SUBALGEBRA DEPTHS WITHIN THE PATH ALGEBRA OF AN ACYCLIC QUIVER

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Abstract. Constraints are given on the depth of diagonal subalgebras in generalized triangular matrix algebras. The depth of the top subalgebra $B \cong A/\operatorname{rad} A$ in a finite, connected, acyclic quiver algebra A over an algebraically closed field $\mathbb{K}$ is then computed. Also the depth of the primary arrow subalgebra $1\mathbb{K} + \operatorname{rad} A = B$ in A is obtained. The two types of subalgebras have depths 3 and 4 respectively, independent of the number of vertices. An upper bound on depth is obtained for the quotient of a subalgebra pair.

1. Introduction

Given a subalgebra pair, one extracts a (minimum) depth from a comparison of n-fold tensor products of the subalgebra pair with one another in a meaningful way. The interesting case is when an $(n+1)$-fold tensor product divides a multiple of the n-fold tensor product in the sense of Krull-Schmidt unique factorization into indecomposable bimodules, or more generally as a bimodule isomorphism with a direct summand. The bimodule structures on the n-fold tensor products are naturally any one of four possibilities as left and right modules over the subalgebra or overalgebra. The least restrictive of these conditions is two-sided over the subalgebra and we fix the depth in the situation mentioned above to be $2n+1$; for mixed bimodules, we have the left and right depth $2n$ conditions [3]. The most stringent condition, as bimodules of the overalgebra, is H-depth $2n-1$ [16], and is useful to ordinary depth gauging as well when the overalgebra has nice bimodules such as a separable algebra (see Proposition 2.1 below).

Comparing the tensor-square of an algebra extension with the overalgebra as mixed bimodules leads to a characterization of the Galois extension [12, 14, 13]. Thus not unexpectedly the depth two condition placed on Hopf subalgebras is equivalent to the normality condition with respect to the adjoint actions [4]. The depth three condition is satisfied by a subalgebra $B \subseteq A$ when, in a suitably nice category of bimodules, A contains all B^e-indecomposables that can possibly appear up to isomorphism in decompositions of tensor products $A \otimes_B \cdots \otimes_B A$ [4, 15]. Semisimple complex subalgebra pairs of each depth $n \in \mathbb{N}$ are noted in [8] via bipartite graphs and inclusion matrices for $K_0(B) \to K_0(A)$.

In the paper [3] it was shown that the depth of a finite group algebra extension is bounded by twice the index of the normalizer of the subgroup in the group. In the papers [7, 8, 3, 9, 10] the depth of certain group algebra extensions are computed; for example, [10] computes the depth of all the subgroups of $PSL(2,q)$ viewed as complex group algebras. In [8] the complex group algebras associated to the permutation groups are shown to have depth $d(S_n, S_{n+1}) = 2n-1$; in [3], this same result is shown to not depend on the ground ring.

It was noted in the paper [15] that a subalgebra B in a finite-dimensional algebra A has finite depth $d(B, A)$ if B^e has finite representation type; below we note that this holds if A^e has finite representation type. In addition it is possible in algebras without involution that a subalgebra having left depth $2n$ may not have right depth $2n$. Moreover, the matrix power inequality characterizing depth n subalgebra pairs of semisimple complex algebras in [7, 8] breaks down in the presence of indecomposables of length greater than one. For these reasons, it becomes interesting to begin a study of depth of subalgebras in path algebras of quivers. A reasonable place to start is with acyclic quivers for whose path algebras there is a classic theorem about which have finite representation type in terms of Dynkin diagrams and the underlying graphs [1]. This paper computes the depth of the top and arrow subalgebras of the path algebra of a finite, connected, acyclic quiver. In Section 3 we note constraints on the depth of a diagonal subalgebra of a generalized matrix ring. We also note an inequality of depth in case the subalgebra contains ideals of the overalgebra, perhaps useful in computing depth of certain subalgebras of bounded quiver algebras. In the last Section 6 of concluding remarks we discuss other subalgebras of certain quiver algebras and their depth.

2. Preliminaries on depth

Given a unital associative ring R and unital R-modules M and N, we say that M divides N and write $M \mid N$ if $N \cong M \oplus *$ as R-module for some (unnamed) complementary module. If there are natural numbers r and s such that $N \mid rM = M \oplus \cdots \oplus M$ and $M \mid sN$, then M and N are H-equivalent (or similar), as R-modules; denoted by $M \sim N$. Note that this is indeed an equivalence relation. In this case their endomorphism rings $\operatorname{End} M_R$ and $\operatorname{End} N_R$ are Morita equivalent with Morita context bimodules $\operatorname{Hom}(M_R, N_R)$ and $\operatorname{Hom}(N_R, M_R)$ (with module actions and Morita pairings given by composition).

If M and N are in a category of finitely generated R-modules having unique factorization into indecomposables, then M and N have the same indecomposable constituents if and only if M and N are H-equivalent modules. If F is an additive endofunctor of the category of R-modules, then $M \sim N$ implies $F(M) \sim F(N)$; which in practice means that H-equivalent bimodules may replace one another in certain H-equivalences of tensor products. In addition, $M \sim N$ and $U \sim V$ implies $M \oplus U \sim N \oplus V$.

Throughout this paper, let A be a unital associative ring and $B \subseteq A$ a subring where $1_B = 1_A$. Note the natural bimodules ${}_BA_B$ obtained by restriction of the natural A-A-bimodule (briefly A-bimodule) A, also to the natural bimodules ${}_BA_A$, ${}_AA_B$ or ${}_BA_B$, which are referred to with no further notation. Equivalently we denote the proper ring extension $A \supseteq B$ occasionally by $A \mid B$. (Often results are valid as well for a ring homomorphism $B \to A$ and its induced bimodules on A.)

Let $C_0(A, B) = B$, and for $n \geq 1$,

$$C_n(A, B) = A \otimes_B \cdots \otimes_B A \quad (n \text{ times } A)$$

For $n \geq 1$, the $C_n(A, B)$ has a natural A-bimodule structure given by $a(a_1 \otimes \cdots \otimes a_n)a' = aa_1 \otimes \cdots \otimes a_n a'$. Of course, this bimodule structure restricts to B-A-, A-B- and B-bimodule structures as we may need them. Let $C_0(A, B)$ denote the natural B-bimodule B itself. Recall from [3, 15] that a subring $B \subseteq A$ has right depth $2n$

if

$$C_{n+1}(A,B) \sim C_n(A,B) \tag{1}$$

as natural A-B-bimodules; left depth $2n$ if the same condition holds as B-A-bimodules; if both left and right conditions hold, it has depth $2n$; and depth $2n+1$ if the same condition holds as B-bimodules. If condition (1) holds in its strongest form as A-A-modules for $n \geq 1$ the subring $B \subseteq A$ is said to have H-depth $2n-1$; H-depth is investigated in [16].

Note that if the subring has left or right depth $2n$, it automatically has depth $2n+1$ by restriction to B-bimodules. Also note that if the subring has depth $2n+1$, it has depth $2n+2$ by tensoring the H-equivalence by $- \otimes_B A$ or $A \otimes_B -$. The *minimum depth* (or just depth when the context makes it clear) is denoted by $d(B,A)$; if $B \subseteq A$ has no finite depth, write $d(B,A) = \infty$. There is hidden in this a subtlety: if there is a subring $B \subseteq A$ of left depth $2n$ but not of right depth $2n$, then it has depth $2n+1$, left and right depth $2n+2$, and nevertheless its minimum depth is $2n$. There is not a published example of such a subring at present (but a search for this must occur outside the class of QF extensions [15, Th. 2.4]). Note too that if $B \subseteq A$ has H-depth $2n-1$, it has depth $2n$ by restriction.

In practice one only need check half of the condition in (1) to establish depth $2n$ or $2n+1$ of a ring extension $A \supseteq B$. This is due to the fact that it is always the case that $C_n(A,B) \,|\, C_{n+1}(A,B)$ for $n \geq 1$ via appropriate face and degeneracy maps in the relative homological bar complex; e.g. the A-A-epimorphism $a_1 \otimes a_2 \mapsto a_1 a_2$ is split by the B-A-monomorphism $a \mapsto 1 \otimes_B a$, whence $C_1(A,B) \,|\, C_2(A,B)$ as B-A-bimodules.

For a k-algebra B let B^e denote $B \otimes_k B^{\rm op}$. For a finite dimensional dimensional algebra A let n_A denote the cardinal number of isomorphism classes of indecomposable finitely generated A-modules. Of course each of the B^e-modules $C_n(A,B)$ are finitely generated when A is a finite dimensional algebra.

Proposition 2.1. *Let $B \subseteq A$ be a subring pair of finite dimensional algebras. If B^e has finite representation type, then $d(B,A) \leq 1 + 2n_{B^e}$. If A^e has finite representation type, then $d(B,A) \leq 2n_{A^e}$. If $A \otimes B^{\rm op}$ has finite representation type, then $d(B,A) \leq 2n_{A \otimes B^{\rm op}}$.*

Proof. If B^e has finite representation type, it is shown in [15] that subring depth $d(B,A)$ is finite based on two basic facts. First, a finitely generated module M over a finite dimensional algebra divides a multiple of another module N if and only if their Krull-Schmidt unique factorization into indecomposable modules possess the indecomposable constituents satisfying $\text{Indec}\,(M) \subseteq \text{Indec}\,(N)$; then M and N are H-equivalent iff $\text{Indec}\,(M) = \text{Indec}\,(N)$. Secondly, from $C_n(A,B) \,|\, C_{n+1}(A,B)$ we obtain $\text{Indec}\, C_n(A,B)$ as sequence of subsets of a finite number of indecomposables that grows with n.

If A^e has finite representation type, then one applies the same argument with growing $\text{Indec}\, C_n(A,B)$, this time as A-A-bimodules, which shows that $C_{N+1}(A,B)$ and $C_N(A,B)$ are H-equivalent after at most $N = n_{A^e}$ steps. Then the minimum H-depth $d_H(B,A) \leq 2N-1$, and one notes by restricting modules that $d(B,A) \leq 2N$. The last statement is proven similarly using the definition of even depth. □

Corollary 2.2. *Suppose $B \subseteq A$ is a subalgebra pair where either A or B is a separable algebra. Then depth $d(B,A)$ is finite.*

3. Constraints on subring depth in triangular matrix rings

Let R and S be unital associative rings. Suppose ${}_SM_R$ is a unital S-R-bimodule as suggested by the notation. There is a triangular matrix ring, denoted by A, associated with this data,

$$A := \begin{pmatrix} R & 0 \\ M & S \end{pmatrix} \tag{2}$$

with the obvious matrix addition and multiplication, which defines a well-known class of examples in the demonstration of independence of axioms in ring theory such as left and right noetherian property of rings.

Note the subring of diagonal matrices in A is isomorphic (and identified) with $R \times S$. The obvious split epimorphism of rings $A \to R \times S$ is denoted by $\pi : \begin{pmatrix} r & 0 \\ m & s \end{pmatrix} \mapsto (r, s)$. The mapping π is of course an isomorphism if $M = 0$. Also note the orthogonal idempotents $e_1 = (1_R, 0)$ and $e_2 = (0, 1_S)$, where $A = e_1A \oplus e_2Ae_1 \oplus Ae_2$.

Let R' be a unital subring of R, and S' a unital subring of S. Then $B := R' \times S'$ is a subalgebra of diagonal matrices in A. We will be interested in the depth $d(B, A)$. At first we will dispose of the case $M = 0$ and note that $d(R' \times S', R \times S) = \max\{d(R', R), d(S', S)\}$. (This proposition should be compared with [8, Prop. 3.15].)

Proposition 3.1. *The depth of a subalgebra of a direct product of rings is given by*

$$d(R' \times S', R \times S) = \max\{d(R', R), d(S', S)\}.$$

Proof. Let $A = R \times S$ and $B = R' \times S'$. Note that the central orthogonal idempotents $e_1, e_2 \in B \subseteq A$. It follows that there is the following isomorphism of n-fold tensor products (any $n \in \mathbb{N}$),

$$C_n(A, B) \cong C_n(R, R') \oplus C_n(S, S') \tag{3}$$

as B-B-, A-B- and B-A-bimodules up to a trivial extension of for example R-module to A-module by $S \cdot x = 0$, all elements x in the module. Such a decomposition holds as well for bimodule homomorphisms between n- and $n+1$-fold tensor products.

Let $2m + 1 \geq \max\{d(R', R), d(S', S)\}$. Then the righthand-side of (3) where $n = m + 1$ divides a multiple of the m-fold tensor product of the same form, then so does the lefthand-side. Hence $d(B, A) \leq 2m + 1$. If both depths $d(R', R)$ and $d(S', S)$ are even, the same argument replacing $2m+1$ with $2m$ suffices to establish $d(B, A) \leq \max\{d(R', R), d(S', S)\}$. Note that the argument works for 0-fold tensor product and depth one case too. The reverse inequality follows from applying the central idempotents to $C_n(A, B) \sim C_{n+1}(A, B)$. □

Next we continue the notation $B = R' \times S'$ and A as the triangular matrix ring formed from the rings R, S and the bimodule ${}_SM_R \neq 0$. Let $\mathcal{M}$ denote a category of modules or bimodules, where left and right subscripts denote the rings in action.

Lemma 3.2. *As abelian categories,*

$${}_B\mathcal{M}_B \cong {}_{R'}\mathcal{M}_{R'} \oplus {}_{R'}\mathcal{M}_{S'} \oplus {}_{S'}\mathcal{M}_{R'} \oplus {}_{S'}\mathcal{M}_{S'}$$

Proof. This isomorphism is induced on objects by ${}_BV_B \mapsto e_1Ve_1 \oplus e_1Ve_2 \oplus e_2Ve_1 \oplus e_2Ve_2$. Conversely, an object (W_1, W_2, W_3, W_4) on the right side is sent to a matrix

$\begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$ with left action by row vectors (r,s) and right action by column vectors $\begin{pmatrix} r' \\ s' \end{pmatrix}$. A B-bimodule homomorphism $f : V \to W$ commutes with e_1, e_2 from left and right, so that f sends e_iVe_j into e_iWe_j for all $i,j = 1,2$. Conversely, a morphism of 2×2 matrices as before commutes with row and column vectors, and so is a B-bimodule homomorphism. □

We now apply the lemma to the B-bimodules, the n-fold tensor products of the triangular matrix ring A over the diagonal subalgebra B.

Lemma 3.3. *For integer $n \geq 1$, $e_1C_n(A,B)e_1 = C_n(R,R')$, $e_1C_n(A,B)e_2 = 0$, $e_2C_n(A,B)e_2 = C_n(S,S')$ and*

$$e_2C_n(A,B)e_1 = \sum_{r=0}^{n-1} \oplus\, C_r(S,S') \otimes_{S'} M \otimes_{R'} C_{n-1-r}(R,R') \tag{4}$$

Proof. For $a_1,\ldots,a_n \in A$, the computations follow from $e_1a_1 \otimes_B \cdots \otimes_B a_n = e_1a_1e_1 \otimes \cdots \otimes_B a_n = \cdots = e_1a_1 \otimes_B \cdots \otimes_B e_1a_n$; moreover, $a_1 \otimes_B \cdots \otimes_B a_ne_2 = a_1 \otimes_B \cdots \otimes_B e_2a_ne_2 = \cdots = a_1e_2 \otimes_B \cdots \otimes_B a_ne_2$; furthermore, $e_1a_1 \otimes_B \cdots \otimes_B a_ne_2 = 0$ by referring to the last computation and noting $e_1Ae_2 = 0$. Naturally, $C_n(e_1A,B) = C_n(R,R')$ since $B = R' \times S'$ and S' acts as zero, so the relative tensor product is given by factoring out by only the nonzero relations; the same is true of $C_n(Ae_2,B) = C_n(S,S')$.

Finally, the last equation follows from $e_2a_1 \otimes_B \cdots \otimes_B a_ne_1 = (e_2a_1e_2 + e_2a_1e_1) \otimes_B \cdots \otimes_B (e_2a_ne_1 + e_1a_ne_1) = \cdots = \sum_{i=1}^n a_1e_2 \otimes_B \cdots \otimes e_2a_ie_1 \otimes_B \cdots \otimes_B e_1a_n$. This follows from cancellations of the type $\cdots \otimes a_ie_1 \otimes_B \cdots \otimes_B e_2a_j \otimes_B \cdots = 0$ since $e_1a_k = e_1a_ke_1$, $a_ke_2 = e_2a_ke_2$ for all $a_k \in A$ and of course $e_1e_2 = 0$. □

Let $d_{\text{odd}}(B,A)$ be the smallest odd number greater than or equal to $d(B,A)$, which we call the odd depth of the subring $B \subseteq A$. If the depth is finite and already odd, then $d_{\text{odd}}(B,A) = d(B,A)$, and otherwise $d_{\text{odd}}(B,A) = d(B,A)+1$. In other words, a ring extension $A \,|\, B$ has $d_{\text{odd}}(B,A) = 2n+1$ if the natural B-B-bimodules $C_{n+1}(A,B) \sim C_n(A,B)$ and n is the smallest such natural number.

Theorem 3.4. *The odd depth $d_{\text{odd}}(B,A)$ satisfies the inequalities,*

$$d(B, R \oplus S) \leq d_{\text{odd}}(B,A) \leq d_{\text{odd}}(R',R) + d_{\text{odd}}(S',S) + 1 \tag{5}$$

Proof. If $B \subseteq A$ has depth $2n+1$, then there is $q \in \mathbb{N}$ such that $C_{n+1}(A,B) \oplus V \cong qC_n(A,B)$ for some B-B-bimodule V. It follows that $e_iC_{n+1}(A,B)e_i \oplus e_iVe_i \cong qe_iC_n(A,B)e_i$ for $i = 1,2$, so that $C_{n+1}(R,R') \,|\, qC_n(R,R')$ and $C_{n+1}(S,S') \,|\, qC_n(S,S')$. It follows that $R' \subseteq R$ and $S' \subseteq S$ both have depth $2n+1$. Then $\max\{d(R',R), d(S',S)\} \leq d_{\text{odd}}(B,A)$. This completes the proof of the first of the two inequalities.

Next let $R' \subseteq R$ and $S' \subseteq S$ have depths $2n+1$ and $2m+1$ respectively. This means that for each integer $s \geq 1$ and $r \geq 0$ there is $q \in \mathbb{N}$ such that $C_{n+s}(R,R') \,|\, qC_{n+r}(R,R')$ as B-B-bimodules (and similarly for $S' \subseteq S$). Consider $C_{n+m+2}(A,B)$ as a natural B-B-bimodule. By the lemma, $C_{n+m+2}(A,B) \cong$

$$C_{n+m+2}(R,R') \oplus C_{n+m+2}(S,S') \oplus \sum_{i=0}^{n+m+1} \oplus C_i(S,S') \otimes_{S'} M \otimes_{R'} C_{n+m+1-i}(R,R')$$

which divides as B-B-bimodules (due to the depth hypotheses) a multiple of

$$C_{n+m+1}(R,R') \oplus C_{n+m+1}(S,S') \oplus \sum_{j=0}^{n+m} C_j(S,S') \otimes_{S'} M \otimes_{R'} C_{n+m-j}(R,R'),$$

which is isomorphic to a multiple of $C_{n+m+1}(A,B)$. Hence $B \subseteq A$ has depth $2(n+m+1)+1 = 2n+2m+3$. This establishes that $d(B,A) \leq d_{\text{odd}}(B,A) \leq d_{\text{odd}}(R',R) + d_{\text{odd}}(S',S) + 1$. □

Note that the proof shows that if $R' \subseteq R$ and $S' \subseteq S$ are subrings of finite depth, then so is $B \subseteq A$, and conversely.

3.1. **Quotient Algebras and Depth Bounds.** Let $B \subseteq A$ be an arbitrary algebra extension and let $I \subseteq B$ be an A-ideal. For purposes of expedient notation we write $B_I := B/I$ and similarly for A_I. The main purpose of this section is to give some depth bounds for $B_I \subseteq A_I$ as another algebra extension. It turns out that if $d(B,A)$ is finite, then so is $d(B_I, A_I)$.

Recall that if the extension $B \subseteq A$ has odd depth $2n+1$ (even depth $2n$) then

$$C_{n+1}(A,B) \sim C_n(A,B)$$

as B-bimodules (A-B-bimodules), which is equivalent to saying that there're two B-B-homomorphisms $f : C_{n+1}(A,B) \to mC_n(A,B)$ and $g : mC_n(A,B) \to C_{n+1}(A,B)$ such that $g \circ f = id$.

Lemma 3.5 (π and σ properties)**.** *Suppose that $B \subseteq A$ and $I \subseteq B$ are as above. We define the following maps:*

$$\pi : C_n(A,B) \to C_n(A_I, B_I)$$
$$: a_1 \otimes \ldots \otimes a_n \mapsto \overline{a_1} \otimes \ldots \otimes \overline{a_n}.$$

$$\sigma : C_{n+1}(A,B) \to C_{n+1}(A_I, B_I)$$
$$: a_1 \otimes \ldots \otimes a_{n+1} \mapsto \overline{a_1} \otimes \ldots \otimes \overline{a_{n+1}}.$$

These two maps are well-defined and will be k-linear as well as satisfying

$$\pi(r\heartsuit s) = \overline{r}\pi(\heartsuit)\overline{s} \text{ and } \sigma(r\diamondsuit s) = \overline{r}\sigma(\diamondsuit)\overline{s},$$

$\forall r, s \in R$, $\forall \heartsuit \in C_n(A,B)$ and $\forall \diamondsuit \in C_{n+1}(A,B)$.

As will be necessary in our next result we "raise π to the m^{th} power" in that we define $\pi' : mC_n(A,B) \to mC_n(A_I,B_I)$ in the obvious way:

$$(\heartsuit_i) \mapsto (\pi(\heartsuit_i)).$$

The important thing to note however is that $\pi'(r\heartsuit_i s) = \overline{r}\pi'(\heartsuit_i)\overline{s}$, where $r, s \in R$ and $\heartsuit_i \in mC_n(A,B)$, furthermore π' is k-linear over elements of $mC_n(A,B)$.

Theorem 3.6. *Suppose that $B \subseteq A$ is an algebra extension with depth $2n+1$ ($2n$), suppose also that $I \subseteq B \subseteq A$ is an A-ideal. Then $B_I \subseteq A_I$ also has depth $2n+1$ ($2n$). Indeed we can say $d(B_I, A_I) \leq d(B,A)$.*

Proof. We prove the odd case because it involves B-bimodules and the proof can be extended to the even case with A-B-bimodules. First, because $B \subseteq A$ has depth $2n+1$ we have B-bimodule maps $f : C_{n+1}(A,B) \to mC_n(A,B)$ and g :

$mC_n(A,B) \to C_{n+1}(A,B)$ such that $g \circ f = id$, where $m \geq 1$. We'd like first to find an B_I-bimodule map

$$\widetilde{f} : C_{n+1}(A_I, B_I) \to mC_n(A_I, B_I)$$

and secondly another B_I-bimodule map

$$\widetilde{g} : mC_n(A_I, B_I) \to C_{n+1}(A_I, B_I)$$

such that $\widetilde{g} \circ \widetilde{f} = id$.

We define $\widetilde{f}$ as follows:

$$\widetilde{f}(\overline{a_1} \otimes \ldots \otimes \overline{a_n}) := \pi' \circ f(a_1 \otimes \ldots \otimes a_n) \tag{6}$$

We must show that $\widetilde{f}$ is well-defined, and to that end with some $1 \leq p \leq n$ let $\overline{a_p} = \overline{y}$, that is $a_p = y + t$, for $t \in I$. Thus

$$\begin{aligned}
\widetilde{f}(\overline{a_1} \otimes \ldots \otimes \overline{a_p} \otimes \ldots \otimes \overline{a_n}) &= \pi' f(a_1 \otimes \ldots \otimes y + t \otimes \ldots \otimes a_n) \\
&= \pi' f(a_1 \otimes \ldots \otimes y \otimes \ldots \otimes a_n) + \pi' f(a_1 \otimes \ldots \otimes t \otimes \ldots \otimes a_n) \\
&= \pi' f((a_1 \otimes \ldots \otimes y \otimes \ldots \otimes a_n)) \\
&= \widetilde{f}(\overline{a_1} \otimes \ldots \otimes \overline{y} \otimes \ldots \otimes \overline{a_n})
\end{aligned}$$

since $\pi' f(a_1 \otimes \ldots \otimes a_{p-1} \otimes t \otimes a_{p+1} \otimes \ldots \otimes a_n) = \pi' f(a_1 \otimes \ldots \otimes t_1 \otimes 1 \otimes a_{p+1} \otimes \ldots \otimes a_n)$ etc until we have $\pi'(t_p f(1 \otimes \ldots \otimes 1 \otimes a_{p+1} \otimes \ldots \otimes x_n)) = \overline{t_p}(\pi' f(1 \otimes \ldots \otimes a_n)) = 0$ (where each $t_i \in I$). This all follows because $I \subseteq B$ is an A-ideal with the properties of lemma (3.5) in effect. Repeating such a process over all $1 \leq p \leq n$ the map will be well-defined.

Now we describe $\widetilde{g}$:

$$\widetilde{g}((\overline{a_1} \otimes \ldots \otimes \overline{a_{n+1}})_i) := \sigma \circ g((a_1 \otimes \ldots \otimes a_{n+1})_i) \tag{7}$$

Proving that $\widetilde{g}$ is well-defined is so similar to the (6) case it can be considered a minor exercise. Furthermore we should notice that $\widetilde{g} \circ \pi' = \sigma \circ g$ straight off. Using (6) and (7) we demonstrate that $\widetilde{g} \circ \widetilde{f} = id$:

$$\begin{aligned}
\widetilde{g} \circ \widetilde{f}(\overline{a_1} \otimes \ldots \otimes \overline{a_n}) &= \widetilde{g} \circ \pi' \circ f(a_1 \otimes \ldots \otimes a_n) \\
&= \sigma \circ g \circ f(a_1 \otimes \ldots \otimes a_n) \\
&= \sigma \circ id(a_1 \otimes \ldots \otimes a_n) \\
&= \overline{a_1} \otimes \ldots \otimes \overline{a_n}
\end{aligned}$$

□

Corollary 3.7. *Given a chain of A-ideals $J_0 \subseteq J_1 \subseteq \ldots \subseteq B$ we have*

$$\ldots \leq d(B_{J_1}, A_{J_1}) \leq d(B_{J_0}, A_{J_0}) \leq d(B, A)$$

Proof. The second isomorphism theorem tells us that $(B/J_0)/(J_1/J_0) \cong B/J_1$. Apply our last theorem to see that the depth of $(B/J_0)/(J_1/J_0) \subseteq (A/J_0)/(J_1/J_0)$ is less than or equal to the depth of $(B/J_0) \subseteq (A/J_0)$, but then we're done. □

4. Depth of top subalgebra in path algebra of acyclic quiver

Let $Q = (V, E, s, t)$ denote a finite connected acyclic quiver with vertices V of cardinality $|V| = n$ and oriented edges E such that $|E| < \infty$, where an oriented edge or arrow is denoted by $\alpha : a \to b$, or $(a|\alpha|b) \in E$, where $a = s(\alpha)$ and $b = t(\alpha)$ define the source and target mappings $E \to V$, respectively. Since Q is acyclic, there is no loop in E, i.e., no arrow $\beta \in E$ such that $s(\beta) = t(\beta)$; moreover, there are no other cycles, i.e., paths $(a|\alpha_1, \ldots, \alpha_r|a)$ of length $r > 1$ beginning at a vertex a and ending there (where all $\alpha_i \in E$ and $s(\alpha_{i+1}) = t(\alpha_i)$, $i = 1, \ldots, r-1$).

Let $\mathbb{K}$ be an algebraically closed field and let $A = \mathbb{K}Q$ be the path algebra on the quiver A [1, 17] with basis the set of all paths, including stationary paths denoted by $\varepsilon_a = (a||a)$ for each $a \in V$, such that the product of two basis elements is given by the following concatenation formula:

$$(a|\alpha_1, \ldots, \alpha_r|b)(c|\beta_1, \ldots, \beta_s|d) = \delta_{bc}(a|\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s|d). \tag{8}$$

The product on A is given by this formula and linearization, which clearly makes A into a graded algebra where A_s denotes the $\mathbb{K}$-vector subspace spanned by paths of length s, a complete set of primitive orthogonal idempotents are $\{\varepsilon_a | a \in V\} \in A_0$ and the radical ideal is $\operatorname{rad} A = A_1 \oplus A_2 \oplus \cdots$, also known as the arrow ideal.

There is always a numbering of the vertices from $1, \ldots, n$ such that $(i|\alpha|j) \in E$ implies $i > j$ [17, cor. 8.6]. The vertex n is then a source and 1 a sink. With such a numbering the algebra $A = \mathbb{K}Q$ is embeddable in a lower triangular matrix algebra [1, Lemma 1.12] of the form,

$$A = \begin{pmatrix} \varepsilon_1(\mathbb{K}Q)\varepsilon_1 & 0 & \cdots & 0 \\ \varepsilon_2(\mathbb{K}Q)\varepsilon_1 & \varepsilon_2(\mathbb{K}Q)\varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \varepsilon_n(\mathbb{K}Q)\varepsilon_1 & \varepsilon_n(\mathbb{K}Q)\varepsilon_2 & \cdots & \varepsilon_n(\mathbb{K}Q)\varepsilon_n \end{pmatrix} \tag{9}$$

Note that $\varepsilon_i(\mathbb{K}Q)\varepsilon_i \cong K$ for each $i = 1, \ldots, n$ since there are no cycles. For example, if the quiver Q has no multiple arrows between vertices and its underlying graph is a tree, then there is at most one path between two points $i > j$, so that $\dim \varepsilon_i(\mathbb{K}Q)\varepsilon_j \leq 1$, and $A = \mathbb{K}Q$ is isomorphic to a subalgebra of the full triangular matrix algebra $T_n(\mathbb{K}) = \sum_{n \geq i \geq j \geq 1} \mathbb{K} e_{ij}$ (in terms of matrix units e_{ij}).

Another example: if $Q = (V, E)$ where $V = \{1, 2\}$ and $E = \{\alpha, \beta : 2 \to 1\}$, then

$$A = \mathbb{K}Q = \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K}^2 & \mathbb{K} \end{pmatrix} \tag{10}$$

From the result of the previous section, we note that with $M = \mathbb{K}^2$, and $B = \mathbb{K}\varepsilon_1 + \mathbb{K}\varepsilon_2$, the depth of B in A is bounded by

$$1 \leq d(B, A) \leq 3. \tag{11}$$

For this algebra, one constructs from nilpotent Jordan blocks of order m an infinite sequence of indecomposable A-modules [1, pp. 75-76], a tame Kronecker algebra [2, V111.7]. The algebra $A = \mathbb{K}Q$ has finite representation type if and only if the underlying (multi-) graph of Q is one of the Dynkin diagrams $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7, E_8$: see for example [1, Gabriel's Theorem, 5.10] or [2, VIII.5.2].

Coming back to the algebra A in (9), note that A has n augmentations $\rho_i : A \to \mathbb{K}$ given by $\rho_i(\lambda_1, \ldots, \lambda_n) = \lambda_i$. Let A_i^+ denote $\ker \rho_i$, and for a subalgebra $B \subseteq A$, let B_i^+ denote $\ker \rho_i \cap B$. Denote the n A-simples of dimension one by ${}_{\rho_i}\mathbb{K}$, and the

n^2 A^e-simples by $\mathbb{K}_{ij}$ where $a \cdot 1 \cdot b = \rho_i(a)\rho_j(b)1$ for all $a, b \in A$ and $i, j = 1, \ldots, n$. We have the following

Lemma 4.1. *Suppose $B \subseteq A$ is a subalgebra of an algebra with augmentations $\rho_1, \ldots, \rho_n$. If $B \subseteq A$ has right depth 2, then $AB_i^+ \subseteq B_i^+ A$ for each $i = 1, \ldots, n$. If $B \subseteq A$ has left depth 2, then $B_i^+ A \subseteq AB_i^+$ for each $i = 1, \ldots, n$.*

Proof. We prove the statement about a subalgebra having left depth two, namely, $A \otimes_B A \,|\, qA$ as B-A-bimodules. To this apply the additive functor $- \otimes_{A\, \rho_i} \mathbb{K}$, which results in $A/AB_i^+ \,|\, q\mathbb{K}$ as left B-modules. The annihilator of $q\mathbb{K}$ restricted to B is of course B_i^+, which then also annihilates A/AB_i^+, so $B_i^+ A \subseteq AB_i^+$. This holds for each $i = 1, \ldots, n$. The opposite inclusion is similarly shown to be satisfied by a right depth 2 extension of augmented algebras. □

The next theorem computes the depth $d(B, A)$ of the top subalgebra $A/\mathrm{rad}\, A \cong \mathbb{K}^n$, or subalgebra of diagonal matrices, in the path algebra A of an acyclic quiver as given in (9).

Theorem 4.2. *Suppose the number of vertices $n > 1$ in the quiver Q, $A = \mathbb{K}Q$ and $B = \mathbb{K}^n$. Then depth $d(B, A) = 3$.*

Proof. If the subalgebra in question has depth 1, it has depth 2. But if it has left depth 2, the lemma above applies, so that $B_i^+ A \subseteq AB_i^+$ for each $i = 1, \ldots, n$. Note that AB_i^+ are all the lower triangular matrices of the form in (9) having only 0's on column i; similarly, $B_i^+ A$ are the triangular matrices having only zeroes on row i. It follows that $\varepsilon_j A \varepsilon_i = 0$ for each $j = i + 1, \ldots, n$. But $\varepsilon_j(\mathbb{K}Q)\varepsilon_i$ consists of all the paths from j to i. Since this holds for each i, Q consists of n points with no edges; thus we have contradicted the assumption that Q is connected. The same contradiction is reached assuming $B \subset A$ has right depth 2.

Next it is shown that ${}_BA \otimes_B A_B$ divides a multiple of ${}_BA_B$. Let $\dim \varepsilon_i A \varepsilon_j = n_{ij}$. Then it is clear from (9) and simple matrix arithmetic that ${}_BA_B \cong \oplus_{n \geq i \geq j \geq 1} n_{ij} \mathbb{K}_{ij}$.

Now

$$A \otimes_B A = \oplus_{i,j=1}^n \oplus_{i \geq k \geq j} \; \varepsilon_i A \varepsilon_k \otimes_B \varepsilon_k A \varepsilon_j$$

since each $\varepsilon_j \in B$ and for each $r \neq k$, $\varepsilon_k \varepsilon_r = 0$. It follows that ${}_BA \otimes_B A_B \cong \oplus_{n \geq i \geq j \geq 1} m_{ij} \mathbb{K}_{ij}$ where $m_{ij} = \sum_{i \geq k \geq j} n_{ik} n_{kj}$. Since $n_{ii} = 1$ for each i, it follows that $m_{ij} \geq n_{ij}$; moreover, $n_{ij} = 0$ implies $m_{ij} = 0$, since otherwise there is a path from i to j via some k such that $i \geq k \geq j$.

From the last remark it follows that there is $q \in \mathbb{N}$ such that $A \otimes_B A \,|\, qA$ as B-B-bimodules. Thus the minimum depth $d(B, A) = 3$. □

5. Depth of arrow subalgebra in acyclic quiver algebra

In this section we compute the depth of the primary arrow subalgebra $B = \mathbb{K}1_A \oplus A_1 \oplus A_2 \oplus \cdots = \mathbb{K}1_A + \mathrm{rad}\, A$ in the path algebra A of an acyclic quiver Q, which is of the form

$$A = \begin{pmatrix} \mathbb{K} & 0 & \cdots & 0 \\ \varepsilon_2(\mathbb{K}Q)\varepsilon_1 & \mathbb{K} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \varepsilon_n(\mathbb{K}Q)\varepsilon_1 & \varepsilon_n(\mathbb{K}Q)\varepsilon_2 & \cdots & \mathbb{K} \end{pmatrix} \tag{12}$$

Note that B is a local algebra and augmented algebra with one augmentation $\epsilon : B \to \mathbb{K}$ equal to the canonical quotient map $B \to B/\mathrm{rad}\, B \cong \mathbb{K}$. We denote the B-simple by $\mathbb{K}_\epsilon$ as a pullback module. Again there are n augmentations of A denoted by ρ_i defining n simple A-B-bimodules denoted by ${}_i\mathbb{K}_\epsilon$, $i = 1, \ldots, n$.

Lemma 5.1. *The natural B-B-bimodule A is indecomposable.*

Proof. It suffices to show that $\mathrm{End}\,{}_BA_B$ is a local ring [1, 17]. Let $F \in \mathrm{End}\,{}_BA_B$ and choose an ordered basis of A given by $I = \langle \varepsilon_1, \ldots, \varepsilon_n, \alpha_1, \ldots, \alpha_m \rangle$ where the length of the path α_i is less than or equal to the length of α_{i+1}, all $i = 1, \ldots, m-1$. Consider the matrix with $\mathbb{K}$-coefficients, $M = (M^\alpha_\beta)_{\alpha,\beta \in I}$ of F relative to I; then $F(\alpha) = \sum_{\beta \in I} M^\alpha_\beta \beta$.

Given a path of length $r \geq 1$, $(i|\alpha|j) \in A_r$, note that $F(\alpha) = \alpha F(\varepsilon_j) = F(\varepsilon_i)\alpha$, so that

$$\sum_{\beta \in I} M^\alpha_\beta \beta = \sum_{\gamma \in I} M^{\varepsilon_j}_\gamma \alpha\gamma = \sum_{\delta \in I} M^{\varepsilon_i}_\delta \delta\alpha.$$

It follows that $M^{\varepsilon_j}_\gamma = 0$ for paths $(j|\gamma|k)$ and $M^{\varepsilon_i}_\delta = 0$ for all paths $(\ell|\delta|i)$. Also $M^\alpha_\beta = 0$ for all path $\beta \notin \varepsilon_i A \varepsilon_j$, i.e. not a path from i to j. Finally deduce that $M^\alpha_\beta = 0$ if $\beta \in \varepsilon_i A \varepsilon_j$ but $\beta \neq \alpha$ and $M^\alpha_\alpha = M^{\varepsilon_i}_{\varepsilon_i} = M^{\varepsilon_j}_{\varepsilon_j}$.

For $i \neq j$ and $\alpha \in \varepsilon_k A \varepsilon_i$, note that $\alpha F(\varepsilon_j) = F(\alpha\varepsilon_j) = 0$, so that $\sum_{\beta \in I} M^{\varepsilon_j}_\beta \alpha\beta = 0$ implies $M^{\varepsilon_j}_\beta = 0$ whenever $s(\beta) = i$. In particular, $M^{\varepsilon_j}_{\varepsilon_i} = 0$. It follows that the set of $F \in \mathrm{End}\,{}_BA_B$ has the form of a triangular matrix algebra with constant diagonal, like B, and is a local algebra. □

Theorem 5.2. *The depth of the primary arrow subalgebra B in the path algebra A defined above is $d(B, A) = 4$.*

Proof. We first compute $A \otimes_B A$ and show $d(B, A) > 3$. Note that two paths of nonzero length, α, β where $s(\alpha) = i$ satisfy $\alpha \otimes_B \beta = \varepsilon_i \otimes_B \alpha\beta$, which is zero unless $t(\alpha) = s(\beta)$. It follows that

$$A \otimes_B A = \oplus_{i=1}^n \mathbb{K}\,\varepsilon_i \otimes_B \varepsilon_i \oplus_{i=2}^n \oplus_{j=1}^{i-1} \varepsilon_i \otimes_B \varepsilon_i A \varepsilon_j \oplus_{i \neq j} \mathbb{K}\,\varepsilon_i \otimes_B \varepsilon_j.$$

It is obvious that the first two summations above are isomorphic as B-B-bimodules to ${}_BA_B$. Note that when $i \neq j$, for all paths α, β,

$$\alpha\varepsilon_i \otimes_B \varepsilon_j = 0 = \varepsilon_i \otimes_B \varepsilon_j \beta$$

since $\alpha\varepsilon_i \in B$ is either zero or a path ending at i, whence $\alpha\varepsilon_i\varepsilon_j = 0$. It follows that $A \otimes_B A \cong A \oplus n(n-1)_\epsilon \mathbb{K}_\epsilon$ as B-B-bimodules; moreover, as A-B-bimodules, we note for later reference

$$ {}_AA \otimes_B A_B \cong {}_AA_B \oplus \oplus_{i=1}^n (n-1)_i \mathbb{K}_\epsilon \tag{13}$$

By lemma, ${}_BA_B$ is an indecomposable, but the B-B-bimodule $A \otimes_B A$ contains another nonisomorphic indecomposable, in fact ${}_\epsilon\mathbb{K}_\epsilon$, so that as B-bimodules, $A \otimes_B A \oplus * \ncong qA$ for any multiple q by Krull-Schmidt.

Now we establish that the subalgebra $B \subseteq A$ has right depth 4 by comparing (13) with the computation below:

$$A \otimes_B A \otimes_B A = \oplus_{i=1}^n \mathbb{K}\,\varepsilon_i \otimes \varepsilon_i \otimes \varepsilon_i \oplus_{i=2}^n \oplus_{j=1}^{i-1} \varepsilon_i \otimes \varepsilon_i \otimes \varepsilon_i A \varepsilon_j \oplus_{i \neq j \neq k} \mathbb{K}\,\varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_k$$

$$\cong A \oplus (n^2 - 1)\,{}_1\mathbb{K}_\epsilon \oplus \cdots \oplus (n^2 - 1)\,{}_n\mathbb{K}_\epsilon$$

as A-B-bimodules, where $i \neq j \neq k$ symbolizes $i \neq j$, $j \neq k$ or $i \neq k$. It is clear that since no new bimodules appear in a decomposition of ${}_AA \otimes_B A \otimes_B A_B$ as compared with ${}_AA \otimes_B A_B$, that there is $q \in \mathbb{N}$ (in fact $q = n + 1$ will do) such that $A \otimes_B A \otimes_B A \mid qA \otimes_B A$ as A-B-bimodules. It follows that the minimum depth $d(B, A) = 4$. □

It is easy to see from the proof that as natural B-A bimodules $A \otimes_B A \otimes_B A \mid (n+1)A \otimes_B A$ for very similar reasons. Note the general fact that ${}_A A_B$ or ${}_B A_A$ are indecomposable modules if $\mathrm{End}\, {}_A A_B \cong A^B$, the centralizer subalgebra of B in A, is a local algebra.

6. Concluding Remarks

It is well-known and easily computed from (12) that the path algebra $\mathbb{K}Q$ of the quiver

$$Q: \quad n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

is the lower triangular matrix algebra $T_n(\mathbb{K})$. Then we have shown above that for the subalgebras $B_1 = D_n(\mathbb{K})$ equal to the set of diagonal matrices, and $B_2 = U_n(\mathbb{K})$ defined by

$$U_n(\mathbb{K}) = \{ \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ a_{21} & a & 0 & \cdots & 0 \\ a_{31} & a_{32} & a & \cdots & 0 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a \end{pmatrix} | a, a_{ij} \in \mathbb{K} \} \tag{14}$$

the depths are given by $d(D_n(\mathbb{K}), T_n(\mathbb{K})) = 3$ and $d(U_n(\mathbb{K}), T_n(\mathbb{K})) = 4$. Both are not dependent on the order n of matrices.

This situation is different for another interesting series of subalgebras within $T_n(\mathbb{K})$ given by

$$J_n(\mathbb{K}) = \{ \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & & & & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix} | a_1, \ldots, a_n \in \mathbb{K} \} \tag{15}$$

also known as the Jordan algebra. This is isomorphic as algebras to $\mathbb{K}[x]/(x^n)$, a Gorenstein dimension zero local ring. Notice that $U_2(\mathbb{K}) = J_2(\mathbb{K})$, so

$$d(J_2(\mathbb{K}), T_2(\mathbb{K})) = 4.$$

The interesting fact worth mentioning here is that $d(J_3(\mathbb{K}), T_3(\mathbb{K})) \geq 6$. This is based on computations comparing $A \otimes_B A$ and $A \otimes_B A \otimes_B A$ as B-B-bimodules, since a new 2-dimensional indecomposable turns up in the tensor-cube of the ring extension.

The following seems to be an interesting problem not accessible by the techniques of the previous sections:

$$d(J_n(\mathbb{K}), T_n(\mathbb{K})) = \,? \tag{16}$$

6.1. **Acknowledgements.** The authors would like to thank Sebastian Burciu for visiting Porto in May 2012 and discussing topics related to this paper. Research on this paper was partially funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT under the project PE-C/MAT/UI0144/2011.

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