Further results on monoids acting on trees

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ABSTRACT

This paper further develops the theory of arbitrary semigroups acting on trees via elliptic mappings. A key tool is the Lyndon-Chiswell length function L for the semigroup S which allows one to construct a tree T and an action of S on T via elliptic maps. Improving on previous results, the length function of the action will also be L.

1 Introduction

This paper substantially improves and extends the results in [22]. We consider the case of expansions cut down to generators, which is more compatible with geometric semigroup theory [17] and also allows the following major improvement over [22]. A Lyndon-Chiswell function L for the semigroup S with generators X allows one to construct a tree T and an elliptic action of S on T. The action also gives a unique length function L' on S. In [22],L and L' need not be equal. However, in this paper, by cutting to generators X and performing a more refined construction, one obtains that $L = L'$. Unfortunately, this makes the proofs sometimes more difficult and longer than in [22]. Our proofs here occasionally correct some minor errors and misprints in [22] and also just refer to the arguments in [22] when the proofs are the same. Applications of these results to the free Burnside semigroups, see [15, 16, 17], are indicated in Section 9. Full details of the elliptic actions of the free Burnside semigroups will be given in a future paper.

2 Graphs and contractions

Throughout the paper, morphisms and contractions shall be written on the right. Other mappings will be written on the left.

Given a nonempty set X and $n \in \mathbb{N}$, let

$$\mathcal{P}_n(X) = \{Y \subseteq X : |Y| = n\}.$$

As usual, we identify $P_1(X)$ with X to simplify notation.

We define a *graph* to be an ordered pair of the form $G = (X, e)$ where

(G1) X is a nonempty set;

(G2) $e : X \to \mathcal{P}_1(X) \cup \mathcal{P}_2(X)$ is a one-to-one mapping satisfying

$$\forall x, v \in X \ (v \in e(x) \Rightarrow e(v) = v).$$

The elements of $\mathrm{Vert}(G) = e^{-1}(P_1(X))$ are the *vertices* of G and those of $\mathrm{Edge}(G) = e^{-1}(P_2(X))$ are the *edges*. The mapping e fixes the vertices since

$$e(v) = w \in X \Rightarrow e(v) = w = e(w) \Rightarrow v = w = e(v)$$

by (G2) and injectivity of e, and associates to each edge its two *adjacent* vertices. Note that this definition of (unordered) graph excludes loops and multiple edges due to the fact of e being one-to-one.

A *path* in $G = (X, e)$ of length $n \in \mathbb{N}$ is a sequence $p = (v_0, \ldots, v_n)$ in $\mathrm{Vert}(G)$ such that $\{v_{i-1}, v_i\} \in e(\mathrm{Edge}(G))$ for $i = 1, \ldots, n$. We say that p is a path *from* v_0 *to* v_n. If $n = 0$ the path is said to be *trivial.* The graph G is said to be *connected* if, for all $v, w \in \mathrm{Vert}(G)$, there exists a path in G from v to w.

A *cycle* in G is a path of the form $(v_0, \ldots, v_{n-1}, v_n)$ with $n \geq 3$, $v_n = v_0$ and $v_0, \ldots, v_{n-1}$ all distinct. A connected graph with no cycles is said to be a *tree.*

Let $G_i = (X_i, e_i)$ be a graph for $i = 1, 2$. A graph *morphism* $\varphi : G_1 \to G_2$ is a mapping $\varphi : X_1 \to X_2$ such that

(GM1) $(\mathrm{Vert}(G_1))\varphi \subseteq \mathrm{Vert}(G_2)$;

(GM2) $(e_1(x_1))\varphi = e_2(x_1\varphi)$ for every $x_1 \in X_1$.

Note that φ can collapse vertices to edges: for example, every graph has a morphism onto the trivial graph with a single vertex.

Given a connected graph G, we define a distance d on $\mathrm{Vert}(G)$ by taking $d(v, w)$ to be the length of the shortest path from v to w in G. Such a shortest path is said to be a *geodesic* from v to w and d is the *geodesic distance* in G. We write $\mathrm{Geo}(G) = (\mathrm{Vert}(G), d)$. If G is a tree, there is a unique geodesic connecting v and w and $\mathrm{Geo}(G)$ is a *hyperbolic* metric space as considered in [9].

Let $G_i = (X_i, e_i)$ be a graph for $i = 1, 2$ and let $\mathrm{Geo}(G_i) = (\mathrm{Vert}(G_i), d_i)$. A *contraction* $\psi : \mathrm{Geo}(G_1) \to \mathrm{Geo}(G_2)$ is a mapping $\psi : \mathrm{Vert}(G_1) \to \mathrm{Vert}(G_2)$ satisfying

$$\forall v, w \in \mathrm{Vert}(G_1) \ d_2(v\psi, w\psi) \leq d_1(v, w).$$

Proposition 2.1 *[22, Fact 1.4] Let $G_i = (X_i, e_i)$ be a graph for $i = 1, 2$. A mapping $\psi : Vert(G_1) \to Vert(G_2)$ is a contraction if and only if ψ can be extended to a morphism $\overline{\psi} : G_1 \to G_2$. In that case, the extension is unique.*

Let G be a graph. From now on, given a graph G, we shall identify G with its underlying set, and we shall assume that the one-to-one mapping is denoted by e and the geodesic distance by d_G. We denote by $\mathrm{End}(G)$ the monoid of all endomorphisms of G and by $\mathrm{Con}(G)$ the monoid of all contractions of $\mathrm{Geo}(G)$ into itself.

The following result is a straightforward consequence of Proposition 2.1.

Corollary 2.2 *The mapping*

$$\begin{aligned} End(G) &\to Con(G) \\ \varphi &\mapsto \varphi|_{Vert(G)} \end{aligned}$$

is a monoid isomorphism.

Proof. By Proposition 2.1, this mapping is a well-defined bijection. Since $\varphi(\mathrm{Vert}(G)) \subseteq \mathrm{Vert}(G)$ for every $\varphi \in \mathrm{End}(G)$, it follows that

$$(\varphi\varphi')|_{\mathrm{Vert}(G)} = \varphi|_{\mathrm{Vert}(G)}\varphi'|_{\mathrm{Vert}(G)}$$

for all $\varphi, \varphi' \in \mathrm{End}(G)$. Since the restriction of the identity endomorphism is the identity contraction, our mapping is indeed a monoid isomorphism. $\square$

3 Elliptic M-trees

Let G be a graph and let M be a monoid with identity 1. A *(right) action* of M on G is a monoid homomorphism

$$\begin{aligned} \theta : M &\to \mathrm{End}(G) \\ m &\mapsto \theta_m \end{aligned}$$

The action is *faithful* if θ is one-to-one.

To simplify notation, we write $xm = x\theta_m$. With this notation, the action can be equivalently defined through the axioms:

(A1) $(\mathrm{Vert}(G))M \subseteq \mathrm{Vert}(G)$

(A2) $(e(x))m = e(xm)$

(A3) $x(mm') = (xm)m'$

(A4) $x1 = x$

for all $x \in G$ and $m, m' \in M$.

Note that, in view of Corollary 2.2, the action could be equivalently defined as a monoid homomorphism $M \to \mathrm{Con}(G)$.

We are interested in the case of G being a tree, a rooted tree to be more precise. A *rooted tree* is an ordered pair of the form (r_0, T), where T is a tree and $r_0 \in \mathrm{Vert}(T)$. A rooted tree admits a natural representation by levels $0, 1, 2, \ldots$ where we locate at level (or depth) n those vertices lying at distance n from r_0. We write then

$$\mathrm{dep}(v) = d(r_0, v)$$

for $v \in \mathrm{Vert}(T)$. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$. The *depth* of a rooted tree is defined by

$$\mathrm{dep}(r_0, T) = \sup\{\mathrm{dep}(v)\ ; v \in \mathrm{Vert}(T)\} \in \overline{\mathbb{N}}.$$

Given $v \in \mathrm{Vert}(T)$, we define the *degree* of v in (r_0, T) by

$$\deg(v) = \begin{cases} |\{x \in \mathrm{Edge}(T) \mid v \in e(x)\}| & \text{if } v = r_0 \\ |\{x \in \mathrm{Edge}(T) \mid v \in e(x)\}| - 1 & \text{otherwise,} \end{cases}$$

that is, we count the number of outgoing edges if we orient them *away* from the root. A vertex of degree 0 is called a *leaf.* If two vertices v and w are connected by an edge, we say that

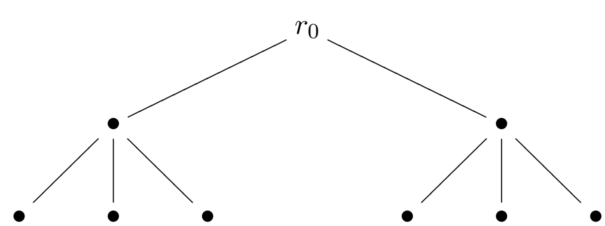
$$v \text{ is } \begin{cases} \text{a } son \text{ of } w & \text{if } \mathrm{dep}(v) = \mathrm{dep}(w) + 1 \\ \text{the } father \text{ of } w & \text{if } \mathrm{dep}(v) = \mathrm{dep}(w) - 1 \end{cases}$$

Note that a father may have many sons, but the father is always unique. All vertices but the root have a father.

We generalize this notion with the obvious terminology. If v_i is a son of v_{i-1} for $i = 1, \ldots, k$, we say that v_k is a *descendant* of v_0 and v_0 an *ancestor* of v_k.

A very important example is given by *rooted uniformly branching trees*:

Example 3.1 *Let $n_1, \ldots, n_l \geq 1$. Up to isomorphism, the rooted uniformly branching tree $(r_0, T(n_l, \ldots, n_1))$ is the rooted tree of depth l such that every vertex of depth $i-1$ has degree n_i $(i = 1, \ldots, l)$. For example, $(r_0, T(3, 2))$ can be pictured by*



We can of course extend this definition to infinite cardinals in the obvious way, as well as considering $T(\ldots, n_2, n_1)$ for an infinite sequence. It is standard to represent $\mathrm{Vert}(T(n_l, \ldots, n_1))$ as

$$\{r_0\} \cup (\bigcup_{i=1}^{l} X_i \ldots \times X_1)$$

with $|X_i| = n_i$ for every i.

Let $(r_0, T), (r'_0, T')$ be rooted trees. An *elliptic* contraction $\varphi : (r_0, T) \to (r'_0, T')$ is a depth-preserving contraction, that is, a contraction $\varphi : \mathrm{Vert}(T) \to \mathrm{Vert}(T')$ satisfying

$$\forall v \in \mathrm{Vert}(T)\ \mathrm{dep}(v\varphi) = \mathrm{dep}(v).$$

In view of Proposition 2.1, a bijective elliptic contraction extends to an isomorphism of rooted trees.

Lemma 3.2 *Let (r_0, T) be a rooted tree and let $\varphi : Vert(T) \to Vert(T')$ be a mapping. Then φ is an elliptic contraction from (r_0, T) into (r'_0, T') if and only if*

(i) $r_0\varphi = r'_0$;

(ii) if $v \in Vert(T)$ is the father of w, then $v\varphi$ is the father of $w\varphi$.

Proof. Assume that φ is an elliptic contraction. Then (i) holds trivially and (ii) follows from φ preserving depth and being the restriction of a tree morphism by Proposition 2.1.

Assume now that φ satisfies conditions (i) and (ii). We extend φ to $\overline{\varphi} : T \to T'$ as follows. Given $x \in \mathrm{Edge}(T)$, we may write $e(x) = \{v, w\}$ and assume that v is the father of w. By (ii), it follows that $v\varphi$ is the father of $w\varphi$ and so there exists some $x' \in \mathrm{Edge}(T')$ such that $e(x') = \{v\varphi, w\varphi\}$. We define $x\overline{\varphi} = x'$.

It follows from the definition that $\overline{\varphi} : T \to T'$ is a morphism. By Proposition 2.1, φ is a contraction. By (i), φ preserves depth 0. By (ii) and induction, φ preserves depth n for each $n \in \{0, \dots, \mathrm{dep}(r_0, T)\}$. $\square$

The set of all elliptic contractions on (r_0, T) is denoted by $\mathrm{Ell}(r_0, T)$. This is a monoid under composition and is termed the *elliptic product* on (r_0, T).

Wreath products constitute as we shall see important examples of elliptic products. A *partial transformation monoid* is an ordered pair of the form (X, M), where X is a nonempty set and M is a submonoid of the monoid $P(X)$ of all partial transformations of X. If M is a submonoid of the monoid $M(X)$ of all full transformations of X, we say that (X, M) is a *transformation monoid*.

Throughout the paper, given a direct product of the form $X = X_l \times \ldots \times X_1$ and $i \in \{1, \dots, l\}$, we shall denote by $\pi_i : X \to X_i$ the projection on the ith component, and by $\pi_{[i,1]} : X \to X_i \times \ldots \times X_1$ the projection on the last i components.

Assume that $X = \cup_{i=1}^{l}(X_i \times \ldots \times X_1)$. For $i = 1, \dots, l$, we define an equivalence relation $\equiv_i$ on X by

$$(x_j, \dots, x_1) \equiv_i (x'_k, \dots, x'_1) \quad \text{if} \quad (i \leq j, k \quad \text{and} \quad x_i = x'_i, \dots, x_1 = x'_1).$$

Given $\varphi \in P(X)$, we denote by $\mathrm{dom}\varphi$ the domain of φ.

A mapping $\varphi \in P(X)$ is said to be *sequential* if:

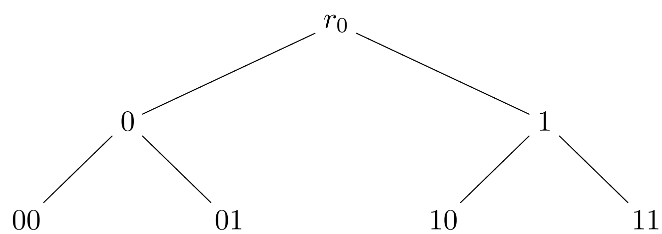
(SQ1) $\forall i \in \{2, \dots, l\}$ $((x_i, \dots, x_1) \in \mathrm{dom}\varphi \Rightarrow (x_{i-1}, \dots, x_1) \in \mathrm{dom}\varphi)$;

(SQ2) $\forall i \in \{1, \dots, l\}$ $\forall (x_i, \dots, x_1) \in \mathrm{dom}\varphi$ $(x_i, \dots, x_1)\varphi \in X_i \times \ldots \times X_1$;

(SQ3) $\forall i \in \{1, \dots, l\}$ $\forall x, x' \in \mathrm{dom}\varphi$ $(x \equiv_i x' \Rightarrow x\varphi \equiv_i x'\varphi)$.

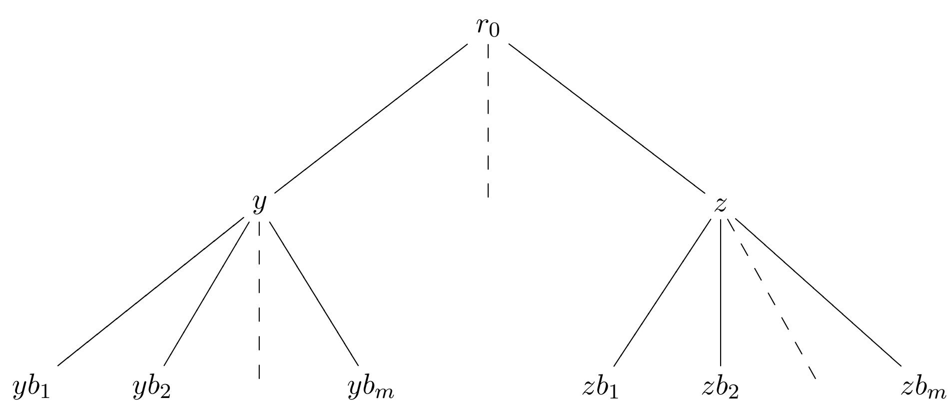
It is immediate that the composition of sequential partial transformations of X is still sequential.

Adjoining a root r_0 provides a natural tree representation for $\cup_{i=1}^{l} X_i \times \ldots \times X_1$. For example, taking $X_2 = X_1 = \{0, 1\}$, we obtain the tree



Given $(a_{i-1}, \ldots, a_1) \in X_{i-1} \times \ldots \times X_1$ ($i \in \{1, \ldots, l\}$), we have $(\cdot, a_{i-1}, \ldots, a_1)\varphi\pi_i \in P(X_i)$.

Graphically, whenever $y\varphi = z$ for $y = (a_{i-1}, \ldots, a_1)$ and $X_i = \{b_1, \ldots, b_m\}$, then we have



in the tree representation and $\{yb_1, \ldots, yb_m\}\varphi \subseteq \{zb_1, \ldots, zb_m\}$. Then $(\cdot, a_{i-1}, \ldots, a_1)\varphi\pi_i$ is the induced partial mapping $\{b_1, \ldots, b_m\} \to \{b_1, \ldots, b_m\}$ (not necessarily injective!).

If $\varphi, \varphi' \in P(X)$ and $(a_{i-1}, \ldots, a_1)\varphi = (a'_{i-1}, \ldots, a'_1)$, it is easy to check [8, 23] that we have

$$(\cdot, a_{i-1}, \ldots, a_1)(\varphi\varphi'\pi_i) = ((\cdot, a_{i-1}, \ldots, a_1)\varphi\pi_i)((\cdot, a'_{i-1}, \ldots, a'_1)\varphi'\pi_i). \tag{1}$$

Given partial transformation monoids $(X_l, M_l), \ldots, (X_1, M_1)$, their *wreath product* is defined by

$$(X_l, M_l) \circ \ldots \circ (X_1, M_1) = (X_l \times \ldots \times X_1, M_l \circ \ldots \circ M_1),$$

where $M_l \circ \ldots \circ M_1$ consists of all $\varphi \in P(X)$ satisfying

(W1) φ is sequential;

(W2) $\varphi\pi_1 \in M_1$

(W2) $(\cdot, a_{i-1}, \ldots, a_1)\varphi\pi_i \in M_i$ for all $i \in \{2, \ldots, l\}$ and $(a_{i-1}, \ldots, a_1) \in \mathrm{dom}\varphi$.

More informally, $M_l \circ \ldots \circ M_1$ consists of those partial self-maps of X "in sequential form with component action in the M_i's". Note that $M_l \circ \ldots \circ M_1$ is a submonoid of $P(X)$ since the composition of sequential mappings is sequential and by (1): if $(\cdot, a_{i-1}, \ldots, a_1)\varphi\pi_i$ and

$(\cdot, a'_{i-1}, \ldots, a'_1)\varphi'\pi_i$ are both in M_i, so is their composition $(\cdot, a_{i-1}, \ldots, a_1)(\varphi\varphi'\pi_i)$. Therefore $(X_l, M_l) \circ \ldots \circ (X_1, M_1)$ is a well-defined partial transformation monoid.

If $(X_l, M_l), \ldots, (X_1, M_1)$ are (full) transformation monoids, their wreath product is a submonoid of $M(X)$. In the case of a wreath product of two monoids with $X_1 = \{a_1, \ldots, a_m\}$, it is common to use the notation $(\beta_1, \ldots, \beta_m)\alpha$ ($\alpha \in M_1, \beta_i \in M_2$) to denote the element of $M_2 \circ M_1$ defined by

$$(x_2, a_i)((\beta_1, \ldots, \beta_m)\alpha) = (x_2\beta_i, a_i\alpha).$$

The wreath product of (partial) transformation monoids is associative, among other properties. See [2, 8, 23] for more details about the wreath product.

Proposition 3.3 *For all nonempty sets $X_l, \ldots, X_1$, the monoids $M(X_l) \circ \ldots \circ M(X_1)$ and $Ell(r_0, T(|X_l|, \ldots, |X_1|))$ are isomorphic.*

Proof. We may write $T = T(|X_l|, \ldots, |X_1|)$ with

$$\mathrm{Vert}(T) = \{r_0\} \cup (\bigcup_{i=1}^{l} X_i \times \ldots \times X_1).$$

We consider

$$\begin{array}{c} \eta : \mathrm{Ell}(r_0, T) \to M(X_l) \circ \ldots \circ M(X_1) \\ \varphi \mapsto \varphi\mid_{X_l \times \ldots \times X_1} . \end{array}$$

Let $\varphi \in \mathrm{Ell}(r_0, T)$. Since elliptic contractions preserve depth, $\eta(\varphi) \in M(X_l \times \ldots \times X_1)$. It follows easily from Lemma 3.2(ii) that $\eta(\varphi)$ is sequential: if $x \equiv_i x'$, then x, x' are descendants of $x\pi_{[i,1]}$ and so $x\varphi, x'\varphi$ are descendants of $x\pi_{[i,1]}\varphi$, yielding $x\varphi \equiv_i x'\varphi$. Since (W2) and (W3) are trivially satisfied due to $\bigcup_{i=1}^{l} X_i \times \ldots \times X_1 \subset \mathrm{dom}\varphi$, η is well defined.

Also by Lemma 3.2, the image of each $v \in \mathrm{Vert}(T)$ by φ determines the images of all its ancestors, hence φ is determined by its restriction to the leafs of T, i.e., $\eta(\varphi)$. Therefore η is one-to-one.

Next let $\psi \in M(X_l) \circ \ldots \circ M(X_1)$. We define $\varphi : \mathrm{Vert}(T) \to \mathrm{Vert}(T)$ by $r_0\varphi = r_0$ and

$$(x_i, \ldots, x_1)\varphi = (x_l, \ldots, x_1)\psi\pi_{[i,1]},$$

the domain extension described before for a sequential map. If $(x_i, \ldots, x_1)$ is a son of $(x_{i-1}, \ldots, x_1)$ ($i = 2, \ldots, l$), then $(x_l, \ldots, x_1)\psi\pi_{[i,1]}$ is a son of $(x_l, \ldots, x_1)\psi\pi_{[i-1,1]}$ and so $(x_i, \ldots, x_1)\varphi$ is a son of $(x_{i-1}, \ldots, x_1)\varphi$. Since $(x_1)\varphi = (x_l, \ldots, x_1)\psi\pi_1$ is always a son of r_0, condition (ii) of Lemma 3.2 holds and so φ is an elliptic contraction. Since $\psi = \eta(\varphi)$, we conclude that η is onto and therefore a bijection.

Since $(X_l \times \ldots \times X_1)\varphi \subseteq X_l \times \ldots \times X_1$ for every elliptic contraction φ, it follows that η is a monoid homomorphism and therefore an isomorphism. □

An *elliptic action* of a monoid M on the rooted tree (r_0, T) is a monoid homomorphism $\theta : M \to \mathrm{Ell}(r_0, T)$. The elliptic action is *faithful* if θ is one-to-one.

We can generalize Proposition 3.3 to the case of arbitrary wreath products of transformation monoids:

Corollary 3.4 *For all transformation monoids $(X_l, M_l), \ldots, (X_1, M_1)$, the monoid $M_l \circ \ldots \circ M_1$ embeds in $Ell(r_0, T(|X_l|, \ldots, |X_1|))$.*

Proof. Write $T = T(|X_l|, \dots, |X_1|)$. We proved in Proposition 3.3 that

$$\begin{aligned} \eta : \mathrm{Ell}(r_0, T) &\to M(X_l) \circ \dots \circ M(X_1) \\ \varphi &\mapsto \varphi\,|_{X_l \times \dots \times X_1} . \end{aligned}$$

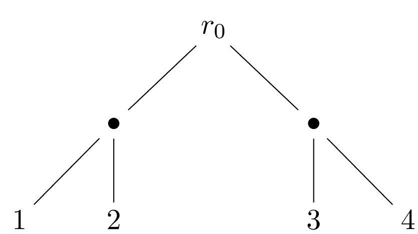
is a monoid isomorphism, its inverse being the mapping η^{-1} that assigns to every $\psi \in M(X_l) \circ \dots \circ M(X_1)$ its natural domain extension $\overline{\psi} : \mathrm{Vert}(T) \to \mathrm{Vert}(T)$. The restriction of η^{-1} to the submonoid $M_l \circ \dots \circ M_1$ of $M(X_l) \circ \dots \circ M(X_1)$ defines a faithful elliptic action of $M_l \circ \dots \circ M_1$ on $\mathrm{Ell}(r_0, T)$ and so $M_l \circ \dots \circ M_1$ embeds in $\mathrm{Ell}(r_0, T(|X_l|, \dots, |X_1|))$. $\square$

However, not all submonoids of $\mathrm{Ell}(r_0, T)$, where T is a rooted uniformly branching tree, can be obtained via wreath products of transformation monoids, as the next example shows.

Example 3.5 *Let M be the submonoid of $M(\{1, 2, 3, 4\})$ given by*

$$M = \{(1234), (2244), (3434), (3444), (4444)\},$$

where $\varphi = (a_1 a_2 a_3 a_4)$ is defined by $i\varphi = a_i$ for $i = 1, \dots, 4$. Then M acts faithfully on $(r_0, T(2, 2))$ by elliptic contractions according to the labelling



and so M embeds in Ell$(r_0, T(2, 2))$. However, M cannot be obtained as $M_2 \circ M_1$ with M_2, M_1 monoids of full transformations since

$$|M_2 \circ M_1| = |M_2|^2 |M_1|$$

and $M_1, M_2 \leq M(\{0, 1\})$ implies $|M_1|, |M_2| \leq 4$.

Let (r_0, T) be a rooted tree. Clearly, every $v \in (r_0, T)$ is determined by the geodesic

$$\alpha = (v = \alpha_l, \dots, \alpha_1, \alpha_0 = r_0).$$

We call such a geodesic a *ray* of (r_0, T). An infinite path of the form $\alpha = (\dots, \alpha_1, \alpha_0 = r_0)$ is also said to be a ray if $\mathrm{dep}(\alpha_i) = i$ for every $i \in \mathbb{N}$. An infinite ray is also called an *end*. We denote by $\mathrm{Ray}(r_0, T)$ the set of all rays of (r_0, T).

Given $\alpha = (\alpha_l, \dots, \alpha_1, \alpha_0) \in \mathrm{Ray}(r_0, T)$, we write $|\alpha| = l$ and $\mathrm{dom}\alpha = \{0, \dots, l\}$. If α is infinite, we write $|\alpha| = \omega$ and $\mathrm{dom}\alpha = \mathbb{N}$. In any case, given $\alpha \in \mathrm{Ray}(r_0, T)$ and $i \in \mathrm{dom}\alpha$, we denote by α_i the vertex of depth i in α.

We define a partial order on $\mathrm{Ray}(r_0, T)$ by

$$\alpha \leq \beta \qquad \text{if} \qquad |\alpha| \leq |\beta| \text{ and } \alpha_i = \beta_i \text{ for every } i \in \mathrm{dom}\alpha.$$

We say that a ray of (r_0, T) is *maximal* if it is maximal for this partial order. Clearly, the maximal rays are either ends or correspond to the leaves of the tree. We denote by $\mathrm{MRay}(r_0, T)$ the set of all maximal rays of (r_0, T).

We say that (r_0, T) is *uniform* if all its maximal rays have the same length $l \in \overline{\mathbb{N}}$. In particular, if (r_0, T) has finite depth l, it is uniform if all its leaves have depth l. If (r_0, T) has infinite depth, it is uniform if it has no leaves at all. The concept of maximal ray constitutes the possible generalization of the concept of leaf to uniform trees of infinite depth.

Assume that $\varphi \in \mathrm{Ell}(r_0, T)$. We extend φ to a mapping $\overline{\varphi} : \mathrm{Ray}(r_0, T) \to \mathrm{Ray}(r_0, T)$ by

$$\alpha\overline{\varphi} = (\ldots, \alpha_2\varphi, \alpha_1\varphi).$$

It follows from Lemma 3.2 that $\overline{\varphi}$ is well defined. Identifying finite rays with vertices as usual, $\overline{\varphi}$ can be seen as an extension of φ. We shall denote $\overline{\varphi}$ by φ when no confusion arises.

As a particular case, if M acts elliptically on (r_0, T), we can extend this action to $\mathrm{Ray}(r_0, T)$ by

$$\alpha m = (\ldots, \alpha_2 m, \alpha_1 m) \qquad (\alpha \in \mathrm{Ray}(r_0, T),\ m \in M).$$

Note that

$$\alpha 1 = \alpha, \quad \alpha(mm') = (\alpha m)m'$$

for all $\alpha \in \mathrm{Ray}(r_0, T)$ and $m, m' \in M$, hence we can properly speak of an action of M on $\mathrm{Ray}(r_0, T)$.

Let (r_0, T) be a uniform rooted tree and let $\alpha \in \mathrm{MRay}(r_0, T)$. An elliptic action of M on (r_0, T) is said to be α-*transitive* if

$$\mathrm{Vert}(T) = \alpha M = \cup_{i \in \mathrm{dom}\alpha} \alpha_i M.$$

If (r_0, T) has finite depth l, it should be clear that the elliptic action of M on (r_0, T) is α-transitive if and only if $\alpha_l M$ is the set of leaves of (r_0, T). Indeed, since the action of M is depth-preserving, only leaves can be sent to leaves. On the other hand, transitivity at the deepest level clearly implies transitivity on the upper levels in view of Lemma 3.2.

An *elliptic M-tree* is a structure of the form $\chi = (r_0, T, \alpha, \theta)$, where

(E1) (r_0, T) is a uniform rooted tree;

(E2) $\alpha \in \mathrm{MRay}(r_0, T)$;

(E3) $\theta : M \to \mathrm{Ell}(r_0, T)$ is an α-transitive action.

We say that χ is a *faithful* elliptic M-tree if θ is one-to-one. We say χ is a *strongly faithful* elliptic M-tree if

$$\alpha m = \alpha m' \Rightarrow m = m' \quad \text{for all } m, m' \in M.$$

We shall omit θ from the representation of χ when no confusion arises from doing so.

Let $\chi = (r_0, T, \alpha)$, $\chi' = (r'_0, T', \alpha')$ be elliptic M-trees. A *morphism* $\varphi : \chi \to \chi'$ of elliptic M-trees is an elliptic contraction $\varphi : (r_0, T) \to (r'_0, T')$ such that:

(EM1) $\alpha\varphi = \alpha'\varphi$;

(EM2) $\forall v \in \mathrm{Vert}(T)\ \forall m \in M\ (vm)\varphi = (v\varphi)m$.

If φ is bijective, we say it is an *isomorphism* of elliptic M-trees.

Given a transformation monoid (X, M) and $x_0 \in X$, we say that M acts transitively on (X, x_0) if $x_0 M = X$. A *pointed transformation monoid* is a triple of the form (X, x_0, M), where M acts transitively on (X, x_0).

Corollary 3.6 *Let* $\ldots, (X_2, x_2, M_2), (X_1, x_1, M_1)$ *be pointed transformation monoids. Then* $(r_0, T(\ldots, |X_2|, |X_1|), (\ldots, x_2, x_1))$ *is a faithful elliptic* $(\ldots \circ M_2 \circ M_1)$*-tree.*

Proof. Axioms (E1) and (E2) are trivially verified. Let $\alpha = (\ldots, x_2, x_1)$. We observed in the proof of Corollary 3.4 that the restriction of η^{-1} as defined in Proposition 3.3 to the submonoid $\ldots \circ M_2 \circ M_1$ of $\ldots \circ M(X_2) \circ M(X_1)$ defines a faithful elliptic action of $\ldots \circ M_2 \circ M_1$ on $\mathrm{Ell}(T(\ldots, |X_2|, |X_1|))$. A straightforward induction on i proves that this action is α-transitive: indeed, it is enough to show that, given $(w_i, \ldots, w_1) \in X_i \times \ldots \times X_1$, there exists $\varphi_i \in M_i \circ \ldots \circ M_1$ such that $(x_i, \ldots, x_1)\varphi_i = (w_i, \ldots, w_1)$. The case $i = 1$ follows from M_1 acting transitively on (X_1, x_1). Assume that $(x_i, \ldots, x_1)\varphi_i = (w_i, \ldots, w_1)$ for some $\varphi_i \in M_i \circ \ldots \circ M_1$. Since $x_{i+1}\xi = w_{i+1}$ for some $\xi \in M_{i+1}$, we can define $\varphi_{i+1} \in M_{i+1} \circ \ldots \circ M_1$ by

$$\varphi_{i+1} = (\xi, \ldots, \xi)\varphi_i.$$

It follows that

$$(x_{i+1}, \ldots, x_1)\varphi_i = (x_{i+1}\xi, w_i, \ldots, w_1) = (w_{i+1}, \ldots, w_1)$$

and so (E3) holds as required. $\square$

Example 3.5 shows also that not all faithful elliptic M-trees on a rooted uniformly branching tree can be obtained via wreath products, the action of M on $(r_0, T(2, 2))$ being obviously α-transitive for the ray defined by the leaf 1.

4 Length functions

Let M be a monoid. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$ have the obvious ordering. A *length function* for M is a function $D : M \times M \to \overline{\mathbb{N}}$ satisfying the axioms

(L1) $D(m, m') = D(m', m)$

(L2) $D(m', m'') \leq D(m, m)$

(L3) $D(m', m'') \leq D(m'm, m''m)$

(L4) $D(m, m'') \geq \min\{D(m, m'), D(m', m'')\}$ (isoperimetric inequality)

for all $m, m', m'' \in M$.

Note that, by (L2), D has a maximum $l \in \overline{\mathbb{N}}$ and

$$D(m, m) = l \quad \text{for every } m \in M. \tag{2}$$

Moreover, for any submonoid M' of M, the restriction of D to $M' \times M'$ is a length function for M'.

We recall that a *quasi-ultrametric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the axioms

(Q1) $d(x, x') = d(x', x)$

(Q2) $d(x, x) = 0$

(Q3) $d(x, x'') \leq \max\{d(x, x'), d(x', x'')\}$

for all $x, x', x'' \in X$.

Bounded length functions can be related to quasi-ultrametrics as follows:

Proposition 4.1 *Let M be a monoid and let $D : M \times M \to \mathbb{N}$ be a bounded function with maximum $l \in \mathbb{N}$. Define $d : M \times M \to \mathbb{N}$ by $d(m, m') = 2l - 2D(m, m')$. Then D is a length function for M if and only if the following conditions hold:*

(i) d is a quasi-ultrametric;

(ii) $d(m'm, m''m) \leq d(m', m'')$ for all $m, m', m'' \in M$.

Proof. It is immediate that (Q1) $\Leftrightarrow$ (L1), (Q2) $\Leftrightarrow$ (L2), (Q3) $\Leftrightarrow$ (L4) and (ii) $\Leftrightarrow$ (L3). $\square$

Let (r_0, T) be a rooted tree and consider the partial order $\leq$ defined on $\mathrm{Ray}(r_0, T)$ in Section 3. It is immediate that $(\mathrm{Ray}(r_0, T), \leq)$ is a $\wedge$-semilattice and $\alpha \wedge \beta$ is defined by

$$\alpha \wedge \beta = \begin{cases} \alpha & \text{if } \alpha = \beta \\ (\alpha_k, \ldots, \alpha_0) & \text{if } k = \max\{i \in \mathbb{N} \mid \alpha_i = \beta_i\}. \end{cases}$$

Thus $(\mathrm{Ray}(r_0, T), \wedge)$ is a semilattice with zero (r_0) (in the semigroup theory sense). Identifying vertices with finite rays, we can say that $\mathrm{Vert}(T)$ is a $\wedge$-subsemilattice of $(\mathrm{Ray}(r_0, T), \leq)$, considering the ancestor partial ordering on $\mathrm{Vert}(T)$:

$$v \leq w \qquad \text{if} \qquad v = w \text{ or } v \text{ is an ancestor of } w.$$

Lemma 4.2 *Let (r_0, T) be a rooted tree and $\alpha \in Ray(r_0, T)$. Write $(\alpha] = \{\beta \in Ray(r_0, T) \mid \beta \leq \alpha\}$. Then:*

(i) $(\alpha]$ is a chain;

(ii) if α is finite, $(\alpha]$ is finite.

Proof. (i) and (ii) follow from

$$(\alpha] = \{\alpha\} \cup \{(\alpha_i, \ldots, \alpha_0) \mid i = 0, \ldots, \mathrm{dom}\alpha\}.$$

$\square$

Lemma 4.3 *Let (r_0, T) be a rooted tree. Then*

$$|\alpha \wedge \alpha''| \geq min\{|\alpha \wedge \alpha'|, |\alpha' \wedge \alpha''|\}$$

for all $\alpha, \alpha', \alpha'' \in Ray(r_0, T)$.

Proof. We have $\alpha \wedge \alpha', \alpha' \wedge \alpha'' \leq \alpha'$. Since $(\alpha']$ is a chain by Lemma 4.2(i) and $\wedge$ is commutative, we may assume that $\alpha \wedge \alpha' \geq \alpha' \wedge \alpha''$. Thus

$$\alpha \wedge \alpha'' \geq \alpha \wedge \alpha' \wedge \alpha'' = (\alpha \wedge \alpha') \wedge (\alpha' \wedge \alpha'') = \alpha' \wedge \alpha''$$

and so

$$|\alpha \wedge \alpha''| \geq |\alpha' \wedge \alpha''| \geq \min\{|\alpha \wedge \alpha'|, |\alpha' \wedge \alpha''|\}$$

as claimed. $\square$

Proposition 4.4 *Let $\chi = (r_0, T, \alpha)$ be an elliptic M-tree and define a mapping $D_\chi : M \times M \to \overline{\mathbb{N}}$ by*

$$D_\chi(m, m') = |\alpha m \wedge \alpha m'|.$$

Then:

(i) D_χ is a length function for M;

(ii) if dep$(r_0, T) = l \in \mathbb{N}$, the quasi-ultrametric d associated to D_χ satisfies

$$d(m, m') = d_T(\alpha_l m, \alpha_l m');$$

(ii) if dep$(r_0, T) = l \in \mathbb{N}$ and χ is strongly faithful, then d is an ultrametric.

Proof. (i) Since $\wedge$ is commutative, axiom (L1) is trivially satisfied.

Since $\alpha, \alpha m$ are maximal rays of (r_0, T), we have

$$\begin{aligned} D_\chi(m', m'') &= |\alpha m' \wedge \alpha m''| \leq \mathrm{dep}(r_0, T) = |\alpha m| \\ &= |\alpha m \wedge \alpha m| = D_\chi(m, m) \end{aligned}$$

and so (L2) holds.

For all $\beta, \beta' \in \mathrm{Ray}(r_0, T)$, we have that

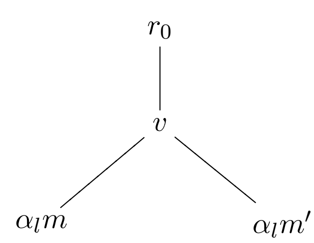
$$\beta_i = \beta'_i \text{ for } i = 0, \ldots, k \qquad \Rightarrow \qquad \beta_i m = \beta'_i m \text{ for } i = 0, \ldots, k.$$

Thus $|\beta \wedge \beta'| \leq |\beta m \wedge \beta' m|$ and so

$$D_\chi(m', m'') = |\alpha m' \wedge \alpha m''| \leq |\alpha m' m \wedge \alpha m'' m| = D_\chi(m' m, m'' m).$$

Thus (L3) holds. Since (L4) follows fromm Lemma 4.3, D_χ is a length function for M.

(ii) Assume that dep$(r_0, T) = l \in \mathbb{N}$. By (L2), l is the maximum value of D_χ. By Proposition 4.1, the associated quasi-metric is defined by $d(m, m') = 2l - 2D_\chi(m, m')$. Let v be the deepest vertex of $\alpha m \wedge \alpha m'$. Since v lies in the geodesics r_0 — $\alpha_l m$, r_0 — $\alpha_l m'$ and $\alpha_l m$ — $\alpha_l m'$,



we obtain

$$\begin{aligned} d(m, m') &= 2l - 2D_\chi(m, m') \\ &= d_T(r_0, \alpha_l m) + d_T(r_0, \alpha_l m') - 2|\alpha m \wedge \alpha m'| \\ &= d_T(r_0, \alpha_l m) - d_T(r_0, v) + d_T(r_0, \alpha_l m') - d_T(r_0, v) \\ &= d_T(\alpha_l m, v) + d_T(\alpha_l m', v) = d_T(\alpha_l m, \alpha_l m'). \end{aligned}$$

(iii) Assume that dep$(r_0, T) = l \in \mathbb{N}$ and χ is strongly faithful. Let $m, m' \in M$ be such that $d(m, m') = 0$. By (ii), we have $d_T(\alpha_l m, \alpha_l m') = 0$ and so $\alpha_l m = \alpha_l m'$. Hence $\alpha m = \alpha m'$ and so $m = m'$ since χ is strongly faithful. Therefore d is an ultrametric. $\square$

In view of Corollary 3.6, it is interesting to analyze the particular case of wreath products. The canonical length function for two mappings $\varphi, \varphi' \in M_l \circ \ldots \circ M_1$ measures the maximum number of components (from right to left) where $(x_l, \ldots, x_1)\varphi$ and $(x_l, \ldots, x_1)\varphi'$ coincide:

Corollary 4.5 *Let* $\ldots, (X_2, x_2, M_2), (X_1, x_1, M_1)$ *be pointed transformation monoids and let* $\chi = (r_0, T(\ldots, |X_2|, |X_1|), (\ldots, x_2, x_1))$ *be the corresponding faithful elliptic* $(\ldots \circ M_2 \circ M_1)$*-tree. Then*

$$D_\chi(\varphi, \varphi') = max\{i : (\ldots, x_2, x_1)\varphi\pi_{[i,1]} = (\ldots, x_2, x_1)\varphi'\pi_{[i,1]}\}.$$

Proof. We prove the finite case for pointed transformation monoids $(X_l, x_l, M_l), \ldots, (X_1, x_1, M_1)$. Let $(x'_l, \ldots, x'_1), (x''_l, \ldots, x''_1) \in \mathrm{Vert}(T(|X_l|, \ldots, |X_1|))$ (we identify r_0 with the empty sequence). Since

$$(x'_l, \ldots, x'_1) \wedge (x''_l, \ldots, x''_1) = (x'_k, \ldots, x'_1)$$

where

$$k = \max\{i \in \{0, \ldots, l\} : (x'_i, \ldots, x'_1) = (x''_i, \ldots, x''_1)\},$$

we have

$$\begin{aligned} D_\chi(\varphi, \varphi') &= |(x_l, \ldots, x_1)\varphi \wedge (x_l, \ldots, x_1)\varphi'| \\ &= \max\{i \in \{0, \ldots, l\} : (x_l, \ldots, x_1)\varphi\pi_{[i,1]} = (x_l, \ldots, x_1)\varphi'\pi_{[i,1]}\}. \end{aligned}$$

□

Lemma 4.6 *Let* χ, χ' *be elliptic* M*-trees. Then* $D_\chi = D_{\chi'}$ *if and only if* $\chi \cong \chi'$*.*

Proof. Let $\chi = (r_0, T, \alpha)$, $\chi' = (r'_0, T', \alpha')$. Assume that $D_\chi = D_{\chi'}$. Note that

$$\mathrm{dep}(r_0, T) = \max D_\chi = \max D_{\chi'} = \mathrm{dep}(r'_0, T').$$

Write $l = \mathrm{dep}(r_0, T)$.

We define a mapping

$$\begin{aligned} \varphi : \mathrm{Vert}(T) &\to \mathrm{Vert}(T') \\ \alpha_i m &\mapsto \alpha'_i m \end{aligned}$$

where $i \in \mathrm{dom}\alpha$ and $m \in M$. Since the action of M on (r_0, T) is α-transitive, we have

$$\{v \in \mathrm{Vert}(T) \mid dep(v) = i\} = \alpha_i M.$$

If $\alpha_i m = \alpha_i m'$, then

$$\begin{aligned} |\alpha' m \wedge \alpha' m'| &= D_{\chi'}(m, m') = D_\chi(m, m') \\ &= |\alpha m \wedge \alpha m'| \geq i, \end{aligned}$$

hence $\alpha'_i m = \alpha'_i m'$. Since

$$\{v \in \mathrm{Vert}(T') \mid dep(v) = i\} = \alpha'_i M,$$

it follows that φ is well defined and onto. By symmetry, φ is also one-to-one.

Clearly, $r_0\varphi = (\alpha_0 \cdot 1)\varphi = \alpha'_0 \cdot 1 = r'_0$. Assume that v is the father of $w = \alpha_i m$. Then $v = \alpha_{i-1} m$ since the action of M on (r_0, T) is elliptical, hence $v\varphi = \alpha'_{i-1} m$ is the father of $w\varphi = \alpha'_i m$. By Lemma 3.2, φ is an elliptic contraction from (r_0, T) onto (r'_0, T') and therefore an isomorphism of rooted trees.

Since

$$\alpha\varphi = (\ldots, \alpha_1 \cdot 1, \alpha_0 \cdot 1)\varphi = (\ldots, \alpha'_1 \cdot 1, \alpha'_0 \cdot 1) = \alpha'$$

and

$$((\alpha_i m)m')\varphi = (\alpha_i mm')\varphi = \alpha'_i mm' = ((\alpha_i m)\varphi)m'$$

for all $m, m' \in M$, axioms (EM1) and (EM2) are satisfied and so $\varphi : \chi \to \chi'$ is an isomorphism of elliptic M-trees.

Conversely, assume that $\varphi : \chi \to \chi'$ is an isomorphism of elliptic M-trees. For all $m, m' \in M$, we have

$$\begin{aligned} D_\chi(m, m') &= |\alpha m \wedge \alpha m'| = |(\alpha m)\varphi \wedge (\alpha m')\varphi| \\ &= |(\alpha\varphi)m \wedge (\alpha\varphi)m'| = |\alpha' m \wedge \alpha' m'| \\ &= D_{\chi'}(m, m'), \end{aligned}$$

hence $D_\chi = D_{\chi'}$ and the lemma holds. $\square$

A proof for the following theorem can be found in [22], but the important role played by the Chiswell construction in it makes it worthwhile to include it here.

Theorem 4.7 *[22, Theorem 1.12] Let M be a monoid and let $D : M \times M \to \overline{\mathbb{N}}$ be a mapping. Then the following conditions are equivalent:*

(i) D is a length function for M;

(ii) $D = D_\chi$ for some elliptic M-tree χ.

Moreover, if the conditions hold, χ is unique up to isomorphism.

Proof. Assume that D is a length function for M. We adapt the important *Chiswell construction* of [5] as follows. By (L2), there exists $l = \max D \in \overline{\mathbb{N}}$. Let

$$P = \begin{cases} \{0, \ldots, l\} \times M & \text{if } l \in \mathbb{N} \\ \mathbb{N} \times M & \text{if } l = \omega \end{cases}$$

and define a relation $\sim$ on P by

$$(k, m) \sim (k', m') \qquad \text{if} \qquad k = k' \text{ and } D(m, m') \geq k.$$

We show that $\sim$ is an equivalence relation on P.

In fact, $\sim$ is reflexive by (2), and symmetric by (L1). Transitivity follows from the isoperimetric inequality (L4). Let $[k, m]$ denote the $\sim$ equivalence class of (k, m). We define a graph T by

$$\mathrm{Vert}(T) = P/\sim,$$

$$\mathrm{Edge}(T) = \{[k, m] \text{ --- } [k+1, m];\ (k, m) \in P,\ k < l\}.$$

It follows from the definitions that

$$\forall m, m' \in M,\ (0, m) \sim (0, m').$$

Let $r_0 = [0, 1]$. Since

$$[k, m] \text{ --- } \ldots \text{ --- } [1, m] \text{ --- } [0, m] = r_0 \tag{3}$$

is a path in T for every $(k, m) \in P$, T is a connected graph.

We show next that

$$[k, m] \text{ --- } [k + 1, m'] \in \mathrm{Edge}(T) \Leftrightarrow (k, m) \sim (k, m') \tag{4}$$

holds for all $k \in \{0, \ldots, l - 1\}$ and $m, m' \in M$. Indeed, if $[k, m]$ — $[k + 1, m'] \in \mathrm{Edge}(T)$ then $[k, m] = [k, m'']$ and $[k + 1, m'] = [k + 1, m'']$ for some $m'' \in M$. Hence $D(m, m'') \geq k$ and $D(m', m'') \geq k + 1$, yielding

$$D(m, m') \geq \min\{D(m, m''), D(m', m'')\} \geq k$$

by (L4). Thus $(k, m) \sim (k, m')$. The converse implication is trivial, therefore (4) holds.

We can prove now that T is a tree. Assume that T has a cycle C and let $[k, m]$ be a vertex in C with k maximum. Let $[k', m']$ and $[k'', m'']$ be its adjacent vertices in C. By maximality of k, we have $k' = k'' = k - 1$, hence $(k - 1, m') \sim (k - 1, m) \sim (k - 1, m'')$ by (4) and so $[k', m'] = [k'', m'']$, contradicting C being a cycle. Therefore T is a tree and so (r_0, T) is a rooted tree.

Clearly, (3) is a ray for every vertex $[k, m]$. If $l \in \mathbb{N}$, then (r_0, T) has finite depth l and it is uniform since $\mathrm{MRay}(r_0, T)$ consists of all paths of the form $([l, m], \ldots, [1, m], [0, m] = r_0)$ with $m \in M$. If $l = \omega$, all rays must have infinite length since $[k + 1, m]$ — $[k, m]$ is an edge for every $(k, m) \in P$, hence (r_0, T) is uniform as well.

We define a mapping $\eta : \mathrm{Vert}(T) \times M \to \mathrm{Vert}(T)$ by

$$\eta([k, m], m') = [k, m]m' = [k, mm'].$$

Note that

$$[k, m] = [k, n] \Rightarrow D(m, n) \geq k \Rightarrow D(mm', nm') \geq k \Rightarrow [k, mm'] = [k, nm']$$

by (L3) and so the mapping is well defined.

Clearly, $r_0 m = r_0$ for every $m \in M$. On the other hand, if $[k - 1, m']$ is the father of $[k, m']$, then $[k - 1, m']m$ is the father of $[k, m']m$ and so η induces a mapping

$$\begin{aligned} \theta : M &\to \mathrm{Ell}(r_0, T) \\ m &\mapsto \eta(\cdot, m) \end{aligned}$$

by Lemma 3.2. Since θ is a monoid homomorphism due to

$$[k, m]1 = [k, m], \quad [k, m](m'm'') = ([k, m]m')m'',$$

it follows that θ is an elliptic action of M on (r_0, T).

Let $\alpha \in \mathrm{Ray}(r_0, T)$ be defined by

$$|\alpha| = l, \quad \alpha_i = [i, 1] \ (i \in \mathrm{dom}\alpha).$$

Since $[i, m] = \alpha_i m$ for every $m \in M$, the action θ is α-transitive. Thus $\chi = (r_0, T, \alpha, \theta)$ is an elliptic M-tree. We show that $D = D_\chi$.

For all $m, m' \in M$, we have

$$\begin{aligned} D_\chi(m, m') &= |\alpha m \wedge \alpha m'| = |([i, m])_i \wedge ([i, m'])_i| = \sup\{i \in \mathrm{dom}\alpha : [i, m] = [i, m']\} \\ &= \sup\{i \in \mathrm{dom}\alpha : D(m, m') \geq i\} = D(m, m') \end{aligned}$$

and so $D = D_\chi$. Therefore (ii) holds.

(ii) $\Rightarrow$ (i). By Proposition 4.4.

The uniqueness of χ up to isomorphism follows from Lemma 4.6. $\square$

We consider now the case of strongly faithful elliptic M-trees. A length function $D : M \times M \to \overline{\mathbb{N}}$ is said to be *strict* if

(L5) $D(m', m'') = D(m, m) \Rightarrow m' = m''$ for all $m, m', m'' \in M$;

Corollary 4.8 *Let M be a monoid and let $D : M \times M \to \overline{\mathbb{N}}$ be a mapping. Then the following conditions are equivalent:*

(i) D is a strict length function for M;

(ii) $D = D_\chi$ for some strongly faithful elliptic M-tree χ.

Moreover, if the conditions hold, χ is unique up to isomorphism.

Proof. (i) $\Rightarrow$ (ii). Assume that (i) holds. By Theorem 4.7, $D = D_\chi$ for the elliptic M-tree $\chi = (r_0, T, \alpha, \theta)$ defined in its proof. We show that χ is strongly faithful. Indeed, let $m, m' \in M$. Suppose that $\alpha m = \alpha m'$. Then $[k, m] = [k, m']$ for every $k \in \mathrm{dom}\alpha$ and so $D(m, m') \geq k$ for every $k \in \mathrm{dom}\alpha$. It follows that $D(m, m') = l = \max D = D(m, m)$ and so $m = m'$ by (L5). Thus χ is strongly faithful.

(ii) $\Rightarrow$ (i). Assume that (ii) holds for $\chi = (r_0, T, \alpha)$. By Theorem 4.7, we only need to show that D_χ satisfies (L5). Suppose that $D(m', m'') = D(m, m)$ for some $m, m', m'' \in M$. Hence

$$\begin{aligned} |\alpha m' \wedge \alpha m''| &= D_\chi(m', m'') = D(m', m'') = D(m, m) = D_\chi(m, m) \\ &= |\alpha m \wedge \alpha m| = |\alpha m| = \mathrm{dep}(r_0, T) \end{aligned}$$

and so $\alpha m' = \alpha m''$. Since χ is strongly faithful, we get $m' = m''$ and so (L5) holds.

The uniqueness of χ up to isomorphism follows from Lemma 4.6. $\square$

We end this section by associating a length function to any wreath product of partial transformation monoids. To simplify notation, we present just the infinite case, the finite one being absolutely similar.

Proposition 4.9 *Let $\ldots, (X_2, M_2), (X_1, M_1)$ be partial transformation monoids and let*

$$(X, M) = \ldots \circ (X_2, M_2) \circ (X_1, M_1) = (\ldots \times X_2 \times X_1, \ \ldots \circ M_2 \circ M_1)$$

be their wreath product. Let $D : M \times M \to \overline{\mathbb{N}}$ be defined by

$$D(\varphi, \psi) = sup\{j \in \mathbb{N} \mid \varphi|_{X_j \times \ldots \times X_1} = \psi|_{X_j \times \ldots \times X_1}\}.$$

Then D is a strict length function for M.

Proof. Axioms (L1), (L2) and (L5) hold trivially.

Let $\varphi, \psi, \mu \in M$. Since $(X_j \times \ldots \times X_1)\theta \subseteq X_j \times \ldots \times X_1$ for every $\theta \in M$, $\varphi|_{X_j \times \ldots \times X_1} = \psi|_{X_j \times \ldots \times X_1}$ implies $(\varphi\mu)|_{X_j \times \ldots \times X_1} = (\psi\mu)|_{X_j \times \ldots \times X_1}$. Thus $D(\varphi, \psi) \leq D(\varphi\mu, \psi\mu)$ and (L3) holds.

Finally, $\varphi|_{X_j \times \ldots \times X_1} \neq \mu|_{X_j \times \ldots \times X_1}$ implies either $\varphi|_{X_j \times \ldots \times X_1} \neq \psi|_{X_j \times \ldots \times X_1}$ or $\psi|_{X_j \times \ldots \times X_1} \neq \mu|_{X_j \times \ldots \times X_1}$, hence $D(\varphi, \mu) \geq \min\{D(\varphi, \psi), D(\psi, \mu)\}$ and (L4) holds. $\square$

5 Expansions

Let $\mathcal{M}$ denote the category of all monoids. A *monoid expansion* is a functor $F : \mathcal{M} \to \mathcal{M}$ preserving surjective morphisms such that there exists a natural transformation η from the functor F to the identity functor with η_M surjective for each $M \in \mathcal{M}$.

That is, F assigns to each monoid M a monoid $F(M)$ and a surjective morphism $\eta_M : F(M) \to M$, and to each monoid homomorphism $\varphi : M \to N$ a monoid homomorphism $F(\varphi) : F(M) \to F(N)$ satisfying:

(E1) if φ is surjective, so is $F(\varphi)$;

(E2) if $\varphi = \mathrm{Id}_M$, then $F(\varphi) = \mathrm{Id}_{F(M)}$;

(E3) if $\varphi : M \to M'$, $\varphi' : M' \to M''$ are morphisms, then $F(\varphi\varphi') = F(\varphi)F(\varphi')$;

(E4) if $\varphi : M \to N$ is a morphism, then the following diagram commutes:

$$\begin{tikzcd}
F(M) \arrow[r, "F(\varphi)"] \arrow[d, "\eta_M"'] & F(N) \arrow[d, "\eta_N"] \\
M \arrow[r, "\varphi"'] & N
\end{tikzcd}$$

Semigroup expansions are defined analogously.

An element $a \in M$ is said to be *aperiodic* if $a^{n+1} = a^n$ for some $n \in \mathbb{N}$. A morphism $\varphi : M \to N$ is said to be *aperiodic* if, whenever $a \in N$ is aperiodic, all elements in $a\varphi^{-1}$ are also aperiodic. The expansion F is said to be *aperiodic* if the morphism η_M is aperiodic for every monoid M.

We define now the Rhodes expansion for monoids, omitting the expansion of morphisms. The reader is referred to [26, 19, 4, 8, 18, 20, 23] for more details.

The *$\mathcal{L}$-preorder* on a monoid M is defined by

$$a \leq_{\mathcal{L}} b \qquad \text{if} \qquad a \in Mb.$$

This preorder is clearly compatible with multiplication on the right:

$$\forall a, b, m \in M \ (a \leq_{\mathcal{L}} b \Rightarrow am \leq_{\mathcal{L}} bm).$$

The Green relation $\mathcal{L}$ can of course be defined by

$$a \,\mathcal{L}\, b \qquad \text{if} \qquad a \leq_{\mathcal{L}} b \text{ and } b \leq_{\mathcal{L}} a.$$

The *strict* $\mathcal{L}$*-order* on M is defined by

$$a <_{\mathcal{L}} b \qquad \text{if} \qquad a \in Mb \text{ and } b \notin Ma,$$

i.e., $<_{\mathcal{L}} = \leq_{\mathcal{L}} \setminus \mathcal{L}$.

The $\mathcal{R}$- and $\mathcal{J}$-versions are defined similarly. In particular,

$$a <_{\mathcal{J}} b \qquad \text{if} \qquad a \in MbM \text{ and } b \notin MaM.$$

Given a finite chain of the form

$$\sigma = (m_k \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m_1 \leq_{\mathcal{L}} m_0)$$

in M, we define a chain

$$\mathrm{lm}(\sigma) = (m_{i_l} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_{i_1} <_{\mathcal{L}} m_{i_0})$$

by keeping the leftmost term in each $\mathcal{L}$-class of terms of σ. Thus

$$\sigma = (m_{i_l} \,\mathcal{L}\, m_{i_l - 1} \,\mathcal{L} \ldots \mathcal{L}\, m_{i_{l-1}+1} <_{\mathcal{L}} m_{i_{l-1}} \ldots m_{i_0+1} <_{\mathcal{L}} m_{i_0} \,\mathcal{L} \ldots \mathcal{L}\, m_0).$$

We define the *Rhodes expansion* $\mathrm{Rh}(M)$ of M to be the set of all finite chains of the form

$$m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = 1$$

with $k \geq 0$ and $m_i \in M$. The product of two chains

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = 1), \quad \tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = 1)$$

is defined by

$$\sigma\tau = \mathrm{lm}(m_k m'_l \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m_1 m'_l \leq_{\mathcal{L}} m_0 m'_l = m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = 1).$$

Note that the product is well defined since $\leq_{\mathcal{L}}$ is right compatible. It turns out that $\mathrm{Rh}(M)$ is a monoid having the trivial chain $(m_0 = 1)$ as identity.

The surjective morphisms $\eta_M : \mathrm{Rh}(M) \to M$ are defined by

$$(m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = 1)\eta_M = m_k.$$

It follows from the definition of $\leq_{\mathcal{L}}$ that the elements of $\mathrm{Rh}(M)$ are precisely the finite chains of the form

$$x_k \ldots x_2 x_1 <_{\mathcal{L}} x_{k-1} \ldots x_2 x_1 <_{\mathcal{L}} \ldots <_{\mathcal{L}} x_2 x_1 <_{\mathcal{L}} x_1 <_{\mathcal{L}} 1$$

with $x_1, \ldots, x_k \in M$. Moreover,

$$(x_k \ldots x_2 x_1 <_{\mathcal{L}} \ldots <_{\mathcal{L}} x_2 x_1 <_{\mathcal{L}} x_1 <_{\mathcal{L}} 1) = (x_k <_{\mathcal{L}} 1) \ldots (x_2 <_{\mathcal{L}} 1)(x_1 <_{\mathcal{L}} 1), \tag{5}$$

hence $\mathrm{Rh}(M)$ is generated (as a monoid) by the chains $m <_{\mathcal{L}} 1$.

Given a set Y, we define a Y*-monoid* to be an ordered pair of the form (M, φ), where M is a monoid and $\varphi : Y^* \to M$ is a surjective morphism. Similarly, we define Y*-semigroup*. A *morphism* from the Y-monoid (M, φ) to the Y-monoid (M', φ') is a monoid morphism $\theta : M \to M'$ such that the diagram

$$\begin{tikzcd} Y^* \arrow[r, "\varphi"] \arrow[d, "\varphi'"'] & M \arrow[dl, "\theta"] \\ M' & \end{tikzcd}$$

commutes. Whenever possible, to simplify notation, we omit the morphism in the representation of Y-monoids, that is, we view Y as a subset of M and φ as canonical.

Clearly, Y-monoids and their morphisms constitute a category, and we can consider expansions in the category of Y-monoids just as we did for the category of monoids. We define now the expansion Rh_Y in the category of Y-monoids, that can be described as the Rhodes expansion *cut-down to the generators* Y.

Indeed, let M be an Y-monoid. We remarked before that $\mathrm{Rh}(M)$ is generated (as a monoid) by the chains $m <_{\mathcal{L}} 1$. We define $\mathrm{Rh}_Y(M)$ to be the submonoid of $\mathrm{Rh}(M)$ generated by the chains $y <_{\mathcal{L}} 1$ $(y \in Y)$. It is shown in [4] that Rh_Y defines an expansion of Y-monoids. We omit the description of the expansion for morphisms.

The Rhodes expansion has many interesting properties that are subsequently inherited by Rh_Y, such as the following:

Proposition 5.1 *[26, 4]*

(i) The Rhodes expansion is aperiodic.

(ii) The Rhodes expansion preserves regularity.

(iii) $\forall \sigma \in Rh(M)\ (\sigma \in E(Rh(M)) \Leftrightarrow \sigma\eta_M \in E(M))$.

The expansion Rh_Y *possesses analogous properties.*

We introduce now another expansion with important properties. Let M be a semigroup and let M^+ denote the free semigroup on (the set) M. Hence

$$M^+ = \{(m_1, \ldots, m_k) \mid k \geq 1,\ m_i \in M\}.$$

Given $(m_1, \ldots, m_k) \in M^+$, let

$$F_3(m_1, \ldots, m_k) = \{(m_1 \ldots m_i, m_{i+1} \ldots m_j, m_{j+1} \ldots m_k) \in M \times M \times M \mid 0 \leq i \leq j \leq k\}.$$

Write

$$\Phi_3(M) = \{F_3(m_1, \ldots, m_k) \mid (m_1, \ldots, m_k) \in M^+\}.$$

We define a multiplication on $\Phi_3(M)$ by

$$F_3(m_1, \ldots, m_k)F_3(m'_1, \ldots, m'_l) = F_3(m_1, \ldots, m_k, m'_1, \ldots, m'_l).$$

By [4, Section 7.2], $M \to \Phi_3(M)$ is part of a semigroup expansion (we omit here the expansion of morphisms). The surjective morphisms $\eta_M : \Phi_3(M) \to M$ are defined by

$$(F_3(m_1, \ldots, m_k))\eta_M = m_1 \ldots m_k.$$

We can also perform the cut-down to generators for this expansion [4]. Indeed, If M is a Y-semigroup, we denote by $\Phi_{3,Y}(M)$ the subsemigroup of $\Phi_3(M)$ generated by the elements of the form

$$F_3(y) = \{(y, 1, 1),\ (1, y, 1),\ (1, 1, y)\} \quad (y \in Y).$$

Then the restriction of the morphism η_M to $\Phi_{3,Y}(M)$ is surjective and $\Phi_{3,Y}(M)$ is part of an expansion of Y-semigroups [4].

We recall that a semigroup M is said to be *finite $\mathcal{J}$-above* if $\{y \in M \mid y \geq_{\mathcal{J}} x\}$ is finite for every $x \in M$.

The following properties make the expansion Φ_3 of great interest. Note that, since $\Phi_{3,Y}(M)$ is a subsemigroup of $\Phi_3(M)$, these properties generalize immmediately to $\Phi_{3,Y}$.

Proposition 5.2 *[4, Propositions 7.8 and 7.9]*

(i) $\Phi_3(M)$ is finite $\mathcal{J}$-above for every semigroup M;

(ii) Φ_3 is aperiodic.

The expansion $\Phi_{3,Y}$ possesses analogous properties.

6 The Holonomy Theorem

Given a semigroup M, we denote by M^I the monoid obtained by adjoining a new identity I to M (*even if M is already a monoid*), see [23, Chapter 1]. We shall consider the Rhodes expansion $\mathrm{Rh}(M^I)$ of the monoid M^I, consisting of all finite chains of the form

$$m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I$$

with $k \geq 0$ and $m_i \in M$ $(i = 1, \ldots, k)$.

We say that a mapping $f : M \to N$ between monoids is *$\leq_{\mathcal{J}}$-preserving* if

(JP1) $f(1) = 1$;

(JP2) $a \leq_{\mathcal{J}} b \Rightarrow f(a) \leq_{\mathcal{J}} f(b)$ for all $a, b \in M$.

It follows that

$$a \mathrel{\mathcal{J}} b \Rightarrow f(a) \mathrel{\mathcal{J}} f(b) \qquad \text{for all } a, b \in M.$$

The important particular case arises for mappings $f : M^I \to \mathbb{N}$, where we consider addition on $\mathbb{N}$. Note that, for all $n, n' \in \mathbb{N}$,

$$n \leq_J n' \Leftrightarrow n \geq n'.$$

Thus $f : M^I \to \mathbb{N}$ is $\leq_J$-preserving if and only if $f(I) = 0$ and

$$\forall m, m', m'' \in M^I,\ f(m'mm'') \geq f(m).$$

Since $\mathbb{N}$ is $\mathcal{J}$-trivial, note that

$$a \,\mathcal{J}\, b \Rightarrow f(a) = f(b) \qquad \text{for all } a, b \in M^I. \tag{6}$$

Let

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = I)$$

be elements of $\mathrm{Rh}(M^I)$. The *maximum $\mathcal{L}$-point of agreement* of σ and τ is defined by $\sigma \wedge_{\mathcal{L}} \tau = m_r$, with

$$r = \max\{i \in \{0, \ldots, \min\{k, l\}\} \mid m_0 = m'_0, \ldots, m_{i-1} = m'_{i-1}, m_i \,\mathcal{L}\, m'_i\}.$$

We present now the Holonomy Theorem in its most abstract version:

Theorem 6.1 (Holonomy Theorem) *Let M be a semigroup and let $f : M^I \to \mathbb{N}$ be $\leq_J$-preserving. Let $D : Rh(M^I) \times Rh(M^I) \to \overline{\mathbb{N}}$ be defined by*

$$D(\sigma, \tau) = \begin{cases} f(\sigma \wedge_{\mathcal{L}} \tau) & \text{if } \sigma \neq \tau \\ 1 + \sup f & \text{if } \sigma = \tau. \end{cases}$$

Then

(i) D is a strict length function for $Rh(M^I)$;

(ii) $D = D_\chi$ for some (unique up to isomorphism) strongly faithful elliptic $Rh(M^I)$-tree χ.

Proof. We show that D satisfies axioms (L1) – (L5).

(L1): Let $\sigma, \tau \in \mathrm{Rh}(M^I)$. Since $(\sigma \wedge_{\mathcal{L}} \tau) \,\mathcal{L}\, (\tau \wedge_{\mathcal{L}} \sigma)$, we have $D(\sigma, \tau) = D(\tau, \sigma)$ in view of (6).

(L2) follows from the definition of D.

(L3): Let $\sigma, \tau, \rho \in \mathrm{Rh}(M^I)$. We start by showing that

$$(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) \leq_{\mathcal{L}} (\sigma \wedge_{\mathcal{L}} \tau). \tag{7}$$

In view of (5), we may assume that $\rho = (m <_{\mathcal{L}} I)$. Write

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = I)$$

and assume that $\sigma \wedge_{\mathcal{L}} \tau = m_r$. Then we may write

$$\sigma\rho = \mathrm{lm}(m_k m \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m_1 m \leq_{\mathcal{L}} m <_{\mathcal{L}} I),$$

$$\tau\rho = \mathrm{lm}(m'_l m \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m'_1 m \leq_{\mathcal{L}} m <_{\mathcal{L}} I).$$

Clearly, $m_r \,\mathcal{L}\, m'_r$ yields $(m_r m) \,\mathcal{L}\, (m'_r m)$ and we also have $m_i m = m'_i m$ for $i \in \{0, \ldots, r-1\}$. Note that $m_r m$ may not be in $\sigma\rho$, but some $m_s m \in \mathcal{L}_{m_r m}$ will, and similarly for $m'_r m$. Hence (7) holds.

Back to checking (L3), we may assume that $\sigma\rho \neq \tau\rho$. Hence $\sigma \neq \tau$ as well. Since f is $\leq_J$-preserving and

$$(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) \leq_{\mathcal{J}} (\sigma \wedge_{\mathcal{L}} \tau)$$

by (7), we get

$$D(\sigma\rho,\tau\rho)=f(\sigma\rho\wedge_{\mathcal{L}}\tau\rho)\geq f(\sigma\wedge_{\mathcal{L}}\tau)=D(\sigma,\tau)$$

since f is $\leq_J$-preserving and $\sigma\rho\neq\tau\rho$. Thus (L3) holds.

(L4): Let $\sigma,\tau,\rho\in\mathrm{Rh}(M^I)$. We show that

$$(\sigma\wedge_{\mathcal{L}}\rho)\leq_{\mathcal{L}}(\sigma\wedge_{\mathcal{L}}\tau)\quad\vee\quad(\sigma\wedge_{\mathcal{L}}\rho)\leq_{\mathcal{L}}(\tau\wedge_{\mathcal{L}}\rho).\tag{8}$$

Write

$$\sigma=(m_k<_{\mathcal{L}}\ldots<_{\mathcal{L}}m_1<_{\mathcal{L}}m_0=I),\quad\tau=(m'_l<_{\mathcal{L}}\ldots<_{\mathcal{L}}m'_1<_{\mathcal{L}}m'_0=I),$$

$$\rho=(m''_p<_{\mathcal{L}}\ldots<_{\mathcal{L}}m''_1<_{\mathcal{L}}m''_0=I),$$

$$(\sigma\wedge_{\mathcal{L}}\rho)=m_r,\quad(\sigma\wedge_{\mathcal{L}}\tau)=m_s,\quad(\tau\wedge_{\mathcal{L}}\rho)=m'_t.$$

Suppose that $m_r\not\leq_{\mathcal{L}}m_s$ and $m_r\not\leq_{\mathcal{L}}m'_t$. Then $r<s$ and $m''_r\;\mathcal{L}\;m_r\not\leq_{\mathcal{L}}m'_t\;\mathcal{L}\;m''_t$ yields $r<t$ as well. For $i=0,\ldots,r$, we have $m_i=m'_i=m''_i$ since $r<s,t$. Moreover, $m_{r+1}\;\mathcal{L}\;m'_{r+1}\;\mathcal{L}\;m''_{r+1}$ since $r+1\leq s,t$, contradicting $(\sigma\wedge_{\mathcal{L}}\rho)=m_r$. Therefore $m_r\leq_{\mathcal{L}}m_s$ or $m_r\leq_{\mathcal{L}}m'_t$, and so (8) holds.

To prove (L4), we may assume that σ,τ,ρ are all distinct. Without loss of generality, we may assume by (8) that $(\sigma\wedge_{\mathcal{L}}\rho)\leq_{\mathcal{L}}(\sigma\wedge_{\mathcal{L}}\tau)$, hence $(\sigma\wedge_{\mathcal{L}}\rho)\leq_{\mathcal{J}}(\sigma\wedge_{\mathcal{L}}\tau)$ and so

$$D(\sigma,\rho)=f(\sigma\wedge_{\mathcal{L}}\rho)\geq f(\sigma\wedge_{\mathcal{L}}\tau)=D(\sigma,\tau)\geq\min\{D(\sigma,\tau),D(\tau,\rho)\}$$

by (JP2). Thus (L4) holds.

(L5): Assume that $D(\tau,\rho)=D(\sigma,\sigma)$ for some $\sigma,\tau,\rho\in\mathrm{Rh}(M^I)$ with $\tau\neq\rho$. It follows that

$$1+\sup f=D(\sigma,\sigma)=D(\tau,\rho)=f(\tau\wedge_{\mathcal{L}}\rho)\in\mathbb{N},$$

a contradiction. Thus $D(\tau,\rho)=D(\sigma,\sigma)$ imples $\tau=\rho$ and (L5) holds.

Therefore D is a strict length function for $\mathrm{Rh}(M^I)$ and so $D=D_\chi$ for some (unique up to isomorphism) strongly faithful elliptic $\mathrm{Rh}(M^I)$-tree χ by Corollary 4.8. $\square$

A preordered set $(X,\leq)$ is said to be *upper finite* if every subset of the form $[x)=\{y\in X\mid y\geq x\}$ is finite. In particular, if $(X,\leq)$ is upper finite, every nonempty subset of X must contain a maximal element.

To show how to obtain all $\leq_J$-preserving mappings $f:M^I\to\mathbb{N}$ when M is finite $\mathcal{J}$-above, we introduce the concept of weight function in a more general setting. Given an upper finite partially ordered set $(P,\leq)$ with maximum I, a *weight function* $w:P\to\mathbb{N}$ is any function satisfying $w(I)=0$. Given $w:P\to\mathbb{N}$, let $h_w:P\to N$ be defined by

$$h_w(p)=\max\{\sum_{i=0}^{n}w(p_i)\mid p=p_n<\ldots<p_1<p_0=I\text{ is a chain in }P\}.$$

Since $(P,\leq)$ is upper finite, h_w is well defined.

Proposition 6.2 (Dedekind inversion). *Let $(P,\leq)$ be an upper finite partially ordered set $(P,\leq)$ with maximum I. Then the correspondence $\mu:w\mapsto h_w$ defines a bijection between the set of all weight functions $w:P\to\mathbb{N}$ and all order-reversing mappings $h:P\to\mathbb{N}$ satisfying $h(I)=0$.*

Proof. Let $w : P \to \mathbb{N}$ be a weight function. Assume that $q \leq p$ in P. Since any chain $p = p_n < \ldots < p_1 < p_0 = I$ in P extends to a chain

$$q \leq p = p_n < \ldots < p_1 < p_0 = I,$$

we get $h_w(q) \geq h_w(p)$ and so h_w is order-reversing. Since the only ascending chain starting at I is the trivial chain and $w(I) = 0$, we have $h_w(I) = 0$. Thus μ is well defined.

Suppose that $w, w' : P \to \mathbb{N}$ are distinct weight functions. Take a maximal element p from the set

$$\{x \in P \mid w(x) \neq w'(x)\}.$$

Since P is upper finite, there exist such maximal elements. Assume that $w(p) < w'(p)$. Given a chain $p = p_n < \ldots < p_1 < p_0 = I$, we have $w(p_i) = w'(p_i)$ for $i = 0, \ldots, n-1$ by maximality of p, hence

$$\sum_{i=0}^{n} w(p_i) = w(p) + \sum_{i=0}^{n-1} w'(p_i) < \sum_{i=0}^{n} w'(p_i)$$

and so $h_w(p) < h_{w'}(p)$. Thus μ is one-to-one.

Finally, take $h : P \to \mathbb{N}$ order-reversing satisfying $h(I) = 0$. We define a weight function $w : P \to \mathbb{N}$ as follows. Given $p \in P \setminus \{I\}$, let

$$\overline{p} = \{q \in P \mid p < q \text{ and there exists no } r \in P \text{ such that } p < r < q\}$$

denote the set of all elements of P covering p. Since P is upper finite, $\overline{p}$ is nonempty. We define

$$w(p) = h(p) - \max\{h(q) \mid q \in \overline{p}\}.$$

Since h is order-reversing, $w(p) \geq 0$ and so w is a well-defined weight function. We show that $h = h_w$.

Let $p \in P \setminus \{I\}$. We show that

$$h_w(q) = h(q) \text{ for every } q \in \overline{p} \Rightarrow h_w(p) = h(p). \tag{9}$$

Indeed, assume the hypothesis and let $p = p_n < \ldots < p_1 < p_0 = I$ be a chain in P with $h_w(p) = \sum_{i=0}^{n} w(p_i)$. By maximality of $\sum_{i=0}^{n} w(p_i)$, we may assume that $p_{n-1} \in \overline{P}$. Moreover, $h_w(p_{n-1}) = \sum_{i=0}^{n-1} w(p_i)$ must be maximal among $\{h_w(q) \mid q \in \overline{p}\}$. It follows that

$$\begin{aligned} h_w(p) &= \textstyle\sum_{i=0}^{n} w(p_i) = h_w(p_{n-1}) + w(p) \\ &= \max\{h_w(q) \mid q \in \overline{p}\} + w(p) \\ &= \max\{h(q) \mid q \in \overline{p}\} + w(p) \\ &= h(p) \end{aligned}$$

and so (9) holds.

Suppose that $h \neq h_w$. Since P is upper finite, we can take a maximal element p from the set $\{x \in P \mid h(x) \neq h_w(x)\}$. Since $h_w(I) = 0 = h(I)$, we have $p \neq I$. By maximality of p, we must have $h_w(q) = h(q)$ for every $q \in \overline{p}$. But then $h_w(p) = h(p)$ by (9), a contradiction. Therefore $h_w = h$ and so μ is onto as required. $\square$

Throughout the paper, we consider the set $M/\mathcal{J}$ of all $\mathcal{J}$-classes of a semigroup M partially ordered by

$$\mathcal{J}_a \leq \mathcal{J}_b \text{ if } a \leq_{\mathcal{J}} b.$$

Corollary 6.3 *Let M be a finite $\mathcal{J}$-above semigroup. Then the $\leq_J$-preserving mappings $f : M^I \to \mathbb{N}$ are defined by*

$$f(m) = h_w(\mathcal{J}_m)$$

for some weight function $w : M^I/\mathcal{J} \to \mathbb{N}$.

Proof. Clearly, (6) implies that the $\leq_J$-preserving mappings $f : M^I \to \mathbb{N}$ must be those of the form

$$f(m) = h(\mathcal{J}_m)$$

for some $\leq_J$-preserving mapping $h : M^I/\mathcal{J} \to \mathbb{N}$. Since $M^I/\mathcal{J}$ is an upper finite partially ordered set, the claim follows from Proposition 6.2. □

The following is a straighforward corollary from Theorem 6.1 and Corollary 6.3.

Corollary 6.4 *Let M be a finite $\mathcal{J}$-above semigroup and let $w : M^I/\mathcal{J} \to \mathbb{N}$ be a weight function. Let $f_w : M^I \to \mathbb{N}$ be defined by*

$$f_w(m) = max\{\sum_{i=0}^{n} w(\mathcal{J}_{m_i}) \mid m = m_n <_{\mathcal{J}} \ldots <_{\mathcal{J}} m_1 <_{\mathcal{J}} m_0 = I \text{ is a chain in } M^I\}.$$

Let $D : Rh(M^I) \times Rh(M^I) \to \overline{\mathbb{N}}$ be defined by

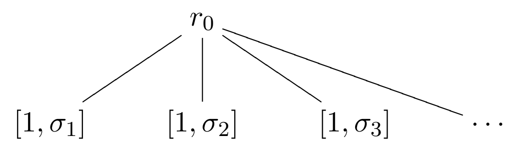
$$D(\sigma, \tau) = \begin{cases} f_w(\sigma \wedge_{\mathcal{L}} \tau) & \text{if } \sigma \neq \tau \\ 1 + sup f_w & \text{if } \sigma = \tau. \end{cases}$$

Then

(i) D is a strict length function for $Rh(M^I)$;

(ii) $D = D_\chi$ for some (unique up to isomorphism) strongly faithful elliptic $Rh(M^I)$-tree χ.

Example 6.5 *[22, Example 2.8(a)] Let M be a finite $\mathcal{J}$-above semigroup and let $w : M^I/\mathcal{J} \to \mathbb{N}$ be the null weight function. Then $f_w : M^I \to \overline{\mathbb{N}}$ is the null function and so the induced length function $D : Rh(M^I) \times Rh(M^I) \to \mathbb{N}$ is induced by the strongly faithful elliptic M-tree χ whose underlying tree can be depicted by*



if $Rh(M^I) = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$

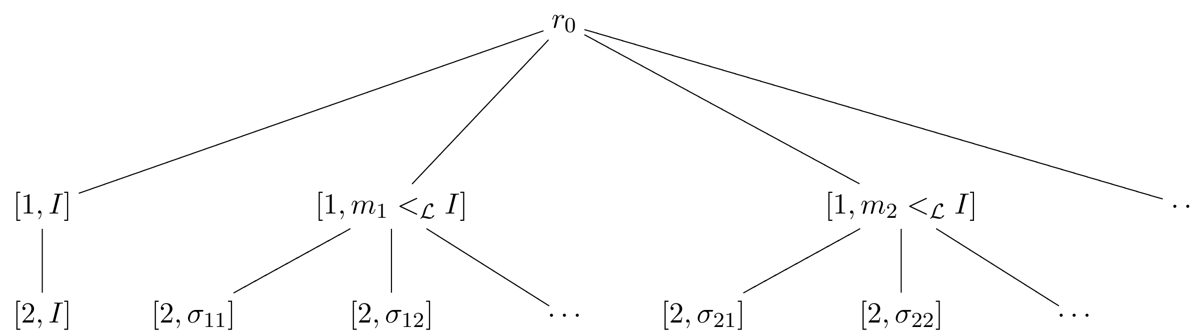
Example 6.6 *Let M be a finite $\mathcal{J}$-above monoid and let $w : M^I/\ \mathcal{J} \to \mathbb{N}$ be the weight function defined by*

$$w(\mathcal{J}_m) = \begin{cases} 1 & \textit{if } m\ \mathcal{J}\ 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Then $f_w : M^I \to \overline{\mathbb{N}}$ is defined by

$$f_w(m) = \begin{cases} 0 & \textit{if } m = I \\ 1 & \textit{otherwise} \end{cases}$$

and so the induced length function $D : Rh(M^I) \times Rh(M^I) \to \overline{\mathbb{N}}$ is induced by the elliptic M-tree χ whose underlying tree can be depicted by



where $M/\ \mathcal{L} = \{\mathcal{L}_{m_1}, \mathcal{L}_{m_2}, \ldots\}$ and $\{\sigma_{i1}, \sigma_{i2}, \ldots\}$ denotes the set of all $(\ldots <_{\mathcal{L}} m_i' <_{\mathcal{L}} I) \in Rh(M^I)$ with $m_i'\ \mathcal{L}\ m_i$.

Proof. It is immediate that f_w must be of the claimed form since all $m \in M^I$ except I satisfy $m \leq_{\mathcal{J}} 1$. Hence, for

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \tau = (m_l' <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1' <_{\mathcal{L}} m_0' = I),$$

we have

$$D(\sigma, \tau) = \begin{cases} 2 & \text{if } \sigma = \tau \\ 1 & \text{if } \sigma \neq \tau \text{ and } k, l > 0 \text{ and } m_1\ \mathcal{L}\ m_1' \\ 0 & \text{otherwise} \end{cases}$$

since $D(\sigma, \tau) = 1$ if and only if $\sigma \neq \tau$ and $\sigma \wedge_{\mathcal{L}} \tau \neq I$.

Following the Chiswell construction in the proof of Theorem 4.7, the underlying tree T of the elliptic M-tree χ induced by χ has vertex set

$$\text{Vert}(T) = \{r_0\} \cup \{[i, \sigma];\ i = 1, 2;\ \sigma \in \text{Rh}(M^I)\}.$$

We have $[1, \sigma] = [1, \tau]$ if and only if $D(\sigma, \tau) \geq 1$, hence

$$\{[1, I], [1, m_1 <_{\mathcal{L}} I], [1, m_2 <_{\mathcal{L}} I], \ldots\}$$

constitutes a full set of representatives for the classes $[1, \sigma]$. Clearly, $[2, \sigma] = [2, \tau]$ if and only if $\sigma = \tau$ and so the tree is the claimed one. $\square$

Given a finite $\mathcal{J}$-above semigroup M, we define a mapping $h_{\mathcal{J}} : M \to \mathbb{N}$ by

$$h_{\mathcal{J}}(m) = \max\{k \in \mathbb{N} : \text{ there exists a chain } m = m_0 <_{\mathcal{J}} \ldots <_{\mathcal{J}} m_k \text{ in } M\}.$$

We say that $h_{\mathcal{J}}$ is the (Dedekind) *$\mathcal{J}$-height function* of M [3]. Since the $\mathcal{J}$-class of I contains only I and lies above all the others, it is immediate that M^I has also a Dedekind $\mathcal{J}$-height function $h'_{\mathcal{J}}$, satisfying

$$h'_{\mathcal{J}}(m) = \begin{cases} h_{\mathcal{J}}(m) + 1 & \text{if } m \in M \\ 0 & \text{if } m = I \end{cases}$$

Proposition 6.7 *Let M be a finite $\mathcal{J}$-above semigroup and let $h_{\mathcal{J}}$ be the $\mathcal{J}$-height function of M^I. Then:*

(i) $h_{\mathcal{J}}$ is $\leq_{\mathcal{J}}$-preserving;

(ii) $h_{\mathcal{J}} = f_w$ for the weight function $w : M^I/\mathcal{J} \to \mathbb{N}$ defined by

$$w(\mathcal{J}_m) = \begin{cases} 1 & \text{if } m \in M \\ 0 & \text{if } m = I. \end{cases}$$

Proof. (i) Immediate.

(ii) We have $h_{\mathcal{J}}(I) = 0 = h_w(I)$. Given $m \in M$,

$$\begin{aligned} h_{\mathcal{J}}(m) &= \max\{k \in \mathbb{N} : \text{ there exists a chain } m = m_0 <_{\mathcal{J}} \ldots <_{\mathcal{J}} m_k \text{ in } M^I\} \\ &= \max\{k \in \mathbb{N} : \text{ there exists a chain } m = m_0 <_{\mathcal{J}} \ldots <_{\mathcal{J}} m_k = I \text{ in } M^I\} \\ &= \max\{\textstyle\sum_{i=0}^k w(\mathcal{J}_{m_i}) : \text{ there exists a chain } m = m_0 <_{\mathcal{J}} \ldots <_{\mathcal{J}} m_k = I \text{ in } M^I\} \\ &= f_w(m). \end{aligned}$$

□

The mapping $h_{\mathcal{J}}$ will play the most important role as a $\leq_{\mathcal{J}}$-preserving mapping in forthcoming sections.

We can use the expansion Φ_3 to avoid the finite $\mathcal{J}$-above requirement in Corollary 6.4:

Corollary 6.8 *Let M be a semigroup and let $w : (\Phi_3(M))^I/\mathcal{J} \to \mathbb{N}$ be a weight function. Let $f_w : (\Phi_3(M))^I \to \mathbb{N}$ be defined by*

$$f_w(x) = max\{\sum_{i=0}^{n} w(\mathcal{J}_{x_i}) \mid x = x_n <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I \text{ is a chain in } (\Phi_3(M))^I\}.$$

Let $D : Rh((\Phi_3(M))^I) \times Rh((\Phi_3(M))^I) \to \overline{\mathbb{N}}$ be defined by

$$D(\sigma, \tau) = \begin{cases} f_w(\sigma \wedge_{\mathcal{L}} \tau) & \text{if } \sigma \neq \tau \\ 1 + sup f_w & \text{if } \sigma = \tau. \end{cases}$$

Then

(i) D is a strict length function for $Rh((\Phi_3(M))^I)$;

(ii) $D = D_\chi$ for some (unique up to isomorphism) strongly faithful elliptic $Rh((\Phi_3(M))^I)$-tree χ;

(iii) the canonical surjective morphism $Rh((\Phi_3(M))^I) \to M$ is aperiodic.

Proof. (i) and (ii) follow from Corollary 6.4 since $\Phi_3(M)$ is finite $\mathcal{J}$-above by Proposition 5.2(i).

For (iii), we can decompose the canonical morphism $\mathrm{Rh}((\Phi_3(M))^I) \to M$ as the composition

$$\mathrm{Rh}((\Phi_3(M))^I) \xrightarrow{\eta_{(\Phi_3(M))^I}} (\Phi_3(M))^I \xrightarrow{\varphi} \Phi_3(M) \xrightarrow{\eta'_M} M.$$

The morphisms $\eta_{(\Phi_3(M))^I}$ and η'_M are aperiodic by Propositions 5.1(i) and 5.2(ii). Since φ is trivially aperiodic and the composition of aperiodic morphisms is aperiodic, the result follows. $\square$

7 Stable monoids and the Zeiger encoding

We start by introducing some well-known concepts and results. For details, the reader is referred to [6, 23].

A semigroup M is said to be *stable* if the following conditions hold for all $a, x \in M$:

(S1) $ax \; \mathcal{J} \; a \Rightarrow ax \; \mathcal{R} \; a$;

(S2) $xa \; \mathcal{J} \; a \Rightarrow xa \; \mathcal{L} \; a$.

It folows easily that [6, 23]

$$\text{if } M \text{ is stable, then } \; <_{\mathcal{R}} \subseteq <_J \; \text{ and } \; <_{\mathcal{L}} \subseteq <_J \; . \tag{10}$$

The following lemma will turn out to be quite useful:

Lemma 7.1 *Let M be stable and let $a, b, c \in M$ satisfy $a <_{\mathcal{L}} b \, \mathcal{R} \, bc$. Then $a \, \mathcal{R} \, ac <_{\mathcal{L}} bc$.*

Proof. Clearly, $a <_{\mathcal{L}} b$ yields $ac \leq_{\mathcal{L}} bc$. Since

$$ac \leq_{\mathcal{J}} a <_{\mathcal{L}} b \, \mathcal{R} \, bc,$$

it follows from (10) that $ac <_{\mathcal{J}} bc$ and so $ac <_{\mathcal{L}} bc$.

On the other hand, $b \, \mathcal{R} \, bc$ yields $b = bcx$ for some $x \in M$. Since $a <_{\mathcal{L}} b$, we get $a = acx$ and so $a \, \mathcal{R} \, ac$. $\square$

Every finite $\mathcal{J}$-above semigroup is stable, a fact that will be thoroughly used throughout the paper.

Assume that M is stable. Then the Green relations $\mathcal{J}$ and $\mathcal{D}$ on M coincide. Given a $\mathcal{J}$-class J of a monoid, we can always define a semigroup structure $(J^0, *)$ on $J^0 = J \cup \{0\}$ by taking

$$a * b = \begin{cases} ab & \text{if } a, b, ab \in J \\ 0 & \text{otherwise.} \end{cases}$$

If the monoid is stable, the semigroup J^0 defined above is completely 0-simple and can thus be given a *Rees matrix coordinatization*: there exist nonempty sets A, B, a group G

and a $(B \times A)$-matrix C with entries in $G \cup \{0\}$ such that $J^0 \cong M^0(G, A, B, C)$, where $M^0(G, A, B, C) = (A \times G \times B) \cup \{0\}$ is the semigroup with zero defined by

$$(a, g, b)(a', g', b') = \begin{cases} (a, gC(b, a')g', b') & \text{if } C(b, a') \in G \\ 0 & \text{if } C(b, a') = 0. \end{cases}$$

The Green relations in $M^0(G, A, B, C)$ are characterized by

$$\begin{aligned} (a, g, b) \ \mathcal{R} \ (a', g', b') &\Leftrightarrow a = a', \\ (a, g, b) \ \mathcal{L} \ (a', g', b') &\Leftrightarrow b = b'. \end{aligned}$$

We shall need the detailed construction of the Rees matrix semigroup, so we present it briefly. For more details, see [6, 23].

We fix a $\mathcal{H}$-class H in J and $h_0 \in H$. Let A (respectively B) be the set of $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in J. For every $a \in A$, we fix $\widehat{a} \in a \cap \mathcal{L}_{h_0}$. For every $b \in B$, we fix also $\widehat{b} \in b \cap \mathcal{R}_{h_0}$. Finally, we fix $e_a, \overline{e}_a, f_b, \overline{f}_b \in M$ such that

$$e_a\widehat{a} = h_0, \quad \overline{e}_a h_0 = \widehat{a}, \quad \widehat{b} f_b = h_0, \quad h_0 \overline{f}_b = \widehat{b}.$$

Let $\mathrm{Stab}(H) = \{x \in M \mid Hx = H\}$ and define an equivalence relation on $\mathrm{Stab}(H)$ by

$$[x] = [y] \quad \text{if } h_0 x = h_0 y.$$

Then the quotient

$$G = \{[x] \mid x \in \mathrm{Stab}(H)\}$$

is the *Schützenberger group* of H. For each $h \in H$, we fix some $\widetilde{h} \in M$ such that $h_0\widetilde{h} = h$. By the well-known Green's Lemma [6, 23], $\widetilde{h} \in \mathrm{Stab}(H)$. Then there exists some $(B \times A)$-matrix C with entries in $G \cup \{0\}$ such that

$$\begin{aligned} J^0 &\to M^0(G, A, B, C) \\ u &\mapsto \begin{cases} (\mathcal{R}_u = a, [\widetilde{e_a u f_b}], \mathcal{L}_u = b) & \text{if } u \in J \\ 0 & \text{if } u = 0 \end{cases} \end{aligned}$$

is a semigroup isomorphism.

Throughout this section, we assume that M is a fixed stable semigroup. Hence M^I is a stable monoid. We fix a coordinatization (assuming equality to simplify notation)

$$\mathcal{J}_m^0 = M^0(G_m, A_m, B_m, C_m)$$

for every $m \in M^I$. We assume that 1 denotes the identity in every group and $1 \in A_m, B_m$ for every $m \in M^I$. If $m \ \mathcal{J} \ m'$, we assume of course that $(G_m, A_m, B_m, C_m) = (G_{m'}, A_{m'}, B_{m'}, C_{m'})$.

We fix mappings

$$\begin{aligned} M^I &\to M^I \\ m &\mapsto m^* \end{aligned} \qquad\qquad \begin{aligned} M^I &\to M^I \\ m &\mapsto m^\# \end{aligned}$$

defined as follows. If $m = (a, g, b) \in \mathcal{J}_m^0$, $m^*, m^\#$ satisfy

$$mm^* = (a, 1, 1), \quad (a, 1, 1)m^\# = m.$$

The existence of such elements follows from $(a, g, b) \ \mathcal{R} \ (a, 1, 1)$. Note that $I^* = I^\# = I$.

Lemma 7.2 *For all $m, m' \in M$,*

(i) $mm^*m^{\#} = m$;

(ii) $m \,\mathcal{R}\, m' \Leftrightarrow mm^* = m'm'^*$.

Proof. (i) is trivial. Assume now that $m \,\mathcal{R}\, m'$. Then we may write $m = (a, g, b)$, $m' = (a, g', b')$ as elements of the same $\mathcal{J}$-class. Thus $mm^* = (a, 1, 1) = m'm'^*$. Conversely, assume that $mm^* = m'm'^*$. By (i), we get $m = mm^*m^{\#} = m'm'^*m^{\#}$ and so $m \leq_{\mathcal{R}} m'$. By symmetry, it follows that $m \,\mathcal{R}\, m'$ and so (ii) holds. $\square$

We define $F_{\mathcal{J}}(M^I)$ to be the set of all finite chains of the form

$$n_k <_{\mathcal{J}} \ldots <_{\mathcal{J}} n_1 <_{\mathcal{J}} n_0 = I$$

with $k \geq 0$ and $n_i \in M^I$.

Lemma 7.3 *Given* $(m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_2 <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I) \in Rh(M^I)$, *let*

$$x_0 = I, \quad x_i = m_i m^*_{i-1} \; (i = 1, \ldots, k).$$

Then:

(i) $x_i \,\mathcal{R}\, m_i$ *for* $i = 0, \ldots, k$;

(ii) $x_i <_{\mathcal{L}} x_{i-1}x^*_{i-1}$ *for* $i = 1, \ldots, k$;

(iii) $x_i <_{\mathcal{J}} x_{i-1}$ *for* $i = 1, \ldots, k$;

(iv) $m_i = x_i m^{\#}_{i-1}$ *for* $i = 1, \ldots, k$.

Proof. We have $x_0 = I = m_0$. Let $i \in \{1, \ldots, k\}$. Since $m_i <_{\mathcal{L}} m_{i-1}$, we may write $m_i = ym_{i-1}$ for some $y \in M^I$. Now $m_{i-1} \;\mathcal{R}\; m_{i-1}m^*_{i-1}$ yields

$$x_i = m_i m^*_{i-1} = ym_{i-1}m^*_{i-1} \;\mathcal{R}\; ym_{i-1} = m_i.$$

Thus (i) holds.

Since $m_i <_{\mathcal{L}} m_{i-1}$, we have $m_i m^*_{i-1} \leq_{\mathcal{L}} m_{i-1}m^*_{i-1}$. Since

$$m_i m^*_{i-1} \leq_{\mathcal{J}} m_i <_{\mathcal{J}} m_{i-1} \;\mathcal{R}\; m_{i-1}m^*_{i-1}$$

by (10), we obtain $m_i m^*_{i-1} <_{\mathcal{L}} m_{i-1}m^*_{i-1}$ and so

$$x_i = m_i m^*_{i-1} <_{\mathcal{L}} m_{i-1}m^*_{i-1} = x_{i-1}x^*_{i-1}$$

by (i) and Lemma 7.2(ii). Thus (ii) holds.

Since $x_{i-1}x^*_{i-1} \;\mathcal{R}\; x_{i-1}$, (ii) implies (iii) in view of (10). Finally, $m_i = ym_{i-1}$ yields

$$m_i = ym_{i-1} = ym_{i-1}m^*_{i-1}m^{\#}_{i-1} = m_i m^*_{i-1}m^{\#}_{i-1} = x_i m^{\#}_{i-1}$$

and (iv) holds as well. $\square$

Lemmas 7.2 and 7.3 will be used so thoroughly for the remainder of the paper that we shall often omit a specific reference to them.

We can now define a mapping $\epsilon : \mathrm{Rh}(M^I) \to F_{\mathcal{J}}(M^I)$ by

$$\epsilon(m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I) = (x_k <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I)$$

taking $x_0 = I$ and $x_i = m_i m_{i-1}^*$ for $i = 1, \ldots, k$.

Note that ϵ is sequential in the sense that if

$$\epsilon(m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I) = (x_k <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I)$$

and $k > 0$, then

$$\epsilon(m_{k-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I) = (x_{k-1} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I).$$

The mapping ϵ is known as the *Zeiger encoding map* and plays an essential role in the next section to ensure the Zeiger property of the wreath product.

Proposition 7.4 *The mapping $\epsilon : Rh(M^I) \to F_{\mathcal{J}}(M^I)$ is one-to-one.*

Proof. By definition, ϵ preserves chain length. Take

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \tau = (m'_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = I)$$

such that

$$\epsilon(\sigma) = (x_k <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I) = \epsilon(\tau).$$

We show that $m_i = m'_i$ for $i = 0, \ldots, k$ by induction on i. The case $i = 0$ being trivial, assume that $i \in \{1, \ldots, k\}$ and $m_{i-1} = m'_{i-1}$. By Lemma 7.3(ii) and the induction hypothesis, we get

$$m_i = x_i m_{i-1}^{\#} = x_i (m'_{i-1})^{\#} = m'_i.$$

It follows that $\sigma = \tau$ and so ϵ is one-to-one. $\square$

The next result will reveal in the next section the adequacy of the encoding map ϵ to deal with the product in $\mathrm{Rh}(M^I)$.

Theorem 7.5 *Let $\sigma, \tau \in Rh(M^I)$ with*

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \sigma\tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = I),$$

$$\epsilon(\sigma) = (x_k <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I), \quad \epsilon(\sigma\tau) = (x'_l <_{\mathcal{J}} \ldots <_{\mathcal{J}} x'_1 <_{\mathcal{J}} x'_0 = I).$$

Assume that $m_k \,\mathcal{R}\, m'_l$. If

$$\sigma' = (m_{k+p} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \epsilon(\sigma') = (x_{k+p} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_1 <_{\mathcal{J}} x_0 = I),$$

then

$$\epsilon(\sigma'\tau) = (x_{k+p} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_{k+1} <_{\mathcal{J}} x'_l <_{\mathcal{J}} \ldots <_{\mathcal{J}} x'_1 <_{\mathcal{J}} x'_0 = I).$$

Proof. Let $y \in M^I$ denote the leftmost term in τ. Then $m'_l = m_k y$. Since $m'_l \,\mathcal{R}\, m_k$, we can apply successively Lemma 7.1 to get

$$\sigma'\tau = (m_{k+p}y <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_{k+1}y <_{\mathcal{L}} m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0)$$

and

$$m_{k+i}y \,\mathcal{R}\, m_{k+i} \quad (i = 0, \ldots, p). \tag{11}$$

Since ϵ is sequential, we obtain

$$\epsilon(\sigma'\tau) = (m_{k+p}y(m_{k+p-1}y)^* <_{\mathcal{J}} \ldots <_{\mathcal{J}} m_{k+1}y(m_k y)^* <_{\mathcal{J}} x'_l <_{\mathcal{J}} \ldots <_{\mathcal{J}} x'_0).$$

Now (11) and Proposition 7.2(ii) yield

$$m_{k+i-1}y(m_{k+i-1}y)^* = m_{k+i-1}m^*_{k+i-1} \quad (i = 1, \ldots, p).$$

Thus $m_{k+i} <_{\mathcal{L}} m_{k+i-1}$ yields

$$m_{k+i}y(m_{k+i-1}y)^* = m_{k+i}m^*_{k+i-1} = x_{k+i}$$

for $i = 1, \ldots, p$ and the lemma is proved. $\square$

We complete the section with a straightforward consequence of Green's Lemma [6, 23], to be used in the next section.

Proposition 7.6 *Let J be a $\mathcal{J}$-class of a stable monoid M with $J^0 = M^0(G, A, B, C)$. Let $u = (a, g, b) \in J$ and $v \in M$ be such that $uv \,\mathcal{R}\, u$ and let*

$$\begin{aligned} \varphi : G &\to G \\ h &\mapsto ((a, h, b)v)\pi_2. \end{aligned}$$

Then φ is bijective and there exists some $g_0 \in G$ such that $h\varphi = hg_0$ for every $h \in G$.

Proof. By Green's Lemma, we have a bijection

$$\begin{aligned} \mathcal{H}_u &\to \mathcal{H}_{uv} \\ u' &\mapsto u'v, \end{aligned}$$

hence φ is well defined. Let H be the fixed $\mathcal{H}$-class in the construction $M^0(G, A, B, C)$ and consider all the distinguished elements introduced there. Our Rees matrix representation restricts to bijections

$$\begin{aligned} \mathcal{H}_u &\to G \\ u' &\mapsto [\widetilde{e_a u' f_b}] \end{aligned} \qquad \begin{aligned} \mathcal{H}_{uv} &\to G \\ u'v &\mapsto [\widetilde{e_a u' v f_{b'}}] \end{aligned}$$

where $a = \mathcal{R}_u = \mathcal{R}_{uv}$, $b = \mathcal{L}_u$ and $b' = \mathcal{L}_{uv}$. Hence we obtain a diagram

$$\begin{tikzcd}
\mathcal{H}_u \arrow[rrr] \arrow[ddd] & & & \mathcal{H}_{uv} \arrow[ddd] \\
& u' \arrow[r] \arrow[dd] & u'v \arrow[dd] & \\
& & & \\
G & [\widetilde{e_a u' f_b}] & [\widetilde{e_a u' v f_{b'}}] & G
\end{tikzcd}$$

We must show that the mapping

$$\varphi : [\widetilde{e_a u' f_b}] \mapsto [\widetilde{e_a u' v f_{b'}}],$$

a composition of bijections, can be defined by right multiplication. We show that $h_0 \overline{f}_b v f_{b'} \in H$ and

$$[\widetilde{e_a u' v f_{b'}}] = [\widetilde{e_a u' f_b}][\widetilde{h_0 \overline{f}_b v f_{b'}}] \tag{12}$$

for every $u' \in \mathcal{H}_u$. Indeed, take $u_0 \in \mathcal{H}_u$ such that $e_a u_0 f_b = h_0$. Since $e_a u_0 = e_a u_0 f_b \overline{f}_b$, we have

$$h_0 \overline{f}_b v f_{b'} = e_a u_0 f_b \overline{f}_b v f_{b'} = e_a u_0 v f_{b'} \in H.$$

Moreover, we have

$$\forall h \in H \ \forall w \in M \ (hw \in H \Rightarrow [\widetilde{hw}] = [\widetilde{h}][\widetilde{h_0 w}]). \tag{13}$$

Indeed, we have $h_0 w \in H$ by Green's Lemma. Since $h_0 \widetilde{h_0 w} = h_0 w$, we obtain $h \widetilde{h_0 w} = hw$ and so (13) follows from $h_0 \widetilde{h} \widetilde{h_0 w} = h \widetilde{h_0 w} = hw = h_0 \widetilde{hw}$.

Finally, making $h = e_a u' f_b$ and $w = \overline{f}_b v f_{b'}$ in (13), we get

$$[\widetilde{e_a u' v f_{b'}}] = [e_a u' \widetilde{f_b \cdot \overline{f}_b} v f_{b'}] = [\widetilde{e_a u' f_b}][\widetilde{h_0 \overline{f}_b v f_{b'}}]$$

and so (13) holds as claimed. $\square$

8 From elliptic M-trees to wreath products

Let (r_0, T) be a rooted tree and let X be a nonempty set. Given $v \in \mathrm{Vert}(T)$, let $\mathrm{Sons}(v)$ denote the (possibly empty) set of sons of v. A mapping $f : \mathrm{Vert}(T) \backslash \{r_0\} \to X$ is said to be *locally injective* if, for every $v \in \mathrm{Vert}(T)$, $f|_{\mathrm{Sons}(v)}$ is injective.

For $i = 1, 2, \ldots$, let $\mathrm{Vert}_i(T)$ denote the set of all $v \in \mathrm{Vert}(T)$ having depth i.

Theorem 8.1 *Let M be a semigroup and assume that $\theta : M^I \to Ell(r_0, T)$ is a faithful elliptic action of M^I on a uniform rooted tree (r_0, T). Let $f : (Vert(T)) \setminus \{r_0\} \to \cup_{i \geq 1} X_i$ be locally injective with $f(Vert_i(T)) \subseteq X_i$ for $i \geq 1$. Then:*

(i) if (r_0, T) has finite depth l, then M^I embeds in the wreath product

$$(X_l, P(X_l)) \circ \ldots \circ (X_2, P(X_2)) \circ (X_1, P(X_1));$$

(ii) if (r_0, T) has infinite depth, then M^I embeds in the infinite wreath product

$$\ldots \circ (X_3, P(X_3)) \circ (X_2, P(X_2)) \circ (X_1, P(X_1)).$$

Proof. We prove the finite depth case, the infinite case being analogous.

Write $X = \cup_{i=1}^{l} (X_i \times \ldots \times X_1)$. Associating vertices with rays as usual, we define a mapping

$$\begin{aligned} \psi : \mathrm{Ray}(r_0, T) \setminus \{(r_0)\} &\to X \\ (v_i, \ldots, v_1, r_0) &\mapsto (f(v_i), \ldots, f(v_1)). \end{aligned}$$

Suppose that $(v_i, \ldots, v_1, v_0 = r_0), (v'_i, \ldots, v'_1, v_0 = r_0) \in \mathrm{Ray}(r_0, T)$ are distinct. Let

$$k = \min\{j \in \{1, \ldots, i\} : v_j \neq v'_j\}.$$

By minimality of k, v_k and v'_k must have $v_{k-1} = v'_{k-1}$ as their common father. Since f is locally injective, it follows that $f(v_k) \neq f(v'_k)$, thus $(v_i, \ldots, v_1, r_0)\psi \neq (v'_i, \ldots, v'_1, r_0)\psi$ and so ψ is one-to-one.

Let

$$\begin{aligned} \Psi : M^I &\to (X_l, P(X_l)) \circ \ldots \circ (X_1, P(X_1)) \\ m &\mapsto \Psi_m \end{aligned}$$

be defined by

$$x\Psi_m = \begin{cases} x\psi^{-1}\theta_m\psi & \text{if } m \in M \\ x & \text{if } m = I \end{cases} \qquad (x \in X).$$

Clearly, $\Psi_m \in P(X)$. To show that $\Psi_m \in (X_l, P(X_l)) \circ \ldots \circ (X_1, P(X_1))$, we only need to check that Ψ_m is sequential. We may assume that $m \in M$. Since $\mathrm{dom}\Psi_m = \mathrm{im}\psi$, (SQ1) holds. Since θ is an elliptic action, (SQ2) holds as well.

Let $(x_j, \ldots, x_1), (x'_k, \ldots, x'_1) \in \mathrm{dom}\Psi_m = \mathrm{im}\psi$ and suppose that $(x_j, \ldots, x_1) \equiv_i (x'_k, \ldots, x'_1)$ with $1 \leq i \leq j, k$. Write

$$\begin{aligned} (x_j, \ldots, x_1) &= (v_j, \ldots, v_1, r_0)\psi = (f(v_j), \ldots, f(v_1)) \\ (x'_k, \ldots, x'_1) &= (v'_k, \ldots, v'_1, r_0)\psi = (f(v'_k), \ldots, f(v'_1)). \end{aligned}$$

Since $(x_j, \ldots, x_1) \equiv_i (x'_k, \ldots, x'_1)$, we have $f(v_i) = f(v'_i), \ldots, f(v_1) = f(v'_1)$. Since f is locally injective, we obtain successively $v_1 = v'_1, \ldots, v_i = v'_i$. Thus $(v_j, \ldots, v_1, r_0)\theta_m \equiv_{i+1} (v'_k, \ldots, v'_1, r_0)\theta_m$ and so $(v_j, \ldots, v_1, r_0)\theta_m\psi \equiv_i (v'_k, \ldots, v'_1, r_0)\theta_m\psi$, that is,

$$(x_j, \ldots, x_1)\Psi_m = (x_j, \ldots, x_1)\psi^{-1}\theta_m\psi \equiv_i (x'_k, \ldots, x'_1)\psi^{-1}\theta_m\psi = (x'_k, \ldots, x'_1)\Psi_m.$$

Thus (SQ3) holds. Therefore Ψ_m is sequential and so $\Psi_m \in (X_l, P(X_l)) \circ \ldots \circ (X_1, P(X_1))$.

We show next that Ψ is a monoid homomorphism. It suffices to show that $\Psi_{mm'} = \Psi_m\Psi_{m'}$ for all $m, m' \in M$. Since θ is an action and ψ is injective, we obtain

$$\Psi_{mm'} = \psi^{-1}\theta_{mm'}\psi = \psi^{-1}\theta_m\theta_{m'}\psi = \psi^{-1}\theta_m\psi\psi^{-1}\theta_{m'}\psi = \Psi_m\Psi_{m'}.$$

Thus Ψ is a monoid homomorphism.

It remains to show that Ψ is one-to-one. Let $m, m' \in M^I$. We show that

$$m \neq m' \Rightarrow \psi^{-1}\theta_m\psi \neq \psi^{-1}\theta_{m'}\psi. \tag{14}$$

Indeed, since θ is one-to-one, $m \neq m'$ implies that there exists some $v \in \mathrm{Vert}(T)$ such that $v\theta_m \neq v\theta_{m'}$. Taking the geodesic $(v = v_i, \ldots, v_1, r_0) \in \mathrm{Ray}(r_0, T)$, it follows that $(v_i, \ldots, v_1, r_0)\theta_m \neq (v_i, \ldots, v_1, r_0)\theta_{m'}$. Let

$$(x_i, \ldots, x_1) = (v_i, \ldots, v_1, r_0)\psi.$$

Then

$$(x_i, \ldots, x_1)\psi^{-1}\theta_m = (v_i, \ldots, v_1, r_0)\theta_m \neq (v_i, \ldots, v_1, r_0)\theta_{m'} = (x_i, \ldots, x_1)\psi^{-1}\theta_{m'}.$$

Since ψ is one-to-one, we obtain $(x_i,\dots,x_1)\psi^{-1}\theta_m\psi \neq (x_i,\dots,x_1)\psi^{-1}\theta_{m'}\psi$ and so (14) holds.

Assume that $m \neq m'$. Now (14) implies that $\Psi_m \neq \Psi_{m'}$ if $m, m' \in M$. For the remaining cases, we may assume that $m = I$. If ψ is onto, then $\Psi_I = \psi^{-1}\theta_I\psi$ and so (14) also implies that $\Psi_I \neq \Psi_{m'}$. Otherwise, we have $\Psi_I \neq \Psi_{m'}$ since $\mathrm{dom}\Psi_I = X \supset \mathrm{im}\psi = \mathrm{dom}\Psi_{m'}$. Therefore Ψ is one-to-one. $\square$

By Theorem 8.1, we know that when a monoid M^I acts faithfully by elliptic contractions on a uniform rooted tree, then M^I embeds into a (possibly infinite) wreath product $\ldots \circ (X_2, M_2) \circ (X_1, M_1)$ of partial transformation monoids. The question is *how small can the M_i's be made* (where *small* is used in the sense of division). We start with a series of lemmas.

Lemma 8.2 *Let $\sigma, \tau, \rho \in Rh(M^I)$.*

(i) If $h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) = h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \rho)$, then $(\sigma \wedge_{\mathcal{L}} \tau) = (\sigma \wedge_{\mathcal{L}} \rho)$.

(ii) $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \leq_{\mathcal{L}} (\sigma \wedge_{\mathcal{L}} \tau)$.

Proof. (i) Assume that $h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) = h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \rho)$. If $(\sigma \wedge_{\mathcal{L}} \tau) \neq (\sigma \wedge_{\mathcal{L}} \rho)$, we may assume that $(\sigma \wedge_{\mathcal{L}} \tau) <_{\mathcal{L}} (\sigma \wedge_{\mathcal{L}} \rho)$ and so $(\sigma \wedge_{\mathcal{L}} \tau) <_{\mathcal{J}} (\sigma \wedge_{\mathcal{L}} \rho)$ by (10), contradicting $h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) = h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \rho)$. Therefore $(\sigma \wedge_{\mathcal{L}} \tau) = (\sigma \wedge_{\mathcal{L}} \rho)$.

(ii) Write

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I), \quad \tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I).$$

In view of (5), we may assume that $\rho = (n <_{\mathcal{L}} I)$. Hence

$$\rho\sigma = \mathrm{lm}(nm_k \leq_{\mathcal{L}} m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0),$$

$$\rho\tau = \mathrm{lm}(nm'_l \leq_{\mathcal{L}} m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0).$$

The claim follows at once. $\square$

Next we define

$$V(M^I) = \{\, (\sigma,\tau) \in \mathrm{Rh}(M^I) \times \mathrm{Rh}(M^I) \mid \forall \rho \in \mathrm{Rh}(M^I) \\ (\rho\sigma \wedge_{\mathcal{L}} \rho\tau)\, \mathcal{L}\, (\sigma \wedge_{\mathcal{L}} \tau) \;\Rightarrow\; (\rho\sigma \wedge_{\mathcal{L}} \rho\tau)\, \mathcal{R}\, (\rho\tau \wedge_{\mathcal{L}} \rho\sigma)\}.$$

Note that $(\sigma,\tau) \in V(M^I)$ implies in particular that $(\sigma \wedge_{\mathcal{L}} \tau)\, \mathcal{R}\, (\tau \wedge_{\mathcal{L}} \sigma)$.

Next we define

$$W(M^I) = \{m \in M^I \mid \mathcal{L}_m = \mathcal{H}_m\}.$$

Lemma 8.3 *$W(M^I)$ is a union of $\mathcal{J}$-classes of M^I.*

Proof. Let $m \in W(M^I)$. Since our monoid M is finite $\mathcal{J}$-above, we have $\mathcal{J} = \mathcal{D}$ and so it suffices to show that $\mathcal{L}_m \cup \mathcal{R}_m \subseteq W(M^I)$.

Assume that $m'\, \mathcal{L}\, m$. Then $m' \in \mathcal{L}_m = \mathcal{H}_m$ and so $\mathcal{L}_{m'} = \mathcal{L}_m = \mathcal{H}_m = \mathcal{H}_{m'}$. Thus $m' \in W(M^I)$.

Finally, assume that $m'\, \mathcal{R}\, m$ and take $u \in \mathcal{L}_{m'}$. Write $m' = mx$ and $m = m'y$. Then $m'\, \mathcal{L}\, u$ yields $m'y\ \mathcal{L}\ uy$ and so $uy \in \mathcal{L}_m = \mathcal{H}_m$. By Green's Lemma, we get $uyx\ \mathcal{H}\ mx = m'$. Since $u\, \mathcal{L}\, m' = m'yx$ yields $uyx = u$, we obtain $u\, \mathcal{H}\, m'$ and so $m' \in W(M^I)$. $\square$

The next lemma provides an alternative characterization of $V(M^I)$:

Lemma 8.4 *Let $\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$, $\tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0)$ and $(\sigma \wedge_{\mathcal{L}} \tau) = m_i$. Then $(\sigma, \tau) \in V(M^I)$ if and only if $m_i \,\mathcal{R}\, m'_i$ and one of the following conditions holds:*

(V1) $i = k = l$;

(V2) $i < k, l$;

(V3) $i = k < l$ and $m_i \in W(M^I)$;

(V4) $i = l < k$ and $m'_i \in W(M^I)$;

Proof. Since $(\sigma \wedge_{\mathcal{L}} \tau) = m_i$, we have $(\tau \wedge_{\mathcal{L}} \sigma) = m'_i$. Taking $\rho = (I)$ in the condition defining $V(M^I)$, it becomes clear that $m_i \,\mathcal{R}\, m'_i$ is a necessary condition for $(\sigma, \tau) \in V(M^I)$.

Assume that $m_i \,\mathcal{R}\, m'_i$ and one of conditions (V1)–(V4) holds. Write $\rho = (n_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)$ and assume that $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\mathcal{L}\; (\sigma \wedge_{\mathcal{L}} \tau)$. We have

$$\rho\sigma = \mathrm{lm}(n_p m_k \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} n_1 m_k \leq_{\mathcal{L}} m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0),$$

$$\rho\tau = \mathrm{lm}(n_p m'_l \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} n_1 m'_l \leq_{\mathcal{L}} m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0).$$

It should be clear that if (V2) holds, then $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) = m_i$ and $(\rho\tau \wedge_{\mathcal{L}} \rho\sigma) = m'_i$, hence $(\sigma, \tau) \in V(M^I)$.

Since $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\mathcal{L}\; (\sigma \wedge_{\mathcal{L}} \tau)$ if and only if $(\rho\tau \wedge_{\mathcal{L}} \rho\sigma) \;\mathcal{L}\; (\tau \wedge_{\mathcal{L}} \sigma)$, it follows that $V(M^I)$ is a symmetric relation. Thus we may assume that $i = k$. Let

$$j = \max\{r \in \{0, \ldots, p\} \mid n_r m_k \;\mathcal{L}\; m_k\}.$$

Since $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\mathcal{L}\; (\sigma \wedge_{\mathcal{L}} \tau) = m_k$, we have $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) = n_j m_k$.

Assume first that $k < l$ (case (V3)). Then $m_k \in W(M^I)$ and so $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\mathcal{L}\; (\sigma \wedge_{\mathcal{L}} \tau) = m_k$ yields

$$(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\mathcal{H}\; m_k \,\mathcal{R}\, m'_k = (\rho\tau \wedge_{\mathcal{L}} \rho\sigma).$$

Hence $(\sigma, \tau) \in V(M^I)$.

It remains to be considered the case $k = l$ (case (V1)). Then $m'_k \,\mathcal{H}\, m_k$ and so $n_j m_k \;\mathcal{L}\; m_k$ yields $n_j m'_k \;\mathcal{L}\; m'_k$. By symmetry, we obtain $(\rho\tau \wedge_{\mathcal{L}} \rho\sigma) = n_j m'_k$. Since $m'_k \;\mathcal{H}\; m_k$ implies $n_j m_k \;\mathcal{R}\; n_j m'_k$, it follows that $(\sigma, \tau) \in V(M^I)$ also in this case.

To prove the converse implication, we assume that the necessary condition $m_i \;\mathcal{R}\; m'_i$ holds but none of the conditions (V1)–(V4) is satisfied. By symmetry, we may assume that $i = k < l$ and $m_k \notin W(M^I)$. Then there exists some $n \in \mathcal{L}_{m_k} \setminus \mathcal{H}_{m_k}$, say $n = x m_k$. Let $\rho = (x <_{\mathcal{L}} I)$. It follows easily that $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) = x m_k = n$ and $(\rho\tau \wedge_{\mathcal{L}} \rho\sigma) = m'_k$. Since $m'_k \,\mathcal{R}\, m_k$ and $n \not\mathcal{R}\, m_k$, we get $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\not\mathcal{R}\; (\rho\tau \wedge_{\mathcal{L}} \rho\sigma)$ and so $(\sigma, \tau) \notin V(M^I)$ as required. $\square$

Corollary 8.5 *Let $\sigma, \tau \in Rh(M^I)$ be such that $(\sigma \wedge_{\mathcal{L}} \tau) \in W(M^I)$. Then $(\sigma, \tau) \in V(M^I)$.*

Proof. Let $\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$, $\tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0)$ and $(\sigma \wedge_{\mathcal{L}} \tau) = m_i$. Then $(\tau \wedge_{\mathcal{L}} \sigma) = m'_i$. Since $m'_i \,\mathcal{L}\, m_i \in W(M^I)$, it follows that $m'_i \,\mathcal{H}\, m_i$. Hence also $m'_i \in W(M^I)$ by Lemma 8.3. Now we obtain $(\sigma, \tau) \in V(M^I)$ by Lemma 8.4. $\square$

We define a mapping $H : \mathrm{Rh}(M^I) \times \mathrm{Rh}(M^I) \to \overline{\mathbb{N}}$ by

$$H(\sigma,\tau) = \begin{cases} 2\sup h_{\mathcal{J}} + 2 & \text{if } \sigma = \tau \\ 2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) + 1 & \text{if } \sigma \neq \tau \text{ and } (\sigma,\tau) \in V(M^I) \\ 2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) & \text{otherwise.} \end{cases}$$

If M is an Y-semigroup, then M^I is an Y-monoid. We denote by H_Y the restriction of H to $\mathrm{Rh}_Y(M^I) \times \mathrm{Rh}_Y(M^I)$.

Lemma 8.6 *Let M be a finite $\mathcal{J}$-above Y-semigroup. Then*

(i) H is a strict length function for $Rh(M^I)$;

(ii) H_Y is a strict length function for $Rh_Y(M^I)$;

(iii) $H_Y = D_\chi$ for some (unique up to isomorphism) strongly faithful elliptic $Rh_Y(M^I)$-tree χ.

Proof. (i) We show that H satisfies axioms (L1) – (L5). By Corollary 6.4 and Proposition 6.7, we may consider the length function $D : \mathrm{Rh}(M^I) \times \mathrm{Rh}(M^I) \to \overline{\mathbb{N}}$ defined by

$$D(\sigma,\tau) = \begin{cases} \sup h_{\mathcal{J}} + 1 & \text{if } \sigma = \tau \\ h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) & \text{otherwise.} \end{cases}$$

Clearly, $H'(\sigma,\tau) = 2D(\sigma,\tau)$ defines also a length function for $\mathrm{Rh}(M^I)$. We shall make use of H' and perform the necessary adaptations.

Axiom (L1) follows from $V(M^I)$ being a symmetric relation (see the proof of Lemma 8.4). Axioms (L2) and (L5) can be verified for H straightforwardly as in the proof of Theorem 6.1. We concentrate our efforts on (L3) and (L4).

(L3) Let $\sigma,\tau,\rho \in \mathrm{Rh}(M^I)$ and assume that $\sigma \neq \tau$. By (5), we may assume that $\rho = (m <_{\mathcal{L}} I)$. By (7), we have $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) \leq_{\mathcal{J}} (\sigma \wedge_{\mathcal{L}} \tau)$. If $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) <_{\mathcal{J}} (\sigma \wedge_{\mathcal{L}} \tau)$, then

$$H(\sigma\rho,\tau\rho) \geq 2h_{\mathcal{J}}(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) > 2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) + 1 \geq H(\sigma,\tau).$$

Thus we may assume that

$$(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) \;\mathcal{J}\; (\sigma \wedge_{\mathcal{L}} \tau). \tag{15}$$

It suffices to show that

$$(\sigma,\tau) \in V(M^I) \Rightarrow (\sigma\rho,\tau\rho) \in V(M^I). \tag{16}$$

Assume that $(\sigma,\tau) \in V(M^I)$ and write

$$\sigma = (m_k <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_1 <_{\mathcal{L}} m_0 = I), \quad \tau = (m'_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_1 <_{\mathcal{L}} m'_0 = I).$$

Then

$$\sigma\rho = \mathrm{lm}(m_k m \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m_1 m <_{\mathcal{L}} m <_{\mathcal{L}} I),$$

$$\tau\rho = \mathrm{lm}(m'_l m \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m'_1 m <_{\mathcal{L}} m <_{\mathcal{L}} I).$$

We use Lemma 8.4. In particular, we know that $(\sigma \wedge_{\mathcal{L}} \tau) \;\mathcal{H}\; (\tau \wedge_{\mathcal{L}} \sigma)$.

Suppose first that (σ, τ) satisfies (V1). Then $(\sigma \wedge_{\mathcal{L}} \tau) = m_k \; \mathcal{H} \; m'_k = (\tau \wedge_{\mathcal{L}} \sigma)$ and $m_k m \; \mathcal{L} \; m'_k m$ yields $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) = m_k m$ and $(\tau\rho \wedge_{\mathcal{L}} \sigma\rho) = m'_k m$. Now (15) yields $m_k m \; \mathcal{J} \; m_k$ and therefore $m_k m \; \mathcal{R} \; m_k$ by (S1). Similarly, $m'_k m \; \mathcal{R} \; m'_k$. It follows that $m_k m \; \mathcal{R} \; m'_k m$ and $(\sigma\rho, \tau\rho)$ satisfies (V1), thus (16) holds in this case.

Suppose next that (σ, τ) satisfies (V2). Then $(\sigma \wedge_{\mathcal{L}} \tau) = m_i$ implies $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) = m_i m$. Indeed, It is clear that $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) = m_j m$ for some $j \geq i$ since $m_i m \; \mathcal{L} \; m'_i m$ and $m_r m = m'_r m$ for $r < i$. However, if $j > i$, then (10) yields $m_j m \leq_{\mathcal{J}} m_j <_{\mathcal{J}} m_i$, contradicting (15). Hence $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) = m_i m$. Similarly, $(\tau\rho \wedge_{\mathcal{L}} \sigma\rho) = m'_i m$. Similarly to the preceding case, we get $m_i m \; \mathcal{R} \; m_i \; \mathcal{R} \; m'_i \; \mathcal{R} \; m'_i m$. Moreover, $(\sigma\rho, \tau\rho)$ satisfies (V2), thus (16) holds in this case as well.

Finally, we assume that (σ, τ) satisfies (V3) (the case (V4) is dual). Similarly to the preceding cases, we get $(\sigma\rho \wedge_{\mathcal{L}} \tau\rho) = m_k m$, $(\tau\rho \wedge_{\mathcal{L}} \sigma\rho) = m'_k m$ and $m_k m \; \mathcal{R} \; m'_k m$. Now (15) is equivalent to $m_k m \; \mathcal{J} \; m_k$ and so $m_k \in W(M^I)$ yields $m_k m \in W(M^I)$ by Lemma 8.3, hence $(\sigma\rho, \tau\rho)$ satisfies (V3). Thus (16) holds and (L3) is satisfied.

(L4): Let $\sigma, \tau, \rho \in \mathrm{Rh}(M^I)$. We may assume that σ, τ, ρ are all distinct. Since H' is a length function, we have

$$H'(\sigma, \rho) \geq \min\{H'(\sigma, \tau), H'(\tau, \rho)\}.$$

Since $H(x, y) = H'(x, y)$ or $H(x, y) = H'(x, y) + 1$ for all $x, y \in \mathrm{Rh}(M^I)$, we may assume that

$$H'(\sigma, \rho) = \min\{H'(\sigma, \tau), H'(\tau, \rho)\}. \tag{17}$$

By (L1), we may further assume that

$$H'(\sigma, \rho) = H'(\sigma, \tau). \tag{18}$$

If $H(\sigma, \tau) = H'(\sigma, \tau)$ we are done, hence assume also that $H(\sigma, \tau) = H'(\sigma, \tau) + 1$, that is,

$$(\sigma, \tau) \in V(M^I).$$

Similarly, we may assume that

$$H'(\sigma, \rho) = H'(\tau, \rho) \Rightarrow (\tau, \rho) \in V(M^I), \tag{19}$$

otherwise $H(\sigma, \rho) \geq H(\tau, \rho)$. In view of (17), to prove (L4) it suffices to show that $(\sigma, \rho) \in V(M^I)$.

Assume first that $H'(\sigma, \rho) = H'(\tau, \rho)$. By (19), we have $(\tau, \rho) \in V(M^I)$. Moreover, $H'(\sigma, \rho) = H'(\sigma, \tau) = H'(\tau, \rho)$ and Lemma 8.2(i) yield

$$(\sigma \wedge_{\mathcal{L}} \rho) = (\sigma \wedge_{\mathcal{L}} \tau) \; \mathcal{R} \; (\tau \wedge_{\mathcal{L}} \sigma) = (\tau \wedge_{\mathcal{L}} \rho) \; \mathcal{R} \; (\rho \wedge_{\mathcal{L}} \tau) = (\rho \wedge_{\mathcal{L}} \sigma).$$

We discuss now the cases (V1)–(V4).

If (σ, τ) satisfies (V1), then (τ, ρ) must satisfy either (V1) or (V3), and so (σ, ρ) satisfies (V1) or (V3) accordingly in view of Lemma 8.3.

If (σ, τ) satisfies (V2), then (τ, ρ) must satisfy either (V2) or (V4), and so (σ, ρ) satisfies (V2) or (V4) accordingly.

If (σ, τ) satisfies (V3), then (τ, ρ) must satisfy either (V2) or (V4). In the first case, (σ, ρ) satisfies (V3). In the latter, (σ, ρ) satisfies (V1) or (V3).

Finally, if (σ, τ) satisfies (V4), then (τ, ρ) must satisfy either (V1) or (V3). In the first case, (σ, ρ) satisfies (V4) by Lemma 8.3. In the latter, (σ, ρ) satisfies (V2). This completes the discussion of the case $H'(\sigma, \rho) = H'(\tau, \rho)$.

It remains to be considered the case $H'(\sigma, \rho) < H'(\tau, \rho)$. By (18) and Lemma 8.2(i), we have

$$(\sigma \wedge_{\mathcal{L}} \rho) = (\sigma \wedge_{\mathcal{L}} \tau) \tag{20}$$

and so

$$(\tau \wedge_{\mathcal{L}} \sigma) \; \mathcal{L} \; (\rho \wedge_{\mathcal{L}} \sigma). \tag{21}$$

Since $H'(\tau, \sigma) = H'(\sigma, \tau) = H'(\sigma, \rho) < H'(\tau, \rho)$, $\tau \wedge_{\mathcal{L}} \sigma$ must be a term of ρ with $\tau \wedge_{\mathcal{L}} \sigma >_{\mathcal{L}} \rho \wedge_{\mathcal{L}} \tau$. Hence (21) yields

$$(\tau \wedge_{\mathcal{L}} \sigma) = (\rho \wedge_{\mathcal{L}} \sigma). \tag{22}$$

Since $(\sigma, \tau) \in V(M^I)$, it follows from (20) and (22) that

$$(\sigma \wedge_{\mathcal{L}} \rho) \;\; \mathcal{R} \;\; (\rho \wedge_{\mathcal{L}} \sigma).$$

We discuss now the cases (V1)–(V4).

Clearly, $H'(\sigma, \rho) < H'(\tau, \rho)$ implies that (σ, τ) must satisfy either (V2) or (V3). It is easy to see that (σ, ρ) satisfies necessarily the same condition, hence $(\sigma, \rho) \in V(M^I)$ and so (L4) holds.

Therefore H is a strict length function for $\mathrm{Rh}(M^I)$.

(ii) Since $\mathrm{Rh}_Y(M^I)$ is a submonoid of $\mathrm{Rh}(M^I)$, the restriction of H to $\mathrm{Rh}_Y(M^I) \times \mathrm{Rh}_Y(M^I)$ is a length function for $\mathrm{Rh}_Y(M^I)$.

(iii) We get $H_Y = D_\chi$ for some (unique up to isomorphism) strongly faithful elliptic $\mathrm{Rh}_Y(M^I)$-tree χ by Corollary 4.8. $\square$

Throughout the remaining part of this section, we assume that M, $H_Y = D_\chi$ for $\chi = (r_0, T, \alpha, \theta)$ are fixed. Moreover, we may assume that χ is obtained by the Chiswell construction according to the proofs of Theorem 4.7 and Corollary 4.8.

We say that $[n, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I] = v$ is a *minimal representation* of $v \in \mathrm{Vert}(T)$ if $v \neq [n, m_i <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]$ for every $i < l$.

The following lemma helps to establish that an $\mathcal{L}$-chain belongs to $\mathrm{Rh}_Y(M^I)$:

Lemma 8.7 *Let* $(m_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \in Rh_Y(M^I)$.

(i) For every $i < l$, $(m_i <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \in Rh_Y(M^I)$.

(ii) If $m'_l \; \mathcal{L} \; m_l$, *then* $(m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \in Rh_Y(M^I)$.

Proof. (i) If $(m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) = (y_r <_{\mathcal{L}} I) \ldots (y_1 <_{\mathcal{L}} I)$, then

$$(m_i <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) = (y_s <_{\mathcal{L}} I) \ldots (y_1 <_{\mathcal{L}} I) \tag{23}$$

for $s = \max\{j < r \mid y_j \ldots y_1 \; \mathcal{L} \; m_i\}$.

(ii) Write

$$\sigma = (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0), \quad \tau = (m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

Since $m'_l \mathcal{L} m_l$, we have $m'_l = y_r \dots y_1 m_l$ for some $y_1, \dots, y_r \in Y$. Since $\sigma \in \mathrm{Rh}_Y(M^I)$, it follows that

$$\begin{aligned}&\mathrm{lm}(y_r \dots y_1 m_l \leq_{\mathcal{L}} y_{r-1} \dots y_1 m_l \leq_{\mathcal{L}} \dots \leq_{\mathcal{L}} y_1 m_l \leq_{\mathcal{L}} m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0)\\ &\qquad = (y_r <_{\mathcal{L}} I) \dots (y_1 <_{\mathcal{L}} I)\sigma \in \mathrm{Rh}_Y(M^I).\end{aligned}$$

Since $m'_l = y_r \dots y_1 m_l \mathcal{L} m_l$, we obtain

$$\mathrm{lm}(y_r \dots y_1 m_l \leq_{\mathcal{L}} y_{r-1} \dots y_1 m_l \leq_{\mathcal{L}} \dots \leq_{\mathcal{L}} y_1 m_l \leq_{\mathcal{L}} m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0) = \tau$$

and so $\tau \in \mathrm{Rh}_Y(M^I)$. □

Lemma 8.8 *(i) Let $v = [2k, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I] \in$ Vert(T) and $i \in \{0, \dots, l-1\}$. Then $v = [2k, m_i <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$ if and only if $h_{\mathcal{J}}(m_i) \geq k$.*

(ii) Let $v = [2k+1, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I] \in$ Vert(T) and $i \in \{0, \dots, l-1\}$. Then $v = [2k+1, m_i <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$ if and only if $h_{\mathcal{J}}(m_i) > k$ or

$$h_{\mathcal{J}}(m_i) = k \text{ and } m_i \in W(M^I).$$

Proof. Let

$$\sigma = (m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0), \quad \tau = (m_i <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0).$$

Then $(\sigma \wedge_{\mathcal{L}} \tau) = m_i = (\tau \wedge_{\mathcal{L}} \sigma)$.

(i) We have

$$[2k, \sigma] = [2k, \tau] \Leftrightarrow H_Y(\sigma, \tau) \geq 2k \Leftrightarrow 2h_{\mathcal{J}}(m_i) \geq 2k \Leftrightarrow h_{\mathcal{J}}(m_i) \geq k.$$

(ii) Assume first that $m_i \in W(M^I)$. Then

$$H_Y(\sigma, \tau) = 2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) + 1 = 2h_{\mathcal{J}}(m_i) + 1$$

and so

$$[2k+1, \sigma] = [2k+1, \tau] \Leftrightarrow H_Y(\sigma, \tau) \geq 2k+1 \Leftrightarrow 2h_{\mathcal{J}}(m_i) + 1 \geq 2k+1 \Leftrightarrow h_{\mathcal{J}}(m_i) \geq k.$$

If $m_i \notin W(M^I)$, then $H_Y(\sigma, \tau) = 2h_{\mathcal{J}}(m_i)$ and so

$$[2k+1, \sigma] = [2k+1, \tau] \Leftrightarrow H_Y(\sigma, \tau) \geq 2k+1 \Leftrightarrow 2h_{\mathcal{J}}(m_i) \geq 2k+1 \Leftrightarrow h_{\mathcal{J}}(m_i) > k.$$

□

We immediately obtain

Corollary 8.9 *(i) $v = [2k, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I] \in$ Vert(T) is in minimal representation if and only if $h_{\mathcal{J}}(m_{l-1}) < k$.*

(ii) $v = [2k+1, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I] \in$ Vert(T) is in minimal representation if and only if $h_{\mathcal{J}}(m_{l-1}) < k$ or

$$h_{\mathcal{J}}(m_{l-1}) = k \text{ and } m_{l-1} \notin W(M^I).$$

Given $m \in M^I$, let

$$Y_m = \{y \in Y \mid ym <_{\mathcal{L}} m\}.$$

For every $y \in Y_m$, there exists a unique $b \in B_{ym}$ such that $(1,1,b) \; \mathcal{L} \; ym$. We denote by Q_m the set of all such b when y takes values in Y_m.

Lemma 8.10 *For every $m \in M^I$, $Y_m = Y_{mm^*}$ and*

$$\begin{aligned} \varphi : Q_m &\to Q_{mm^*} \\ b &\mapsto ((1,1,b)m^*)\pi_3 \end{aligned}$$

is a bijection.

Proof. Let $m \in M^I$. We always have $ym \leq_{\mathcal{L}} m$ and $ymm^* \leq_{\mathcal{L}} mm^*$. Since $m = mm^*m^\sharp$, it is immediate that $ym \; \mathcal{L} \; m$ if and only if $ymm^* \; \mathcal{L} \; mm^*$, hence $Y_m = Y_{mm^*}$.

Let $b \in Q_m$. Then $(1,1,b) \; \mathcal{L} \; ym$ for some $y \in Y_m$ and so $(1,1,b)m^* \; \mathcal{L} \; ymm^*$. Since $y \in Y_m = Y_{mm^*}$, it follows that $\varphi(b) \in Q_{mm^*}$. Thus φ is well defined.

Suppose now that $\varphi(b) = \varphi(c)$. Then $(1,1,b)m^* \; \mathcal{L} \; (1,1,c)m^*$ and so $(1,1,b)m^*m^\sharp \; \mathcal{L} \; (1,1,c)m^*m^\sharp$. Since $(1,1,b) <_{\mathcal{L}} m$, we get $(1,1,b)m^*m^\sharp = (1,1,b)$. Similarly, $(1,1,c)m^*m^\sharp = (1,1,c)$ and so $(1,1,b) \; \mathcal{L} \; (1,1,c)$. Thus $b = c$ and φ is one-to-one.

Finally, let $c \in Q_{mm^*}$. Then $(1,1,c) \; \mathcal{L} \; ymm^*$ for some $y \in Y_{mm^*} = Y_m$. It follows that $(1,1,c)m^\sharp \; \mathcal{L} \; ymm^*m^\sharp = ym$. Write $b = ((1,1,c)m^\sharp)\pi_3$. Then $b \in Q_m$. We show that $\varphi(b) = c$. It suffices to show that $(1,1,b)m^* \; \mathcal{L} \; (1,1,c)$. Now $(1,1,b) \; \mathcal{L} \; (1,1,c)m^\sharp$ yields $(1,1,b)m^* \; \mathcal{L} \; (1,1,c)m^\sharp m^*$. Since $(1,1,c) \; \mathcal{L} \; ymm^*$, we get $(1,1,c)m^\sharp m^* = (1,1,c)$ and so $(1,1,b)m^* \; \mathcal{L} \; (1,1,c)$ as required. Thus φ is onto and therefore a bijection. $\square$

Given $m = (a,g,b) \in M^I$, define

$$A'_m = \{a' \in A_m \mid Y_{(a',g,b)} \neq \emptyset\}.$$

For every $k \in \mathbb{N}$, let

$$U_0(k) = \{m \in W(M^I) : h_{\mathcal{J}}(m) = k \text{ and } |A_m| > 1\},$$

$$U_1(k) = \{m \in M^I \setminus W(M^I) : h_{\mathcal{J}}(m) = k \text{ and } |A_m| + |A'_m| > 1\},$$

$$U_2(k) = \{m \in W(M^I) : h_{\mathcal{J}}(m) = k \text{ and } |G_m|(1 + |Q_m|) > 1\},$$

$$U_3(k) = \{m \in M^I \setminus W(M^I) : h_{\mathcal{J}}(m) = k \text{ and } |G_m| > 1\},$$

$$U_4(k) = \{m \in M^I \setminus W(M^I) : h_{\mathcal{J}}(m) = k \text{ and } |G_m| \cdot |Q_m| > 1\}.$$

Lemma 8.11 *$U_i(k)$ is a union of $\mathcal{R}$-classes of M^I for $i = 0,2,3,4$.*

Proof. The claim follows from Lemmas 8.3 and 8.10 and

$$m \; \mathcal{R} \; m' \Rightarrow A'_m = A'_{m'}. \tag{24}$$

We prove that (24) holds. Indeed, assume that $m = (a,g,b)$ and $m' = (a,g',b')$ are $\mathcal{R}$-related. By Lemma 8.10, we have $Y_{(a',g,b)} = Y_{(a',g',b')}$ for every $a' \in A_m = A_{m'}$. Hence $A'_m = A'_{m'}$ and (24) holds as required. $\square$

We discuss now the cases when a vertex has more than one son. For every $m \in M^I$ with $Y_m \neq \emptyset$, we fix an arbitrary element $\gamma_m \in Y_m m$.

Lemma 8.12 *Let $v = [2k, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I] \in Vert(T)$ be in minimal representation and $2k < dep(r_0, T)$. Then $|Sons(v)| > 1$ if and only if $m_l \in U_0(k) \cup U_1(k)$. In that case, if $m_l = (a, g, b)$, then*

$$Sons(v) = \begin{cases} Sons_1(v) & \text{if } m_l \in U_0(k) \\ Sons_1(v) \cup Sons_2(v) & \text{if } m_l \in U_1(k) \end{cases}$$

with

$$\begin{aligned} Sons_1(v) &= \{[2k+1, (a', g, b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ a' \in A_{m_l}\}, \\ Sons_2(v) &= \{[2k+1, \gamma_{(a',g,b)} <_{\mathcal{L}} (a', g, b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ a' \in A'_{m_l}\} \end{aligned}$$

and the represented elements are all distinct in each case.

Proof. Write $\sigma = (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. Since $2k < \mathrm{dep}(r_0, T)$, we have $|\mathrm{Sons}(v)| \geq 1$. It follows from (4) that $|\mathrm{Sons}(v)| = 1$ if and only if

$$[2k, \sigma] = [2k, \tau] \Rightarrow [2k+1, \sigma] = [2k+1, \tau] \tag{25}$$

for every $\tau \in \mathrm{Rh}_Y(M^I)$.

Suppose first that $h_{\mathcal{J}}(m_l) < k$. Since

$$2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) + 1 \leq 2h_{\mathcal{J}}(m_l) + 1 \leq 2(k-1) + 1 < 2k,$$

then $[2k, \sigma] = [2k, \tau]$ implies $\sigma = \tau$ and so (25) holds. Thus $|\mathrm{Sons}(v)| = 1$.

Suppose now that $h_{\mathcal{J}}(m_l) > k$. Assume that $[2k, \sigma] = [2k, \tau]$ with $\sigma \neq \tau$. Then $2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) + 1 \geq H_Y(\sigma, \tau) \geq 2k$. Since $[2k, \sigma]$ is a minimal representation, it follows that $h_{\mathcal{J}}(m_{l-1}) < k$ and so $(\sigma \wedge_{\mathcal{L}} \tau) = m_l$. Thus

$$H_Y(\sigma, \tau) \geq 2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) = 2h_{\mathcal{J}}(m_l) \geq 2k + 2$$

and so $[2k+1, \sigma] = [2k+1, \tau]$. Therefore (25) holds and $|\mathrm{Sons}(v)| = 1$.

Therefore we assume that $h_{\mathcal{J}}(m_l) = k$ and write $m_l = (a, g, b)$.

Suppose first that $m_l \in W(M^I)$. We show that $\mathrm{Sons}(v) = \mathrm{Sons}_1(v)$. For every $a' \in A_{m_l}$, we have $(a', g, b)\ \mathcal{L}\ (a, g, b) = m_l$. Let $\tau = ((a', g, b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. By Lemma 8.7(ii), $\tau \in \mathrm{Rh}_Y(M^I)$. On the other hand, $(\sigma \wedge_{\mathcal{L}} \tau) = m_l$ and so $H_Y(\sigma, \tau) \geq 2h_{\mathcal{J}}(m_l) = 2k$. Hence $[2k, \tau] = v$ and we conclude by (4) that $[2k+1, \tau] \in \mathrm{Sons}(v)$.

Conversely, assume that $[2k+1, \zeta] \in \mathrm{Sons}(v)$ is in minimal representation. We show that $[2k+1, \zeta]$ is of the claimed form and we may assume that $\zeta \neq \sigma$. By (4), we have $[2k, \sigma] = [2k, \zeta]$ and so $H_Y(\sigma, \zeta) \geq 2k$. Hence $(\sigma \wedge_{\mathcal{L}} \zeta) = m_l \in W(M^I)$ and so $(\zeta \wedge_{\mathcal{L}} \sigma) =\in W(M^I)$ by Lemma 8.3. By Corollary 8.9(ii), we get $\zeta = (m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ with $m'_l\ \mathcal{L}\ m_l$. If $m'_l = (a', g', b')$, it follows that $b' = b$ and we may (if $g' \neq g$) replace g' by g to get $\zeta' = ((a', g, b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ since $(a', g, b)\ \mathcal{R}\ (a', g', b)$ and case (V1) of Lemma 8.4 imply $(\zeta, \zeta') \in V(M^I)$. Hence

$$H_Y(\zeta, \zeta') = 2h_{\mathcal{J}}((a', g', b)) + 1 = 2k + 1$$

and so $[2k+1,\zeta] = [2k+1,\zeta']$. Thus $\mathrm{Sons}(v) = \mathrm{Sons}_1(v)$.

Finally, given $\rho = ((a'',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ with $a'' \neq a'$, then

$$(\tau \wedge_{\mathcal{L}} \rho) = (a',g,b) \not\mathcal{R} \ (a'',g,b) = (\rho \wedge_{\mathcal{L}} \tau)$$

and so $H_Y(\tau,\rho) = 2h_{\mathcal{J}}((a',g,b)) = 2k$. Thus $[2k+1,\tau] \neq [2k+1,\rho]$ and so the elements in $\mathrm{Sons}_1(v)$ are all distinct. In particular, $|\mathrm{Sons}(v)| = |A_{m_l}|$ and so $|\mathrm{Sons}(v)| > 1$ if and only if $m_l \in U_0(k)$.

Assume now that $m_l \notin W(M^I)$. We show that $\mathrm{Sons}(v) = \mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v)$. We pass the inclusion $\mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v) \subseteq \mathrm{Sons}(v)$, a straightforward adaptation of the preceding case, and move straight to the converse inclusion. Let $[2k+1,\zeta] \in \mathrm{Sons}(v)$ and assume that $\zeta \neq \sigma$. By (4), we have $[2k,\sigma] = [2k,\zeta]$ and so $H_Y(\sigma,\zeta) \geq 2k$. Hence $(\sigma \wedge_{\mathcal{L}} \zeta) = m_l$ and so by Corollary 8.9(ii) we must have

$$\zeta = (m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \qquad \text{or} \qquad \zeta = (m'_{l+1} <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$$

with $m_l \ \mathcal{L} \ m'_l = (\zeta \wedge_{\mathcal{L}} \sigma)$. The discussion of the first case is analogous to the case $m_l \in W(M^I)$, hence we assume that $\zeta = (m'_{l+1} <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ and $m'_l = (a',g',b)$. Let

$$\zeta' = (\gamma_{(a',g,b)} <_{\mathcal{L}} (a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

Since $\zeta \in \mathrm{Rh}_Y(M^I)$, it follows from the maximality of s in (23) that $Y_{m'_l} \neq \emptyset$. Since $m'_l \ \mathcal{R} \ (a',g,b)$, it follows from Lemma 8.10 that $Y_{(a',g,b)} \neq \emptyset$ and so $a' \in A'_m$. Thus $[2k+1,\zeta'] \in \mathrm{Sons}_2(v)$. Finally, either $h_{\mathcal{J}}(\zeta \wedge_{\mathcal{L}} \zeta') > k$, or $(\zeta \wedge_{\mathcal{L}} \zeta') = m'_l$ and so $(\zeta,\zeta') \in V(M^I)$ through case (V2) of Lemma 8.4. In any case, it follows that $H_Y(\zeta,\zeta') \geq 2k+1$ and so $[2k+1,\zeta] = [2k+1,\zeta'] \in \mathrm{Sons}_2(v)$. Thus $\mathrm{Sons}(v) = \mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v)$.

For uniqueness, we only have to care about distinguishing $[2k+1,\zeta]$ from $[2k+1,\zeta']$ for

$$\zeta = ((a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0), \quad \zeta' = (\gamma_{(a',g,b)} <_{\mathcal{L}} (a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0),$$

the remaining cases following the same argument of the case $m_l \in W(M^I)$.

Since $m_l \notin W(M^I)$, then $(a',g,b) \notin W(M^I)$ by Lemma 8.3 and so $(\zeta,\zeta') \notin V(M^I)$ by Lemma 8.4. Hence

$$H(\zeta,\zeta') = 2h_{\mathcal{J}}(\zeta \wedge_{\mathcal{L}} \zeta') = 2h_{\mathcal{J}}((a',g,b)) = 2k$$

and so $[2k+1,\zeta] \neq [2k+1,\zeta']$. Thus the elements in $\mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v)$ are all distinct. In particular, $|\mathrm{Sons}(v)| = |A_{m_l}| + |A'_{m_l}|$ and so $|\mathrm{Sons}(v)| > 1$ if and only if $m_l \in U_1(k)$. □

Note that $v = [2k, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ with $m_l \in U_0(k) \cup U_1(k)$ implies $2k < \mathrm{dep}(r_0,T)$ and so $|\mathrm{Sons}(v)| > 1$.

Lemma 8.13 *Let $v = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I] \in Vert(T)$ be in minimal representation and $2k+1 < dep(r_0,T)$. Then $|Sons(v)| > 1$ if and only if $m_l \in U_2(k) \cup U_3(k)$ or $m_{l-1} \in U_4(k)$. In that case,*

$$Sons(v) = \begin{cases} Sons_1(v) \cup Sons_2(v) & \text{if } m_l \in U_2(k) \\ Sons_1(v) & \text{if } m_l \in U_3(k) \\ Sons_3(v) & \text{if } m_{l-1} \in U_4(k) \end{cases}$$

with

$Sons_1(v) = \{[2k+2, m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ m'_l \in \mathcal{H}_{m_l}\},$
$Sons_2(v) = \{[2k+2, (1,1,b') <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ m'_l \in \mathcal{H}_{m_l},\ b' \in Q_{m'_l}\}$
$Sons_3(v) = \{[2k+2, (1,1,b') <_{\mathcal{L}} m'_{l-1} <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ m'_{l-1} \in \mathcal{H}_{m_{l-1}},\ b' \in Q_{m'_{l-1}}\}$

and the represented elements are all distinct in each case.

Proof. Write $\sigma = (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. Since $2k+1 < \mathrm{dep}(r_0, T)$, we have $|\mathrm{Sons}(v)| \geq 1$. By (4), $|\mathrm{Sons}(v)| = 1$ if and only if

$$[2k+1, \sigma] = [2k+1, \tau] \Rightarrow [2k+2, \sigma] = [2k+2, \tau] \tag{26}$$

for every $\tau \in \mathrm{Rh}_Y(M^I)$.

The case $h_{\mathcal{J}}(m_l) < k$ is discussed analogously to the proof of Lemma 8.12.

Assume next that $h_{\mathcal{J}}(m_l) = k$ and $m_l \in W(M^I)$. We show that $\mathrm{Sons}(v) = \mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v)$.

Let $m'_l \in \mathcal{H}_{m_l}$ and write $\tau = (m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. By Lemma 8.7(ii), $\tau \in \mathrm{Rh}_Y(M^I)$. Since $(\sigma \wedge_{\mathcal{L}} \tau) = m_l \in W(M^I)$, we get $(\sigma, \tau) \in V(M^I)$ by Corollary 8.5, hence $H_Y(\sigma, \tau) = 2h_{\mathcal{J}}(m_l) + 1 = 2k+1$. By (4), we conclude that $[2k+2, \tau] \in \mathrm{Sons}(v)$.

Assume now that $b' \in Q_{m'}$. Then $(1,1,b')\ \mathcal{L}\ ym'_l$ for some $y \in Y$ such that $ym'_l <_{\mathcal{L}} m'_l$. Write

$$\rho = ((1,1,b') <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0), \quad \rho' = (ym'_l <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

It is immediate that $\rho' = (y <_{\mathcal{L}} I)\tau \in \mathrm{Rh}_Y(M^I)$. Since $(1,1,b')\ \mathcal{L}\ ym'_l$, it follows from Lemma 8.7(ii) that $\rho \in \mathrm{Rh}_Y(M^I)$ as well. Now we have $(\tau \wedge_{\mathcal{L}} \rho) = m'_l \in W(M^I)$ by Lemma 8.3 and so Corollary 8.5 yields

$$H_Y(\tau, \rho) = 2h_{\mathcal{J}}(m'_l) + 1 = 2h_{\mathcal{J}}(m_l) + 1 = 2k+1.$$

Hence $[2k+1, \rho] = [2k+1, \tau] = [2k+1, \sigma]$ and so $[2k+2, \rho] \in \mathrm{Sons}(v)$ as well.

Conversely, let $[2k+2, \zeta] \in \mathrm{Sons}(v)$ be a minimal representation. We show that $[2k+2, \zeta]$ is of the claimed form and we may assume that $\zeta \neq \sigma$. By (4), we have $[2k+1, \sigma] = [2k+1, \zeta]$ and so $H_Y(\sigma, \zeta) \geq 2k+1$. It follows that $(\sigma \wedge_{\mathcal{L}} \zeta) = m_l$. Let $m'_l = (\zeta \wedge_{\mathcal{L}} \sigma)$. Then $m_l\ \mathcal{H}\ m'_l$ since $m_l \in W(M^I)$, and $\zeta = (\ldots m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. If m'_l is the leftmost term of ζ, we are done. Otherwise, it follows from Corollary 8.9(i) and $h_{\mathcal{J}}(m'_l) = k$ that $\zeta = (m'_{l+1} <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ for some m'_{l+1}. Assume that $\zeta = (y_r <_{\mathcal{L}} I) \ldots (y_1 <_{\mathcal{L}} I)$ with $y_1, \ldots, y_r \in Y$. Then

$$(m'_{l+1} <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) = \mathrm{lm}(y_r \ldots y_1 \leq_{\mathcal{L}} y_{r-1} \ldots y_1 \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} y_1 <_{\mathcal{L}} I).$$

Let $s = \max\{j < r \mid y_j \ldots y_1 = m'_l\}$. Then $y_{s+1} \ldots y_1 <_{\mathcal{L}} y_s \ldots y_1$ since otherwise, by maximality of s, m'_l would not be the leftmost element in its $\mathcal{L}$-class. Moreover, $y_{s+1}m'_l = y_{s+1} \ldots y_1\ \mathcal{L}\ y_r \ldots y_1 = m'_{l+1}$. Let

$$\zeta' = (y_{s+1} <_{\mathcal{L}} I) \ldots (y_1 <_{\mathcal{L}} I) = (y_{s+1}m'_l <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$$

and write $y_{s+1}m'_l = (a', g', b')$,

$$\zeta'' = ((1,1,b') <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

Clearly, $\zeta' \in \mathrm{Rh}_Y(M^I)$ and so $\zeta'' \in \mathrm{Rh}_Y(M^I)$ by Lemma 8.7(ii). Moreover, $m'_{l+1}\ \mathcal{L}\ y_{s+1}m'_l\ \mathcal{L}\ (1,1,b')$ yields

$$H_Y(\zeta,\zeta'') \geq 2h_{\mathcal{J}}(m'_{l+1}) \geq 2k+2$$

and so $[2k+2,\zeta] = [2k+2,\zeta'']$. Since $y_{s+1} \in Y_{m'_l}$ and $b' \in Q_{m'_l}$, this completes the proof of $\mathrm{Sons}(v) = \mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v)$.

Finally, suppose that $[2k+2,\tau]$ and $[2k+2,\rho]$ are two sons of the described form with $\tau \neq \rho$. Then $(\tau \wedge_{\mathcal{L}} \rho) = m'_l$ for some $m'_l\ \mathcal{H}\ m_l$. It follows that

$$H_Y(\tau,\rho) \leq 2h_{\mathcal{J}}(m'_l) + 1 = 2h_{\mathcal{J}}(m_l) + 1 = 2k+1$$

and so $[2k+2,\tau] \neq [2k+2,\rho]$. Thus the claimed elements of $\mathrm{Sons}(v)$ are all distinct.

By Lemma 8.10, we have $|\mathrm{Sons}(v)| = |G_{m_l}|(1+|Q_{m_l}|)$ and so $|\mathrm{Sons}(v)| > 1$ if and only if $m_l \in U_2(k)$.

Assume next that $h_{\mathcal{J}}(m_l) = k$ and $m_l \notin W(M^I)$. We show that $\mathrm{Sons}(v) = \mathrm{Sons}_1(v)$.

Let $m'_l \in \mathcal{H}_{m_l}$ and write $\tau = (m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. By Lemma 8.7(ii), $\tau \in \mathrm{Rh}_Y(M^I)$. Since $(\sigma \wedge_{\mathcal{L}} \tau) = m_l\ \mathcal{R}\ m'_l = (\tau \wedge_{\mathcal{L}} \sigma)$, we are in case (V1) of Lemma 8.4 and so $(\sigma,\tau) \in V(M^I)$. Hence $H_Y(\sigma,\tau) = 2h_{\mathcal{J}}(m_l)+1 = 2k+1$. By (4), we conclude that $[2k+2,\tau] \in \mathrm{Sons}(v)$.

Conversely, let $[2k+2,\zeta] \in \mathrm{Sons}(v)$. We show that $[2k+2,\zeta]$ is of the claimed form and we may assume that $\zeta \neq \sigma$. By (4), we have $[2k+1,\sigma] = [2k+1,\zeta]$ and so $H_Y(\sigma,\zeta) \geq 2k+1$. It follows that $(\sigma \wedge_{\mathcal{L}} \zeta) = m_l$ and $(\sigma,\zeta) \in V(M^I)$. Since $m_l \notin W(M^I)$, it follows from Lemma 8.4 that (σ,ζ) must be in case (V1),and so $\zeta = (m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ with $m'_l\ \mathcal{R}\ m_l$. Since $m_l = (\sigma \wedge_{\mathcal{L}} \tau)\ \mathcal{L}\ (\tau \wedge_{\mathcal{L}} \sigma) = m'_l$, we get $[2k+2,\zeta] \in \mathrm{Sons}_1(v)$.

Proving that the elements of $\mathrm{Sons}_1(v)$ are distinct is similar to the preceding case. Therefore $|\mathrm{Sons}(v)| = |G_{m_l}|$ and so $|\mathrm{Sons}(v)| > 1$ if and only if $m_l \in U_3(k)$.

We consider now the case $h_{\mathcal{J}}(m_l) > k$. Suppose first that $h_{\mathcal{J}}(m_{l-1}) < k$ and take $[2k+1,\sigma] = [2k+1,\tau]$ with $\sigma \neq \tau$. then $2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) + 1 \geq H_Y(\sigma,\tau) \geq 2k+1$ and so $(\sigma \wedge_{\mathcal{L}} \tau) = m_l$. Thus

$$H_Y(\sigma,\tau) \geq 2h_{\mathcal{J}}(\sigma \wedge_{\mathcal{L}} \tau) = 2h_{\mathcal{J}}(m_l) \geq 2k+2$$

and so $[2k+2,\sigma] = [2k+2,\tau]$. Therefore (26) holds and $|\mathrm{Sons}(v)| = 1$.

Since v is in minimal representation, we may assume now by Corollary 8.9(ii) that $h_{\mathcal{J}}(m_{l-1}) = k$ and $m_{l-1} \notin W(M^I)$. We show that $\mathrm{Sons}(v) = \mathrm{Sons}_3(v)$.

Let $m'_{l-1} \in \mathcal{H}_{m_{l-1}}$ and $b' \in Q_{m'_{l-1}}$. Then $(1,1,b')\ \mathcal{L}\ ym'_{l-1}$ for some $y \in Y$ such that $ym'_{l-1} <_{\mathcal{L}} m'_{l-1}$. Write

$$\rho = ((1,1,b') <_{\mathcal{L}} m'_{l-1} <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0),$$

$$\rho' = (ym'_{l-1} <_{\mathcal{L}} m'_{l-1} <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

It is immediate that

$$\rho' = (y <_{\mathcal{L}} I)(m'_{l-1} <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \in \mathrm{Rh}_Y(M^I).$$

Since $(1,1,b')\ \mathcal{L}\ ym'_{l-1}$, it follows from Lemma 8.7(ii) that $\rho \in \mathrm{Rh}_Y(M^I)$ as well. The case $(\sigma \wedge_{\mathcal{L}} \rho) = m_l$ is straightforward, hence we assume that $(\sigma \wedge_{\mathcal{L}} \rho) = m_{l-1}$. Thus

$(\rho \wedge_{\mathcal{L}} \sigma) = m'_{l-1} \, \mathcal{H} \, m_{l-1}$ and we are in case (V2) of Lemma 8.4, yielding $(\sigma, \rho) \in V(M^I)$. It follows that $H_Y(\sigma, \rho) = 2h_{\mathcal{J}}(m_{l-1}) + 1 = 2k + 1$ and so $[2k+1, \rho] = [2k+1, \sigma]$. Therefore $[2k+2, \rho] \in \mathrm{Sons}(v)$.

Conversely, let $[2k+2, \zeta] \in \mathrm{Sons}(v)$ be in minimal representation. We show that $[2k+2, \zeta] \in \mathrm{Sons}_3(v)$. By (4), we have $[2k+1, \sigma] = [2k+1, \zeta]$ and so $H_Y(\sigma, \zeta) \geq 2k+1$. It follows that $(\sigma \wedge_{\mathcal{L}} \zeta) = m_l$ or else

$$(\sigma \wedge_{\mathcal{L}} \zeta) = m_{l-1} \qquad \text{and} \qquad (\sigma, \zeta) \in V(M^I). \tag{27}$$

Suppose that (27) holds. Let $m'_{l-1} = (\zeta \wedge_{\mathcal{L}} \sigma)$. Then $m_{l-1} \, \mathcal{H} \, m'_{l-1}$ and $\zeta = (\ldots m'_{l-1} <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. If m'_{l-1} is the leftmost term of ζ, then (σ, ζ) would be in case (V4) of Lemma 8.4 and so $m'_{l-1} \in W(M^I)$, contradicting $m_{l-1} \notin W(M^I)$ in view of Lemma 8.3. On the other hand, since $[2k+2, \zeta]$ is in minimal representation, it follows from Corollary 8.9(i) and $h_{\mathcal{J}}(m'_{l-1}) = k$ that $\zeta = (m'_l <_{\mathcal{L}} m'_{l-1} <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ for some m'_l. Now the proof that $[2k+2, \zeta] \in \mathrm{Sons}_3(v)$ is completely analogous to the case $h_{\mathcal{J}}(m_l) = k$ and $m_l \in W(M^I)$, and is therefore omitted. The same arguments hold for the case $(\sigma \wedge_{\mathcal{L}} \zeta) = m_l$, which is actually simpler. Therefore $\mathrm{Sons}(v) = \mathrm{Sons}_3(v)$.

Proving that the elements of $\mathrm{Sons}_1(v)$ are distinct is similar to the preceding case. By Lemma 8.10, we have $|\mathrm{Sons}(v)| = |G_{m_l}| \cdot |Q_{m_l}|$ and so $|\mathrm{Sons}(v)| > 1$ if and only if $m_{l-1} \in U_4(k)$. $\square$

Note that $v = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ with $m_l \in U_2(k)$ implies $2k+1 < \mathrm{dep}(r_0, T)$ and so $|\mathrm{Sons}(v)| > 1$.

Lemma 8.14 *Let $v = [i, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I] \in Vert(T)$ be in minimal representation and let $\sigma = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ be such that $m_j m'_p \, \mathcal{R} \, m_j$ for some $j \in \{0, \ldots, l-1\}$. Then $v\sigma = [i, (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)\sigma]$ is in minimal representation.*

Proof. By successive application of Lemma 7.1, we get

$$(m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)\sigma = (m_l m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_j m'_p <_{\mathcal{L}} \ldots)$$

and $m_{l-1} m'_p \, \mathcal{R} \, m_{l-1}$. Hence $h_{\mathcal{J}}(m_{l-1} m'_p) = h_{\mathcal{J}}(m_{l-1})$. By Lemma 8.3, we also have $m_{l-1} \in W(M^I)$ if and only if $m_{l-1} m'_p \in W(M^I)$. Thus $[i, (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)\sigma]$ is in minimal representation by Corollary 8.9. $\square$

Assume that $\delta = \mathrm{dep}(r_0, T)$. For commodity, we assume for the remaining part of this section that $\delta \in \mathbb{N}$, the infinite case being absolutely similar. We take two new symbols $\downarrow, *$. For every $k \in \mathbb{N}$ such that $2k+1 \leq \delta$, let

$$X_{2k+1} = \{\downarrow\} \cup \Big(\bigcup_{m \in U_0(k) \cup U_1(k)} A_m \Big) \cup \Big(\bigcup_{m \in U_1(k)} (A'_m \times \{*\}) \Big).$$

For every $k \in \mathbb{N}$ such that $2k+2 \leq \delta$, let

$$\begin{aligned} X_{2k+2} = \{\downarrow\} &\cup (\textstyle\bigcup_{m \in U_2(k)} (G_m \times (\{*\} \cup Q_{mm^*}))) \\ &\cup (\textstyle\bigcup_{m \in U_3(k)} (G_m \times \{*\})) \cup (\textstyle\bigcup_{m \in U_4(k)} (G_m \times Q_{mm^*})). \end{aligned}$$

A very important remark: in view of Lemma 8.10 and (24), *we assume the union over $m \in U_i(k)$ to be disjoint over distinct $\mathcal{R}$-classes*, e.g.: if $m, m' \in U_2(k)$ are R-related, i.e.

$mm^* = m'(m')^*$, then $G_m \times (\{*\} \cup Q_{mm^*})) = G_{m'} \times (\{*\} \cup Q_{m'(m')^*}))$. Otherwise, they are disjoint.

If M is finitely generated, then the X_i turn out to be finite:

Lemma 8.15 *If Y is finite, then all X_i are finite.*

Proof. It is enough to show that each set

$$E_k = \{m \in M^I \mid h_{\mathcal{J}}(m) = k\}$$

is finite. Since M is finite $\mathcal{J}$-above, this follows easily by induction on k from $E_0 = \{I\}$ and

$$E_k \subseteq \bigcup_{i=0}^{k-1} \bigcup_{x \in Y E_{i-1}} \mathcal{J}_x \; . \tag{28}$$

Indeed, if $m = y_s \dots y_1 \in E_k$ with $y_i \in Y$, take

$$r = \max\{j \in \{0, \dots, s\} \mid m <_{\mathcal{J}} y_j \dots y_1\}.$$

Let $n = y_r \dots y_1$. Then $n \in E_i$ for some $i \in \{0, \dots, k-1\}$ and $m \in \mathcal{J}_{y_{r+1}n}$, hence (28) holds and so does the lemma. $\square$

In view of Lemmas 8.12 and 8.13, we define a mapping $f : (\mathrm{Vert}(T)) \setminus \{r_0\} \to \cup_{i=1}^{\delta} X_i$ as follows. Let $v \in \mathrm{Vert}(T)$ and let $w \in \mathrm{Sons}(v)$.

(F1) If $\mathrm{Sons}(v) = \{w\}$, let $f(w) = \downarrow$.

(F2) If $v = [2k, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$ with $m_l = (a, g, b) \in U_0(k) \cup U_1(k)$ and $w = [2k + 1, (a', g, b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$, let $f(w) = a'$.

(F3) If $v = [2k, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$ with $m_l = (a, g, b) \in U_1(k)$ and $w = [2k + 1, \gamma_{(a',g,b)} <_{\mathcal{L}} (a', g, b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$, let $f(w) = (a', *)$.

(F4) If $v = [2k + 1, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I]$ with $m_l = (a, g, b) \in U_2(k) \cup U_3(k)$ and $w = [2k + 2, (a, g', b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$, write

$$\epsilon((a, g', b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0) = (x_l <_{\mathcal{J}} \dots <_{\mathcal{J}} x_0).$$

If $x_l = (a_1, g_1, b_1)$, let $f(w) = (g_1, *)$.

(F5) If $v = [2k + 1, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I]$ with $m_l = (a, g, b) \in U_2(k)$ and $w = [2k + 2, (1, 1, b') <_{\mathcal{L}} (a, g', b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$, write

$$\epsilon((1, 1, b') <_{\mathcal{L}} (a, g', b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0) = (x_{l+1} <_{\mathcal{J}} \dots <_{\mathcal{J}} x_0).$$

If $x_l = (a_1, g_1, b_1)$ and $x_{l+1} = (a_2, g_2, b_2)$, let $f(w) = (g_1, b_2)$.

(F6) If $v = [2k + 1, m_l <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0 = I]$ with $m_{l-1} = (a, g, b) \in U_4(k)$ and $w = [2k + 2, (1, 1, b') <_{\mathcal{L}} (a, g', b) <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0]$, write

$$\epsilon((1, 1, b') <_{\mathcal{L}} (a, g', b) <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \dots <_{\mathcal{L}} m_0) = (x_l <_{\mathcal{J}} \dots <_{\mathcal{J}} x_0).$$

If $x_{l-1} = (a_1, g_1, b_1)$ and $x_l = (a_2, g_2, b_2)$, let $f(w) = (g_1, b_2)$.

Note that $w = [i,\sigma] \Rightarrow f(w) \in X_i$ in all cases: this holds trivially if $|\mathrm{Sons}(v)| = 1$. If $i = 2k+1$ and $v = [2k, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]$ with $m_l \in U_0(k)$, then $f(w) \in A_{m_l} \in X_{2k+1} = X_i$; if $m_l \in U_1(k)$, then $f(w) \in A_{m_l} \cup (A'_{m_l} \times \{*\}) \subseteq X_{2k+1} = X_i$.

Finally, assume that $i = 2k+2$ and $v = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ with $m_l \in U_2(k)$. If $\epsilon(\sigma) = [x_l <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0]$, then $f(w) \in G_{x_l} \times \{*\} = G_{m_l} \times \{*\}$ by Lemma 7.3(i). Thus $f(w) \in X_{2k+2} = X_i$.

Assume now that $\epsilon(\sigma) = [x_{l+1} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0 = I]$, $x_l = (a_1, g_1, b_1)$ and $x_{l+1} = (a_2, g_2, b_2)$. Then $f(w) = (g_1, b_2)$. Clearly, $g_1 \in G_{x_l} = G_{m_l}$ by Lemma 7.3(i). We show that $b_2 \in Q_{m_l m_l^*}$. By Lemma 8.13, we may assume that

$$\sigma = ((1,1,b') <_{\mathcal{L}} m'_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$$

with $m'_l \in \mathcal{H}_{m_l}$ and $b' \in Q_{m'_l}$. Hence $(1,1,b') \;\mathcal{L}\; y m'_l <_{\mathcal{L}} m'_l$ for some $y \in Y$ and so $(1,1,b')(m'_l)^* \;\mathcal{L}\; y m'_l (m'_l)^* = y m_l m_l^*$ by Lemma 7.2(ii). Thus $(1,1,b_2) \;\mathcal{L}\; x_{l+1} = (1,1,b')(m'_l)^* \;\mathcal{L}\; y m_l m_l^*$. Since $m'_l \;\mathcal{R}\; m_l m_l^*$, $y m'_l <_{\mathcal{L}} m'_l$ implies $y m_l m_l^* <_{\mathcal{L}} m_l m_l^*$ by Lemma 8.10 and so $b_2 \in Q_{m_l m_l^*}$. Thus $f(w) \in X_{2k+2} = X_i$ as claimed.

The discussion of the cases arising from $U_3(k)$ and $U_4(k)$ is analogous and can be omitted.

Clearly, for all $\sigma \in \mathrm{Rh}_Y(M^I)$ and $v \in \mathrm{Vert}(T)$, the elliptic action θ induces a mapping

$$\begin{aligned} \theta^v_\sigma : \mathrm{Sons}(v) &\to \mathrm{Sons}(v\sigma) \\ w &\mapsto w\sigma. \end{aligned}$$

Lemma 8.16 *Let $v = [2k, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ with $m_l \in U_1(k)$ and let $\sigma = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I) \in Rh_Y(M^I)$. Then*

(i) $f|_{Sons(v)}$ is one-to-one;

(ii) $f(Sons(v)) = A_{m_l} \cup (A'_{m_l} \times \{*\})$;

(iii) $|(Sons(v))\sigma| > 1$ if and only if $m_l m'_p \;\mathcal{R}\; m_l$; in this case $f(w\sigma) = f(w)$ for every $w \in Sons(v)$ and θ^v_σ is a permutation;

(iv) $|(Sons(v))\sigma| = 1$ if and only if $m_l m'_p <_{\mathcal{J}} m_l$; in this case θ^v_σ is constant.

Proof. Writing $m_l = (a,g,b)$, then

$$\begin{aligned} \mathrm{Sons}(v) = &\{[2k+1, (a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ a' \in A_{m_l}\} \\ &\cup \{[2k+1, \gamma_{(a',g,b)} <_{\mathcal{L}} (a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ a' \in A'_{m_l}\} \end{aligned}$$

by Lemma 8.12 and these elements are all distinct. Since

$$f([2k+1, (a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]) = a',$$

$$f([2k+1, \gamma_{(a',g,b)} <_{\mathcal{L}} (a',g,b) <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]) = (a',*),$$

(i) and (ii) follow.

We may write

$$(m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)\sigma = (m_l m'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)$$

for some $n_0, \ldots, n_t \in M^I$. Since $(a', g, b) \mathrel{\mathcal{L}} m_l$, we get $(a', g, b)m'_p \mathrel{\mathcal{L}} m_l m'_p$ and so

$$[2k+1, (a', g, b) <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]\sigma = [2k+1, (a', g, b)m'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0].$$

Writing $\zeta = (\gamma_{(a',g,b)} <_{\mathcal{L}} (a', g, b) <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$, we get also

$$[2k+1, \zeta]\sigma = [2k+1, \mathrm{lm}(\gamma_{(a',g,b)} m'_p \leq_{\mathcal{L}} (a', g, b)m'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)].$$

Suppose that $m_l m'_p \not\mathrel{\mathcal{R}} m_l$. Since $m_l m'_p \leq_{\mathcal{J}} m_l$, it follows from (10) that $m_l m'_p <_{\mathcal{J}} m_l$ and so $h_{\mathcal{J}}(m_l m'_p) > k$. Then $(a', g, b)m'_p \mathrel{\mathcal{L}} m_l m'_p$ yields $h_{\mathcal{J}}((a', g, b)m'_p) > k$ and it follows easily that $|(\mathrm{Sons}(v))\sigma| = 1$.

Conversely, assume that $m_l m'_p \mathrel{\mathcal{R}} m_l$. Since $(a', g, b) \mathrel{\mathcal{L}} m_l$, we get $(a', g, b)m'_p \mathrel{\mathcal{R}} (a', g, b)$ and so

$$f([2k+1, (a', g, b) <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]\sigma) = a' = f([2k+1, (a', g, b) <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]).$$

Moreover, if $a' \in A'_{m_l}$ and $w = [2k+1, \zeta]$, Lemma 7.1 yields

$$w\sigma = [2k+1, \gamma_{(a',g,b)} m'_p <_{\mathcal{L}} (a', g, b)m'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0].$$

Assume that $m_l m'_p = (a, g', b')$ so that $v\sigma = [2k, (a, g', b') <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)]$. Since $w\sigma \in \mathrm{Sons}(v\sigma)$, it follows from Lemma 8.12 that

$$w\sigma = [2k+1, (a', g', b') <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0] \text{ for some } a' \in A_{(a,g',b')} \tag{29}$$

or

$$w\sigma = [2k+1, \gamma_{(a',g',b')} <_{\mathcal{L}} (a', g', b') <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0] \text{ for some } a' \in A'_{(a,g',b')}. \tag{30}$$

If (29) holds, then

$$H_Y(\gamma_{(a',g,b)} m'_p <_{\mathcal{L}} (a', g, b)m'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0, (a', g', b') <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0) \geq 2k+1$$

and so this pair belongs to $V(M^I)$, yielding $(a', g', b') \in W(M^I)$ by Lemma 8.4. Since

$$(a', g', b') \mathrel{\mathcal{L}} (a, g', b') = m_l m'_p \mathrel{\mathcal{R}} m_l \notin W(M^I),$$

this contradicts Lemma 8.3. Hence (30) holds and so $f(w\sigma) = (a', *) = f(w)$. Thus $|(\mathrm{Sons}(v))\sigma| > 1$ and also $f(w\sigma) = f(w)$ for every $w \in \mathrm{Sons}(v)$. Since $A_{m_l} = A_{m_l m'_p}$ and $A'_{m_l} = A'_{m_l m'_p}$ by (24), we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sons}(v) \arrow[rr, "\theta^v_\sigma"] \arrow[dr, "f_1"'] & & \mathrm{Sons}(v\sigma) \arrow[dl, "f_2"] \\
& A_{m_l} \cup (A'_{m_l} \times \{*\}) &
\end{tikzcd}$$

where f_1 and f_2 are the corresponding restrictions of f. Since f_1 and f_2 are bijective by (i) and (ii), θ^v_σ must be bijective as well. Thus (iii) holds.

We have $m_l m'_p \leq_{\mathcal{J}} m_l$. By (iii) and (S1), $|(\mathrm{Sons}(v))\sigma| = 1$ if and only if $m_l m'_p \not\mathrel{\mathcal{J}} m_l$ and therefore $m_l m'_p <_{\mathcal{J}} m_l$. It is straightforward to check that θ^v_σ is constant. $\square$

The proof of the following lemma is a simplification of the preceding one and is therefore omitted.

Lemma 8.17 *Let* $v = [2k, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ *with* $m_l \in U_0(k)$ *and let* $\sigma = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I) \in Rh_Y(M^I)$. *Then*

(i) $f|_{Sons(v)}$ *is one-to-one;*

(ii) $f(Sons(v)) = A_{m_l}$*;*

(iii) $|(Sons(v))\sigma| > 1$ *if and only if* $m_l m'_p \ \mathcal{R}\ m_l$*; in this case* $f(w\sigma) = f(w)$ *for every* $w \in Sons(v)$ *and* θ^v_σ *is a permutation;*

(iv) $|(Sons(v))\sigma| = 1$ *if and only if* $m_l m'_p <_{\mathcal{J}} m_l$*; in this case* θ^v_σ *is constant.*

Lemma 8.18 *Let* $v = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ *with* $m_l \in U_2(k)$ *and let* $\zeta = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I) \in Rh(M^I)$. *Then*

(i) $f|_{Sons(v)}$ *is one-to-one;*

(ii) $f(Sons(v)) = G_{m_l} \times (\{*\} \cup Q_{m_l m_l^*})$*;*

(iii) $|(Sons(v))\zeta| > 1$ *if and only if* $m_l m'_p \ \mathcal{R}\ m_l$*; in this case* $f(Sons(v\zeta)) = f(Sons(v))$ *and* θ^v_ζ *is a permutation;*

(iv) $|(Sons(v))\zeta| = 1$ *if and only if* $m_l m'_p <_{\mathcal{J}} m_l$*; in this case* θ^v_ζ *is constant.*

Proof. Writing $m_l = (a, g, b)$, it follows from Lemma 8.13 that $\mathrm{Sons}(v) = \mathrm{Sons}_1(v) \cup \mathrm{Sons}_2(v)$ with

$$\begin{aligned}
&\mathrm{Sons}_1(v) = \{[2k+2, r <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ r \in \mathcal{H}_{m_l}\},\\
&\mathrm{Sons}_2(v) = \{[2k+2, (1,1,b') <_{\mathcal{L}} r <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ r \in \mathcal{H}_{m_l},\ b' \in Q_r\}
\end{aligned}$$

and these elements are all distinct.

Let
$$\sigma = (r <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \in \mathrm{Sons}_1(v).$$

If $\epsilon(\sigma) = (x_l <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0 = I)$ and $x_l = (a_1, g_1, b_1)$, then $f([2k+2, \sigma]) = (g_1, *) \in G_{x_l} \times \{*\}$. Note that $x_l \ \mathcal{R}\ r \ \mathcal{H}\ m_l$ by Lemma 7.3(i) and so $f([2k+2, \sigma]) \in G_{m_l} \times \{*\} \subseteq X_{2k+2}$. By Green's Lemma, the mapping

$$\begin{aligned}
\mathcal{H}_{m_l} &\to \mathcal{H}_{m_l m^*_{l-1}}\\
r &\mapsto r m^*_{l-1}
\end{aligned}$$

is a bijection and so $f|_{\mathrm{Sons}_1(v)}$ is one-to-one and

$$f(\mathrm{Sons}_1(v)) = G_{m_l} \times \{*\}. \tag{31}$$

Next let
$$\tau = ((1,1,b') <_{\mathcal{L}} r <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \in \mathrm{Sons}_2(v)$$

with $r \in \mathcal{H}_{m_l}$ and $b' \in Q_r$. If $\epsilon(\tau) = (x_{l+1} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0 = I)$, $x_l = (a_1, g_1, b_1)$ and $x_{l+1} = (1, 1, b')r^* = (a_2, g_2, b_2)$, then $f([2k+2, \tau]) = (g_1, b_2)$. We fix $r \in \mathcal{H}_{m_l}$ and write

$$\mathrm{Sons}_{2,r}(v) = \{[2k+2, (1, 1, b') <_{\mathcal{L}} r <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0];\ b' \in Q_r\}.$$

In view of the preceding case, to complete the proof of (i) and (ii) it suffices to show that $f|_{\mathrm{Sons}_{2,r}(v)}$ is one-to-one and

$$f(\mathrm{Sons}_{2,r}(v)) = \{g_1\} \times Q_{m_l m_l^*}. \tag{32}$$

Indeed, since $rr^* = m_l m_l^*$ by Proposition 7.2(ii), the mapping

$$\begin{aligned} \varphi : Q_r &\to Q_{m_l m_l^*} \\ b &\mapsto ((1, 1, b)r^*)\pi_3 \end{aligned}$$

is a bijection by Lemma 8.10. Thus (32) holds and so

$$f(\mathrm{Sons}_2(v)) = G_{m_l} \times Q_{m_l m_l^*}. \tag{33}$$

In view of (31), (32) and the partial injectivity results obtained, (i) and (ii) hold.

Assume now that $|(\mathrm{Sons}(v))\zeta| > 1$. Suppose that $h_{\mathcal{J}}(m_l m'_p) > k$. Then $h_{\mathcal{J}}(rm'_p) > k$ for every $r \in \mathcal{H}_{m_l}$ due to $rm'_p \ \mathcal{L}\ m_l m'_p$. Since the rm'_p would then be all $\mathcal{L}$-equivalent, we would get $|(\mathrm{Sons}(v))\zeta| = 1$, a contradiction. Thus $h_{\mathcal{J}}(m_l m'_p) = k = h_{\mathcal{J}}(m_l)$ and so $m_l m'_p \ \mathcal{J}\ m_l$. By (S1), we get $m_l m'_p \ \mathcal{R}\ m_l$.

Conversely, assume that $m_l m'_p \ \mathcal{R}\ m_l$. Then $m_l = m_l m'_p z$ for some $z \in M^I$. Taking a minimal representation

$$v\zeta = [2k+1, m_l m'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0]$$

for some $n_0, \ldots, n_t \in M^I$, it follows easily from $m_l = m_l m'_p z$ that the elements $[2k+2, rm'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0]$ and $[2k+2, (1, 1, b')m'_p <_{\mathcal{L}} rm'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0]$ of $(\mathrm{Sons}(v))\zeta$ are all distinct, hence $|(\mathrm{Sons}(v))\zeta| = |\mathrm{Sons}(v)| > 1$. Moreover, applying (i) and (ii) to v and $v\zeta$, we have

$$|\mathrm{Sons}(v)| = |G_{m_l}| \cdot (1 + |Q_{m_l m_l^*}|),$$

$$|\mathrm{Sons}(v\zeta)| = |G_{m_l m'_p}| \cdot (1 + |Q_{m_l m'_p (m_l m'_p)^*}|).$$

Since $m_l m'_p \ \mathcal{R}\ m_l$, we get $G_{m_l m'_p} = G_{m_l}$ and also $m_l m'_p (m_l m'_p)^* = m_l m_l^*$ by Proposition 7.2(ii). Thus $|\mathrm{Sons}(v\zeta)| = |\mathrm{Sons}(v)| > 1$. Still applying (i) and (ii) to v and $v\zeta$, we get

$$f(\mathrm{Sons}(v\zeta)) = G_{m_l} \times (\{*\} \cup Q_{m_l m_l^*}) = f(\mathrm{Sons}(v)).$$

Furthermore, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sons}(v) \arrow[rr, "\theta_\zeta^v"] \arrow[dr, "f_1"'] & & \mathrm{Sons}(v\zeta) \arrow[dl, "f_2"] \\ & G_{m_l} \times (\{*\} \cup Q_{m_l m_l^*}) & \end{tikzcd}$$

where f_1 and f_2 are the corresponding restrictions of f. Since f_1 and f_2 are bijective by (i) and (ii), θ^v_ζ must be bijective as well.

The proof of (iv) is analogous to the proof of Lemma 8.16(iv). □

The proofs of the following two lemmas constitute straightforward adaptations of the proof of Lemma 8.18 and can therefore be omitted.

Lemma 8.19 *Let* $v = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ *with* $m_l \in U_3(k)$ *and let* $\zeta = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I) \in Rh(M^I)$. *Then*

(i) $f|_{Sons(v)}$ *is one-to-one;*

(ii) $f(Sons(v)) = G_{m_l} \times \{*\}$;

(iii) $|(Sons(v))\zeta| > 1$ *if and only if* $m_l m'_p \;\mathcal{R}\; m_l$*; in this case* $f(Sons(v\zeta)) = f(Sons(v))$ *and* θ^v_ζ *is a permutation;*

(iv) $|(Sons(v))\zeta| = 1$ *if and only if* $m_l m'_p <_{\mathcal{J}} m_l$*; in this case* θ^v_ζ *is constant.*

Lemma 8.20 *Let* $v = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I]$ *with* $m_{l-1} \in U_4(k)$ *and let* $\zeta = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I) \in Rh(M^I)$. *Then*

(i) $f|_{Sons(v)}$ *is one-to-one;*

(ii) $f(Sons(v)) = G_{m_{l-1}} \times Q_{m_{l-1}m^*_{l-1}}$;

(iii) $|(Sons(v))\zeta| > 1$ *if and only if* $m_{l-1} m'_p \;\mathcal{R}\; m_{l-1}$*; in this case* $f(Sons(v\zeta)) = f(Sons(v))$ *and* θ^v_ζ *is a permutation;*

(iv) $|(Sons(v))\zeta| = 1$ *if and only if* $m_{l-1} m'_p <_{\mathcal{J}} m_{l-1}$*; in this case* θ^v_ζ *is constant.*

Given a set X, we write

$$S(X) = \{\varphi \in M(X) : \varphi \text{ is a permutation of } X\}.$$

$$K(X) = \{\varphi \in P(X) : |X\varphi| \leq 1\}.$$

It is immediate that both $S(X) \cup K(X)$ and $\{\mathrm{Id}_X\} \cup K(X)$ constitute submonoids of $P(X)$.

In the main result of the paper, we construct an embedding

$$\begin{aligned} \varphi : \mathrm{Rh}(M^I) &\to \Pi_{i=1}^{\delta}(X_i, M_i) = \ldots \circ (X_2, M_2) \circ (X_1, M_1) \\ \sigma &\mapsto \varphi_\sigma \end{aligned}$$

into an iterated wreath product of partial transformation semigroups where M_{2k+1} is a submonoid of $\{\mathrm{Id}_{X_{2k+1}}\} \cup K(X_{2k+1})$ and M_{2k+2} is a submonoid of $S(X_{2k+2}) \cup K(X_{2k+2})$. Furthermore, we shall prove that this embedding has the *Zeiger property*: if

$$(\cdot, x_{2k+1}, \ldots, x_1)\varphi_\sigma \pi_{2k+2} \in S(X_{2k+2}) \setminus K(X_{2k+2}),$$

then any local mapping of the form $(\cdot, x_{q-1}, \ldots, x_1)\varphi_\sigma \pi_q$ for $2k+2 \leq q-1 < \delta$ must be the identity.

Theorem 8.21 *Let M be a finite $\mathcal{J}$-above Y-semigroup and let $\delta = 2 + 2sup\{h_{\mathcal{J}}(m) \mid m \in M\} \in \overline{\mathbb{N}}$. Then there exists an embedding φ of $Rh_Y(M^I)$ into the iterated wreath product of partial transformation semigroups $\Pi_{i=1}^{\delta}(X_i, M_i) = \ldots \circ (X_2, M_2) \circ (X_1, M_1)$ such that:*

(i) M_{2k+1} is a submonoid of $\{Id_{X_{2k+1}}\} \cup K(X_{2k+1})$ for $2k+1 \leq \delta$.

(ii) M_{2k+2} is a submonoid of $S(X_{2k+2}) \cup K(X_{2k+2})$ for $2k+2 \leq \delta$; if $\{R_\lambda \mid \lambda \in \Lambda\}$ is the set of all $\mathcal{R}$-classes of M contained in $U_2(k) \cup U_3(k) \cup U_4(k)$, then

$$M_{2k+2} \cap S(X_{2k+2}) \cong \oplus_{\lambda \in \Lambda} G'_\lambda, \tag{34}$$

where G'_λ is a subgroup of G_λ.

(iii) φ has the Zeiger property.

Moreover, if Y is finite, then the X_i (and consequently the M_i) are all finite.

Proof. For commodity, we assume that $\delta \in \mathbb{N}$, the infinite case being absolutely similar.

We consider the length function H_Y and we assume that $H_Y = D_\chi$ for $\chi = (r_0, T, \alpha, \theta)$, χ being obtained by the Chiswell construction. Let X_i and f be defined as before for $i = 1, \ldots, \delta$. Write $X = \bigcup_{i=1}^{\delta}(X_i \times \ldots \times X_1)$. By Theorem 8.1 and Lemmas 8.16–8.20(i), there exists an injective monoid homomorphism

$$\begin{aligned} \Psi : \mathrm{Rh}_Y(M^I) &\to (X_\delta, P(X_\delta)) \circ \ldots \circ (X_1, P(X_1)) \\ \sigma &\mapsto \Psi_\sigma \end{aligned}$$

defined by

$$x\Psi_\sigma = \begin{cases} x\psi^{-1}\theta_\sigma\psi & \text{if } \sigma \neq (I) \\ x & \text{if } \sigma = (I), \end{cases} \qquad (x \in X).$$

where

$$\begin{aligned} \psi : \mathrm{Ray}(r_0, T) &\to X \\ (v_i, \ldots, v_1, r_0) &\mapsto (f(v_i), \ldots, f(v_1)). \end{aligned}$$

Given $\sigma \in \mathrm{Rh}_Y(M^I) \setminus \{I\}$, we extend Ψ_σ to a mapping $\varphi_\sigma \in P(X)$ by taking

$$\begin{aligned} \mathrm{dom}\varphi_\sigma = \mathrm{im}\psi \cup (\cup_{i=1}^{\delta}\{\, (x_i, f(v_{i-1}), \ldots, f(v_1)) \in X : (v_{i-1}, \ldots, v_1) \in \mathrm{Ray}(r_0, T) \\ \text{and } |(\mathrm{Sons}(v_{i-1}))\sigma| > 1\}) \end{aligned}$$

and

$$(x_i, f(v_{i-1}), \ldots, f(v_1))\varphi_\sigma = (x_i, (f(v_{i-1}), \ldots, f(v_1))\Psi_\sigma)$$

if $(x_i, f(v_{i-1}), \ldots, f(v_1)) \notin \mathrm{im}\psi$. Since ψ is one-to-one, φ_σ is well-defined. Being an extension of Ψ_σ, it is easy to see that φ_σ inherits some of its properties, namely being sequential. Moreover, it follows from Lemmas 8.16–8.20(iii) that

$$(\mathrm{dom}\varphi_\sigma \setminus \mathrm{im}\psi)\varphi_\sigma \, \cap \, \mathrm{im}\psi = \emptyset. \tag{35}$$

Taking $\varphi_I = \Psi_I = \mathrm{Id}_X$, we define

$$\begin{aligned} \varphi : \mathrm{Rh}_Y(M^I) &\to P(X) \\ \sigma &\mapsto \varphi_\sigma. \end{aligned}$$

We show that φ is a monoid homomorphism.

Since φ_I is the identity and $\text{dom}\varphi_{\sigma\tau} \subseteq \text{dom}\varphi_\sigma$, we only have to take $\sigma, \tau \in \text{Rh}_Y(M^I) \backslash \{I\}$ and show that

$$x\varphi_\sigma\varphi_\tau = x\varphi_{\sigma\tau} \tag{36}$$

holds for every $x \in \text{dom}\varphi_\sigma$. Since $\Psi_\sigma \subseteq \varphi_\sigma$ is a homomorphism, (36) holds for $x \in \text{im}\psi$. Assume now that

$$x = (x_i, f(v_{i-1}), \ldots, f(v_1)) \in \text{dom}\varphi_\sigma \setminus \text{im}\psi.$$

Hence $|(\text{Sons}(v_{i-1}))\sigma| > 1$. Write $(f(v_{i-1}), \ldots, f(v_1))\Psi_\sigma = (f(v'_{i-1}), \ldots, f(v'_1))$. In particular, $v_{i-1}\sigma = v'_{i-1}$.

Assume first that $|(\text{Sons}(v'_{i-1}))\tau| \leq 1$. Then $v_{i-1}\sigma = v'_{i-1}$ yields $(\text{Sons}(v_{i-1}))\sigma \subseteq \text{Sons}(v'_{i-1})$ since the action is elliptical and so $|(\text{Sons}(v_{i-1}))\sigma\tau| \leq 1$ as well. Thus $x \notin \text{dom}\varphi_{\sigma\tau}$. On the other hand, $x\varphi_\sigma = (x, f(v'_{i-1}), \ldots, f(v'_1)) \notin \text{im}\psi$ by (35) and so $x\varphi_\sigma \notin \text{dom}\varphi_\tau$. Thus (36) holds in this case.

Finally, assume that $|(\text{Sons}(v'_{i-1}))\tau| > 1$. Write

$$(f(v_{i-1}), \ldots, f(v_1))\Psi_\sigma\Psi_\tau = (f(v'_{i-1}), \ldots, f(v'_1))\Psi_\tau = (f(v''_{i-1}), \ldots, f(v''_1)).$$

Then

$$x\varphi_\sigma\varphi_\tau = (x, f(v'_{i-1}), \ldots, f(v'_1))\varphi_\tau = (x, f(v''_{i-1}), \ldots, f(v''_1))$$

by (35). On the other hand, in view of Lemmas 8.16–8.20(iii), $|(\text{Sons}(v_{i-1}))\sigma| > 1$ and $|(\text{Sons}(v'_{i-1}))\tau| > 1$ together yield $|(\text{Sons}(v_{i-1}))\sigma\tau| > 1$. Since $x \notin \text{im}\psi$ and Ψ is a homomorphism, we obtain

$$x\varphi_{\sigma\tau} = (x, (f(v_{i-1}), \ldots, f(v_1))\Psi_{\sigma\tau}) = (x, f(v''_{i-1}), \ldots, f(v''_1)) = x\varphi_\sigma\varphi_\tau$$

and so (36) holds as well in this case. Thus φ is a monoid homomorphism.

We show next that φ is one-to-one. Given distinct $\sigma, \tau \in \text{Rh}_Y(M^I) \setminus \{I\}$, we have $\Psi_\sigma \neq \Psi_\tau$ by Theorem 8.1. Since $\text{dom}\Psi_\sigma = \text{im}\psi = \text{dom}\Psi_\tau$, it follows that $\varphi_\sigma \neq \varphi_\tau$ as well. To show that $\varphi_\sigma \neq \varphi_I$, it suffices now to show that Ψ_σ is not one-to-one. Indeed, using the Chiswell construction and by Lemma 8.13, we have $|\text{Sons}(v)| > 1$ for $v = [1, I]$ since $h_{\mathcal{J}}(I) = 0$ and $Q_I \neq \emptyset$. However, for $\sigma = (n_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)$ with $p > 0$, we have $In_p = n_p <_{\mathcal{J}} I$ and so θ^v_σ is constant by Lemma 8.18(iv). Thus Ψ_σ is not one-to-one and so φ is indeed one-to-one.

We proceed now to discuss the local mappings. Let $(x_{i-1}, \ldots, x_1) \in X_{i-1} \times \ldots \times X_1$ and write $\xi = (\cdot, x_{i-1}, \ldots, x_1)\varphi_\sigma \in P(X_i)$. We assume $\sigma \neq (I)$. Assume that $\xi \notin K(X_i)$. In particular, ξ is not the empty map and so $(x_{i-1}, \ldots, x_1) = (f(v_{i-1}), \ldots, f(v_1))$ for some $(v_{i-1}, \ldots, v_1) \in \text{Ray}(r_0, T)$. Let $\xi' = \xi|_{\text{im}\psi}$. It follows from the definition of φ_σ that $|\text{Sons}(v_{i-1})\sigma| > 1$, otherwise $\xi = \xi' \in K(X_i)$. By Lemmas 8.16–8.20, it follows that $\xi' \in S(X'_i)$ for some $X'_i \subset X_i$ and so $\xi \in S(X_i)$ by definition of φ_σ.

If i is odd, then $\xi = \text{Id}_{X_i}$ by Lemmas 8.16(iii) and 8.17(iii), thus we can take M_i to be a submonoid of $\{\text{Id}_{X_i}\} \cup K(X_i)$ and (i) holds.

Assume now that $i = 2k + 2$ is even. We can take M_i to be the submonoid of $S(X_i) \cup K(X_i)$ generated by the local mappings ξ. Write $\sigma = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0 = I)$ with $v_{i-1} = [2k+1, m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0]$ in minimal representation. Since $\xi \notin K(X_i)$, then $|\text{Sons}(v_{i-1})| > 1$ and so, by Lemma 8.13, either $m_l \in U_2(k) \cup U_3(k)$ or $m_{l-1} \in U_4(k)$.

We consider first the case $m_l \in U_3(k)$. By Lemma 8.19(iii), ξ' permutes $G_{m_l} \times \{*\}$. We show that there exists some $g_0 \in G_{m_l}$ such that

$$(h,*)\xi = (hg_0,*) \text{ for every } h \in G_{m_l}. \tag{37}$$

Indeed, by Lemma 7.3(i) we may write $m_l = (a,g,b)$ and $x_l = (a,g_1,b_1)$. Write also

$$(m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)\sigma = (m_l m_p' <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0). \tag{38}$$

Given $h \in G_{m_l}$, take $r = (a,h,b_1)m_{l-1}^{\#}$, $\tau = (r <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ and $w = [2k+2,\tau]$. We claim that

$$w \in \text{Sons}(v_{i-1}) \qquad \text{and} \qquad f(w) = (h,*). \tag{39}$$

Indeed, $(a,h,b_1)\ \mathcal{H}\ x_l$ yields $r\ \mathcal{L}\ x_l m_{l-1}^{\#} = m_l$ by Lemma 7.3(iv). On the other hand, $(a,h,b_1)m_{l-1}^{\#}\ \mathcal{L}\ m_l\ \mathcal{R}\ x_l\ \mathcal{H}\ (a,h,b_1)$ yields $r\ \mathcal{R}\ (a,h,b_1)$ by (S1) and so $r\ \mathcal{R}\ x_l\ \mathcal{R}\ m_l$. Thus $r \in \mathcal{H}_{m_l}$. Moreover,

$$r m_{l-1}^* = (a,h,b_1)m_{l-1}^{\#}m_{l-1}^* = (a,h,b_1)$$

since $(a,h,b_1)\ \mathcal{L}\ x_l$ and $x_l m_{l-1}^{\#} m_{l-1}^* = m_l m_{l-1}^* = x_l$ by Lemma 7.3(iv). Thus (39) holds. Now, since $m_l m_p'\ \mathcal{L}\ r m_p'$, it follows from (38) that $\tau\sigma = (r m_p' <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)$ and so

$$\epsilon(\tau\sigma) = (r m_p' n_t^* <_{\mathcal{J}} \ldots). \tag{40}$$

Thus

$$(h,*)\xi = ((r m_p' n_t^*)\pi_2, *) = (((a,h,b_1)m_{l-1}^{\#} m_p' n_t^*)\pi_2, *).$$

Let $y = m_{l-1}^{\#} m_p' n_t^*$. Since

$$(a,g_1,b_1)m_{l-1}^{\#} m_p' n_t^* = x_l m_{l-1}^{\#} m_p' n_t^* = m_l m_p' n_t^*$$

and $m_l m_p' n_t^*\ \mathcal{R}\ m_l m_p'\ \mathcal{R}\ m_l\ \mathcal{R}\ x_l = (a,g_1,b_1)$ by Lemma 7.3(i) and (40), it follows from Proposition 7.6 that there exists some $g_0 \in G_{m_l}$ such that

$$\forall h \in G_{m_l},\ ((a,h,b_1)y)\pi_2 = hg_0.$$

Thus (37) holds.

We consider next the case $m_{l-1} \in U_4(k)$. By Lemma 8.20(iii), ξ' permutes $G_{m_{l-1}} \times Q_{m_{l-1}m_{l-1}^*}$. We show that there exists some $g_0 \in G_{m_{l-1}}$ such that

$$(h,c)\xi' = (hg_0,c) \text{ for all } h \in G_{m_{l-1}} \text{ and } c \in Q_{m_{l-1}m_{l-1}^*}. \tag{41}$$

Since $m_{l-1}m_p'\ \mathcal{R}\ m_{l-1}$ by Lemma 8.20(iii), $m_l <_{\mathcal{L}} m_{l-1}$ yields $m_l m_p'\ \mathcal{R}\ m_l$ by Lemma 7.1 and so we may assume that

$$(m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)\sigma = (m_l m_p' <_{\mathcal{L}} m_{l-1}m_p' <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0). \tag{42}$$

Let $h \in G_{m_{l-1}}$ and $c \in Q_{m_{l-1}m_{l-1}^*}$. Let $r = (a,h,b_1)m_{l-2}^{\sharp}$ and $(1,1,b')\ \mathcal{L}\ (1,1,c)r^{\sharp}$. Let

$$\tau = ((1,1,b') <_{\mathcal{L}} r <_{\mathcal{L}} m_{l-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$$

and $w = [2k+2, \tau]$. We claim that

$$w \in \mathrm{Sons}(v_{i-1}) \qquad \text{and} \qquad f(w) = (h, c). \tag{43}$$

Indeed, the proof of (39) can be easily adapted to show that $r \,\mathcal{H}\, m_{l-1}$ and $f(w) = (h, \ldots)$ (if indeed $w \in \mathrm{Sons}(v_{i-1})$). Since $c \in Q_{m_{l-1}m^*_{l-1}}$, we have $(1,1,c) \,\mathcal{L}\, ym_{l-1}m^*_{l-1}$ for some $y \in Y_{m_{l-1}m^*_{l-1}}$. Hence

$$(1,1,b') \,\mathcal{L}\, (1,1,c)r^\sharp \,\mathcal{L}\, ym_{l-1}m^*_{l-1}r^\sharp = yrr^*r^\sharp = yr.$$

Since $yr \,\mathcal{L}\, r$ would imply $ym_{l-1}m^*_{l-1} \,\mathcal{L}\, m_{l-1}m^*_{l-1}$ in view of $r \,\mathcal{R}\, m_{l-1}m^*_{l-1}$, contradicting $y \in Y_{m_{l-1}m^*_{l-1}}$, we get $yr <_{\mathcal{L}} r$ and so $y \in Y_r$. Thus $b' \in Q_r$ and so $w \in \mathrm{Sons}(v_{i-1})$ by Lemma 8.13. Now

$$(1,1,b')r^* \,\mathcal{L}\, yrr^* = ym_{l-1}m^*_{l-1} \,\mathcal{L}\, (1,1,c),$$

hence $f(w) = (h,c)$ and so (43) holds.

Now $(1,1,b') <_{\mathcal{L}} r \,\mathcal{L}\, m_{l-1}$ yields $(1,1,b')m'_p <_{\mathcal{L}} rm'_p$ by Lemma 7.1. Similarly to the preceding case, it follows easily from (42) that

$$\tau\sigma = ((1,1,b')m'_p <_{\mathcal{L}} rm'_p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)$$

and so

$$\epsilon(\tau\sigma) = ((1,1,b')m'_p(rm'_p)^* <_{\mathcal{J}} rm'_pn^*_t <_{\mathcal{J}} \ldots <_{\mathcal{J}} I).$$

Since $(1,1,b') <_{\mathcal{L}} r$, we may write $(1,1,b') = zr$ for some $z \in M$. Since $m_{l-1}m'_p \,\mathcal{R}\, m_{l-1}$ and $r \,\mathcal{L}\, m_{l-1}$, we get $rm'_p \,\mathcal{R}\, r$ and so Lemma 7.2(ii) yields

$$(1,1,b')m'_p(rm'_p)^* = zrm'_p(rm'_p)^* = zrr^* = (1,1,b')r^*.$$

Hence the leftmost term in $\epsilon(\tau\sigma)$ is the same as in $\epsilon(\tau)$ and so $(h,c)\xi' = (\ldots, c)$. A straightforward adaptation of the proof of (37) completes the proof of (41).

Similarly, in the case $m_l \in U_2(k)$ we show that there exists some $g_0 \in G_{m_l}$ such that

$$(h,c)\xi' = (hg_0, c) \text{ for all } h \in G_{m_l} \text{ and } c \in \{*\} \cup Q_{m_lm^*_l}. \tag{44}$$

Indeed, by (31) and (33), ξ' is the (disjoint) union of a permutation ξ'_1 of $G_{m_l} \times \{*\}$ with a permutation ξ'_2 of $G_{m_l} \times Q_{m_lm^*_l}$. A straightforward combination of the two preceding cases yields (44).

Write

$$K = \begin{cases} G_{m_l} \times (\{*\} \cup Q_{m_lm^*_l}) & \text{if } m_l \in U_2(k) \\ G_{m_l} \times \{*\} & \text{if } m_l \in U_3(k) \\ G_{m_{l-1}} \times Q_{m_{l-1}m^*_{l-1}} & \text{if } m_{l-1} \in U_4(k). \end{cases}$$

By (37), (41) and (44), each local map $\xi \in M_i \cap S(X_i)$ can be decomposed as a disjoint union of permutations $\xi = \xi' \cup \xi''$ where

$$\begin{aligned} \xi' : K &\to K \\ (h,c) &\mapsto (hg_0, c), \end{aligned}$$

for some $g_0 \in G_{m_l}$ ($G_{m_{l-1}}$ if $m_{l-1} \in U_4(k)$), and ξ'' is the identity mapping on $X_i \setminus K$.

For every $\lambda \in \Lambda$, take $m \in R_\lambda$ and

$$K_\lambda = \begin{cases} G_m \times (\{*\} \cup Q_{mm^*}) & \text{if } m \in U_2(k) \\ G_m \times \{*\} & \text{if } m \in U_3(k) \\ G_m \times Q_{mm^*} & \text{if } m \in U_4(k). \end{cases}$$

Note that K_λ is well defined in view of Lemmas 8.3, 8.10 and 8.11. Write

$$S_\lambda(X_{2k+2}) = \{\varphi \in S(X_{2k+2}) \mid \varphi|_{X_{2k+2} \setminus K_\lambda} = \mathrm{Id}\}.$$

Since the sets K_λ are disjoint subsets of X_{2k+2}, we can view $S_\lambda(X_{2k+2})$ as a direct sum of its subgroups $S_\lambda(X_{2k+2})$. We show that

$$M_{2k+2} \cap S(X_{2k+2}) = \oplus_{\lambda \in \Lambda} (M_{2k+2} \cap S_\lambda(X_{2k+2})). \tag{45}$$

Indeed, the union

$$\begin{aligned} X_{2k+2} = \{\downarrow\} &\cup (\textstyle\bigcup_{m \in U_2(k)} (G_m \times (\{*\} \cup Q_{mm^*}))) \\ &\cup (\textstyle\bigcup_{m \in U_3(k)} (G_m \times \{*\})) \cup (\textstyle\bigcup_{m \in U_4(k)} (G_m \times Q_{mm^*})). \end{aligned}$$

is supposed to be disjoint over distinct $\mathcal{R}$-classes, and the decomposition $\xi = \xi' \cup \xi''$ shows that every local map ξ belongs indeed to a unique $S_\lambda(X_{2k+2})$. Since M_{2k+2} is by definition generated by the local maps ξ, it follows that $M_{2k+2} \cap S(X_{2k+2}) \subseteq \oplus_{\lambda \in \Lambda} (M_{2k+2} \cap S_\lambda(X_{2k+2}))$. The opposite inclusion is trivial, hence (45) holds.

It follows from the decomposition $\xi = \xi' \cup \xi''$, (37), (41) and (44) that we can take $M_{2k+2} \cap S_\lambda(X_{2k+2}) \cong G'_\lambda$ for some subgroup G'_λ of G_λ, hence

$$M_{2k+2} \cap S(X_{2k+2}) \cong \oplus_{\lambda \in \Lambda} G'_\lambda$$

and (ii) holds.

Finally, we prove that φ has the Zeiger property. Let $\sigma = (m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} m'_0) \in \mathrm{Rh}_Y(M^I)$. We may assume that $p > 0$. Suppose that $\xi = (\cdot, f(v_{2k+1}), \ldots, f(v_1))\varphi_\sigma \pi_{2k+2} \in S(X_{2k+2}) \setminus K(X_{2k+2})$ and $(v_{q-1}, \ldots, v_1, r_0) \in \mathrm{Ray}(r_0, T)$ with $2k+1 < q-1 < \delta$. Let $\xi' = (\cdot, f(v_{q-1}), \ldots, f(v_1))\varphi_\sigma \pi_q$. We show that ξ' is the identity mapping by induction on q. Assume the claim holds for q' whenever $2k+1 < q'-1 < q-1$.

Let $v_{i-1} = [2k+1, \tau]$ in minimal representation, with $\tau = (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$. Since $\xi \in S(X_{2k+2}) \setminus K(X_{2k+2})$, we have either $m_l \in U_2(k) \cup U_3(k)$ or $m_{l-1} \in U_4(k)$ by Lemma 8.13. Let

$$d = \begin{cases} l & \text{if } m_l \in U_2(k) \cup U_3(k) \\ l-1 & \text{if } m_{l-1} \in U_4(k) \end{cases}$$

By Lemmas 8.18–8.20(iii), we have $m_d m'_p \;\mathcal{R}\; m_d$. Write $v_{q-1} = [q-1, \rho]$ in minimal representation. Since v_{q-1} must be a descendant of v_{2k+1}, it follows from (4) that $H(\tau \wedge_{\mathcal{L}} \rho) \geq 2k+1$ and so either $h_{\mathcal{J}}(\tau \wedge_{\mathcal{L}} \rho) > k$ or $(\tau, \rho) \in V(M^I)$. Hence

$$\rho = (n_{l'} <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d <_{\mathcal{L}} m_{d-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0) \tag{46}$$

for some n_j. By Lemma 8.4, we have $n_d = (\rho \wedge_{\mathcal{L}} \tau) \;\mathcal{H}\; (\tau \wedge_{\mathcal{L}} \rho) = m_d$. Since $m_d m'_p \;\mathcal{R}\; m_d$, Lemma 7.1 yields

$$n_j m'_p \;\mathcal{R}\; n_j \quad (j = d, \ldots, l'), \tag{47}$$

$$\rho\sigma = (n_{l'}m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d m'_p <_{\mathcal{L}} \ldots). \tag{48}$$

Assume first that $|\mathrm{Sons}(v_{q-1})| = 1$. Then we must have $\mathrm{dom}\xi' = \{\downarrow\}$. Suppose that $|\mathrm{Sons}(v_{q-1}\sigma)| > 1$. Suppose further that $l' = d$. Then

$$v_{q-1}\sigma = [q-1, \rho\sigma] = [q-1, n_d m'_p <_{\mathcal{L}} \ldots].$$

Since $h_{\mathcal{J}}(n_d m'_p) = h_{\mathcal{J}}(n_d) = h_{\mathcal{J}}(m_d) = k$ and $2k+2 \leq q-1$, it follows from Lemmas 8.12 and 8.13 that $|\mathrm{Sons}(v_{q-1}\sigma)| = 1$, a contradiction. Hence $l' > d$. By (47), (48) and Lemma 8.14, $v_{q-1} = [q-1, \rho]$ being in minimal representation implies that so it is $v_{q-1}\sigma = [q-1, \rho\sigma]$. It follows that for q odd (respectively even) we have $n_{l'}m'_p \in U_0(k') \cup U_1(k')$ for $k' = \frac{q-1}{2}$ (respectively $n_{l'}m'_p \in U_2(k') \cup U_3(k')$ or $n_{l'-1}m'_p \in U_4(k')$ for $k' = \frac{q-2}{2}$).

Suppose first that $n_{l'}m'_p \in U_0(k')$. Then $h_{\mathcal{J}}(n_{l'}m'_p) = k'$ and $|A_{n_{l'}m'_p}| > 1$. Since $n_{l'}m'_p \;\mathcal{R}\; n_{l'}$ by (47), we get $h_{\mathcal{J}}(n_{l'}) = k'$ and $|A_{n_{l'}}| > 1$ and so $n_{l'} \in U_0(k')$. By Lemma 8.12, this contradicts $|\mathrm{Sons}(v_{q-1})| = 1$. The case $n_{l'}m'_p \in U_1(k')$ is analogous.

Assume now that $n_{l'}m'_p \in U_2(k')$. Then $h_{\mathcal{J}}(n_{l'}m'_p) = k'$ and $|G_{n_{l'}m'_p}|(1 + |Q_{n_{l'}m'_p}|) > 1$. By Lemma 8.10, we have

$$|Q_{n_{l'}m'_p}| = |Q_{n_{l'}m'_p(n_{l'}m'_p)^*}|, \quad |Q_{n_{l'}}| = |Q_{n_{l'}n^*_{l'}}|. \tag{49}$$

Since $n_{l'}m'_p \;\mathcal{R}\; n_{l'}$ by (47), we get $n_{l'}m'_p(n_{l'}m'_p)^* = n_{l'}n^*_{l'}$, hence $h_{\mathcal{J}}(n_{l'}) = k'$ and (49) yields $|G_{n_{l'}}|(1 + |Q_{n_{l'}}|) > 1$ and thus $n_{l'} \in U_2(k')$. By Lemma 8.13, this contradicts $|\mathrm{Sons}(v_{q-1})| = 1$ as well.

The cases $n_{l'}m'_p \in U_3(k')$ and $n_{l'-1}m'_p \in U_4(k')$ are analogous and can be omitted. Therefore we may conclude that $|\mathrm{Sons}(v_{q-1}\sigma)| = 1$ and so $\downarrow \xi' = \downarrow$.

We assume now that $|\mathrm{Sons}(v_{q-1})| \neq 1$. Since $q - 1 < \delta$, it follows that $|\mathrm{Sons}(v_{q-1})| > 1$. Clearly, if $l' = d$, then $h_{\mathcal{J}}(n_{l'}) = h_{\mathcal{J}}(m_d) = k$ and so, since $q-1 \geq 2k+2$, v_{q-1} has a unique son by Lemmas 8.12 and 8.13, a contradiction. Therefore $l' > d$. Now (47) yields $n_{l'}m'_p \;\mathcal{R}\; n_{l'}$, which implies $|(\mathrm{Sons}(v_{q-1}))\sigma| > 1$ by Lemmas 8.16–8.20(iii). Thus $\xi' \in S(X_i)$ by definition of φ_σ. If q is odd, we obtain $\xi' = \mathrm{Id}$ by Lemmas 8.16(iii) and 8.17(iii), hence we may assume that $q = 2k'+2$ with $k < k'$.

Since ξ' is the identity anyway for all the other cases, it suffices to prove that

$$(f(v_q), \ldots, f(v_1))\varphi_\sigma = (f(v_q), \ldots, f(v_1))$$

whenever $(v_q, \ldots, v_1) \in \mathrm{Ray}(r_0, T)$, that is,

$$(v_q, \ldots, v_1)\sigma\psi = (f(v_q), \ldots, f(v_1)).$$

By the induction hypothesis, we have

$$(f(v_{q-1}), \ldots, f(v_1))\varphi_\sigma = (f(v_{q-1}), \ldots, f(v_1)),$$

hence it is enough to show that

$$f(v_q\sigma) = f(v_q). \tag{50}$$

Since $|\mathrm{Sons}(v_{q-1})| > 1$, it follows from Lemma 8.13 that either $n_{l'} \in U_2(k') \cup U_3(k')$ or $n_{l'-1} \in U_4(k')$.

We consider first the case $n_{l'} \in U_2(k')$. Since $h_{\mathcal{J}}(n_{l'}) = k'$, we may replace in (46) $n_{l'}$ by any element in its $\mathcal{H}$-class. Indeed, if

$$\eta = (r <_{\mathcal{L}} n_{l'-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d <_{\mathcal{L}} m_{d-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$$

with $r \;\mathcal{H}\; n_{l'}$, then $(\rho, \eta) \in V(M^I)$ by Lemma 8.4 (case (V1)) and so $H(\rho, \eta) = 2h_{\mathcal{J}}(n_{l'})+1 = 2k'+1$ yields $[q-1, \rho] = [q-1, \eta]$.

Thus we may assume by Lemma 8.13 that either

$$v_q = [2k'+2, \rho] \qquad \text{or} \qquad v_q = [2k'+2, \rho']$$

with

$$\rho' = (n_{l'+1} <_{\mathcal{L}} n_{l'} <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d <_{\mathcal{L}} m_{d-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

Write

$$\epsilon(\rho) = (x_{l'} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0), \quad \epsilon(\rho\sigma) = (x'_s <_{\mathcal{J}} \ldots <_{\mathcal{J}} x'_0).$$

Assume first that $v_q = [2k'+2, \rho]$. Writing $n_{l'} = zn_{l'-1}$, it follows from $l' > d$, (48) and (47) that

$$x'_s = (n_{l'}m'_p)(n_{l'-1}m'_p)^* = zn_{l'-1}m'_p(n_{l'-1}m'_p)^* = zn_{l'-1}n^*_{l'-1} = n_{l'}n^*_{l'-1} = x_{l'},$$

hence $f(v_q\sigma) = (g, *) = f(v_q)$ for the same $g \in G_{m_{l'}}$. Assume now that $v_q = [2k'+2, \rho']$. Since ϵ is sequential, we may write

$$\epsilon(\rho') = (x_{l'+1} <_{\mathcal{J}} x_{l'} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0)$$

for some $x_{l'+1} \in M$. Since $n_{l'} \;\mathcal{R}\; n_{l'}m'_p$ by (47), it follows from Theorem 7.5 that

$$\epsilon(\rho'\sigma) = (x_{l'+1} <_{\mathcal{J}} x'_s <_{\mathcal{J}} \ldots <_{\mathcal{J}} x'_0).$$

Since $x'_s = x_{l'}$ as before, it follows that $f(v_q\sigma) = (g, b) = f(v_q\sigma)$ for the same $g \in G_{m_{l'}}$ and $b \in Q_{m_{l'}m^*_{l'}}$. Therefore (50) holds in this case.

The case $n_{l'} \in U_3(k')$ being actually a simplification of the preceding case, we may assume now that $n_{l'-1} \in U_4(k')$. Since $h_{\mathcal{J}}(n_{l'-1}) = k'$, we may replace in (46) $n_{l'-1}$ by any element in its $\mathcal{H}$-class. Indeed, if

$$\eta = (n_{l'} <_{\mathcal{L}} r <_{\mathcal{L}} n_{l'-2} <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_l <_{\mathcal{L}} m_{l-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$$

with $r \;\mathcal{H}\; n_{l'}$, then $(\rho, \eta) \in V(M^I)$ by Lemma 8.4 (case (V2)) and so $H(\rho, \eta) = 2h_{\mathcal{J}}(n_{l'})+1 = 2k'+1$ yields $[q-1, \rho] = [q-1, \eta]$.

Thus we may assume by Lemma 8.13 that $v_q = [2k'+2, \rho']$ with

$$\rho' = (r <_{\mathcal{L}} n_{l'-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d <_{\mathcal{L}} m_{d-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0).$$

Let $\rho'' = (n_{l'-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d <_{\mathcal{L}} m_{d-1} <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0)$ and

$$\epsilon(\rho'') = (x_{l'-1} <_{\mathcal{J}} \ldots <_{\mathcal{J}} x_0), \quad \epsilon(\rho''\sigma) = (x'_s <_{\mathcal{J}} \ldots <_{\mathcal{J}} x'_0).$$

Since $h_{\mathcal{J}}(n_{l'-1}) = k' > k = h_{\mathcal{J}}(n_d)$, we have $l'-1 > d$.

Similarly to the preceding case, we have

$$n_j m'_p \,\mathcal{R}\, n_j \quad (j = d, \ldots, l'-1),$$

$$\rho''\sigma = (n_{l'-1}m'_p <_{\mathcal{L}} n_{l'-2}m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d m'_p <_{\mathcal{L}} \ldots).$$

and

$$\rho'\sigma = (rm'_p <_{\mathcal{L}} n_{l'-1}m'_p <_{\mathcal{L}} n_{l'-2}m'_p <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_d m'_p <_{\mathcal{L}} \ldots).$$

Now we get $x'_s = x_{l'-1}$ as in the preceding case. Since ϵ is sequential, we now repeat the argument of the preceding case to reach (50) as well. Therefore (iii) is proved.

The final claim follows from Lemma 8.15. □

We can show that, by computing the length function naturally associated by Proposition 4.9 to the wreath product in Theorem 8.21, we recover the original length function H_Y. We need a further lemma.

Lemma 8.22 *For all $\sigma, \tau, \rho \in Rh(M^I)$, $H(\rho\sigma, \rho\tau) \geq H(\sigma, \tau)$.*

Proof. Let $\sigma, \tau, \rho \in \mathrm{Rh}(M^I)$ and assume that $\sigma \neq \tau$. By Lemma 8.2(ii), we have $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \leq_{\mathcal{L}} (\sigma \wedge_{\mathcal{L}} \tau)$. If $(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) <_{\mathcal{L}} (\sigma \wedge_{\mathcal{L}} \tau)$, then

$$H(\rho\sigma, \rho\tau) \geq H'(\rho\sigma, \rho\tau) > H'(\sigma, \tau)$$

yields $H(\rho\sigma, \rho\tau) \geq H(\sigma, \tau)$. Hence we may assume that

$$(\rho\sigma \wedge_{\mathcal{L}} \rho\tau) \;\mathcal{L}\; (\sigma \wedge_{\mathcal{L}} \tau) \tag{51}$$

It suffices to show that

$$(\sigma, \tau) \in V(M^I) \Rightarrow (\rho\sigma, \rho\tau) \in V(M^I). \tag{52}$$

Indeed, let $\mu \in \mathrm{Rh}(M^I)$ and assume that $(\mu\rho\sigma \wedge_{\mathcal{L}} \mu\rho\tau) \;\mathcal{L}\; (\rho\sigma \wedge_{\mathcal{L}} \rho\tau)$. Then $(\mu\rho\sigma \wedge_{\mathcal{L}} \mu\rho\tau) \;\mathcal{L}\; (\sigma \wedge_{\mathcal{L}} \tau)$ by (51). Since $(\sigma, \tau) \in V(M^I)$, it follows that $(\mu\rho\sigma \wedge_{\mathcal{L}} \mu\rho\tau) \;\mathcal{R}\; (\mu\rho\tau \wedge_{\mathcal{L}} \mu\rho\sigma)$ and so $(\rho\sigma, \rho\tau) \in V(M^I)$. Thus (52) holds and so does the lemma. □

Corollary 8.23 *Let $D : \Pi_{i=1}^{\delta}(X_i, M_i) \times \Pi_{i=1}^{\delta}(X_i, M_i) \to \overline{\mathbb{N}}$ be the length function defined by*

$$D(\mu, \nu) = sup\{j \mid \mu|_{X_j \times \ldots \times X_1} = \nu|_{X_j \times \ldots \times X_1}\}.$$

Then $D(\varphi_\sigma, \varphi_\tau) = H_Y(\sigma, \tau)$ for all $\sigma, \tau \in Rh_Y(M^I)$.

Proof. Write $\widehat{X_j} = X_j \times \ldots \times X_1$. Note that

$$\widehat{X_j} = (\widehat{X_j} \cap \mathrm{im}\psi) \cup (\widehat{X_j} \setminus \mathrm{im}\psi). \tag{53}$$

We show by induction on j that

$$\varphi_\sigma|_{\widehat{X_j}} = \varphi_\tau|_{\widehat{X_j}} \Leftrightarrow \varphi_\sigma|_{\widehat{X_j} \cap \mathrm{im}\psi} = \varphi_\tau|_{\widehat{X_j} \cap \mathrm{im}\psi} \tag{54}$$

holds for all $\sigma, \tau \in \mathrm{Rh}_Y(M^I)$ and $j \in \mathbb{N}$. The case $j = 0$ being trivial, assume that $j > 0$ and (54) holds for $j-1$. Let $x = (x_j, \ldots, x_1) \in \widehat{X_j} \setminus \mathrm{im}\psi$ and assume that

$$\varphi_\sigma|_{\widehat{X_j} \cap \mathrm{im}\psi} = \varphi_\tau|_{\widehat{X_j} \cap \mathrm{im}\psi}. \tag{55}$$

We must show that either $x \notin \mathrm{dom}\varphi_\sigma \cup \mathrm{dom}\varphi_\tau$ or else $x\varphi_\sigma = x\varphi_\tau$.

Suppose first that $x \in \mathrm{dom}\varphi_\sigma \setminus \mathrm{im}\psi$. Then $(x_{j-1}, \ldots, x_1) = (f(v_{j-1}), \ldots, f(v_1))$ for some $(v_{j-1}, \ldots, v_1, r_0) \in \mathrm{Ray}(r_0, T)$ such that $|(\mathrm{Sons}(v_{j-1}))\sigma| > 1$. Then $(f(v_{j-1}), \ldots, f(v_1)) \in \mathrm{im}\psi$ and since $\varphi_\sigma, \varphi_\tau$ are sequential, (55) yields $\varphi_\sigma|_{\widehat{X_{j-1}} \cap \mathrm{im}\psi} = \varphi_\tau|_{\widehat{X_{j-1}} \cap \mathrm{im}\psi}$ and $(\mathrm{Sons}(v_{j-1}))\sigma = (\mathrm{Sons}(v_{j-1}))\tau$. Hence $|(\mathrm{Sons}(v_{j-1}))\tau| = |(\mathrm{Sons}(v_{j-1}))\sigma| > 1$ and

$$x\varphi_\tau = (x_j, (x_{j-1}, \ldots, x_1)\varphi_\tau) = (x_j, (x_{j-1}, \ldots, x_1)\varphi_\sigma) = x\varphi_\sigma.$$

The case $x \in \mathrm{im}\psi$ follows directly from (55). By symmetry, we get $\varphi_\sigma|_{\widehat{X_j}} = \varphi_\tau|_{\widehat{X_j}}$. Thus (54) holds.

Now it suffices to show that

$$\varphi_\sigma|_{\widehat{X_j}} = \varphi_\tau|_{\widehat{X_j}} \Leftrightarrow H_Y(\sigma, \tau) \geq j. \tag{56}$$

Indeed, $\varphi_\sigma|_{\widehat{X_j} \cap \mathrm{im}\psi} = \varphi_\tau|_{\widehat{X_j} \cap \mathrm{im}\psi}$ if and only if $(v_j, \ldots, v_1, r_0)\theta_\sigma\psi = (v_j, \ldots, v_1, r_0)\theta_\tau\psi$ for every $(v_j, \ldots, v_1, r_0) \in \mathrm{Ray}(r_0, T)$. Since ψ is one-to-one, this is equivalent to

$$\forall (v_j, \ldots, v_1, r_0) \in \mathrm{Ray}(r_0, T) \qquad (v_j, \ldots, v_1, r_0)\theta_\sigma = (v_j, \ldots, v_1, r_0)\theta_\tau. \tag{57}$$

The vertices of T with depth j are precisely those of the form $[j, \rho]$ with $\rho \in \mathrm{Rh}_Y(M^I)$. Since θ_σ and θ_τ are sequential, (57) is equivalent to

$$\forall \rho \in \mathrm{Rh}_Y(M^I)\ [j, \rho]\sigma = [j, \rho]\tau$$

and so to

$$\forall \rho \in \mathrm{Rh}_Y(M^I)\ H_Y(\rho\sigma, \rho\tau) \geq j.$$

By Lemma 8.22, the latter is equivalent to $H_Y(\sigma, \tau) \geq j$ and so (56) holds as required. □

We present now some further corollaries of Theorem 8.21.

Corollary 8.24 *Let M be a Y-semigroup and let $\delta = sup\{h_{\mathcal{J}}(u) \mid u \in \Phi_{3,Y}(M)\} \in \overline{\mathbb{N}}$. Then there exists an embedding φ of $Rh((\Phi_{3,Y}(M))^I)$ into an iterated wreath product of full transformation semigroups $\Pi_{i=1}^{\delta}(X_i, M_i) = \ldots \circ (X_2, M_2) \circ (X_1, M_1)$ such that:*

(i) M_{2k+1} is a submonoid of $\{Id_{X_{2k+1}}\} \cup K(X_{2k+1})$ for $2k+1 \leq \delta$.

(ii) M_{2k+2} is a submonoid of $S(X_{2k+2}) \cup K(X_{2k+2})$ for $2k+2 \leq \delta$; if $\{R_\lambda \mid \lambda \in \Lambda\}$ is the set of all $\mathcal{R}$-classes of $\Phi_{3,Y}(M)$ contained in $U_2(k) \cup U_3(k) \cup U_4(k)$, then

$$M_{2k+2} \cap S(X_{2k+2}) \cong \oplus_{\lambda \in \Lambda} G'_\lambda,$$

where G'_λ is a subgroup of G_λ.

(iii) φ has the Zeiger property.

Furthermore, if Y is finite, then the X_i (and consequently the M_i) are all finite, and the canonical morphism $\eta : Rh((\Phi_{3,Y}(M))^I) \to M$ is aperiodic.

Proof. The existence of φ and its properties follow from Proposition 5.2(i) and Theorem 8.21. The aperiodicity of η follows from Propositions 5.1(i) and 5.2(ii) since the composition of aperiodic morphisms is clearly aperiodic. □

Let $G = \langle A \rangle$ be an infinite group generated by $A = A \cup A^{-1}$. The *Cayley graph* $\Gamma(G, A)$ is the directed labeled graph defined by

$V(\Gamma(G, A)) = G;$

$E(\Gamma(G, A)) = \{(g, a, h) \in G \times A \times G \mid ga = h\}.$

The *Munn-Margolis-Meakin expansion* $M_3(G, A)$ (see [13, 21, 23] is defined by

$$M_3(G, A) = \{(\gamma, g);\ \gamma \text{ is a finite connected subgraph of } \Gamma(G, A) \text{ and } 1, g \in \gamma\}.$$

With the binary operation

$$(\gamma, g)(\gamma', g') = (\gamma \cup g\gamma', gg'),$$

$M_3(G, A)$ is a E-unitary inverse A-monoid [13]. Moreover, the morphism

$$\begin{aligned} \alpha : M_3(G, A) &\to G \\ (\gamma, g) &\mapsto g \end{aligned}$$

provides the maximal group homomorphic image of $M_3(G, A)$.

Since a finite graph can have only finitely many subgraphs, it is easy to see that $M_3(G, A)$ is finite $\mathcal{J}$-above as well.

We recall that a semigroup M is *orthodox* if it is regular and the subset $E(M)$ of all idempotents of M constitutes a subsemigroup of M. A monoid M is said to be an *orthodox covering* of a group G if M is orthodox and there exists an onto homomorphism $\varphi : M \to G$ such that $1\varphi^{-1} = E(M)$.

Corollary 8.25 *Let Let $G = \langle A \rangle$ be an infinite group. Then $Rh_A(M_3(G, A))$ is an orthodox covering of G and there exists an embedding φ of $Rh_A(M_3(G, A))$ into an iterated wreath product of full transformation semigroups $\Pi_{i=1}^{\infty}(X_i, M_i) = \ldots \circ (X_2, M_2) \circ (X_1, M_1)$ such that:*

(i) M_i is a finite submonoid of $\{1_{X_i}\} \cup K(X_i)$ for i odd.

(ii) M_i is a finite submonoid of $S(X_i) \cup K(X_i)$ for i even; the local groups are then finite subgroups of G.

(iii) φ has the Zeiger property.

Proof. Note that $(M_3(G, A))\backslash\{(\{1\}, 1)\}$ is an A-semigroup and $(M_3(G, A)) \cong ((M_3(G, A))\backslash \{(\{1\}, 1)\})^I$. Since G is infinite, it follows easily that $M_3(G, A)$ has arbitrarily long $\mathcal{J}$-chains and so $\sup\{h_{\mathcal{J}}(u) \mid u \in M_3(G, A)\} = \omega$. Since $M_3(G, A)$ is finite $\mathcal{J}$-above, the existence of φ and its properties follow from Theorem 8.21 and its proof, since any local group must be the Schützenberger group of some $\mathcal{J}$-class and therefore a (group) $\mathcal{H}$-class since $M_3(G, A)$ is inverse. It follows that such a group must be a finite subgroup of G (see [13] for more details).

By Proposition 5.1(ii), $\mathrm{Rh}_A(M_3(G, A))$ is regular. We consider the canonical morphisms $\eta : \mathrm{Rh}_A(M_3(G, A)) \to M_3(G, A)$ and $\alpha : M_3(G, A) \to G$. Clearly, $1\alpha^{-1} = E(M_3(G, A))$. By Proposition 5.1(iii),

$$1(\eta\alpha)^{-1} = 1\alpha^{-1}\eta^{-1} = (E(M_3(G, A)))\eta^{-1} = E(\mathrm{Rh}_A(M_3(G, A)))$$

and so $\mathrm{Rh}_A(M_3(G, A))$ is an orthodox covering of G. $\square$

9 Free Burnside monoids

Given $p, q \geq 1$, let $\mathcal{B}(p,q)$ denote the variety of semigroups defined by the identity $x^{p+q} = x^p$. Given a set X, we denote by $B_X(p,q)$ the free $\mathcal{B}(p,q)$-semigroup on X. Clearly, $B_X(p,q)$ can be defined by the semigroup presentation

$$\langle X \mid u^{p+q} = u^p \ (u \in X^+)\rangle. \tag{58}$$

We say that $B_X(p,q)$ is a *free Burnside semigroup*. The corresponding *free Burnside monoid* $B_X^I(p,q)$ can be obtained by adjoining an identity to $B_X(p,q)$. For details on $B_X(p,q)$, the reader is referred to [15, 17, 16], [7] and [10].

Lemma 9.1 *For all $p, q \geq 1$, $B_X^I(p,q) \cong Rh_X(B_X^I(p,q))$.*

Proof. Take the canonical surjective morphism $\eta : \mathrm{Rh}_X(B_X^I(p,q)) \to B_X^I(p,q)$. Since $B_X(p,q)$ is presented by (58), it suffices to show that $\sigma^{p+q} = \sigma^p$ for every $\sigma \in \mathrm{Rh}(B_X^I(p,q))$. Let

$$\sigma = (m_l <_{\mathcal{L}} \ldots <_{\mathcal{L}} m_0 = I)$$

and write $\sigma^p = (m_l^p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0)$. Then

$$\sigma^{p+q} = \mathrm{lm}(m_l^{p+q} \leq_{\mathcal{L}} \ldots \leq_{\mathcal{L}} m_l^p <_{\mathcal{L}} n_t <_{\mathcal{L}} \ldots <_{\mathcal{L}} n_0).$$

Since $m_l^{p+q} = m_l^p$, it follows that $\sigma^{p+q} = \sigma^p$ as required. □

Proposition 9.2 *[15] For all $p, q \geq 1$, $B_X(p,q)$ is finite $\mathcal{J}$-above and its maximal subgroups are cyclic.*

Clearly, if $|X| \leq 1$ then $B_X(p,q)$ is finite. From now on, we assume that $|X| > 1$. Then $B_X(p,q)$ has infinite $\mathcal{J}$-chains [15, 16]. Now Theorem 8.21 yields

Theorem 9.3 *Let $p, q \geq 1$ and X be a finite set with $|X| > 1$. Then there exists an embedding φ of $B_X^I(p,q)$ into an iterated wreath product of finite partial transformation semigroups $\Pi_{i=1}^{\infty}(X_i, M_i) = \ldots \circ (X_2, M_2) \circ (X_1, M_1)$ such that:*

(i) M_{2k+1} is a finite submonoid of $\{Id_{X_{2k+1}}\} \cup K(X_{2k+1})$ for every k.

(ii) M_{2k+2} is a finite submonoid of $S(X_{2k+2}) \cup K(X_{2k+2})$ for every k; if $\{R_\lambda \mid \lambda \in \Lambda\}$ is the set of all $\mathcal{R}$-classes of M contained in $U_2(k) \cup U_3(k) \cup U_4(k)$, then

$$M_{2k+2} \cap S(X_{2k+2}) \cong \oplus_{\lambda \in \Lambda} G'_\lambda, \tag{59}$$

where G'_λ is a subgroup of G_λ. Therefore $M_{2k+2} \cap S(X_{2k+2})$ is a finite Abelian group.

(iii) φ has the Zeiger property.

A future paper will apply the results of this paper to elliptic actions of the free Burnside semigroups.

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