\infty e^{-2x \cosh^2(u/2)} p_n(x) \frac{\sinh u}{u} dudx = \int_0^1 \int_0^\infty K_t(x) e^{-x} p_n(x) dx dt. \end{aligned}$$

Consequently, we derived the identity for all $n \in \mathbb{N}$

$$(-1)^n (2n)! (2^{2n+1} - 1) \frac{\zeta(2n+1)}{(2\pi)^{2n}} = \int_0^1 \int_0^\infty K_t(x) e^{-x} p_n(x) dx dt. \quad (3.1)$$

In the meantime, integrals (1.28), (3.1) have relationships with integrals, involving the Bernoulli polynomials owing to the following representations proved in [1]

$$B_{2n+1} \left(\frac{1-t}{2} \right) = -\frac{2n+1}{2^{2n+1}\pi} \sin \pi t \int_0^\infty K_t(x) e^{-x} p_n(x) dx, \quad |t| < 1, \quad (3.2)$$

$$B_{2n+1} \left(\frac{1-i\tau}{2} \right) = \frac{2n+1}{2^{2n+1}\pi i} \sinh \pi \tau \int_0^\infty K_{i\tau}(x) e^{-x} p_n(x) dx, \quad \tau \in \mathbb{R}. \quad (3.3)$$

Hence, integrating in (3.2) with respect to $t \in (0, 1)$ and taking into account (3.1) we derive the identity

$$(-1)^{n+1} (2n+1)! (2 - 2^{-2n}) \frac{\zeta(2n+1)}{(2\pi)^{2n+1}} = \int_0^1 B_{2n+1} \left(\frac{1-t}{2} \right) \frac{dt}{\sin \pi t}, \quad n \geq 1. \quad (3.4)$$

Moreover, using (2.6), (2.7), (2.8), after integration by parts with elementary substitutions in (3.4) and elimination of the integrated terms, we write it in the form

$$(-1)^{n+1}(2n)! (2 - 2^{-2n}) \frac{\zeta(2n+1)}{2^{2n+1}\pi^{2n}} = \int_0^{1/2} B_{2n}(t) \log(\cot \pi t) dt, \quad n \geq 1. \quad (3.5)$$

One can find a similar identity, for instance, in [18]. The integral in (3.4) can be reduced via properties for the Bernoulli polynomials to certain integrals considered recently in [19]. Furthermore, appealing to the addition formula for the Bernoulli polynomials [8], Vol. I

$$B_n(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k} \quad (3.6)$$

the integral (3.4) can be represented as a linear combination of the moment integrals of $1/\sin t$, which we denote by I_n following [19]

$$I_n = \int_0^{\pi/2} \frac{t^n}{\sin t} dt, \quad n \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \int_0^1 B_{2n+1} \left(\frac{1-t}{2} \right) \frac{dt}{\sin \pi t} &= \int_0^{1/2} B_{2n+1} \left(\frac{1-t}{2} \right) \frac{dt}{\sin \pi t} + \int_0^{1/2} B_{2n+1} \left(\frac{t}{2} \right) \frac{dt}{\sin \pi t} \\ &= \frac{1}{\pi} \sum_{m=0}^n (2\pi)^{2(m-n)-1} \binom{2n+1}{2m} \left[B_{2m} - B_{2m} \left(\frac{1}{2} \right) \right] I_{2(n-m)+1} - \frac{2n+1}{(2\pi)^{2n+1}} I_{2n}. \end{aligned}$$

But

$$B_{2m} \left(\frac{1}{2} \right) = - (1 - 2^{1-2m}) B_{2m}.$$

Consequently, taking this value into account, we substitute the previous sum into (3.4) and after simplifications arrive at the Ramanujan- type identity (cf. [19]) for zeta -values at odd integers ($n \in \mathbb{N}$)

$$\begin{aligned} &(-1)^{n+1}(2n+1)! (1 - 2^{-2n-1}) \zeta(2n+1) + \left(n + \frac{1}{2} \right) I_{2n} \\ &= \sum_{m=0}^{n-1} \binom{2n+1}{2m+1} B_{2(n-m)} \pi^{2(n-m)-1} (2^{2(n-m)} - 1) I_{2m+1}. \end{aligned} \quad (3.7)$$

In particular, letting $n = 1, 2$ and using the well-known formula $I_1 = 2G$, where G is the Catalan constant, we get, respectively,

$$\frac{7}{2} \zeta(3) + I_2 = 2\pi G,$$

$$I_4 - \frac{93}{2}\zeta(5) = 2\pi(I_3 - \pi^2 G).$$

Remark 1. We note, that the latter identities can be also obtained from corresponding equalities in Example I in [19].

In the same manner one can obtain a finite sum representation of the zeta-values at odd integers in terms of the moment integrals of $1/\sin^2 t$. In this case our starting point will be Ito's identity [18]

$$(-1)^n (2n)! \frac{\zeta(2n+1)}{(2\pi)^{2n}} = \int_0^1 B_{2n}(t) \log(\sin \pi t) dt, \quad n \geq 1. \quad (3.8)$$

Indeed, with the use of (2.8) we have,

$$\begin{aligned} \int_0^1 B_{2n}(t) \log(\sin \pi t) dt &= \int_0^{1/2} B_{2n}(t) \log(\sin \pi t) dt \\ &+ \int_0^{1/2} B_{2n}(1-t) \log(\sin \pi t) dt = 2 \int_0^{1/2} B_{2n}(t) \log(\sin \pi t) dt. \end{aligned}$$

Hence, appealing to (2.4) in the right-hand side of the latter equality and integrating twice by parts in the obtained integral, we find

$$\begin{aligned} \int_0^1 B_{2n}(t) \log(\sin \pi t) dt &= 2 \sum_{m=0}^n \binom{2n}{2m} B_{2m} \int_0^{1/2} t^{2(n-m)} \log(\sin \pi t) dt \\ -2n \int_0^{1/2} t^{2n-1} \log(\sin \pi t) dt &= \frac{\pi^{-2n} M_{2n+1}}{2n+1} - \sum_{m=0}^n \binom{2n}{2m} \frac{B_{2m} \pi^{-2(n-m)-1} M_{2(n+1-m)}}{(2(n-m)+1)(n-m+1)}, \end{aligned}$$

where

$$M_n = \int_0^{\pi/2} \frac{t^n}{\sin^2 t} dt, \quad n \geq 2.$$

Thus combining with (3.8), we derived the identity (compare with (3.7))

$$\begin{aligned} &(-1)^{n+1} (2(n+1))! 2^{-2n-1} \zeta(2n+1) + (n+1) M_{2n+1} \\ &= \sum_{m=0}^n \binom{2(n+1)}{2(m+1)} B_{2(n-m)} \pi^{2(n-m)-1} M_{2(m+1)}, \quad n \geq 1. \end{aligned} \quad (3.9)$$

Appealing to relations (2.5.4.7) in [7], Vol. I, we have the values

$$M_2 = \pi \log 2, \quad M_4 = \frac{\pi^3}{2} \log 2 - \frac{9\pi}{4} \zeta(3).$$

Therefore, letting $n = 1$ in (3.9), we get, for instance,

$$\frac{21}{8}\zeta(3) + M_3 = \frac{3\pi^2}{4}\log 2.$$

Further, returning to (3.3), we multiply its both sides by $\tau/\sinh \pi\tau$ and integrate over \mathbb{R}_+ . Hence

$$\begin{aligned} \int_0^\infty \tau B_{2n+1} \left(\frac{1-i\tau}{2} \right) \frac{d\tau}{\sinh \pi\tau} &= \frac{2n+1}{2^{2n+1} \pi i} \int_0^\infty \int_0^\infty \tau K_{i\tau}(x) e^{-x} p_n(x) dx d\tau \\ &= \frac{2n+1}{2^{2n+1} \pi i} \int_0^1 \int_0^\infty K_t(x) e^{-x} p_n(x) x dx dt, \quad n \in \mathbb{N}_0. \end{aligned} \quad (3.10)$$

On the other hand, calling relation (1.20), we deduce from (3.1) and (3.10)

$$\begin{aligned} &\sum_{k=0}^n \binom{2n+1}{2k} \frac{2^{2k+1}}{2k+1} \int_0^\infty i\tau B_{2k+1} \left(\frac{1-i\tau}{2} \right) \frac{d\tau}{\sinh \pi\tau} \\ &= \frac{1}{\pi} \sum_{k=0}^n \binom{2n+1}{2k} \int_0^1 \int_0^\infty K_t(x) e^{-x} p_k(x) x dx dt \\ &= -\frac{1}{\pi} \int_0^1 \int_0^\infty K_t(x) e^{-x} p_{n+1}(x) dx dt = (-1)^n (2(n+1))! (2 - 2^{-2(n+1)}) \frac{\zeta(2n+3)}{\pi^{2n+3}} \end{aligned}$$

or

$$\begin{aligned} &\sum_{k=0}^n \frac{2^{2k}}{(2k+1)!(2(n-k)+1)!} \int_0^\infty i\tau B_{2k+1} \left(\frac{1-i\tau}{2} \right) \frac{d\tau}{\sinh \pi\tau} \\ &= (-1)^n (n+1) (2 - 2^{-2(n+1)}) \frac{\zeta(2n+3)}{\pi^{2n+3}}. \end{aligned} \quad (3.11)$$

Hence, recalling (3.6), the integral in (3.11) can be rewritten as follows

$$\begin{aligned} 2^{2k} \int_0^\infty i\tau B_{2k+1} \left(\frac{1-i\tau}{2} \right) \frac{d\tau}{\sinh \pi\tau} &= \sum_{m=0}^k (-1)^{k-m+1} \binom{2k+1}{2m} (2^{2m-1} - 1) B_{2m} \\ &\times \int_0^\infty \frac{\tau^{2(k-m+1)} d\tau}{\sinh \pi\tau} = (2k+1)! \sum_{m=0}^k (-1)^{m+1} (m+1) \frac{(2^{2(k-m)} - 2) (2^{2(m+2)} - 2) B_{2(k-m)}}{(2(k-m))!} \\ &\times \frac{\zeta(2m+3)}{(2\pi)^{2m+3}}. \end{aligned}$$

Substituting the right-hand side of the latter equality in (3.11), we find the identity

$$\begin{aligned}
& (-1)^n (n+1) (2 - 2^{-2(n+1)}) \frac{\zeta(2n+3)}{\pi^{2n+3}} = \sum_{k=0}^n \sum_{m=0}^k \frac{(-1)^{m+1} (m+1)}{(2(n-k)+1)!} \\
& \times \frac{(2^{2(k-m)} - 2) (2 - 2^{-2(m+1)}) B_{2(k-m)}}{(2(k-m))! \pi^{2m+3}} \zeta(2m+3) = \sum_{m=0}^n (-1)^{m+1} (m+1) (2 - 2^{-2(m+1)}) \\
& \times \frac{\zeta(2m+3)}{\pi^{2m+3}} \left(\sum_{k=0}^{n-m} \frac{(2^{2k} - 2) B_{2k}}{(2k)! (2(n-m-k)+1)!} \right). \tag{3.12}
\end{aligned}$$

It would be a great achievement to have here a finite recurrence relation for zeta- values at odd integers. However, unfortunately, this is not the case. In fact, (3.12) yields for all $n \geq 1$

$$\sum_{m=0}^{n-1} (-1)^{m+1} (m+1) (2 - 2^{-2(m+1)}) \frac{\zeta(2m+3)}{\pi^{2m+3}} \left(\sum_{k=0}^{n-m} \frac{(2^{2k} - 2) B_{2k}}{(2k)! (2(n-m-k)+1)!} \right) = 0. \tag{3.13}$$

Theorem 4. *For all $n \in \mathbb{N}$ the following identity holds for Bernoulli numbers*

$$\sum_{k=0}^n \binom{2n+1}{2k} (2^{2k-1} - 1) B_{2k} = 0. \tag{3.14}$$

Proof. In fact, recalling (2.3), we see that (3.14) is equivalent to the equality

$$\sum_{k=0}^n \binom{2n+1}{2k} 2^{2k} B_{2k} = 2n+1,$$

which yields

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} 2^{k-2n-1} B_k = 0.$$

But this is true, because the left-hand side is equal (see (2.4), (2.8)) to $B_{2n+1}(1/2) = 0$. \square

Theorem 4 says that all coefficients in front of zeta-values $\zeta(2m+3)$ in (3.14) are equal to zero. Hence such kind of equalities can be a source to obtain possibly new identities for Bernoulli numbers.

Finally, we will get an integral representation of zeta-values at positive numbers, which is a direct consequence of the formulas (1.3), (1.25), (1.26). Precisely, it has

Theorem 5. *Let $\alpha > 1$, $[\alpha]$ be its integer part and $\{\alpha\}$ be its fractional part. Then the following identities take place, when $[\alpha]$ is even or odd, respectively,*

$$\frac{2^\alpha - 1}{(2\pi)^{\alpha-1}} \Gamma(\alpha) \zeta(\alpha) = (-1)^{[\alpha]/2} \int_0^\infty \int_0^\infty \tau^{\{\alpha\}} K_{i\tau}(x) e^{-x} p_{[\alpha]/2}(x) \frac{dx}{x},$$

$$\frac{2^\alpha - 1}{(2\pi)^{\alpha-1}} \Gamma(\alpha) \zeta(\alpha) = (-1)^{([\alpha]-1)/2} \int_0^\infty \int_0^\infty \tau^{\{\alpha\}-1} K_{i\tau}(x) e^{-x} p_{([\alpha]+1)/2}(x) \frac{dx}{x}.$$

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