AN EXAMPLE OF AN INDECOMPOSABLE MODULE WITHOUT NON-ZERO HOLLOW FACTOR MODULES.

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ABSTRACT. A module is M is called hollow-lifting if every submodule N of M such that M/N is hollow contains a direct summand $D \subseteq N$ such that N/D is a small submodule of M/D. A module M is called lifting if such a direct summand D exists for every submodule N. We construct an indecomposable module M without non-zero hollow factor modules, showing that there are hollow-lifting modules which are not lifting. The existences of such modules had been left open in a recent work by N. Orhan, D. Keskin-Tütüncü and R. Tribak.

The purpose of this note is to give an example of an indecomposable module without non-zero hollow factor modules. The existences of such modules had been left open in a recent work by N. Orhan, D. Keskin-Tütüncü and R. Tribak. Recall that a module M is called hollow if there are no two proper submodules K, L of M whose sum spans the whole module, i.e. $M = K + L$. In [2] the authors where concerned with so-called *hollow-lifting* modules, i.e. modules M that have the lifting property with respect to submodules N of M with M/N being hollow. Any module who does not admit such submodules N would of course be an example of hollow-lifting module.

1. The Example

Fix a field k and a finite set Σ with at least two elements. Denote by Σ^* the set of all words in Σ and denote the empty word by ω. Let R be the set of functions $f : \Sigma^* \to k$. For any word $w \in \Sigma^*$ denote by π_w the function with $\pi_w(u)0$ if $u \neq w$ and $\pi_w(u) = 1$ if $u = w$. R becomes an associative ring with unit by pointwise addition and the convolution product:

$$f * g(w) = \sum_{uv=w} f(u)g(v)$$

for $f, g \in R$ and $w \in \Sigma^*$. The unit of R is π_ω. Let M be the k-vector space with basis Σ^*; the words in Σ. M becomes a left R-module by the following action:

$$f \cdot w = \sum_{uv=w} f(v)u$$

for all $w \in \Sigma^*$ and $f \in R$. For any $m = \lambda_1 w_1 + \cdots \lambda_n w_n \in M$ with $\lambda_i \in k$ and $w_i \in \Sigma^*$ one has

$$f \cdot m = \sum_i \sum_{u_i v_i = w_i} \lambda_i f(v_i) u_i.$$

Note that the R-submodule generated by the empty word ω is an essential simple submodule of M. To see this let $S = R \cdot \omega$. Since

$$f \cdot \omega = \sum_{uv=\omega} f(v)u = f(\omega)\omega,$$

S is 1-dimensional and hence simple. For any non-zero element $m = \lambda_1 w_1 + \cdots \lambda_n w_n \in M$ with $\lambda_1 \neq 0$ we have

$$\pi_{w_1} \cdot m = \sum_i \sum_{u_i v_i = w_i} \lambda_i \pi_{w_1}(v_i) u_i = \pi_{w_1}(w_1)\omega = \omega.$$

Hence $S \subseteq R \cdot m$ for all $0 \neq m \in M$ showing that S is essential in M. Hence M is an indecomposable R-module. Let u be any word and denote by K_u the vector space generated by all words w which do **not** have u as prefix. Then K_u is an R-submodule of M, because if w is a word that does not have w as prefix, then no prefix of w can have u as prefix. Hence for all $f \in R$, $f \cdot w = \sum_{rs=w} f(s)r$ is a linear combination of words not having u as prefix and thus belongs to K_u.

Let $x \neq y$ be two letters and let u be a word. Then $M = K_{ux} + K_{uy}$, because for any word w: if ux is not a prefix of w, then $w \in K_{ux}$. Otherwise ux is prefix of w, say $w = uxv$. Since $x \neq y$, uy cannot be a prefix of w and hence $w \in K_{uy}$. Hence any word belongs either to K_{ux} or to K_{uy}, i.e. $K_{ux} + K_{uy} = M$.

We can conclude with the following

Proposition 1.1. *The indecomposable R-module M has no non-zero hollow factor module.*

Proof: Let N be any proper submodule of M. Take any word u that doesn't belong to N and let x be any letter. Then $N \subseteq K_{ux}$, because if ux were a prefix of a word $w \in N$, say $w = uxv$ then $\pi_{xv} \cdot w = u \in N$ a contradiction. Hence ux is not the prefix of any word of N, i.e. $N \subseteq K_{ux}$. Since $u \in K_{ux} \setminus N$, N is a proper submodule of K_{ux}. Let y be a letter different from x. Analogously we have that N is proper submodule of K_{uy}. As seen above, $M = K_{ux} + K_{uy}$ and hence

$$M/N = K_{ux}/N + K_{uy}/N.$$

This shows that no factor module M/N can be hollow.

By this example we see that there are indecomposable modules without non-zero hollow factor modules. As N. Orhan, D. Keskin-Tütüncü and R. Tribak pointed out in [2, 2.10] using M one can construct a hollow-lifting module which is not lifting. In particular the module M from above is such an example since M is indecomposable but not lifting. On the other hand M is trivially hollow-lifting since it does not have any hollow factor modules.

2. Comments

For all module theoretic notion we refer the reader to [3]. Given a submodule N of M with $S = \mathrm{End}(M)$, denote by

$$\mathrm{An}(N) = \{f \in S \mid (N)f = 0\}.$$

It is not difficult to prove, that for any self-injective self-cogenerator M, if M/N is hollow, then $\mathrm{An}(N)$ is a uniform right ideal of S. More general assumptions are possible. Note that for a self-injective module M with $f \in S$, we have that $\mathrm{Hom}(Im(f), M) = fS$. Identifying $Im(f)$ with $M/Ker(f)$ and $An(N)$ with $\mathrm{Hom}(M/N, M)$ we have $\mathrm{An}(\mathrm{Ker}(f)) = fS$. Call a module M semi-injective if for any endomorphism $f \in S$, $\mathrm{An}(\mathrm{Ker}(f)) = fS$. M is called coretractable if $\mathrm{An}(N) = 0 \Rightarrow N = M$. Obviously any self-injective self-cogenerator is semi-injective and coretractable.

Proposition 2.1. *Let M be a semi-injective coretractable module. If M/N is hollow, then $\mathrm{An}(N)$ is a uniform right ideal of S.*

Proof: The equation $\mathrm{An}(N+L) = \mathrm{An}(N) \cap \mathrm{An}(L)$ always holds. Assume M/N is a non-zero hollow module. Then $\mathrm{An}(N) \neq 0$. Let $f, g \in \mathrm{An}(N)$ and suppose $fS \cap gS = 0$. Then

$$\mathrm{An}(\mathrm{Ker}(f) + \mathrm{Ker}(g)) = \mathrm{An}(\mathrm{Ker}(f)) \cap \mathrm{An}(\mathrm{Ker}(g)) = fS \cap gS = 0.$$

By the coretractability, $\mathrm{Ker}(f) + \mathrm{Ker}(g) = M$, but since $N \subseteq \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$,

$$M/N = \mathrm{Ker}(f)/N + \mathrm{Ker}(g)/N.$$

As M/N was hollow $\mathrm{Ker}(f) = M$ and $f = 0$ or $\mathrm{Ker}(g) = M$ and $g = 0$. Thus $\mathrm{An}(N)$ is a uniform right ideal of S.

Hence if the endomorphism ring $End(M)$ of a self-injective self-cogenerator M does not have any uniform right ideal, then M has no non-zero hollow factor module. This module theoretic version explains the above examples as follows: Let C be the path coalgebra over k with quiver consisting of a single vertex and loops from this vertex to itself for each $x \in \Sigma$. Then C^* is isomorphic to the power series ring in Σ non-commuting indeterminantes. It can be shown that $C^* \simeq R$ and $C = M$ as R-module. A general fact on coalgebra from [1] says that C is left (and right) self-injective self-cogenerator C^*-module and that $C^* \simeq End({}_{C^*}C)^{op}$ as k-algebras. Since C^* has no uniform left ideal, C has no hollow factor module as left C^*-module.

Having an example of a module that does not satisfy a certain property, one might ask to characterise those rings where thos counter examples do not exist:

Problem: Characterize the rings such that all modules have non-trivial hollow factor modules.

Examples of such rings are left perfect rings and left conoetherian rings (e.g. injective hull of simples are artinian). Since any module has a proper non-zero cocyclic factor module, we just need to verify whether every cocyclic R-module has a non-zero hollow factor module. Recall that a module is called cocyclic provided it has an essential simple socle or equivalently it is isomorphic to a submodule of the injective hull of a simple module,.

References

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[3] Wisbauer, R., *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991

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