Finite automata for Schreier graphs of virtually free groups

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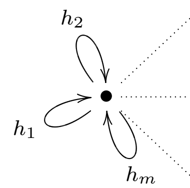
ABSTRACT

The Stallings construction for f.g. subgroups of free groups is generalized by introducing the concept of Stallings section, which allows an efficient computation of the core of a Schreier graph based on edge folding. It is proved that those groups admitting Stallings sections are precisely f.g. virtually free groups, through a constructive approach based on Basse-Serre theory. Complexity issues and applications are also discussed.

1 Introduction

Finite automata became over the years the standard representation of finitely generated subgroups H of a free group F_A. The *Stallings construction* constitutes a simple and efficient algorithm for building an automaton $\mathcal{S}(H)$ which can be used for solving the membership problem of H in F_A and many other applications. This automaton $\mathcal{S}(H)$ is nothing more than the *core automaton* of the Schreier graph (automaton) of H in F_A, whose structure can be described as $\mathcal{S}(H)$ with finitely many infinite trees adjoined. Many features of $\mathcal{S}(H)$ were (re)discovered over the years and were known to Reidemeister, Schreier, and particularly Serre [16]. One of the greatest contributions of Stallings [17] is certainly the algorithm to construct $\mathcal{S}(H)$: taking a finite set of generators $h_1, \ldots, h_m$ of H in reduced form, we start with the so-called flower automaton, where *petals* labelled by the words h_i (and their inverse edges) are glued to a basepoint q_0:



Then we proceed by successively folding pairs of edges of the form $q \xleftarrow{a} p \xrightarrow{a} r$ until no more folding is possible (so we get an inverse automaton). And we will have just built $\mathcal{S}(H)$. For details and applications of the Stallings construction, see [1, 7, 13].

Since $\mathcal{S}(H)$ turns out to be the core of the Schreier graph of $H \leq F_A$, this construction is independent of the finite set of generators of H chosen at the beginning, and of the particular sequence of foldings followed. And the membership problem follows from the fact that $\mathcal{S}(H)$ recognizes all the reduced words representing elements of H... and the reduced words constitute a section for any free group.

Such an approach invites naturally generalizations for further classes of groups. For instance, an elegant geometric construction of Stallings type automata was achieved for amalgams of finite groups by Markus-Epstein [12]. On the other hand, the most general results were obtained by Kapovich, Weidmann and Miasnikov [8] for finite graphs of groups where each vertex group is either polycyclic-by-finite or word-hyperbolic and locally quasi-convex, and where all edge groups are virtually polycyclic. However, the complex algorithms were designed essentially to solve the generalized word problem, and it seems very hard to extend other features of the free group case, either geometric or algorithmic. Our goal in the present paper is precisely to develop a Stallings type approach with some generality which is robust enough to exhibit several prized algorithmic and geometric features, namely in connection with Schreier graphs. Moreover, we succeed on identifying those groups G for which it can be carried on: (finitely generated) virtually free groups.

Which ingredients shall we need to get a Stallings type algorithm? First of all, we need a section S with good properties that may emulate the role played by the reduced words in the free group. In particular, we need a rational language (i.e. recognizable by a finite automaton). We may of course need to be more restrictive than taking all reduced words, if we want our finite automaton to recognize all the representatives of $H \leq_{f.g.} G$ in S. To get inverse automata, it is also convenient to have $S = S^{-1}$

Second, the set S_g of words of S representing a certain $g \in G$ must be at least rational, so we can get a finite automaton to represent each of the generalized petals.

Third, the folding process to be performed in the (generalized) flower automaton (complemented possibly by other identification operations) must ensure in the end that all representatives of elements of H in S are recognized by the automaton. And folding is the automata-theoretic translation of the reduction process $w \to \overline{w}$ taking place in the free group. So we need the condition $S_{g_1 g_2} \subseteq \overline{S_{g_1} S_{g_2}}$, to make sure that the petals (corresponding to the generators of H) carry enough information to produce, after the subsequent folding, all the representatives of elements of H. And this is how we were led to our definition of *Stallings section*.

It is somewhat surprising how much we can get from this concept, that turned out to be more robust than one would expect. Among other features, we can mention independence from the generating set (so we can have Stallings automata for free groups when we consider a non canonical generating set!), or a generalized version of the classical Benois Theorem. We present some applications of the whole theory, believing that many others should follow in due time, as it happened in the free group case.

The paper is structured as follows. In Section 2 we present the required basic concepts. The theory of Stallings sections is presented in Section 3. In Section 4, we discuss the complexity of the generalized Stallings construction in its most favourable version. In Sections 5 and 6 we show that existence of a Stallings section is inherited through free products with amalgamation over finite groups and HNN extensions over finite groups, respectively. In Section 7, we prove that those groups admitting a Stallings section are precisely the finitely generated virtually free groups. In Section 8, we show that we can assume stronger properties for Stallings sections with an eye to applications, namely the characterization of finite index subgroups. Finally, we present some examples in Section 9.

2 Preliminaries

Given a finite alphabet A, we denote by A^* the *free monoid on* A, with 1 denoting the empty word. A subset of a free monoid is called a *language*.

We say that $\mathcal{A} = (Q, q_0, T, E)$ is a (finite) A-automaton if:

- Q is a (finite) set;
- $q_0 \in Q$ and $T \subseteq Q$;
- $E \subseteq Q \times A \times Q$.

A *nontrivial path* in $\mathcal{A}$ is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} p_n$$

with $(p_{i-1}, a_i, p_i) \in E$ for $i = 1, \ldots, n$. Its *label* is the word $a_1 \cdots a_n \in A^+ = A^* \setminus \{1\}$. It is said to be a *successful* path if $p_0 = q_0$ and $p_n \in T$. We consider also the *trivial path* $p \xrightarrow{1} p$ for $p \in Q$. It is successful if $p = q_0 \in T$. The *language* $L(\mathcal{A})$ *recognized by* $\mathcal{A}$ is the set of all labels of successful paths in $\mathcal{A}$. A path of minimal length between two vertices is called a *geodesic*, and so does its label by extension.

The automaton $\mathcal{A} = (Q, q_0, T, E)$ is said to be *deterministic* if, for all $p \in Q$ and $a \in A$, there is at most one edge of the form (p, a, q). We say that $\mathcal{A}$ is *trim* if every $q \in Q$ lies in some successful path.

Given deterministic A-automata $\mathcal{A} = (Q, q_0, T, E)$ and $\mathcal{A}' = (Q', q'_0, T', E')$, a morphism $\varphi : \mathcal{A} \to \mathcal{A}'$ is a mapping $\varphi : Q \to Q'$ such that

- $q_0\varphi = q'_0$ and $T\varphi \subseteq T'$;
- $(p\varphi, a, q\varphi) \in E'$ for every $(p, a, q) \in E$.

It follows that $L(\mathcal{A}) \subseteq L(\mathcal{A}')$ if there is a morphism $\varphi : \mathcal{A} \to \mathcal{A}'$. The morphism $\varphi : \mathcal{A} \to \mathcal{A}'$ is:

- *injective* if it is injective as a mapping $\varphi : Q \to Q'$;
- an *isomorphism* if it is injective, $T' = T\varphi$ and every edge of E' is of the form $(p\varphi, a, q\varphi)$ for some $(p, a, q) \in E$.

The *star* operator on A-languages is defined by

$$L^* = \bigcup_{n \geq 0} L^n,$$

where $L^0 = \{1\}$. A language $L \subseteq A^*$ is said to be *rational* if L can be obtained from finite languages using finitely many times the operators union, product and star (admits a *rational expression*). Alternatively, L is rational if and only if it is recognized by a finite (deterministic) A-automaton $\mathcal{A} = (Q, q_0, T, E)$ [3, Section III]. The definition generalizes to subsets of an arbitrary monoid in the obvious way.

We denote the set of all rational languages $L \subseteq A^*$ by $\mathrm{Rat}\, A^*$. Note that $\mathrm{Rat}\, A^*$, endowed with the product of languages, constitutes a monoid.

In the statement of a result, we shall say that a rational language L is *effectively constructible* if there exists an algorithm to produce from the data implicit in the statement a finite A-automaton $\mathcal{A}$ recognizing L.

It is convenient to summarize some closure and decidability properties of rational languages in the following result (see [3] e.g.). The prefix set of a language $L \subseteq A^*$ is defined as

$$\mathrm{Pref}(L) = \{u \in A^* \mid uA^* \cap L \neq \emptyset\}.$$

A *rational substitution* is a morphism $\varphi : A^* \to \mathrm{Rat}\, B^*$ (where $\mathrm{Rat}\, B^*$ is endowed with the product of languages). Given $K \subseteq A^*$, we denote by $K\varphi$ the language $\cup_{u \in K} u\varphi \subseteq B^*$. Since singletons are rational languages, monoid homomorphisms constitute particular cases of rational substitutions.

Proposition 2.1 *Let A be a finite alphabet and let $K, L \subseteq A^*$ be rational. Then:*

(i) $K \cup L, K \cap L, A^* \setminus L, \mathit{Pref}(L)$ *are rational;*

(ii) if $\varphi : A^* \to \mathit{Rat}\, B^*$ *is a rational substitution, then* $K\varphi$ *is rational;*

(iii) if $\varphi : A^* \to M$ *is a monoid homomorphism and* M *is finite, then* $X\varphi^{-1}$ *is rational for every* $X \subseteq M$.

Moreover, all the constructions are effective, and the inclusion $K \subseteq L$ *is decidable.*

Given an A-automaton $\mathcal{A}$ and $L \subseteq A^*$, we denote by $\mathcal{A} \sqcap L$ the A-automaton obtained by removing from $\mathcal{A}$ all the vertices and edges which do not lie in some successful path labelled by a word in L.

Proposition 2.2 *Let $\mathcal{A}$ be a finite A-automaton and let $L \subseteq A^*$ be a rational language. Then $\mathcal{A} \sqcap L$ is effectively constructible.*

Proof. Write $\mathcal{A} = (Q, q_0, T, E)$ and let $\mathcal{A}' = (Q', q_0', T', E')$ be a finite A-automaton recognizing L. The *direct product*

$$\mathcal{A}'' = (Q \times Q', (q_0, q_0'), T \times T', E'')$$

is defined by

$$E'' = \{((p,p'), a, (q,q')) \mid (p,a,q) \in E,\ (p',a,q') \in E'\}.$$

Let $\mathcal{B}$ denote the *trim part* of $\mathcal{A}''$ (by removing all vertices/edges which are not part of successful paths in $\mathcal{A}''$; this can be done effectively). Then $\mathcal{A} \sqcap L$ can be obtained by projecting into the first component the various constituents of $\mathcal{B}$. $\square$

Given an alphabet A, we denote by A^{-1} a set of *formal inverses* of A, and write $\widetilde{A} = A \cup A^{-1}$. We say that $\widetilde{A}$ is an *involutive alphabet*. We extend ${}^{-1} : A \to A^{-1} : a \mapsto a^{-1}$ to an involution on $\widetilde{A}^*$ through

$$(a^{-1})^{-1} = a, \quad (uv)^{-1} = v^{-1}u^{-1} \quad (a \in A,\ u, v \in \widetilde{A}^*)\,.$$

An automaton $\mathcal{A}$ over an involutive alphabet $\widetilde{A}$ is *involutive* if, whenever (p,a,q) is an edge of $\mathcal{A}$, so is (q, a^{-1}, p). Therefore it suffices to depict just the *positively labelled* edges (having label in A) in their graphical representation.

An involutive automaton is *inverse* if it is deterministic, trim and has a single final state (note that for involutive automata, being trim is equivalent to being connected). If the latter happens to be the initial state, it is called the *basepoint*.

The next result is folklore. For a proof, see [1, Proposition 2.2].

Proposition 2.3 *Given inverse automata $\mathcal{A}$ and $\mathcal{A}'$, then $L(\mathcal{A}) \subseteq L(\mathcal{A}')$ if and only if there exists a morphism $\varphi : \mathcal{A} \to \mathcal{A}'$. Moreover, such a morphism is unique.*

Given an alphabet A, let $\sim$ denote the congruence on $\widetilde{A}^*$ generated by the relation

$$\{(aa^{-1}, 1) \mid a \in \widetilde{A}\}\,. \tag{1}$$

The quotient $F_A = \widetilde{A}^*/\!\sim$ is the *free group on* A. We denote by $\theta : \widetilde{A}^* \to F_A$ the canonical morphism $u \mapsto [u]_\sim$.

Alternatively, we can view (1) as a *confluent* length-reducing rewriting system on $\widetilde{A}^*$, where each word $w \in \widetilde{A}^*$ can be transformed into a unique *reduced* word $\overline{w}$ with no factor of the form aa^{-1}. As a consequence, the equivalence

$$u \sim v \quad \Leftrightarrow \quad \overline{u} = \overline{v} \qquad\qquad (u, v \in \widetilde{A}^*)$$

solves the word problem for F_A. We shall use the notation $R_A = \overline{\widetilde{A}^*}$.

We close this section with the classical Benois Theorem, which relates rational languages with free group reduction:

Theorem 2.4 *[2] If $L \subseteq \widetilde{A}^*$ is rational, then $\overline{L}$ is an effectively constructible rational language.*

3 Stallings sections

Let G be a (finitely generated) group generated by the finite set A. More precisely, we consider an epimorphism $\pi : \widetilde{A}^* \to G$ satisfying

$$a^{-1}\pi = (a\pi)^{-1} \quad \text{for every} \quad a \in A. \tag{2}$$

A homomorphism satisfying condition (2) is said to be *matched*. Note that in this case (2) holds for arbitrary words. For short, we shall refer to a matched epimorphism $\pi : \widetilde{A}^* \to G$ (with A finite) as a *m-epi*.

We shall call a language $S \subseteq \widetilde{A}^*$ a *section* (for π) if $S\pi = G$ and $S^{-1} = S$. For every $X \subseteq G$, we write

$$S_X = X\pi^{-1} \cap S.$$

We say that an effectively constructible rational section $S \subseteq R_A$ is a *Stallings section* for π if, for all $g, h \in G$:

(S1) S_g is an effectively constructible rational language;

(S2) $S_{gh} \subseteq \overline{S_g S_h}$.

Note that (S2) yields immediately

$$S_{g_1 \cdots g_n} \subseteq \overline{S_{g_1} \cdots S_{g_n}} \tag{3}$$

for all $g_1, \ldots, g_n \in G$. Moreover, in (S1) it suffices to consider $S_{a\pi}$ for $a \in A$. Indeed, by (3), and since $S^{-1} = S$ and $S_g\pi = g$ for every $g \in G$, we may write

$$S_{(a_1 \cdots a_n)\pi} = \overline{S_{a_1\pi} \cdots S_{a_n\pi}} \cap S$$

and $S_{a_i^{-1}\pi} = S_{a_i\pi}^{-1}$ for all $a_i \in \widetilde{A}$. Then, by Proposition 2.1 and Theorem 2.4, S_g is a rational language for every $g \in G$; furthermore, it is effectively constructible from $S_{a_1\pi}, \ldots, S_{a_n\pi}$.

Note that if S is a Stallings section, then $S \cup \{1\}$ is also a Stallings section. Indeed, it is easy to see that conditions (S1) and (S2) are still verified: namely, if $gh = 1$, then $1 \in \overline{S_g S_g^{-1}} = \overline{S_g S_h}$ and so $S_{gh} \cup \{1\} \subseteq \overline{S_g S_h}$ as required.

The next result shows that the existence of a Stallings section is independent from the finite set A and the m-epi $\pi : \widetilde{A}^* \to G$ considered:

Proposition 3.1 *Let $\pi : \widetilde{A}^* \to G$ and $\pi' : \widetilde{A'}^* \to G$ be m-epis. Then G has a Stallings section for π if and only if G has a Stallings section for π'.*

Proof. Let $S \subseteq R_A$ be a Stallings section for π. There exists a matched homomorphism $\varphi : \widetilde{A}^* \to \widetilde{A'}^*$ such that $\varphi\pi' = \pi$. Write $S' = \overline{S\varphi}$. By Proposition 2.1(ii) and Theorem 2.4, S' is an effectively constructible rational subset of $R_{A'}$. We claim that

$$S'_g = \overline{S_g\varphi} \tag{4}$$

holds for every $g \in G$.

Indeed, let $u \in S'_g$. Then $u = \overline{v\varphi}$ for some $v \in S$ and $v\pi = v\varphi\pi' = \overline{v\varphi}\pi' = u\pi' = g$. Hence $v \in S_g$ and so $S'_g \subseteq \overline{S_g\varphi}$.

Conversely, let $v \in S_g$. Then $\overline{v\varphi} \in \overline{S\varphi} = S'$ and $\overline{v\varphi}\pi' = v\varphi\pi' = v\pi = g$, hence $\overline{v\varphi} \in S'_g$ and so (4) holds.

Since

$$(S')^{-1} = (\overline{S\varphi})^{-1} = \overline{(S\varphi)^{-1}} = \overline{S^{-1}\varphi} = \overline{S\varphi} = S',$$

it follows from (4) that S' is a section for π'. Moreover, (S1) is inherited by S' from S by Proposition 2.1(ii) and Theorem 2.4. Finally, for all $g, h \in G$, we get

$$\begin{aligned} S'_{gh} &= \overline{S_{gh}\varphi} \subseteq \overline{(\overline{S_g S_h})\varphi} = \overline{(S_g S_h)\varphi} \\ &= \overline{(S_g\varphi)(S_h\varphi)} = \overline{(\overline{S_g\varphi})(\overline{S_h\varphi})} = \overline{S'_g S'_h}, \end{aligned}$$

hence (S2) holds for S' and so S' is a Stallings section for π'. By symmetry, we get the required equivalence. □

Proposition 3.2 *Free groups of finite rank and finite groups have Stallings sections.*

Proof. Let A be a finite set and consider the canonical m-epi $\theta : \widetilde{A}^* \to F_A$. Let $S = R_A = \overline{\widetilde{A}^*}$, which is rational by Theorem 2.4. Since $S_g = \overline{g}$ for every $g \in F_A$, it is immediate that S is a Stallings section for θ.

Assume now that G is finite and $\pi : \widetilde{A}^* \to G$ is a m-epi. We show that $S = R_A$ is a Stallings section for π. For every $g \in G$, we have $S_g = g\pi^{-1} \cap R_A = \overline{g\pi^{-1}}$. Since both $g\pi^{-1}$ and R_A are effectively constructible rational languages, so is their intersection and so (S1) holds. Finally, let $u \in S_{gh}$ and take $v \in S_h$. Then $(uv^{-1})\pi = ghh^{-1} = g$ and so $\overline{uv^{-1}} \in \overline{g\pi^{-1}} = S_g$. Hence $u = \overline{uv^{-1}v} = \overline{\overline{uv^{-1}}v} \in \overline{S_g S_h}$ and (S2) holds as well. Therefore R_A is a Stallings section for π. □

Given a m-epi $\pi : \widetilde{A}^* \to G$ and $H \leqslant G$, we define the *Schreier automaton* $\Gamma(G, H, \pi)$ to be the $\widetilde{A}$-automaton having:

- the right cosets Hg $(g \in G)$ as vertices;
- H as the basepoint;
- edges $Hg \xrightarrow{a} Hg(a\pi)$ for all $g \in G$ and $a \in \widetilde{A}$.

It is immediate that $\Gamma(G, H, \pi)$ is always an inverse $\widetilde{A}$-automaton, but it is infinite unless H has finite index in G. Moreover, $L(\Gamma(G, H, \pi)) = H\pi^{-1}$.

We will prove that $\Gamma(G, H, \pi) \sqcap S$ is an effectively constructible finite inverse automaton when S is a Stallings section for π. The following lemmas pave the way for the construction of $\Gamma(G, H, \pi) \sqcap S$:

Lemma 3.3 *Let $\pi : \widetilde{A}^* \to G$ be a m-epi. Let $\mathcal{A}$ be a trim $\widetilde{A}$-automaton and let $p \xrightarrow{a} q$ be an edge of $\mathcal{A}$ for some $a \in \widetilde{A}$. Let $\mathcal{B}$ be obtained by adding the edge $q \xrightarrow{a^{-1}} p$ to $\mathcal{A}$. Then $(L(\mathcal{B}))\pi \subseteq \langle (L(\mathcal{A}))\pi \rangle$.*

Proof. Write $\mathcal{A} = (Q, q_0, T, E)$. We can factor any $u \in L(\mathcal{B})$ as $u = u_0 a^{-1} u_1 \cdots a^{-1} u_n$, where the a^{-1} label each visit to the new edge. We show that $u\pi \in \langle (L(\mathcal{A}))\pi \rangle$ by induction on n. The case $n = 0$ being trivial, assume that $n \geq 1$ and the claim holds for $n-1$. Writing $v = u_0 a^{-1} u_1 \cdots a^{-1} u_{n-1}$, we have a path in $\mathcal{B}$ of the form

$$q_0 \xrightarrow{v} q \xrightarrow{a^{-1}} p \xrightarrow{u_n} t \in T.$$

Since $\mathcal{A}$ is trim, we have also a path

$$q_0 \xrightarrow{w} p \xrightarrow{a} q \xrightarrow{z} t' \in T$$

in $\mathcal{A}$. By the induction hypothesis, we get $(vz)\pi \in \langle (L(\mathcal{A}))\pi \rangle$ and so

$$u\pi = (va^{-1}u_n)\pi = ((vz)(z^{-1}a^{-1}w^{-1})(wu_n))\pi \in \langle (L(\mathcal{A}))\pi \rangle$$

as claimed. $\square$

Lemma 3.4 *Let $\pi : \widetilde{A}^* \to G$ be a m-epi. Let $\mathcal{A} = (Q, q_0, T, E)$ be a trim $\widetilde{A}$-automaton and let $\mathcal{B}$ be obtained by identifying q_0 with some $t \in T$. Then $(L(\mathcal{B}))\pi \subseteq \langle (L(\mathcal{A}))\pi \rangle$.*

Proof. Let $u \in L(\mathcal{B})$. We can factor it as $u = u_1 \cdots u_n$, where $p_i \xrightarrow{u_i} q_i$ is a path in $\mathcal{A}$ with $p_i, q_i \in \{q_0, t\}$ $(i = 1, \ldots, n)$. In any case, there exist paths

$$q_0 \xrightarrow{v_i} p_i, \quad q_i \xrightarrow{w_i} t \in T$$

in $\mathcal{A}$ with $v_i, w_i \in L(\mathcal{A}) \cup \{1\}$. Since $v_i u_i w_i \in L(\mathcal{A})$, we get $u_i\pi = (v_i^{-1}(v_i u_i w_i) w_i^{-1})\pi \in \langle (L(\mathcal{A}))\pi \rangle$ for every i and so $u\pi \in \langle (L(\mathcal{A}))\pi \rangle$ as well. $\square$

Lemma 3.5 *Let $\pi : \widetilde{A}^* \to G$ be a m-epi. Let $\mathcal{A}$ be an involutive $\widetilde{A}$-automaton and let $p \xrightarrow{w} q$ be a path in $\mathcal{A}$ with $w\pi = 1$. Let $\mathcal{B}$ be obtained by identifying the vertices p and q. Then $L(\mathcal{A}) \subseteq L(\mathcal{B})$ and $(L(\mathcal{B}))\pi = (L(\mathcal{A}))\pi$.*

Proof. The first inclusion is clear. Since $\mathcal{A}$ is involutive, we have also a path $q \xrightarrow{w^{-1}} p$ in $\mathcal{A}$ and $w^{-1}\pi = 1$. Clearly, every $u \in L(\mathcal{B})$ can be lifted to some $v \in L(\mathcal{A})$ by inserting finitely many occurrences of the words w, w^{-1}, that is, we can get factorizations

$$u = u_0 u_1 \cdots u_n \in L(\mathcal{B}), \quad v = u_0 w^{\varepsilon_1} u_1 \cdots w^{\varepsilon_n} u_n \in L(\mathcal{A})$$

with $\varepsilon_1, \ldots, \varepsilon_n \in \{-1,\ 1\}$. Since $u\pi = v\pi$, it follows that $(L(\mathcal{B}))\pi \subseteq (L(\mathcal{A}))\pi$. The opposite inclusion holds trivially. $\square$

Since $(aa^{-1})\pi = 1$ for every $a \in \widetilde{A}$, this same argument proves that:

Lemma 3.6 *Let $\pi : \widetilde{A}^* \to G$ be a m-epi. Let $\mathcal{A}$ be a finite involutive $\widetilde{A}$-automaton and let $\mathcal{B}$ be obtained by successively folding pairs of edges in $\mathcal{A}$. Then $L(\mathcal{A}) \subseteq L(\mathcal{B})$ and $(L(\mathcal{B}))\pi = (L(\mathcal{A}))\pi$.*

The next lemma reveals how the automaton $\Gamma(G, H, \pi) \sqcap S$ can be recognized.

Lemma 3.7 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $H \leqslant_{f.g.} G$. Let $\mathcal{A}$ be a finite inverse $\widetilde{A}$-automaton with a basepoint such that*

$$S_H \subseteq L(\mathcal{A}) \subseteq H\pi^{-1}, \tag{5}$$

$$\text{there is no path } p \xrightarrow{w} q \text{ in } \mathcal{A} \text{ with } p \neq q \text{ and } w\pi = 1. \tag{6}$$

Then $\Gamma(G, H, \pi) \sqcap S \cong \mathcal{A} \sqcap S$.

Proof. Since $\mathcal{A}$ and $\Gamma = \Gamma(G, H, \pi)$ are both inverse automata with a basepoint, and $L(\mathcal{A}) \subseteq H\pi^{-1} = L(\Gamma)$, it follows from Proposition 2.3 that there exists a morphism $\varphi : \mathcal{A} \to \Gamma$. Suppose that $p\varphi = q\varphi$ for some vertices p, q in $\mathcal{A}$. Take geodesics

$$q_0 \xrightarrow{u} p, \quad q_0 \xrightarrow{v} q$$

in $\mathcal{A}$, where q_0 denotes the basepoint. Since $p\varphi = q\varphi$, we have $uv^{-1} \in L(\Gamma) = H\pi^{-1}$. Let $s_0 \in S_{(uv^{-1})\pi} \subseteq S_H$. Then $s_0 \in L(\mathcal{A})$ by (5) and so there is a path $p \xrightarrow{u^{-1}s_0v} q$ in $\mathcal{A}$. Since $(u^{-1}s_0v)\pi = (u^{-1}uv^{-1}v)\pi = 1$, it follows from (6) that $p = q$. Thus φ is injective.

It is immediate that φ restricts to an injective morphism $\varphi' : \mathcal{A} \sqcap S \to \Gamma \sqcap S$. It remains to show that every edge of $\Gamma \sqcap S$ is induced by some edge of $\mathcal{A} \sqcap S$. Assume that $H \xrightarrow{s} H$ is a (successful) path in Γ with $s \in S$. By (5), we have $s \in L(\mathcal{A})$ and the path $q_0 \xrightarrow{s} q_0$ is mapped by φ' onto $H \xrightarrow{s} H$. Since every edge of $\Gamma \sqcap S$ occurs in some path $H \xrightarrow{s} H$, it follows that φ' is an isomorphism. $\square$

Lemma 3.8 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $H \leqslant_{f.g.} G$. Let $\mathcal{A}$ be a finite inverse $\widetilde{A}$-automaton with a basepoint such that $S_H \subseteq L(\mathcal{A}) \subseteq H\pi^{-1}$. It is decidable, given two distinct vertices p, q of $\mathcal{A}$, whether or not there is some path $p \xrightarrow{w} q$ in $\mathcal{A}$ with $w\pi = 1$.*

Proof. Let p, q be distinct vertices of $\mathcal{A}$ and let q_0 denote its basepoint. Take geodesics $q_0 \xrightarrow{u} p$ and $q_0 \xrightarrow{v} q$, and let $s \in S_{(uv^{-1})\pi}$. We claim that there is a path $p \xrightarrow{w} q$ in $\mathcal{A}$ with $w\pi = 1$ if and only if $s \in L(\mathcal{A})$.

Indeed, assume that $p \xrightarrow{w} q$ is such a path. Then $uwv^{-1} \in L(\mathcal{A})$ and so $s\pi = (uv^{-1})\pi = (uwv^{-1})\pi \in H$. Thus $s \in S_H \subseteq L(\mathcal{A})$.

Conversely, assume that $s \in L(\mathcal{A})$. Then there is a path $p \xrightarrow{u^{-1}sv} q$ in $\mathcal{A}$. Since $(u^{-1}sv)\pi = (u^{-1}uv^{-1}v)\pi = 1$, the lemma is proved. $\square$

Theorem 3.9 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $H \leqslant_{f.g.} G$. Then $\Gamma(G, H, \pi) \sqcap S$ is an effectively constructible finite inverse $\widetilde{A}$-automaton with a basepoint such that*

$$S_H \subseteq L(\Gamma(G, H, \pi) \sqcap S) \subseteq H\pi^{-1}. \tag{7}$$

Proof. Assume that $H = \langle h_1, \ldots, h_m \rangle$. For $i = 1, \ldots, m$, let $\mathcal{A}_i = (Q_i, q_i, t_i, E_i)$ be a finite trim $\widetilde{A}$-automaton with a single initial and a single terminal vertex satisfying

$$S_{h_i} \subseteq \overline{L(\mathcal{A}_i)} \subseteq h_i\pi^{-1} \tag{8}$$

(in the next section we shall discuss how to define such an automaton with the lowest possible complexity). Let $\mathcal{B}_0$ be the $\widetilde{A}$-automaton obtained by taking the disjoint union of the $\mathcal{A}_i$ and then identifying all the q_i into a single initial vertex q_0.

Suppose that $q_i \xrightarrow{u} q_i$ is a path in $\mathcal{A}_i$. Take $v \in L(\mathcal{A}_i)$. Then $uv \in L(\mathcal{A}_i) \subseteq h_i\pi^{-1}$ and so $u\pi = (uvv^{-1})\pi = h_ih_i^{-1} = 1$. It follows easily that $(L(\mathcal{B}_0))\pi \subseteq (S_{h_1} \cup \cdots \cup S_{h_m})\pi \subseteq H$.

Let $\mathcal{B}_1$ be the finite trim involutive $\widetilde{A}$-automaton obtained from $\mathcal{B}_0$ by adjoining edges (q, a^{-1}, p) for all edges (p, a, q) in $\mathcal{B}_0$ ($a \in \widetilde{A}$). It follows from Lemma 3.3 that $(L(\mathcal{B}_1))\pi \subseteq \langle (L(\mathcal{B}_0))\pi \rangle \subseteq H$.

Next let $\mathcal{B}_2$ be the $\widetilde{A}$-automaton obtained from $\mathcal{B}_1$ by identifying all terminal vertices with the initial vertex q_0. By Lemma 3.4, we get $(L(\mathcal{B}_2))\pi \subseteq \langle (L(\mathcal{B}_1))\pi\rangle \subseteq H$.

Finally, let $\mathcal{B}_3$ be the finite inverse $\widetilde{A}$-automaton with a basepoint obtained by complete folding of $\mathcal{B}_2$. By Lemma 3.6, we have $(L(\mathcal{B}_3))\pi = (L(\mathcal{B}_2))\pi \subseteq H$ and so $L(\mathcal{B}_3) \subseteq H\pi^{-1}$. Moreover,

$$S_{h_1} \cup \cdots \cup S_{h_m} \subseteq \overline{L(\mathcal{A}_1)} \cup \cdots \cup \overline{L(\mathcal{A}_m)} \subseteq \overline{L(\mathcal{B}_0)} \subseteq \overline{L(\mathcal{B}_3)}$$

and $S^{-1} = S$ yield

$$\overline{(S_{h_1} \cup \cdots \cup S_{h_m} \cup S_{h_1^{-1}} \cup \cdots \cup S_{h_m^{-1}})^*} \subseteq \overline{L(\mathcal{B}_3)}$$

since $\mathcal{B}_3$ is involutive and has a basepoint, and therefore

$$\overline{(S_{h_1} \cup \cdots \cup S_{h_m} \cup S_{h_1^{-1}} \cup \cdots \cup S_{h_m^{-1}})^*} \subseteq L(\mathcal{B}_3)$$

since $\mathcal{B}_3$ is inverse (the language of an inverse automaton is closed under reduction since a word aa^{-1} must label only loops). In view of (3), it follows that $S_h \subseteq L(\mathcal{B}_3)$ for every $h \in H$ and so $S_H \subseteq L(\mathcal{B}_3)$. Therefore (5) holds for $\mathcal{B}_3$.

However, (6) may not hold. Assume that the vertex set Q' of $\mathcal{B}_3$ is totally ordered. By Lemma 3.8, we can decide if that happens, and find all concrete instances

$$J = \{(p, q) \in Q' \times Q' \mid p < q \text{ and there is some path } p \xrightarrow{w} q \text{ in } \mathcal{B}_3 \text{ with } w\pi = 1\}.$$

Let $\mathcal{B}_4$ be the finite inverse $\widetilde{A}$-automaton with a basepoint obtained by identifying all pairs of vertices in J followed by complete folding. Since the existence of a path with label in $1\pi^{-1}$ is preserved through the identification process, it follows from Lemmas 3.5 and 3.6 that $\mathcal{B}_4$ still satisfies (5).

Suppose that there exists a path $p' \xrightarrow{w'} q'$ in $\mathcal{B}_4$ with $p' \neq q'$ and $w'\pi = 1$. We can lift p' and q' to vertices p and q in $\mathcal{B}_3$, respectively. It is straightforward to check that the path $p' \xrightarrow{w'} q'$ can be lifted to a path $p \xrightarrow{w} q$ in $\mathcal{B}_3$ by successively inserting in w' factors of the form:

- aa^{-1} $(a \in \widetilde{A})$ (undoing the folding operations);
- $z \in 1\pi^{-1}$ (undoing the identification arising from $r \xrightarrow{z} s$).

Since $w'\pi = w\pi$, it follows that either $(p, q) \in J$ or $(q, p) \in J$, and so $p' = q'$, a contradiction. Therefore $\mathcal{B}_4$ satisfies (6). Now the theorem follows from Proposition 2.2 and Lemma 3.7. $\square$

We call $\Gamma(G, H, \pi) \sqcap S$ the *Stallings automaton* of H (for a given Stallings section S). Note that $\Gamma(F_A, H, \theta) \sqcap R_A$ is the classical Stallings automaton of $H \leq_{f.g.} F_A$ when we take R_A as Stallings section (for the canonical m-epi θ).

Stallings automata provide a natural decision procedure for the generalized word problem:

Corollary 3.10 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $H \leqslant_{f.g.} G$. Then the following conditions are equivalent for every $g \in G$:*

(i) $g \in H$;

(ii) $S_g \subseteq L(\Gamma(G,H,\pi) \sqcap S)$*;*

(iii) $S_g \cap L(\Gamma(G,H,\pi) \sqcap S) \neq \emptyset$*.*

Therefore the generalized word problem is decidable for G*.*

Proof. (i) $\Rightarrow$ (ii). If $g \in H$, then $S_g \subseteq S_H \subseteq L(\Gamma(G,H,\pi) \sqcap S)$ by Theorem 3.9.

(ii) $\Rightarrow$ (iii). Immediate since $S_g \neq \emptyset$ due to S being a section.

(iii) $\Rightarrow$ (i). Since $S_g \cap L(\Gamma(G,H,\pi) \sqcap S) \subseteq g\pi^{-1} \cap H\pi^{-1}$.

Now decidability follows from (S1) and Theorem 3.9. $\square$

We can also prove the following generalization of Benois Theorem:

Theorem 3.11 *Let* $S \subseteq R_A$ *be a Stallings section for the m-epi* $\pi : \widetilde{A}^* \to G$ *and let* $L \subseteq \widetilde{A}^*$ *be rational. Then* $S_{L\pi}$ *is an effectively constructible rational language.*

Proof. Let $\varphi : \widetilde{A}^* \to \mathrm{Rat}\, \widetilde{A}^*$ be the rational substitution defined by $a\varphi = S_{a\pi}$, for $a \in \widetilde{A}$ (note that $1\varphi = \{1\}$ and, for $u = a_1 \cdots a_n$ $(a_i \in \widetilde{A})$, $u\varphi$ is not $S_{u\pi}$ but just $S_{a_1\pi} \cdots S_{a_n\pi}$). We claim that

$$S_{u\pi} = S \cap \overline{u\varphi} \tag{9}$$

holds for every $u \in L \setminus \{1\}$. Let $u = a_1 \cdots a_n \in L$ $(a_i \in \widetilde{A})$. Then by (3) we get

$$S_{u\pi} = S_{(a_1\pi)\cdots(a_n\pi)} \subseteq \overline{S_{a_1\pi} \cdots S_{a_n\pi}} = \overline{(a_1\varphi) \cdots (a_n\varphi)} = \overline{u\varphi}$$

and so $S_{u\pi} \subseteq S \cap \overline{u\varphi}$.

Since $a\varphi\pi = S_{a\pi}\pi = a\pi$ holds for every $a \in \widetilde{A}$, the inclusion $S \cap \overline{u\varphi} \subseteq S_{u\pi}$ follows from $\overline{u\varphi}\pi = u\varphi\pi = u\pi$. Therefore (9) holds.

Now it becomes clear that

$$S_{L\pi} = S \cap (\cup_{u \in L} \overline{u\varphi}) = S \cap \overline{L\varphi}$$

if $1 \notin L$ and

$$S_{L\pi} = (S \cap \overline{L\varphi}) \cup S_1$$

if $1 \in L$.

Now $L\varphi$ is an effectively constructible rational language by (S1) and Proposition 2.1(ii), and so is $\overline{L\varphi}$ by Theorem 2.4. Since S and S_1 are rational, it follows from Proposition 2.1(i) that $S_{L\pi}$ is rational and effectively constructible. $\square$

A natural question to ask at this stage is if we can identify a Stallings automaton for a given Stallings section S. In the classical case of a free group F_A with $S = R_A$ this is an elementary thing to do: in this case, an $\widetilde{A}$-automaton $\mathcal{A}$ is of the form $\Gamma(F_A, H, \pi) \sqcap R_A = \mathcal{S}_H$ for some $H \leqslant_{f.g.} F_A$ if and only if $\mathcal{A}$ is inverse, has a basepoint, and has no vertex of outdegree one except possibly the basepoint.

Proposition 3.12 *Let* $S \subseteq R_A$ *be a Stallings section for a m-epi* $\pi : \widetilde{A}^* \to G$*. It is decidable, given a finite* $\widetilde{A}$*-automaton* $\mathcal{A}$*, whether or not* $\mathcal{A} \cong \Gamma(G,H,\pi) \sqcap S$ *for some* $H \leqslant_{f.g.} G$*.*

Proof. We may assume that $\mathcal{A}$ is inverse and has a basepoint. Write $\mathcal{A} = (Q, q_0, q_0, E)$. The equality $\mathcal{A} = \mathcal{A} \sqcap S$ is an obvious necessary condition, decidable by Lemma 2.2. Thus we may assume that $\mathcal{A} = \mathcal{A} \sqcap S$ (in particular, $\mathcal{A}$ is trim).

Since $S \subseteq R_A$ and $\mathcal{A}$ is trim, it follows that only the basepoint may have outdegree 1, and so $\mathcal{A} \cong \mathcal{S}(K) \cong \Gamma(F_A, K, \theta) \sqcap R_A$ for some $K \leqslant_{f.g.} F_A$ [1, Proposition 2.12]: the standard algorithm [1, Proposition 2.6] actually computes a finite subset $X \subseteq R_A$ projecting onto a basis $X\theta$ of K. Let $K' = \langle X\pi \rangle \leqslant_{f.g.} G$. We claim that $\mathcal{A} \cong \Gamma(G, H, \pi) \sqcap S$ for some $H \leqslant_{f.g.} G$ if and only if $\mathcal{A} \cong \Gamma(G, K', \pi) \sqcap S$, a decidable condition in view of Theorem 3.9.

The converse implication being trivial, assume that $\mathcal{A} = \Gamma(G, H, \pi) \sqcap S$ for some $H \leqslant_{f.g.} G$. Since words of $1\pi^{-1}$ can only label loops in $\Gamma(G, H, \pi)$, it follows from Lemma 3.7 that we only need to show that

$$S_{K'} \subseteq L(\mathcal{A}) \subseteq K'\pi^{-1}. \tag{10}$$

Since $\mathcal{A} \cong \Gamma(F_A, K, \theta) \sqcap R_A$, it follows from Theorem 3.9 that

$$X \subseteq R_A \cap K\theta^{-1} \subseteq L(\mathcal{A}) \subseteq K\theta^{-1}.$$

Since $K\theta^{-1} \subseteq K'\pi^{-1}$, we get $L(\mathcal{A}) \subseteq K'\pi^{-1}$. Finally, $X \subseteq L(\mathcal{A}) \subseteq H\pi^{-1}$ yields $X\pi \subseteq H$ and so $K' \leqslant H$. Hence

$$S_{K'} \subseteq S_H \subseteq L(\mathcal{A})$$

by (7) and so (10) holds. Thus $\mathcal{A} \cong \Gamma(G, K', \pi) \sqcap S$ and we are done. $\Box$

4 Complexity

In this section we discuss, for a given Stallings section, an efficient way (from the viewpoint of complexity) of constructing the automata $\mathcal{A}_i$ in the proof of Theorem 3.9 and compute an upper bound for the complexity of the construction of the Stallings automata $\Gamma(G, H, \pi) \sqcap S$.

We say that an $\widetilde{A}$-automaton is *uniterminal* if it has a single terminal vertex. It is easy to see that there exist rational languages which fail to be recognized by any uniterminal automaton (e.g. R_A, since regular languages recognizable by uniterminal automata and containing the empty word must have a basepoint and so they are submonoids). However, we can prove the following:

Lemma 4.1 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $g \in G$. Then there exists a finite trim uniterminal $\widetilde{A}$-automaton $\mathcal{C}_g$ satisfying*

$$S_g \subseteq \overline{L(\mathcal{C}_g)} \subseteq g\pi^{-1}.$$

Proof. Let $\mathcal{C} = (Q, i, T, E)$ be the minimum automaton of S_g (or any other finite trim automaton with a single initial vertex recognizing S_g) and let $\mathcal{C}_g$ be obtained by identifying all the terminal vertices of $\mathcal{C}$. Clearly, $\mathcal{C}_g$ is a finite trim uniterminal automaton and $S_g = L(\mathcal{C}) \subseteq L(\mathcal{C}_g)$ yields $S_g = \overline{S_g} \subseteq \overline{L(\mathcal{C}_g)}$. It remains to be proved that $(L(\mathcal{C}_g))\pi = g$.

Let $u \in L(\mathcal{C}_g)$. Then there exists a factorization $u = u_0 u_1 \cdots u_k$ such that

$$i \xrightarrow{u_0} t_0, \quad s_1 \xrightarrow{u_1} t_1, \quad \ldots, \quad s_k \xrightarrow{u_k} t_k$$

are paths in $\mathcal{C}$ with $s_j, t_j \in T$. Take a path $i \xrightarrow{v_j} s_j$ in $\mathcal{C}$, for $j = 1, \ldots, k$. Then $v_j, v_j u_j \in L(\mathcal{C})$ and so $v_j\pi = (v_j u_j)\pi = g$. Hence $u_j\pi = (v_j^{-1} v_j u_j)\pi = g^{-1}g = 1$ and so $u\pi = (u_0 u_1 \cdots u_k)\pi = u_0\pi = g$ since $u_0 \in L(\mathcal{C}) = S_g$. Thus $(L(\mathcal{C}_g))\pi = g$ and so $\overline{L(\mathcal{C}_g)} \subseteq g\pi^{-1}$ as required. $\square$

We introduce next a multiplication of (finite trim) uniterminal automata: given (finite trim) uniterminal $\widetilde{A}$-automata $\mathcal{A} = (Q, i, t, E)$ and $\mathcal{A}' = (Q', i', t', E')$, let $\mathcal{A} * \mathcal{A}' = (Q'', i, t', E'')$ be the (finite trim) uniterminal $\widetilde{A}$-automaton obtained by taking the disjoint union of the underlying graphs of $\mathcal{A}$ and $\mathcal{A}'$ and identifying t with i'.

Lemma 4.2 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $g, g' \in G$. Let $\mathcal{A}$ and $\mathcal{A}'$ be finite trim uniterminal $\widetilde{A}$-automata satisfying*

$$S_g \subseteq \overline{L(\mathcal{A})} \subseteq g\pi^{-1}, \quad S_{g'} \subseteq \overline{L(\mathcal{A}')} \subseteq g'\pi^{-1}.$$

Then

$$S_{gg'} \subseteq \overline{L(\mathcal{A} * \mathcal{A}')} \subseteq (gg')\pi^{-1}.$$

Proof. Since $L(\mathcal{A})L(\mathcal{A}') \subseteq L(\mathcal{A} * \mathcal{A}')$, we get in view of (S2)

$$S_{gg'} \subseteq \overline{S_g S_{g'}} \subseteq \overline{L(\mathcal{A})L(\mathcal{A}')} \subseteq \overline{L(\mathcal{A} * \mathcal{A}')}.$$

Now let $u \in L(\mathcal{A} * \mathcal{A}')$. Then u labels a path in $\mathcal{A} * \mathcal{A}'$ of the form

$$i \xrightarrow{u_0} p \xrightarrow{u_1} p \xrightarrow{u_2} \cdots \xrightarrow{u_{k-1}} p \xrightarrow{u_k} t',$$

where we emphasize all the occurrences of the vertex p obtained through the identification of t and i'. Now it is easy to see that there exist paths $i \xrightarrow{u_0} t$ in $\mathcal{A}$ and $i' \xrightarrow{u_k} t'$ in $\mathcal{A}'$. Moreover, for each $j = 1, \ldots, k-1$, there exists either a path $t \xrightarrow{u_j} t$ in $\mathcal{A}$ or a path $i' \xrightarrow{u_j} i'$ in $\mathcal{A}'$. Now, in view of $(L(\mathcal{A}))\pi = g$ and $(L(\mathcal{A}'))\pi = g'$, we can use the same argument as in the proof of Lemma 4.1 to show that $u_j\pi = 1$ for $j = 1, \ldots, k-1$. Hence $u\pi = (u_0 u_1 \cdots u_k)\pi = (u_0 u_k)\pi = gg'$ and so $\overline{L(\mathcal{A} * \mathcal{A}')} \subseteq (gg')\pi^{-1}$ as required. $\square$

In view of the preceding two lemmas, we can now set an algorithm to construct the automata $\mathcal{A}_i$ in the proof of Theorem 3.9. All we need for a start are the minimum automata of $S_{a\pi}$ for each $a \in A$ (or any other finite trim automaton with a single initial vertex recognizing $S_{a\pi}$; this can be effectively constructed by (S1)). Following the argument in the proof of Lemma 4.1, we may identify all the terminal vertices to get finite trim uniterminal $\widetilde{A}$-automata $\mathcal{C}_{a\pi}$ satisfying

$$S_{a\pi} \subseteq \overline{L(\mathcal{C}_{a\pi})} \subseteq a\pi\pi^{-1}.$$

Note that, since $S^{-1} = S$, we get finite trim uniterminal $\widetilde{A}$-automata $\mathcal{C}_{a^{-1}\pi}$ satisfying

$$S_{a^{-1}\pi} \subseteq \overline{L(\mathcal{C}_{a^{-1}\pi})} \subseteq a^{-1}\pi\pi^{-1}$$

by exchanging the initial and the terminal vertices in $\mathcal{C}_{a\pi}$ and replacing each edge $p \xrightarrow{b} q$ by an edge $q \xrightarrow{b^{-1}} p$.

Now, given $h_i \in G$, we may represent it by some reduced word $a_1 \cdots a_n$ ($a_i \in \widetilde{A}$), and may compute

$$\mathcal{A}_i = ((\cdots(\mathcal{C}_{a_1\pi} * \mathcal{C}_{a_2\pi}) * \mathcal{C}_{a_3\pi}) * \cdots) * \mathcal{C}_{a_n\pi}.$$

By Lemma 4.2, $\mathcal{A}_i$ is a finite trim uniterminal $\widetilde{A}$-automaton satisfying

$$S_{h_i} \subseteq \overline{L(\mathcal{A}_i)} \subseteq h_i \pi^{-1}.$$

What is the maximum size of $\mathcal{A}_i$ relatively to $|h_i|$? What is the time complexity of the algorithm for its construction? Note that we start with only finitely many "atomic" automata $\mathcal{C}_{a\pi}$ ($a \in A$). Hence the number of vertices (edges) in $\mathcal{A}_i$ is a bounded multiple of $|h_i|$, therefore is $O(|h_i|)$, and the time complexity of the construction (disjoint union followed by identification of two vertices, $|h_i| - 1$ times) is also clearly $O(|h_i|)$. This is why we gave ourselves (and the reader) the trouble of constructing the $\mathcal{A}_i$ this way instead of just taking the minimum automaton of S_{h_i}, whatever that may be!

But what is the time complexity of the full algorithm leading to the Stallings automaton $\Gamma(G, H, \pi) \sqcap S$? It is also useful to discuss the complexity of the important intermediate $\mathcal{B}_3$ in the proof of Theorem 3.9 since $\mathcal{B}_3$ suffices for such applications as the generalized word problem: indeed, since $\mathcal{B}_3$ satisfies (5), we may replace $\Gamma(G, H, \pi) \sqcap S$ by $\mathcal{B}_3$ in Corollary 3.10.

Let $n = |h_1| + \cdots + |h_m|$. It follows easily from our previous discussion of the time complexity of the construction of the $\mathcal{A}_i$ that $\mathcal{B}_0$ (and therefore $\mathcal{B}_1$ and $\mathcal{B}_2$) can be constructed in time $O(n)$. Since we get to $\mathcal{B}_3$ through complete folding, the complexity of constructing $\mathcal{B}_3$ is that of the classical Stallings construction in the free group. Touikan proved in [18] that such complexity is $O(n \log^* n)$, where $\log^* n$ denotes the least integer k such that the kth iterate of the log function of n is at most 1 (for most practical purposes, $O(n \log^* n)$ is similar to $O(n)$). Therefore $\mathcal{B}_3$ can be constructed in time $O(n \log^* n)$.

We shall now discuss the complexity of the construction of the Stallings automata:

Theorem 4.3 *Let $S \subseteq R_A$ be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let $H = \langle h_1, \ldots, h_m \rangle \leqslant_{f.g.} G$. Then $\Gamma(G, H, \pi) \sqcap S$ can be constructed in time $O(n^3 \log^* n)$, where $n = |h_1| + \cdots + |h_m|$.*

Proof. We go back to the proof of Theorem 3.9, starting at $\mathcal{B}_3$.

The number of vertices of $\mathcal{B}_3$ is $O(n)$ and therefore we have $O(n^2)$ candidate pairs to J. For each one of these pairs, we must decide whether or not they belong to J. This involves bounding the complexity of the algorithm described in the proof of Lemma 3.8.

Let p, q be distinct vertices of $\mathcal{B}_3$ and let q_0 denote its basepoint. Take geodesics $q_0 \xrightarrow{u} p$ and $q_0 \xrightarrow{v} q$. Clearly, $g = (uv^{-1})\pi$ can be represented by a word of length $O(n)$. It follows from the previous discussion on the complexity of the construction of $\mathcal{A}_i$ that we may construct a finite trim uniterminal $\widetilde{A}$-automaton $\mathcal{C}_g$ satisfying

$$S_g \subseteq \overline{L(\mathcal{C}_g)} \subseteq g\pi^{-1}$$

in time $O(n)$. Performing a complete folding on $\mathcal{C}_g$ (in time $O(n \log^* n)$), we get a finite inverse $\widetilde{A}$-automaton $\mathcal{D}_g$ satisfying

$$S_g \subseteq L(\mathcal{D}_g) \subseteq g\pi^{-1}.$$

Since S is a constant for our problem, we can compute an element $s \in S \cap L(\mathcal{D}_g) = S_g$ in time $O(n)$ and check if $s \in L(\mathcal{B}_3)$ in time $O(n)$. Therefore, by the proof of Lemma 3.8, we can decide whether or not $(p, q) \in J$ in time $O(n \log^* n)$. Since we had $O(n^2)$ candidates to consider, we may compute J in time $O(n^3 \log^* n)$. It is very likely that this upper bound can be improved.

Since $\mathcal{B}_4$ is obtained from $\mathcal{B}_3$ by identifying the pairs in J followed by complete folding, and $\mathcal{B}_3$ has $O(n)$ vertices, then $\mathcal{B}_4$ can be constructed in time $O(n^3 \log^* n)$ in view of Touikan's bound.

For the last step, we must discuss the time complexity of the algorithm in the proof of Proposition 2.2. Note that $\mathcal{B}_4$ has $O(n)$ vertices and therefore (since the alphabet is fixed) $O(n)$ edges. Since S is a constant for our problem, we can build the direct product of $\mathcal{B}_4$ by some deterministic automaton recognizing S in time $O(n)$ and compute its trim part in time $O(n)$ (we have $O(n)$ vertices and $O(n)$ edges), and the final projection can also be performed in linear time. Therefore $\Gamma(G, H, \pi) \sqcap S$ can be constructed in time $O(n^3 \log^* n)$, which means very close to cubic complexity. $\square$

We should stress that the above discussion of time complexity was performed for a fixed Stallings section of a fixed group. But the computation of a Stallings section for a (virtually free) group can be in itself a costly procedure, particularly if it is supported by Bass-Serre theory as in the present case. This will become more evident throughout the next two sections.

5 Amalgamation over finite groups

Given groups H, G_1 and G_2, and isomorphisms $\varphi_j : H \to H_j \leq G_j$ ($j = 1, 2$), the *free product with amalgamation* (amalgam for short) of G_1 and G_2, relative to φ_1 and φ_2, is defined as the quotient of the free product $G_1 * G_2$ by the normal subgroup generated by the elements of the form $(h\varphi_1)(h^{-1}\varphi_2)$ ($h \in H$). It is usually denoted by $G_1 *_H G_2$, whenever a specific reference to the homomorphisms φ_j can be omitted.

The groups G_j embed canonically into $G_1 *_H G_2$, and we shall actually view them as subgroups of their amalgam. In particular, we view $H_1 = H_2$ as a subgroup of $G = G_1 *_H G_2$.

A factorization $g = w_1 \cdots w_n$ is said to be a *reduced form* for $g \in G_1 *_H G_2$ if:

(i) $w_1 \in G_1 \cup G_2$;

(ii) $w_1 \notin H_1 \cup H_2$ if $n > 1$;

(iii) $w_i \in G_j \setminus H_j \Rightarrow w_{i+1} \in G_{j+1} \setminus H_{j+1}$

hold for all $i \in \{1, \ldots, n-1\}$ and $j \in \{1, 2\}$ modulo 2.

Every element of $G_1 *_H G_2$ can be represented by a reduced form, but the representation is not in general unique. However, this representation can be strictly controlled (see e.g [11, Chapter IV]):

Proposition 5.1 *Let $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ be reduced forms of $G_1 *_H G_2$. Then $u = v$ holds in $G_1 *_H G_2$ if and only if one of the following conditions holds:*

(i) $m = n = 1$ and $u_1 = v_1 \in G_1 \cup G_2$;

(ii) $m = n = 1$ and $u_1 = h\varphi_j$, $v_1 = h\varphi_{j+1}$ for some $h \in H$ and $j \in \{1, 2\}$ modulo 2;

(iii) $m = n > 1$ and there exist $z_1, \ldots, z_{n-1} \in H$ and $j \in \{1,2\}$ modulo 2 such that

$$\begin{array}{l}
u_1 = v_1(z_1\varphi_j) \text{ in } G_j,\\
u_2 = (z_1^{-1}\varphi_{j+1})v_2(z_2\varphi_{j+1}) \text{ in } G_{j+1},\\
\quad \ldots\\
u_{n-1} = (z_{n-2}^{-1}\varphi_{j+n-2})v_{n-1}(z_{n-1}\varphi_{j+n-2}) \text{ in } G_{j+n-2},\\
u_n = (z_{n-1}^{-1}\varphi_{j+n-1})v_n \text{ in } G_{j+n-1}.
\end{array}$$

The main theorem of this section is

Theorem 5.2 *Let G_1 and G_2 be groups with Stallings sections and let H be a finite group. Then $G_1 *_H G_2$ has also a Stallings section.*

Proof. Let S (respectively T) be a Stallings section for the m-epi $\pi_1 : \widetilde{A_1}^* \to G_1$ (respectively $\pi_2 : \widetilde{A_2}^* \to G_2$). We assume that $\widetilde{A_1}^* \cap \widetilde{A_2}^* = 1$ and write $A = A_1 \cup A_2$. Let H be a finite group and consider isomorphisms $\varphi_j : H \to H_j \leq G_j$ $(j = 1,2)$. We denote by $G = G_1 *_H G_2$ the amalgam of G_1 and G_2 relative to φ_1 and φ_2. Let $\pi : \widetilde{A}^* \to G$ be the m-epi induced by π_1 and π_2.

Let $B = \{b_h \mid h \in H\}$ be a new alphabet and let $\psi : B^* \to H$ be the homomorphism defined by $b_h\psi = h$ $(h \in H)$. Let $\xi : B^* \to \mathrm{Rat}\,\widetilde{A}^*$ be the rational substitution defined by

$$b_h\xi = S_{h\varphi_1} \cup T_{h\varphi_2} \subseteq h\varphi_1\pi^{-1} = h\varphi_2\pi^{-1}.$$

We define

$$L = 1\psi^{-1}\xi.$$

In the next lemma, we collect some important properties of L:

Lemma 5.3 *(i) L is an effectively constructible rational language;*

(ii) $1 \in L$ and $L\pi = 1$;

(iii) $L^2 = L = L^{-1}$;

(iv) $(b_h\xi)L(b_{h^{-1}}\xi) \subseteq L$ for every $h \in H$.

Proof. (i) Since H is finite, $1\psi^{-1}$ and L are rational and effectively constructible by Proposition 2.1.

(ii) Indeed, if $(b_{h_1} \cdots b_{h_n})\psi = 1$, then $h_1 \cdots h_n = 1$ and so

$$(b_{h_1} \cdots b_{h_n})\xi\pi \subseteq ((h_1\varphi_1\pi^{-1}) \cdots (h_n\varphi_1\pi^{-1}))\pi = (h_1 \cdots h_n)\varphi_1 = 1.$$

(iii) The equality $L^2 = L$ follows from $(1\psi^{-1})^2 = 1\psi^{-1}$. Now let $u \in L$. We may write $u \in (b_{h_1} \cdots b_{h_n})\xi$ with $(b_{h_1} \cdots b_{h_n})\psi = 1$. It follows that $(b_{h_n^{-1}} \cdots b_{h_1^{-1}})\psi = 1$. Since $S^{-1} = S$ and $T^{-1} = T$, we get

$$b_{h^{-1}}\xi = S_{h^{-1}\varphi_1} \cup T_{h^{-1}\varphi_2} = S^{-1}_{h\varphi_1} \cup T^{-1}_{h\varphi_2} = (b_h\xi)^{-1}$$

for every $h \in H$ and so

$$\begin{aligned} u^{-1} &\in ((b_{h_1}\xi) \cdots (b_{h_n}\xi))^{-1} = (b_{h_n}\xi)^{-1} \cdots (b_{h_1}\xi)^{-1}\\ &= (b_{h_n^{-1}}\xi) \cdots (b_{h_1^{-1}}\xi) = (b_{h_n^{-1}} \cdots b_{h_1^{-1}})\xi \subseteq 1\psi^{-1}\xi = L. \end{aligned}$$

Thus $L^{-1} \subseteq L$ and so also $L = (L^{-1})^{-1} \subseteq L^{-1}$. Therefore $L = L^{-1}$.

(iv) Assume that $(b_{h_1} \cdots b_{h_n})\psi = 1$ and $u \in (b_{h_1} \cdots b_{h_n})\xi$. Then $h_1 \cdots h_n = 1$ and so $hh_1 \cdots h_n h^{-1} = 1$. It follows that

$$(b_h\xi)u(b_{h^{-1}}\xi) \subseteq (b_h b_{h_1} \cdots b_{h_n} b_{h^{-1}})\xi \subseteq L.$$

□

Let

$$S' = S \setminus \bigcup_{h \in H} S_{h\varphi_1}, \quad T' = T \setminus \bigcup_{h \in H} T_{h\varphi_2}.$$

Since H is finite and S, T are both Stallings sections, then S', T' are both effectively constructible rational languages. We define

$$V = \overline{LSL} \cup \overline{LTL} \cup \overline{(1 \cup LS')(LT'LS')^*(L \cup LT'L)}. \tag{11}$$

Since L, S, T, S', T' are all effectively constructible rational languages, so is V, in view of Proposition 2.1 and Theorem 2.4. Since S and T are sections for π_1 and π_2, and $1 \in L$, it follows from the representation of amalgams in reduced form that V is a section for π. In particular, note that $(S')^{-1} = S'$, $(T')^{-1} = T'$ and so $V^{-1} = V$. It remains to be proved that V satisfies axioms (S1) and (S2).

Now let $g = g_1 \cdots g_n$ be a reduced form of G. We claim that

$$V_g = \overline{LW^{(1)}_{g_1} \cdots LW^{(n)}_{g_n} L}, \tag{12}$$

where $W^{(i)} = S$ if $g_i \in G_1$ and $W^{(i)} = T$ if $g_i \in G_2$. In particular, $V_g = \overline{LS_gL} = \overline{LT_gL}$ if $g \in G_1 \cap G_2 = H_1 = H_2$.

We prove two cases, the others are similar:

<u>Case</u> $n = 1$ and $g_1 \in G_1$:

We must prove that $V_g = \overline{LS_{g_1}L}$. Indeed, it is immediate that $\overline{LS_{g_1}L} \subseteq V$ and $\overline{LS_{g_1}L}\pi = (LS_{g_1}L)\pi = S_{g_1}\pi = g_1 = g$, hence $\overline{LS_{g_1}L} \subseteq V_g$. Conversely, if $u \in V_g$, it is clear from Proposition 5.1 and $L\pi = 1$ that we must have $u \in \overline{LSL} \cup \overline{LTL}$. If $u \in \overline{LS_xL}$ for some $x \in G_1$, then $g_1 = g = u\pi = x$ and so $u \in \overline{LS_{g_1}L}$. Hence we assume that $u \in \overline{LT_yL}$ for some $y \in G_2$. It follows that $g_1 = g = u\pi = y$ and so $g \in G_1 \cap G_2 = H_1 = H_2$. We can then write $g_1 = h\varphi_1$ and $y = h\varphi_2$ for some $h \in H$. By Lemma 5.3, we get

$$u \in \overline{LT_{h\varphi_2}L} \subseteq \overline{LT_{h\varphi_2}S_{h^{-1}\varphi_1}S_{h\varphi_1}L} \subseteq \overline{L^2S_{h\varphi_1}L} = \overline{LS_{g_1}L}$$

and so (12) holds in this case.

<u>Case</u> $n = 2k$ and $g_1 \in G_1 \setminus H_1$:

We must prove that

$$V_g = \overline{LS_{g_1}LT_{g_2} \cdots LS_{g_{2k-1}}LT_{g_{2k}}L}.$$

Indeed, it is immediate that $\overline{LS_{g_1}LT_{g_2} \cdots LS_{g_{2k-1}}LT_{g_{2k}}L} \subseteq V$ and

$$\begin{aligned}\overline{LS_{g_1}LT_{g_2} \cdots LS_{g_{2k-1}}LT_{g_{2k}}L}\pi &= (LS_{g_1}LT_{g_2} \cdots LS_{g_{2k-1}}LT_{g_{2k}}L)\pi \\ &= (S_{g_1}T_{g_2} \cdots S_{g_{2k-1}}T_{g_{2k}})\pi = g_1 \cdots g_{2k} = g,\end{aligned}$$

hence $\overline{LS_{g_1}LT_{g_2}\cdots LS_{g_{2k-1}}LT_{g_{2k}}L} \subseteq V_g$. Conversely, if $u \in V_g$, it is clear from Proposition 5.1 and $L\pi = 1$ that we must have $u \in \overline{LS_{g'_1}LT_{g'_2}\cdots LS_{g'_{2k-1}}LT_{g'_{2k}}L}$ where

$$\begin{aligned} g'_1 &= g_1(h_1\varphi_1),\\ g'_2 &= (h_1^{-1}\varphi_2)g_2(h_2\varphi_2),\\ &\cdots\\ g'_{2k-1} &= (h_{2k-2}^{-1}\varphi_1)g_{2k-1}(h_{2k-1}\varphi_1),\\ g'_{2k} &= (h_{2k-1}^{-1}\varphi_2)g_{2k} \end{aligned}$$

for some $h_1, \dots, h_{2k-1} \in H$. Since S and T satisfy (S2) and by Lemma 5.3(iv), we get

$$\begin{aligned} u &\in \overline{LS_{g'_1}LT_{g'_2}\cdots LS_{g'_{2k-1}}LT_{g'_{2k}}L}\\ &\subseteq \overline{LS_{g_1}S_{h_1\varphi_1}LT_{h_1^{-1}\varphi_2}T_{g_2}\cdots T_{h_{2k-2}\varphi_2}LS_{h_{2k-2}^{-1}\varphi_1}S_{g_{2k-1}}S_{h_{2k-1}\varphi_1}LT_{h_{2k-1}^{-1}\varphi_2}T_{g_{2k}}L}\\ &\subseteq \overline{LS_{g_1}LT_{g_2}\cdots LS_{g_{2k-1}}LT_{g_{2k}}L} \end{aligned}$$

and so $V_g = \overline{LS_{g_1}LT_{g_2}\cdots LS_{g_{2k-1}}LT_{g_{2k}}L}$ as claimed.

The other cases are absolutely similar, therefore (12) holds. Since S and T satisfy (S1), and by Proposition 2.1, Theorem 2.4 and Lemma 5.3(i), V_g is an effectively constructible rational language for every $g \in G$. Therefore (S1) holds for V.

As a consequence of (12), we have

$$V_g = \overline{V_{g_1\cdots g_i}V_{g_{i+1}\cdots g_n}}$$

whenever $g = g_1\cdots g_n$ is a reduced factorization. In particular, $V_{gg'} = \overline{V_gV_{g'}}$ holds if $g = g_1\cdots g_n$ and $g' = g'_1\cdots g'_m$ are reduced factorizations with $g_n \in G_1\backslash H_1$ and $g'_1 \in G_2\backslash H_2$, or vice-versa. We shall refer to this case as the favourable case.

Given $g \in G$, let $||g||$ denote the number n of components in a reduced form $g_1\cdots g_n$ of g. Let $g, g' \in G$. We prove that

$$V_{gg'} \subseteq \overline{V_gV_{g'}} \tag{13}$$

by induction on $k = ||g|| + ||g'||$.

If $||g|| = ||g'|| = 1$, we may assume that $g, g' \in G_1$ or $g, g' \in G_2$, otherwise we have the favourable case and we are done. Without loss of generality, we may assume that $g, g' \in G_1$. Then (12) and (S2) for S yield

$$V_{gg'} = \overline{LS_{gg'}L} \subseteq \overline{LS_gS_{g'}L} \subseteq \overline{LS_gL^2S_{g'}L} = \overline{V_gV_{g'}}.$$

Therefore (13) holds for $k = 2$.

Assume now that $||g|| + ||g'|| > 2$ and (13) holds for smaller values of $||g|| + ||g'||$. Let $g = g_1\cdots g_n$ and $g' = g'_1\cdots g'_m$ be reduced decompositions of g and g'.

We do not have to consider the favourable case, hence we may assume that $g_n, g'_1 \in G_1$ or $g_n, g'_1 \in G_2$. By symmetry, we may assume that $g_n, g'_1 \in G_1$ and $n > 1$. Write $x = g_1\cdots g_{n-1}$ and $y = g_ng'_1\cdots g'_m$. Then $||x|| + ||y|| < ||g|| + ||g'||$ and so the induction hypothesis yields

$$V_{gg'} = V_{xy} \subseteq \overline{V_xV_y}.$$

Suppose first that $m = 1$. Then (S2) for S yields

$$V_y = \overline{LS_{g_ng'_1}L} \subseteq \overline{LS_{g_n}S_{g'_1}L} \subseteq \overline{LS_{g_n}L^2S_{g'_1}L} = \overline{LS_{g_n}LV_{g'}}$$

and so in view of (12) we get

$$V_{gg'} \subseteq \overline{V_x V_y} \subseteq \overline{V_x L S_{g_n} L V_{g'}} = \overline{V_g V_{g'}}.$$

Now suppose that $m > 1$. Then $V_y \subseteq \overline{V_{g_n g'_1} V_{g'_2 \cdots g'_m}}$ by the induction hypothesis and so

$$V_{gg'} \subseteq \overline{V_x V_y} \subseteq \overline{V_x V_{g_n g'_1} V_{g'_2 \cdots g'_m}} \subseteq \overline{V_x V_{g_n} V_{g'_1} V_{g'_2 \cdots g'_m}} = \overline{V_g V_{g'}}$$

by the favourable case. Thus (13) holds and so (S2) holds for V. Therefore V is a Stallings section for π and the theorem is proved. $\square$

6 HNN extensions over finite groups

Given a subgroup H of a group K and a monomorphism $\varphi : H \to K$, the *HNN extension* $HNN(K, H, \varphi)$ is the group defined by the relative presentation

$$\langle K, t \mid tht^{-1} = h\varphi \ (h \in H) \rangle,$$

that is, is the quotient of the free product $K * F_{\{t\}}$ by the normal subgroup generated by the elements of the form $tht^{-1}(h^{-1}\varphi)$ $(h \in H)$. For details, the reader is referred to [11, Chapter IV].

We use the standard notation $H_1 = H$ and $H_{-1} = H\varphi$. A given factorization of g, $g = w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_n} w_n$, is said to be a *reduced form* for $g \in HNN(K, H, \varphi)$ if:

(i) $w_i \in K$;

(ii) $\varepsilon_i \in \{-1, 1\}$;

(iii) $\varepsilon_{i+1} = -\varepsilon_i \Rightarrow w_i \notin H_{\varepsilon_i}$

hold for every possible i. In particular, 1 is a reduced form.

Every element of $HNN(K, H, \varphi)$ can be represented by a reduced form, but the representation is not in general unique. However, this representation becomes clear as a consequence of the classical Britton's Lemma, which we choose to present in the following form:

Proposition 6.1 *Let $g = u_0 t^{\varepsilon_1} u_1 \cdots t^{\varepsilon_n} u_n$ be a reduced form of $HNN(K, H, \varphi)$. The alternative reduced forms for g in $HNN(K, H, \varphi)$ are obtained by replacing each occurrence of t by some element of $\cup_{h \in H} (h\varphi) t h^{-1}$.*

In particular, 1 is the unique reduced form for the identity and so both K and $F_{\{t\}}$ embed canonically into $HNN(K, H, \varphi)$.

Theorem 6.2 *Let K be a group with a Stallings section and let $\varphi : H \to K$ be a monomorphism for some finite subgroup H of K. Then $HNN(K, H, \varphi)$ has also a Stallings section.*

Proof. Let S be a Stallings section for the m-epi $\eta : \widetilde{A}^* \to K$. Write $G = HNN(K, H, \varphi)$, $B = A \cup \{b\}$ and let $\pi : \widetilde{B}^* \to G$ be the m-epi defined by $a\pi = a\eta$ $(a \in \widetilde{A})$ and $b\pi = t$.

Let $C = \{c_h \mid h \in H\}$ be a new alphabet and let $\psi : C^* \to H$ be the homomorphism defined by $c_h \psi = h$ $(h \in H)$. Let $\xi : C^* \to \mathrm{Rat}\, \widetilde{B}^*$ be the rational substitution defined by $c_h \xi = S_h \cup b^{-1} S_{h\varphi} b$. We define

$$L = 1\psi^{-1}\xi = \{(c_{h_1} \cdots c_{h_n})\xi \mid h_1 \cdots h_n = 1\}.$$

The next lemma summarizes some important properties of L:

Lemma 6.3 *(i) L is an effectively constructible rational language;*

(ii) $1 \in L$ and $L\pi = 1$;

(iii) $L^2 = L = L^{-1}$;

(iv) $(c_h\xi)L(c_{h^{-1}}\xi) \subseteq L$ for every $h \in H$;

(v) $L \subseteq ((H\eta^{-1})b^{-1}(H\varphi\eta^{-1})b)^*(H\eta^{-1})$.

Proof. Since

$$c_h\xi\pi = (S_h \cup b^{-1}S_{h\varphi}b)\pi = S_h\pi = S_h\eta = h$$

for every $h \in H$, the proof of Lemma 5.3 can be used with straightforward adaptations to prove (i)-(iv).

On the other hand, since $1 \in H\eta^{-1}$ we have

$$L = 1\psi^{-1}\xi \subseteq C^*\xi \subseteq ((H\eta^{-1}) \cup b^{-1}(H\varphi\eta^{-1})b)^* \subseteq ((H\eta^{-1})b^{-1}(H\varphi\eta^{-1})b)^*(H\eta^{-1})$$

and so (v) holds as well. □

Now let

$$N = (S\widetilde{b})^*S \setminus \widetilde{B}^*(bS_Hb^{-1} \cup b^{-1}S_{H\varphi}b)\widetilde{B}^*$$

denote the set of all words in $(S\widetilde{b})^*S$ representing reduced forms of G. Let $\alpha : \widetilde{B}^* \to \mathrm{Rat}\,\widetilde{B}^*$ be the rational substitution defined by $a\alpha = a$ ($a \in \widetilde{A}$) and $b\alpha = bL$, $b^{-1}\alpha = L^{-1}b^{-1} = Lb^{-1}$. We claim that

$$V = \overline{N\alpha}$$

is a Stallings section for π.

By Theorem 3.11 and Lemma 6.3, the languages S, S_H, $S_{H\varphi}$ and L are all rational and effectively constructible. By Proposition 2.1 and Theorem 2.4, so are N, $N\alpha$ and V. Since $N\pi = G$ and $1 \in L$, it follows that $V\pi = G$. Note that $S^{-1} = S$ yields $N^{-1} = N$, and together with $L^{-1} = L$, this yields $V^{-1} = V$. Thus V is a section for π.

Let $g = u_0t^{\varepsilon_1}u_1 \cdots t^{\varepsilon_n}u_n$ be a reduced form of G. We claim that

$$V_g = \overline{(S_{u_0}b^{\varepsilon_1}S_{u_1} \cdots b^{\varepsilon_n}S_{u_n})\alpha}. \tag{14}$$

Since $L\pi = 1$, we have $\alpha\pi = \pi$ and so

$$\overline{(S_{u_0}b^{\varepsilon_1}S_{u_1} \cdots b^{\varepsilon_n}S_{u_n})\alpha}\pi = (S_{u_0}b^{\varepsilon_1}S_{u_1} \cdots b^{\varepsilon_n}S_{u_n})\pi = u_0t^{\varepsilon_1}u_1 \cdots t^{\varepsilon_n}u_n = g,$$

hence

$$\overline{(S_{u_0}b^{\varepsilon_1}S_{u_1} \cdots b^{\varepsilon_n}S_{u_n})\alpha} \subseteq V \cap g\pi^{-1} = V_g.$$

Conversely, let $w \in V_g$. Then there exists a reduced form $v_0t^{\delta_1}v_1 \cdots t^{\delta_m}v_m$ of G such that $w \in \overline{(S_{v_0}b^{\delta_1}S_{v_1} \cdots b^{\delta_m}S_{v_m})\alpha}$. Then

$$g = w\pi = v_0t^{\delta_1}v_1 \cdots t^{\delta_m}v_m$$

and it follows from Proposition 6.1 that $m = n$, $\delta_i = \varepsilon_i$ for $i = 1, \ldots, n$, and $v_0 t^{\varepsilon_1} v_1 \cdots t^{\varepsilon_n} v_n$ can be obtained from $u_0 t^{\varepsilon_1} u_1 \cdots t^{\varepsilon_n} u_n$ by replacing each occurrence of t by some element of $\cup_{h \in H} (h\varphi) t h^{-1}$. Assume that t^{ε_i} is replaced by $((h_i\varphi) t h_i^{-1})^{\varepsilon_i}$. For $i = 1, \ldots, n$, write

$$x_i = \begin{cases} h_i & \text{if } \varepsilon_i = -1 \\ h_i\varphi & \text{if } \varepsilon_i = 1 \end{cases} \qquad y_i = \begin{cases} h_i & \text{if } \varepsilon_i = 1 \\ h_i\varphi & \text{if } \varepsilon_i = -1 \end{cases}$$

and also $y_0 = x_{n+1} = 1$. Then

$$v_i = y_i^{-1} u_i x_{i+1}$$

for $i = 0, \ldots, n$.

Moreover, we claim that

$$\overline{(S_{x_i} b^{\varepsilon_i} S_{y_i^{-1}})\alpha} \subseteq \overline{b^{\varepsilon_i}\alpha}. \tag{15}$$

Assume that $\varepsilon_i = 1$. Then

$$\overline{(S_{x_i} b^{\varepsilon_i} S_{y_i^{-1}})\alpha} = \overline{S_{x_i} b L S_{y_i^{-1}}} = \overline{S_{h_i\varphi} b L S_{h_i^{-1}}} = \overline{b b^{-1} S_{h_i\varphi} b L S_{h_i^{-1}}} \subseteq \overline{b(c_{h_i}\xi) L (c_{h_i^{-1}}\xi)} \subseteq \overline{bL} = \overline{b^{\varepsilon_i}\alpha}$$

by Lemma 6.3(iv).

Similarly, (15) holds for $\varepsilon_i = -1$. Hence

$$\begin{aligned} w &\in \overline{(S_{v_0} b^{\varepsilon_1} S_{v_1} \cdots b^{\varepsilon_n} S_{v_n})\alpha} \\ &= \overline{(S_{u_0 x_1} b^{\varepsilon_1} S_{y_1^{-1} u_1 x_2} \cdots S_{y_{n-1}^{-1} u_{n-1} x_n} b^{\varepsilon_n} S_{y_n^{-1} u_n})\alpha} \\ &\subseteq \overline{(S_{u_0} S_{x_1} b^{\varepsilon_1} S_{y_1^{-1}} S_{u_1} S_{x_2} \cdots S_{y_{n-1}^{-1}} S_{u_{n-1}} S_{x_n} b^{\varepsilon_n} S_{y_n^{-1}} S_{u_n})\alpha} \\ &\subseteq \overline{(S_{u_0} b^{\varepsilon_1} S_{u_1} \cdots b^{\varepsilon_n} S_{u_n})\alpha} \end{aligned}$$

and so (14) holds.

Since K has a Stallings section, has decidable generalized word problem and so we can effectively compute a reduced form for any given element of G. Therefore (S1) follows from (14).

If $g \in G$ has a reduced form $g = w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_n} w_n$, we write $||g|| = n$. We show that $V_{gg'} \subseteq \overline{V_g V_{g'}}$ for all $g, g' \in G$ by induction on $||g|| + ||g'||$.

Let $g = w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_n} w_n$ and $g' = w'_0 t^{\varepsilon'_1} w'_1 \cdots t^{\varepsilon'_m} w'_m$ be reduced forms. If

$$gg' = w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_n} w_n w'_0 t^{\varepsilon'_1} w'_1 \cdots t^{\varepsilon'_m} w'_m \tag{16}$$

is a reduced form, then

$$\begin{aligned} V_{gg'} &= \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_n} S_{w_n w'_0} b^{\varepsilon'_1} S_{w'_1} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha} \\ &\subseteq \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_n} S_{w_n} S_{w'_0} b^{\varepsilon'_1} S_{w'_1} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha} \\ &\subseteq \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_n} S_{w_n})\alpha (S_{w'_0} b^{\varepsilon'_1} S_{w'_1} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha} = \overline{V_g V_{g'}}, \end{aligned}$$

hence we may assume that (16) is not a reduced form and $V_{g_1 g_2} \subseteq \overline{V_{g_1} V_{g_2}}$ whenever $||g_1|| + ||g_2|| < ||g|| + ||g'||$. In particular, $n, m > 0$ and either

$$\varepsilon_n = -\varepsilon'_1 = 1, \quad w_n w'_0 \in H \tag{17}$$

or

$$\varepsilon_n = -\varepsilon'_1 = -1, \quad w_n w'_0 \in H\varphi.$$

The second case being analogous, we assume that (17) holds. Let $h = w_n w'_0$. Then

$$gg' = (w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_{n-1}} w_{n-1})((h\varphi) w'_1 t^{\varepsilon'_2} w'_2 \cdots t^{\varepsilon'_m} w'_m).$$

Since $g_1 = w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_{n-1}} w_{n-1}$ and $g_2 = (h\varphi) w'_1 t^{\varepsilon'_2} w'_2 \cdots t^{\varepsilon'_m} w'_m$ are both reduced forms, the induction hypothesis yields

$$\begin{aligned} V_{gg'} = V_{g_1 g_2} &\subseteq \overline{V_{g_1} V_{g_2}} = \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_{n-1}} S_{w_{n-1}})\alpha (S_{(h\varphi)w'_1} b^{\varepsilon'_2} S_{w'_2} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha} \\ &\subseteq \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_{n-1}} S_{w_{n-1}} S_{h\varphi} S_{w'_1} b^{\varepsilon'_2} S_{w'_2} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha}. \end{aligned}$$

Now

$$\overline{S_{h\varphi}\alpha} = S_{h\varphi} \subseteq \overline{b(b^{-1} S_{h\varphi} b S_{h^{-1}}) S_h b^{-1}} \subseteq \overline{bLS_h L b^{-1}} = \overline{(bS_h b^{-1})\alpha} \subseteq (bS_{w_n} S_{w'_0} b^{-1})\alpha$$

yields

$$\begin{aligned} V_{gg'} &\subseteq \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_{n-1}} S_{w_{n-1}} S_{h\varphi} S_{w'_1} b^{\varepsilon'_2} S_{w'_2} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha} \\ &\subseteq \overline{(S_{w_0} b^{\varepsilon_1} S_{w_1} \cdots b^{\varepsilon_{n-1}} S_{w_{n-1}} b^{\varepsilon_n} S_{w_n} S_{w'_0} b^{\varepsilon'_1} S_{w'_1} b^{\varepsilon'_2} S_{w'_2} \cdots b^{\varepsilon'_m} S_{w'_m})\alpha} = \overline{V_g V_{g'}} \end{aligned}$$

and so (S2) holds. Therefore V is a Stallings section for π. □

7 Virtually free groups

Recall that a *pushdown A-automaton* is a sextuple of the form $\mathcal{A} = (Q, q_0, T, D, d_0, \delta)$, where Q and D are finite sets, $q_0 \in Q$, $T \subseteq Q$, $d_0 \in D$ and δ is a finite subset of

$$Q \times (A \cup 1) \times D \times Q \times D^*.$$

A *configuration* of $\mathcal{A}$ is an element of $Q \times D^*$. The pair (q_0, d_0) is the *initial configuration*. If $(q, a, d, p, u) \in \delta$, we write

$$(q, vd) \underset{a}{\longmapsto} (p, vu)$$

for every $v \in D^*$. We call this relation an *elementary transition*. If we have a sequence

$$(q_0, w_0) \underset{a_1}{\longmapsto} (q_1, w_1) \underset{a_2}{\longmapsto} \cdots \underset{a_n}{\longmapsto} (q_n, w_n)$$

for some $n \geq 0$, we write

$$(q_0, w_0) \overset{*}{\underset{a_1 \cdots a_n}{\longmapsto}} (q_n, w_n)$$

and we refer to it as a *transition*. The language *accepted* by $\mathcal{A}$ (by final states) is defined by

$$L(\mathcal{A}) = \{w \in A^* \mid (q_0, d_0) \overset{*}{\underset{w}{\longmapsto}} (t, u) \text{ for some } t \in T \text{ and } u \in D^*\}.$$

A language $L \subseteq A^*$ is *context-free* if $L = L(\mathcal{A})$ for some pushdown A-automaton $\mathcal{A}$. For details on pushdown automata, the reader is referred to [6, Chapter 6].

Recall that a group is *virtually free* if it has a free subgroup of finite index. Some recent papers involving virtually free groups include [5, 9, 10].

We can now prove the main theorem of the paper:

Theorem 7.1 *A finitely generated group has a Stallings section if and only if it is virtually free.*

Proof. It is known that finitely generated virtually free groups are, up to isomorphism, the fundamental groups of graphs of groups where the graph, the vertex groups and the edge groups are all finite [15, Theorem 7.3]. Moreover, they can be obtained from finite groups by finitely many successive applications of free products with amalgamation over finite groups and HNN extensions over finite groups [4, Chapter 1, Example 3.5 (vi)]. Since finite groups have Stallings sections by Proposition 3.2, it follows from Theorems 5.2 and 6.2 that every finitely generated virtually free group has a Stallings section.

Conversely, assume that S is a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$. We show that the *word problem submonoid* $1\pi^{-1}$ is context-free. By Muller and Schupp's Theorem [14], this implies that G is virtually free.

By the remark following the definition of Stallings section in Section 3, we can assume that $1 \in S_1$.

For every $a \in \widetilde{A}$, let $\mathcal{A}^a = (Q^a, q_0^a, T^a, E^a)$ be a finite automaton recognizing $S_{a\pi}$. We define a pushdown $\widetilde{A}$-automaton $\mathcal{A} = (Q, q_0, t, D, d_0, \delta)$ by $Q = (\cup_{a\in\widetilde{A}} Q^a) \cup \{q_0, t\}$, $D = \widetilde{A} \cup \{d_0\}$ and

$$\begin{aligned}\delta = &\{(q_0, 1, d_0, t, 1)\} \cup \{(q_0, a, d_0, q_0^a, d_0) \mid a \in \widetilde{A}\}\\ &\cup \{(p^a, 1, d_0, q^a, d_0 b) \mid (p^a, b, q^a) \in E^a,\ a, b \in \widetilde{A}\}\\ &\cup \{(p^a, 1, c, q^a, \overline{cb}) \mid (p^a, b, q^a) \in E^a,\ a, b, c \in \widetilde{A}\}\\ &\cup \{(t^a, b, d, q_0^b, d) \mid t^a \in T^a,\ a, b \in \widetilde{A},\ d \in D\}\\ &\cup \{(t^a, 1, d_0, t, 1) \mid t^a \in T^a,\ a \in \widetilde{A}\}.\end{aligned}$$

We shall prove that $1\pi^{-1} = L(\mathcal{A})$. First of all we note that $(p^a, b, q^a) \in E^a$ implies

$$(p^a, d_0 v) \underset{1}{\vdash} (q^a, d_0 \overline{vb})$$

for all $b \in \widetilde{A}$ and $v \in R_A$, hence

$$\text{If } p^a \xrightarrow{u} q^a \text{ is a path in } \mathcal{A}^a, \text{ then } (p^a, d_0 v) \underset{1}{\overset{*}{\vdash}} (q^a, d_0 \overline{vu}) \tag{18}$$

holds for all $a \in \widetilde{A}$ and $v \in R_A$.

Assume now that $a_1 \cdots a_n \in 1\pi^{-1}$, with $a_1, \ldots, a_n \in \widetilde{A}$. We may assume that $n > 0$. Then $1 \in S_1 = S_{(a_1\cdots a_n)\pi} \subseteq \overline{S_{a_1\pi} \cdots S_{a_n\pi}}$ and so there exist $u_i \in S_{a_i\pi} = L(\mathcal{A}^{a_i})$ such that $\overline{u_1 \cdots u_n} = 1$. It follows from (18) that

$$(q_0^{a_i}, d_0 \overline{u_1 \cdots u_{i-1}}) \underset{1}{\overset{*}{\vdash}} (t^{a_i}, d_0 \overline{u_1 \cdots u_i})$$

for some $t^{a_i} \in T^{a_i}$. Hence

$$\begin{aligned}(q_0, d_0) &\underset{a_1}{\vdash} (q_0^{a_1}, d_0) \underset{1}{\overset{*}{\vdash}} (t^{a_1}, d_0 u_1) \underset{a_2}{\vdash} (q_0^{a_2}, d_0 u_1) \underset{1}{\overset{*}{\vdash}} (t^{a_2}, d_0 \overline{u_1 u_2}) \underset{a_3}{\vdash} \cdots\\ &\underset{a_{n-1}}{\vdash} (q_0^{a_{n-1}}, d_0 \overline{u_1 \cdots u_{n-2}}) \underset{1}{\overset{*}{\vdash}} (t^{a_{n-1}}, d_0 \overline{u_1 \cdots u_{n-1}}) \underset{a_n}{\vdash} (q_0^{a_n}, d_0 \overline{u_1 \cdots u_{n-1}})\\ &\underset{1}{\overset{*}{\vdash}} (t^{a_n}, d_0 \overline{u_1 \cdots u_n}) = (t^{a_n}, d_0) \underset{1}{\vdash} (t, 1)\end{aligned}$$

and so $a_1 \cdots a_n \in L(\mathcal{A})$. Thus $1\pi^{-1} \subseteq L(\mathcal{A})$.

Conversely, let $a_1 \cdots a_n \in L(\mathcal{A})$, with $a_1, \ldots, a_n \in \widetilde{A}$. We may assume that $n > 0$. It follows easily that there exists a sequence of transitions of the form

$$\begin{aligned}(q_0, d_0) = (p_0, d_0 w_0) &\underset{a_1}{\vdash\!\!-} (q_0^{a_1}, d_0 w_0) \overset{*}{\underset{1}{\vdash\!\!-}} (p_1, d_0 w_1) \underset{a_2}{\vdash\!\!-} \cdots \underset{a_n}{\vdash\!\!-} (q_0^{a_n}, d_0 w_{n-1}) \overset{*}{\underset{1}{\vdash\!\!-}} (p_n, d_0 w_n) \\ &= (p_n, d_0) \underset{1}{\vdash\!\!-} (t, 1)\end{aligned}$$

for some $w_0, \ldots, w_n \in R_A$. Now, for $i = 1, \ldots, n$, we must have $p_i \in T^{a_i}$ and $w_i = \overline{w_{i-1} u_i}$ for some $u_i \in L(\mathcal{A}^{a_i}) = S_{a_i \pi}$. Hence

$$1 = w_n = \overline{w_0 u_1 \cdots u_n} = \overline{u_1 \cdots u_n} \in \overline{S_{a_1 \pi} \cdots S_{a_n \pi}}$$

and so $1 \in (S_{a_1 \pi} \cdots S_{a_n \pi})\pi = (a_1 \cdots a_n)\pi$. Thus $L(\mathcal{A}) \subseteq 1\pi^{-1}$ and so $1\pi^{-1} = L(\mathcal{A})$. Therefore $1\pi^{-1}$ is context-free and so G is virtually free. $\square$

8 Sections with good properties

Having established that finitely generated virtually free groups are precisely the groups with a Stallings section, we have now the possibility of imposing stronger conditions on their Stallings sections, with the purpose of allowing further applications of the Stallings automata $\Gamma(G, H, \pi) \sqcap S$.

The technique is simple. Suppose that:

- every finite group has a Stallings section with property P;
- if G_1 and G_2 have Stallings sections with property P and H is a finite group, then $G_1 *_H G_2$ has also a Stallings section with property P;
- if K has a Stallings section with property P and H is a finite subgroup of K, then $HNN(K, H, \varphi)$ has also a Stallings section with property P.

Then, in view of [4, Chapter 1, Example 3.5 (vi)], every finitely generated virtually free group has a Stallings section with property P.

A good example is given by the concept of extendable Stallings section, which will turn out to be useful to characterize finite index subgroups.

Let S be a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$. We say that S is *extendable* if, for every $u \in S$, there exists some $v \in R_A$ such that $uv^* \subseteq S$ and

$$u \in \mathrm{Pref}(S_{(uv^n u^{-1})\pi}) \text{ for almost all } n \in \mathbb{N}. \tag{19}$$

In order to prove the next result, we consider the following condition on a Stallings section S for $\pi : \widetilde{A}^* \to G$:

(N) If G is not torsion-free, then $S_1 \neq 1$.

Proposition 8.1 *Every finitely generated virtually free group has an extendable Stallings section.*

Proof. In fact, we show that such a group always has an extendable Stallings section satisfying condition (N).

Following the script previously described, we start by considering a m-epi $\pi : \widetilde{A}^* \to G$ with G finite. Let $S = R_A$ and take $v = 1$ for every $u \in S$. Hence $uv^* \subseteq S$. Since $S_g = \overline{g\pi^{-1}}$ for every $g \in G$, we claim that $\mathrm{Pref}(S_g) = R_A$:

Let $w \in R_A$ and take $a \in \widetilde{A}$ such that $wa \in R_A$. Since G is finite, there exists some $m \in \mathbb{N}$ such that every element of G can be represented by some word of length $< m$. In particular, there exists some $z \in R_A$ such that $((a^{-m}w^{-1})\pi)g = z\pi$ and $|z| < m$. Hence $(wa^m z)\pi = g$ and so $\overline{wa^m z} \in \overline{g\pi^{-1}} = S_g$. Since $wa^m \in R_A$ and $|z| < m$, we get $w \in \mathrm{Pref}(S_g)$ and so $\mathrm{Pref}(S_g) = R_A$.

Therefore (19) holds and so R_A is an extendable Stallings section for $\pi : \widetilde{A}^* \to G$ when G is finite. Moreover, if G is nontrivial, then $S_1 = \overline{1\pi^{-1}}$ contains nonempty words and so condition (N) holds for S.

Next, assume that S (respectively T) is an extendable Stallings section for the m-epi $\pi_1 : \widetilde{A_1}^* \to G_1$ (respectively $\pi_2 : \widetilde{A_2}^* \to G_2$), satisfying condition (N). We assume that $\widetilde{A_1}^* \cap \widetilde{A_2}^* = 1$ and write $A = A_1 \cup A_2$. Let H be a finite group and consider isomorphisms $\varphi_j : H \to H_j \leq G_j$ $(j = 1, 2)$. Let $G = G_1 *_H G_2$ be the amalgam of G_1 and G_2 relative to φ_1 and φ_2, and let $\pi : \widetilde{A}^* \to G$ be the m-epi induced by π_1 and π_2. We may assume that $H_1 < G_1$ and $H_2 < G_2$, otherwise $G \cong G_2$ or $G \cong G_1$. We claim that the Stallings section V (for π) defined in the proof of Theorem 5.2 is also extendable. We use all the notation introduced in that proof.

Let $u \in V$. Without loss of generality, we may assume that either $u = 1$ or the last letter of u is in $\widetilde{A_1}$. Let $u\pi = g_1 \cdots g_m$ be a reduced form of $u\pi$.

Suppose first that $g_m \in G_1 \setminus H_1$. Take $w \in T'$ and $z \in S'$, and write $v = wz$. Then $uv^* \subseteq V$. We claim that

$$u \in \mathrm{Pref}(V_{(uv^n u^{-1})\pi}) \text{ if } n \geq \frac{m}{2} + 1. \tag{20}$$

Indeed, a simple induction on ℓ shows that if $x = x_1 \cdots x_{k+\ell+1}$ and $y = y_1 \cdots y_\ell$ are reduced forms in G, then xy has a reduced form $x_1 \cdots x_k z_1 \cdots z_r$:

We may assume that $y_1 \in G_1$. If $y_1 \in H_1$, then $\ell = 1$ and $xy = x_1 \cdots x_{k+1}(x_{k+2}y_1)$ is a reduced form and we are done. Hence we may assume that $y_1, x_{k+\ell+1} \in G_1 \setminus H_1$ and $x_{k+\ell+1}y_1 = h\varphi_1$ for some $h \in H$. Then $(h\varphi_2\, y_2)y_3 \cdots y_\ell$ is a reduced form. If $\ell = 1$, then $xy = x_1 \cdots x_k(x_{k+1}(h\varphi_2))$ is a reduced form and we are done. If $\ell > 1$, we reduce the problem to the case $\ell - 1$ by considering the product $xy = (x_1 \cdots x_{k+\ell})((h\varphi_2\, y_2)y_3 \cdots y_\ell)$. Thus xy has a reduced form $x_1 \cdots x_k z_1 \cdots z_r$ as claimed.

In particular, taking $x = (uv^n)\pi$and $y = u^{-1}\pi$, it follows that $(uv^n u^{-1})\pi$ has a reduced form $g_1 \cdots g_m w \cdots$ if $n \geq \frac{m}{2} + 1$. Since $g_m \in G_1 \setminus H_1$ and uw is reduced, it follows easily from (12) that $V_{(uv^n u^{-1})\pi}$ must contain some word $uw \cdots$ Thus (20) holds.

Suppose next that $g_m \in G_2 \setminus H_2$. We can of course assume that $u \neq 1$. Since the last letter of u is in $\widetilde{A_1}$, it follows from (12) that $H \neq 1$. Since S and T satisfy condition (N), it follows that there exist $z_1 \in S_1 \setminus \{1\}$ and $w_1 \in T_1 \setminus \{1\}$. Take $w \in T'$ and $z \in S'$, and write $v = w_1 z w z_1$. Since $w_1, z_1 \in L$, we have $uv^* \subseteq V$. Similarly to the preceding case, (20) holds.

Finally, we are left with the case $m = 1$ and $g_1 \in H_1$ (equal to H_2 in G). Let $v = 1$. Then $uv^* \subseteq V$ trivially and $(uv^n u^{-1})\pi = 1$, hence it suffices to show that $u \in \mathrm{Pref}(V_1)$. By (12) and Lemma 5.3, we have $V_1 = \overline{LS_1L} = \overline{L}$ and $u \in \overline{LS_{h\varphi_1}L}$ for some $h \in H$. Let $w \in T_{h^{-1}\varphi_2}$. Then $u \in \mathrm{Pref}(uw)$ and $uw = \overline{uw} \in \overline{LS_{h\varphi_1}LT_{h^{-1}\varphi_2}} \subseteq \overline{L} = V_1$ by Lemma 5.3. Therefore V is extendable.

Suppose now that $V_1 = \{1\}$. Since $V_1 = \overline{L}$, it follows that H is trivial and $S_1 = T_1 = \{1\}$. Since S and T satisfy condition (N), it follows that G_1 and G_2 are torsion-free and so G is a free product of torsion-free groups, hence torsion-free. Therefore V satisfies condition (N).

Finally, assume that S is an extendable Stallings section for the m-epi $\eta : \widetilde{A}^* \to K$ satisfying condition (N). Let $\varphi : H \to K$ be a monomorphism for some finite subgroup H of K. Write $G = HNN(K, H, \varphi)$, $B = A \cup \{b\}$ and let $\pi : \widetilde{B}^* \to G$ be the m-epi defined by $a\pi = a\eta$ ($a \in \widetilde{A}$) and $b\pi = t$.

We claim that the Stallings section V (for π) defined in the proof of Theorem 6.2 is also extendable. We use all the notation introduced in that proof.

We start by proving the following lemma:

Lemma 8.2 *Let $k_0t^{\varepsilon_1}k_1 \cdots t^{\varepsilon_m}k_m$ be a reduced form of G with $m \geq 1$ and let*

$$\begin{aligned} P = \{ & z_0w_1b^{\varepsilon_1}z_1w_2b^{\varepsilon_2}z_2 \cdots w_mb^{\varepsilon_m}z_m \mid z_0 \in (k_0H_{-\varepsilon_1})\eta^{-1}, \\ & z_i \in (H_{\varepsilon_i}k_iH_{-\varepsilon_{i+1}})\eta^{-1} \text{ for } i = 1, \ldots, n-1,\ z_m \in (H_{\varepsilon_m}k_m)\eta^{-1}, \\ & w_j \in (b^{\varepsilon_j}(H_{\varepsilon_j}\eta^{-1})b^{-\varepsilon_j}(H_{-\varepsilon_j}\eta^{-1}))^*\}. \end{aligned}$$

Then P is closed under (partial) free group reduction.

Proof. Let $u = z_0w_1b^{\varepsilon_1}z_1w_2b^{\varepsilon_2}z_2 \cdots w_mb^{\varepsilon_m}z_m$ be an element of P of the described form. Suppose first that aa^{-1} is a factor of u for some $a \in \widetilde{A}$. Then aa^{-1} is either a factor of some z_i or a factor of some w_j, and it follows from the definitions that we may cancel aa^{-1} and remain inside P.

Thus we are left to discuss the case of cancellations involving the letter b. Suppose first that we have a factor $b^{\varepsilon_j}b^{-\varepsilon_j}$ to cancel in w_j. Write $w_j = x_1 \cdots x_r$, with $x_i = b^{\varepsilon_j}x'_ib^{-\varepsilon_j}x''_i$ ($x'_i \in H_{\varepsilon_j}\eta^{-1}, x''_i \in H_{-\varepsilon_j}\eta^{-1}$), and assume that $x'_\ell = 1$. Cancelling our factor yields $x_1 \cdots x_{\ell-1}x''_\ell x_{\ell+1} \cdots x_r$. If $\ell > 1$, we can incorporate x''_ℓ into $x_{\ell-1}$ in view of $(H_{-\varepsilon_j}\eta^{-1})^2 \subseteq H_{-\varepsilon_j}\eta^{-1}$. If $\ell = 1$, we can incorporate x''_ℓ into z_{j-1} by the same reason. The case of cancellations $b^{-\varepsilon_j}b^{\varepsilon_j}$ inside w_j is discussed similarly.

Suppose now that $b^{-\varepsilon_j}$ is the last letter of w_j and cancels with its right neighbour b^{ε_j}. Then we replace $w_jb^{\varepsilon_j} = x_1 \cdots x_rb^{\varepsilon_j}$ by $x_1 \cdots x_{r-1}b^{\varepsilon_j}x'_r$. Since x'_r can be absorbed by z_j, the claim holds also in this case.

Finally, we note that we can never have $z_j = 1$ when $\varepsilon_j = -\varepsilon_{j+1}$: otherwise, we would get

$$1 = z_j \in (H_{\varepsilon_j}k_jH_{\varepsilon_j})\eta^{-1}$$

and so $k_j \in H_{\varepsilon_j}$, impossible since $\varepsilon_j = -\varepsilon_{j+1}$ and $k_0t^{\varepsilon_1}k_1 \cdots t^{\varepsilon_m}k_m$ is a reduced form. □

Back to the proof of Proposition 8.1, let $u \in V$. Assume that $u\pi = k_0t^{\varepsilon_1}k_1 \cdots t^{\varepsilon_m}k_m$ is a reduced form of $u\pi$. If $m = 0$, then $u \in V_{u\pi} = S_{u\eta}$ and it follows that $uv^* \subseteq V$ for $v = b$. What if $m > 0$? Then it is clear that $k_0t^{\varepsilon_1}k_1 \cdots t^{\varepsilon_m}k_mt^{\varepsilon_m n}$ is a reduced form for every $n \geq 0$. We claim that $ub^{\varepsilon_m n} \subseteq V$.

Indeed, let P be defined as in Lemma 8.2. It follows from (14) that

$$u \in \overline{(S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots t^{\varepsilon_m}S_{k_m})\alpha}$$

and it is immediate that $S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots t^{\varepsilon_m}S_{k_m} \subseteq P$. Since

$$b^{-1}\alpha = Lb^{-1} \subseteq (H\eta^{-1})(b^{-1}(H\varphi\eta^{-1})b(H\eta^{-1}))^*b^{-1}$$

by Lemma 6.3(v), and the factors z_{j-1} may absorbe factors from $H\eta^{-1}$ on the right when $\varepsilon_j = -1$, it follows that we may replace b^{-1} by $b^{-1}\alpha$ in $S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots t^{\varepsilon_m}S_{k_m}$ and remain inside P.

Similarly,

$$b\alpha = bL \subseteq b((H\eta^{-1})b^{-1}(H\varphi\eta^{-1})b)^*(H\eta^{-1}) = (b(H\eta^{-1})b^{-1}(H\varphi\eta^{-1}))^*b(H\eta^{-1})$$

and the factors z_j may absorbe factors from $H\eta^{-1}$ on the left when $\varepsilon_j = 1$, it follows that $(S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots b^{\varepsilon_m}S_{k_m})\alpha \subseteq P$. Hence $\overline{(S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots t^{\varepsilon_m}S_{k_m})\alpha} \subseteq P$ by Lemma 8.2 and so $u \in P$.

As a consequence, we may write $u = xb^{\varepsilon_m}y$ with $y \in R_A$. Since $k_0t^{\varepsilon_1}k_1\cdots t^{\varepsilon_m}k_mt^{\varepsilon_m n}$ is a reduced form for every $n \in \mathbb{N}$, it follows that

$$\overline{(S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots b^{\varepsilon_m}S_{k_m})\alpha b^{\varepsilon_m n}} \subseteq \overline{(S_{k_0}b^{\varepsilon_1}S_{k_1}\cdots b^{\varepsilon_m}S_{k_m}b^{\varepsilon_m n})\alpha} \subseteq V$$

for every $n \in \mathbb{N}$ and so $\overline{ub^{\varepsilon_m n}} \in V$. Since $u = xb^{\varepsilon_m}y$, taking $v = b^{\varepsilon_m}$ we get $uv^n = ub^{\varepsilon_m n} = \overline{ub^{\varepsilon_m n}} \in V$ for every $n \in \mathbb{N}$ as claimed.

We continue now by showing that in any case

$$u \in \mathrm{Pref}(V_{(uv^nu^{-1})\pi}) \text{ if } n > m. \tag{21}$$

Indeed, since $uv \in R_B$, it suffices to show that $(uv^nu^{-1})\pi$ has a reduced form $k_0t^{\varepsilon_1}k_1\cdots t^{\varepsilon_m}k_mt^{\varepsilon_m}\cdots$ if $n > m$ ($k_0t\cdots$ if $m = 0$), and we may use induction on m. The case $m = 0$ being obvious, assume that $m > 0$ and the claim holds for $m-1$. We assume that $v = t$, the other case being analogous. If $k_0t^{\varepsilon_1}k_1\cdots t^{\varepsilon_m}k_mt^nk_m^{-1}t^{-\varepsilon_m}k_{m-1}^{-1}\cdots t^{-\varepsilon_1}k_0^{-1}$ is not itself a reduced form, then $k_m^{-1} \in H$ and $\varepsilon_m = 1$, hence we may write

$$(uv^nu^{-1})\pi = k_0t^{\varepsilon_1}k_1\cdots t^{\varepsilon_m}k_mt^{n-1}k'_mk_{m-1}^{-1}t^{-\varepsilon_{m-1}}k_{m-2}^{-1}\cdots t^{-\varepsilon_1}k_0^{-1}$$

for some $k'_m \in H\varphi$. Since $n-1 > m-1$, the induction hypothesis applied to the product $k_0t^{\varepsilon_1}k_1\cdots t^{\varepsilon_{m-1}}k_{m-1}(k'_m)^{-1}$ yields now the required result. Thus (21) holds and V is extendable.

It remains to show that V satisfies condition (N). Suppose that $V_1 = \{1\}$. Since $V_1 = S_1$ and S satisfies condition (N), it follows that K is torsion-free, and so H is trivial. Hence G is the free product of K by the infinite cyclic group $F_{\{t\}}$. Being a free product of torsion-free groups, G is itself torsion-free, therefore V satisfies condition (N). $\square$

We can now derive the following application of the concept of extendable Stallings section:

Theorem 8.3 *Let S be an extendable Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$ and let H be a finitely generated subgroup of G. Then the following conditions are equivalent:*

(i) H has finite index in G;

(ii) $S \subseteq$ Pref(S_H);

(iii) every word of S labels a path out of the basepoint of $\Gamma(G,H,\pi) \sqcap S$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $u \in S \setminus \mathrm{Pref}(S_H)$. Since S is extendable, there exist some $v \in R_A$ and $m \in \mathbb{N}$ such that $uv^* \subseteq S$ and $u \in \mathrm{Pref}(S_{(uv^n u^{-1})\pi})$ for $n \geq m$. We claim that

$$H(uv^j)\pi \neq H(uv^i)\pi \text{ if } j \geq i+m. \tag{22}$$

Indeed, assume that $j \geq i+m$. If $H(uv^j)\pi = H(uv^i)\pi$, then $(uv^{j-i}u^{-1})\pi \in H$ and so

$$u \in \mathrm{Pref}(S_{(uv^{j-i}u^{-1})\pi}) \subseteq \mathrm{Pref}(S_H),$$

a contradiction. Therefore (22) holds and so H has infinite index in G.

(ii) $\Rightarrow$ (iii). Since $S_H \subseteq L(\Gamma(G,H,\pi) \sqcap S)$ by Theorem 3.9.

(iii) $\Rightarrow$ (i). Assume now that every word of S labels a path out of the basepoint q_0 of $\mathcal{A} = \Gamma(G,H,\pi) \sqcap S$. Let Q denote the vertex set of $\mathcal{A}$. For every $q \in Q$, fix a path $q_0 \xrightarrow{w_q} q$. We claim that

$$G = \bigcup_{q \in Q} H(w_q\pi). \tag{23}$$

Indeed, let $g \in G$, and take $u \in S_g$. Then there is a path in $\mathcal{A}$ of the form $q_0 \xrightarrow{u} q$ for some $q \in Q$. Hence $uw_q^{-1} \in L(\mathcal{A}) \subseteq H\pi^{-1}$ by Theorem 3.9 and so $g = u\pi \in H(w_q\pi)$. Thus (23) holds and so H has finite index in G. $\square$

A natural question to ask is whether or not one could replace condition (S2) in the definition of Stallings section by the stronger condition

(S2') $S_{gh} = \overline{S_g S_h}$ for all $g,h \in G$.

However, we can prove that this condition can only be assumed in the simplest cases:

Proposition 8.4 *The following conditions are equivalent for a group G:*

(i) there exist a m-epi $\pi : \widetilde{A}^* \to G$ and a Stallings section S for π satisfying (S2');

(ii) G is either finite or free of finite rank;

(iii) R_A is a Stallings section for some m-epi $\pi : \widetilde{A}^* \to G$.

Proof. (i) $\Rightarrow$ (ii). Let S be a Stallings section S for $\pi : \widetilde{A}^* \to G$ satisfying (S2'). Then $S_1^{-1} = S_1 = \overline{S_1^2}$ and so we can view $(S_1, \circ)$ as a subgroup of $(R_A, \circ) \cong F_A$, where $u \circ v = \overline{uv}$. The same holds for $(S, \circ)$ since $S^{-1} = S = \overline{S^2}$, and $(S_1, \circ)$ is then a subgroup of $(S, \circ)$. Now $(S, \circ)$ must be free by Nielsen's Theorem. Since S, being a Stallings section, is rational, so is $(S, \circ)$ (a rational expression for S as a subset of $\widetilde{A}^*$ translates through reduction to a rational expression for S as a subset of $(R_A, \circ)$). The same happens with S_1, so it follows from Anisimov and Seifert's Theorem [1, Theorem 3.1] that both $(S, \circ)$ and $(S_1, \circ)$ are finitely generated groups. Hence $(S, \circ)$ is a free group of finite rank.

For every $u \in S$, we have

$$\overline{uS_1u^{-1}} \subseteq \overline{S_{u\pi}S_1S_{u^{-1}\pi}} = S_1,$$

hence $(S_1, \circ)$ is a finitely generated normal subgroup of the free group $(S, \circ)$. By [11, Proposition 3.12], $(S_1, \circ)$ is either trivial or has finite index in $(S, \circ)$. On the other, we claim that

$$\overline{uS_1} = \overline{vS_1} \Leftrightarrow u\pi = v\pi \tag{24}$$

holds for all $u, v \in S$. The direct implication follows from $S_1\pi = 1$. Conversely, assume that $u\pi = v\pi$. Then $\overline{v^{-1}u} \in \overline{S_{v^{-1}\pi}S_{u\pi}} = S_1$ and so $u \in \overline{vS_1}$ and $\overline{uS_1} \subseteq \overline{VS_1}$. By symmetry, we get $\overline{uS_1} = \overline{VS_1}$ and so (24) holds.

It is now straightforward to check that

$$\begin{aligned}(S, \circ)/(S_1, \circ) &\to G \\ \overline{uS_1} &\mapsto u\pi\end{aligned}$$

is a group isomorphism. Hence either $G \cong (S, \circ)$ is a free group of finite rank, or $G \cong (S, \circ)/(S_1, \circ)$ is a finite group.

(ii) $\Rightarrow$ (iii). Immediate from the proof of Proposition 3.2.

(iii) $\Rightarrow$ (i). Assume that $S = R_A$ is a Stallings section for the m-epi $\pi : \widetilde{A}^* \to G$. Let $u \in S_g$ and $v \in S_h$ for some $g, h \in G$. Since $\overline{uv}\pi = (uv)\pi = gh$, we get $\overline{uv} \in S_{gh}$ and so $\overline{S_gS_h} \subseteq S_{gh}$. Therefore $S_{gh} = \overline{S_gS_h}$ and so R_A satisfies (S2'). □

9 Examples

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