TORELLI THEOREM FOR THE DELIGNE–HITCHIN MODULI SPACE

INDRANIL BISWAS, TOMÁS L. GÓMEZ, NORBERT HOFFMANN, AND MARINA LOGARES

Abstract. Fix integers $g \geq 3$ and $r \geq 2$, with $r \geq 3$ if $g = 3$. Given a compact connected Riemann surface X of genus g, let $\mathcal{M}_{\mathrm{DH}}(X)$ denote the corresponding $\mathrm{SL}(r, \mathbb{C})$ Deligne–Hitchin moduli space. We prove that the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines (up to isomorphism) the unordered pair $\{X, \overline{X}\}$, where $\overline{X}$ is the Riemann surface defined by the opposite almost complex structure on X.

1. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. We denote by $X_{\mathbb{R}}$ the C^∞ real surface underlying X. Let $\overline{X}$ be the Riemann surface defined by the almost complex structure $-J_X$ on $X_{\mathbb{R}}$; here J_X is the almost complex structure of X.

Fix an integer $r \geq 2$. The main object of this paper is the $\mathrm{SL}(r, \mathbb{C})$ Deligne–Hitchin moduli space

$$\mathcal{M}_{\mathrm{DH}}(X)$$

associated to X. This moduli space $\mathcal{M}_{\mathrm{DH}}(X)$ is a complex analytic variety of complex dimension $1 + 2(r^2 - 1)(g - 1)$, which comes with a natural holomorphic map

$$\mathcal{M}_{\mathrm{DH}}(X) \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

We briefly describe $\mathcal{M}_{\mathrm{DH}}(X)$ here; for the target group $\mathrm{GL}(r, \mathbb{C})$ instead of $\mathrm{SL}(r, \mathbb{C})$, its construction is carried out in [Si1, page 7].

- The fiber of $\mathcal{M}_{\mathrm{DH}}(X)$ over $\lambda = 0 \in \mathbb{C} \subset \mathbb{CP}^1$ is the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$ of semistable $\mathrm{SL}(r, \mathbb{C})$ Higgs bundles (E, θ) over X (see section 2 for details).
- The fiber of $\mathcal{M}_{\mathrm{DH}}(X)$ over any $\lambda \in \mathbb{C}^* \subset \mathbb{CP}^1$ is canonically biholomorphic to the moduli space $\mathcal{M}_{\mathrm{conn}}(X)$ of holomorphic $\mathrm{SL}(r, \mathbb{C})$ connections (E, ∇) over X. In fact the restriction of $\mathcal{M}_{\mathrm{DH}}(X)$ to $\mathbb{C} \subset \mathbb{CP}^1$ is the moduli space
$$\mathcal{M}_{\mathrm{Hod}}(X) \longrightarrow \mathbb{C}$$
of λ-connections over X for the group $\mathrm{SL}(r, \mathbb{C})$ (see section 3 for details).
- The fiber of $\mathcal{M}_{\mathrm{DH}}(X)$ over $\lambda = \infty \in \mathbb{CP}^1$ is the moduli space $\mathcal{M}_{\mathrm{Higgs}}(\overline{X})$ of Higgs bundles over $\overline{X}$. Indeed, the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ is constructed by glueing $\mathcal{M}_{\mathrm{Hod}}(X)$ to the analogous moduli space
$$\mathcal{M}_{\mathrm{Hod}}(\overline{X}) \longrightarrow \mathbb{C}$$
of λ-connections over $\overline{X}$. One identifies the fiber of $\mathcal{M}_{\mathrm{Hod}}(X)$ over $\lambda \in \mathbb{C}^*$ with the fiber of $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ over $1/\lambda \in \mathbb{C}^*$, using that holomorphic connections over X and over $\overline{X}$ both correspond to representations of $\pi_1(X_{\mathbb{R}})$ (see section 4 for details).

This construction of $\mathcal{M}_{\mathrm{DH}}(X)$ is due to Deligne [De]. In [Hi2], Hitchin constructed the twistor space for the hyper–Kähler structure of the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$; the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ is identified with this twistor space (see [Si1, page 8]).

We note that while both $\mathcal{M}_{\mathrm{Hod}}(X)$ and $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ are complex algebraic varieties, the moduli space $\mathcal{M}_{\mathrm{DH}}(X)$ does not have any natural algebraic structure.

If we replace X by $\overline{X}$, then the isomorphism class of the Deligne–Hitchin moduli space clearly remains unchanged. In fact, there is a canonical holomorphic isomorphism of $\mathcal{M}_{\mathrm{DH}}(X)$ with $\mathcal{M}_{\mathrm{DH}}(\overline{X})$ over the automorphism of $\mathbb{CP}^1$ defined by $\lambda \longmapsto 1/\lambda$.

We prove the following theorem (see Theorem 4.1):

Theorem 1.1. *Assume that $g \geq 3$, and if $g = 3$, then assume that $r \geq 3$. The isomorphism class of the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines uniquely the isomorphism class of the unordered pair of Riemann surfaces $\{X, \overline{X}\}$.*

In other words, if $\mathcal{M}_{\mathrm{DH}}(X)$ is biholomorphic to the Deligne–Hitchin moduli space $\mathcal{M}_{\mathrm{DH}}(Y)$ for another compact connected Riemann surface Y, then $Y \cong X$ or $Y \cong \overline{X}$.

This paper is organized as follows. Section 2 deals with Higgs bundles; we also obtain a Torelli theorem for them (see Corollary 2.5). Building on that, section 3 deals with λ-connections, and also contains a Torelli theorem for them (see Corollary 3.5). Finally, section 4 deals with the Deligne–Hitchin moduli space; here we prove our main result.

2. Higgs bundles

Let X be a compact connected Riemann surface of genus g, with $g \geq 3$. Fix an integer $r \geq 2$. If $g = 3$, then we assume that $r \geq 3$. Let

$$\mathcal{M}_{r,\mathcal{O}_X}$$

denote the moduli space of semistable $\mathrm{SL}(r,\mathbb{C})$-bundles on X. So $\mathcal{M}_{r,\mathcal{O}_X}$ parameterizes all S-equivalence classes of semistable vector bundles E over X of rank r together with an isomorphism $\bigwedge^r E \cong \mathcal{O}_X$. $\mathcal{M}_{r,\mathcal{O}_X}$ is known to be an irreducible normal complex projective variety of dimension $(r^2-1)(g-1)$. Let

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \subset \mathcal{M}_{r,\mathcal{O}_X}$$

be the open subvariety parameterizing stable $\mathrm{SL}(r,\mathbb{C})$ bundles on X. This open subvariety coincides with the smooth locus of $\mathcal{M}_{r,\mathcal{O}_X}$ according to [NR1, page 20, Theorem 1].

Lemma 2.1. *The holomorphic cotangent bundle*

$$T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$$

does not admit any nonzero holomorphic section.

Proof. Fix a point $x_0 \in X$, and consider the Hecke correspondence

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \xleftarrow{\;q\;} \mathcal{P} \xrightarrow{\;p\;} \mathcal{U} \subseteq \mathcal{M}_{r,\mathcal{O}_X(x_0)}$$

defined as follows:

- $\mathcal{M}_{r,\mathcal{O}_X(x_0)}$ denotes the moduli space of stable vector bundles F over X of rank r together with an isomorphism $\bigwedge^r F \cong \mathcal{O}_X(x_0)$.
- $\mathcal{U} \subseteq \mathcal{M}_{r,\mathcal{O}_X(x_0)}$ denotes the locus of all F for which every subbundle $F' \subset F$ with $0 < \mathrm{rank}(F') < r$ has negative degree; such vector bundles F are called $(0,1)$-stable (see [NR2, page 306, Definition 5.1], [BBGN, page 563]).
- $p : \mathcal{P} \longrightarrow \mathcal{U}$ is the $\mathbb{P}^{r-1}$-bundle whose fiber over any vector bundle $F \in \mathcal{U}$ parameterizes all hyperplanes H in the fiber F_{x_0}.
- $q : \mathcal{P} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ sends any vector bundle $F \in \mathcal{U}$ and hyperplane $H \subsetneq F_{x_0}$ to the vector bundle E given by the short exact sequence
$$0 \longrightarrow E \longrightarrow F \longrightarrow F_{x_0}/H \longrightarrow 0$$
of coherent sheaves on X; here the quotient sheaf F_{x_0}/H is supported at x_0.

As $\mathcal{M}_{r,\mathcal{O}_X(x_0)}$ is a smooth unirational projective variety (see [Se, page 53]), it does not admit any nonzero holomorphic 1-form. The subset $\mathcal{U} \subseteq \mathcal{M}_{r,\mathcal{O}_X(x_0)}$ is open due to [BBGN, page 563, Lemma 2], and the conditions on r and g ensure that the codimension of the complement $\mathcal{M}_{r,\mathcal{O}_X(x_0)} \setminus \mathcal{U}$ is at least two. Hence also
$$H^0(\mathcal{U}, T^*\mathcal{U}) = 0$$
due to Hartog's theorem. Since $H^0(\mathbb{P}^{r-1}, T^*\mathbb{P}^{r-1}) = 0$, any relative holomorphic 1-form on the $\mathbb{P}^{r-1}$-bundle $p : \mathcal{P} \longrightarrow \mathcal{U}$ vanishes identically. Thus we conclude
$$H^0(\mathcal{P}, T^*\mathcal{P}) = 0.$$
The same follows for $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, because the algebraic map $q : \mathcal{P} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ is dominant. $\square$

We denote by K_X the canonical line bundle on X. Let
$$\mathcal{M}_{\mathrm{Higgs}}(X)$$
denote the moduli space of semistable $\mathrm{SL}(r,\mathbb{C})$ Higgs bundles over X. So $\mathcal{M}_{\mathrm{Higgs}}(X)$ parameterizes all S-equivalence classes of semistable pairs (E,θ) consisting of a vector bundle E over X of rank r together with an isomorphism $\bigwedge^r E \cong \mathcal{O}_X$, and a Higgs field $\theta : E \longrightarrow E \otimes K_X$ with $\mathrm{trace}(\theta) = 0$. $\mathcal{M}_{\mathrm{Higgs}}(X)$ is an irreducible normal complex algebraic variety of dimension $2(r^2-1)(g-1)$ according to [Si3, Theorem 11.1].

There is a natural embedding
$$\iota : \mathcal{M}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\mathrm{Higgs}}(X) \tag{2.1}$$
defined by $E \longmapsto (E, 0)$. We denote by
$$\mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X) \subset \mathcal{M}_{\mathrm{Higgs}}(X)$$
the open locus of Higgs bundles (E,θ) whose underlying vector bundle E is stable. Let
$$\mathrm{pr}_E : \mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X) \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \tag{2.2}$$
be the forgetful map defined by $(E,\theta) \longmapsto E$. One has a canonical isomorphism
$$\mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X) \xrightarrow{\sim} T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \tag{2.3}$$

of varieties over $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, because holomorphic cotangent vectors to a point $E \in \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ correspond, via deformation theory and Serre duality, to Higgs fields $\theta : E \longrightarrow E \otimes K_X$ with $\mathrm{trace}(\theta) = 0$. In particular, $\mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X)$ is contained in the smooth locus

$$\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X).$$

We recall that the *Hitchin map*

$$H : \mathcal{M}_{\mathrm{Higgs}}(X) \longrightarrow \bigoplus_{i=2}^{r} H^0(X, K_X^{\otimes i}) \tag{2.4}$$

is defined by sending each Higgs bundle (E, θ) to the characteristic polynomial of θ.

The multiplicative group $\mathbb{C}^*$ acts on the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$ as follows:

$$t \cdot (E, \theta) = (E, t\theta). \tag{2.5}$$

On the other hand, $\mathbb{C}^*$ acts on the Hitchin space $\bigoplus_{i=2}^{r} H^0(X, K_X^{\otimes i})$ as

$$t \cdot (v_2, \cdots, v_i, \cdots, v_r) = (t^2 v_2, \cdots, t^i v_i, \cdots, t^r v_r),$$

where $v_i \in H^0(X, K_X^{\otimes i})$ and $i \in \{2, \ldots, r\}$. The Hitchin map H in (2.4) intertwines these two actions of $\mathbb{C}^*$. Note that no nonzero holomorphic function on the Hitchin space is homogenous of degree 1 for this action. The reason is that we are considering $\mathrm{SL}(r, \mathbb{C})$ Higgs bundles, and hence only Higgs fields $\theta : E \longrightarrow E \otimes K_X$ with $\mathrm{trace}(\theta) = 0$.

Lemma 2.2. *The holomorphic tangent bundle*

$$T\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$$

does not admit any nonzero holomorphic section.

Proof. The proof of [Hi1, page 110, Theorem 6.2] carries over to this situation as follows. A holomorphic section s of $T\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ provides (by contraction) a holomorphic function

$$f : T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathbb{C} \tag{2.6}$$

on the total space of the cotangent bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, which is linear on the fibers. Under the above isomorphism (2.3), it corresponds to a function on $\mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X)$. It can be shown that the complement of $\mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X)$ has codimension at least two in $\mathcal{M}_{\mathrm{Higgs}}(X)$. Since the latter is normal, the function f in (2.6) thus extends to a holomorphic function

$$\widetilde{f} : \mathcal{M}_{\mathrm{Higgs}}(X) \longrightarrow \mathbb{C},$$

for example by [Sc, page 90, Korollar 2]. Because f is linear on the fibers, $\widetilde{f}$ is homogenous of degree 1 for the action (2.5) of $\mathbb{C}^*$.

On the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$, the Hitchin map (2.4) is proper by [Ni, Theorem 6.1], and its fibers are connected. Therefore, the function $\widetilde{f}$ is constant on the fibers of the Hitchin map. Hence $\widetilde{f}$ comes from a holomorphic function on the Hitchin space, which is still homogenous of degree 1. Since the only such function is identically zero, this implies $\widetilde{f} = 0$, and consequently also $f = 0$ and $s = 0$. □

Corollary 2.3. *The restriction of the holomorphic tangent bundle*

$$T\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}} \longrightarrow \mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$$

to $\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ *does not admit any nonzero holomorphic section.*

Proof. Using Lemma 2.2, it suffices to show that the normal bundle of the embedding

$$\iota : \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$$

has no nonzero holomorphic sections. The isomorphism (2.3) allows us to identify this normal bundle with $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$. Now the claim follows from Lemma 2.1. □

The next step is to show that the above property uniquely characterizes the subvariety $\iota(\mathcal{M}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Higgs}}(X)$. This will follow from the following fact.

Proposition 2.4. *Let* Z *be an irreducible component of the fixed point locus*

$$\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathbb{C}^*} \subseteq \mathcal{M}_{\mathrm{Higgs}}(X). \tag{2.7}$$

Then $\dim(Z) \leq (r^2-1)(g-1)$, *with equality only for* $Z = \iota(\mathcal{M}_{r,\mathcal{O}_X})$.

Proof. The $\mathbb{C}^*$-equivariance of the Hitchin map H in (2.4) implies

$$\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathbb{C}^*} \subseteq H^{-1}(0),$$

because 0 is the only fixed point in the Hitchin space. We recall that $H^{-1}(0)$ is called the *nilpotent cone*. The irreducible components of $H^{-1}(0)$ are parameterized by the conjugacy classes of the nilpotent elements in the Lie algebra $\mathrm{sl}(r,\mathbb{C})$, and each irreducible component of $H^{-1}(0)$ is of dimension $(r^2-1)(g-1)$ [La].

Thus $\dim(Z) \leq (r^2-1)(g-1)$, and if equality holds, then Z is an irreducible component of the nilpotent cone $H^{-1}(0)$. A result due to Simpson [Si3, Lemma 11.9] implies that the only irreducible component of $H^{-1}(0)$ contained in the fixed point locus (2.7) is the image $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ of the embedding (2.1). □

Corollary 2.5. *The isomorphism class of the complex analytic space* $\mathcal{M}_{\mathrm{Higgs}}(X)$ *determines uniquely the isomorphism class of the Riemann surface* X.

In other words, if $\mathcal{M}_{\mathrm{Higgs}}(X)$ is biholomorphic to $\mathcal{M}_{\mathrm{Higgs}}(Y)$ for another compact connected Riemann surface Y of the same genus g, then $Y \cong X$.

Proof. Let $Z \subset \mathcal{M}_{\mathrm{Higgs}}(X)$ be a closed analytic subset with the following three properties:

- Z is irreducible and has complex dimension $(r^2-1)(g-1)$.
- The smooth locus $Z^{\mathrm{sm}} \subseteq Z$ lies in the smooth locus $\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X)$.
- The restriction of the holomorphic tangent bundle $T\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ to the subspace $Z^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ has no nonzero holomorphic section.

By Corollary 2.3, the image $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ of the embedding ι in (2.1) has these properties.

The action (2.5) of $\mathbb{C}^*$ on $\mathcal{M}_{\mathrm{Higgs}}(X)$ defines a holomorphic vector field

$$\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}} \longrightarrow T\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}.$$

We have assumed that any such holomorphic vector field on $\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ vanishes on Z^{sm}. It follows that the stabilizer of each point in $Z^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X)$ has nontrivial tangent space at $1 \in \mathbb{C}^*$, and hence has to be the whole group $\mathbb{C}^*$.

This shows that the fixed point locus $\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathbb{C}^*} \subseteq \mathcal{M}_{\mathrm{Higgs}}(X)$ contains Z^{sm}, and hence also contains its closure Z in $\mathcal{M}_{\mathrm{Higgs}}(X)$. Due to Proposition 2.4, this can only happen for $Z = \iota(\mathcal{M}_{r,\mathcal{O}_X})$; in particular, $Z \cong \mathcal{M}_{r,\mathcal{O}_X}$.

We have just shown that the isomorphism class of $\mathcal{M}_{\mathrm{Higgs}}(X)$ determines the isomorphism class of $\mathcal{M}_{r,\mathcal{O}_X}$. The latter determines the isomorphism class of X due to a theorem of Kouvidakis and Pantev [KP, page 229, Theorem E]. □

Remark 2.6. In [BG], an analogous Torelli theorem is proved for Higgs bundles (E, θ) such that the rank and the degree of the underlying vector bundle E are coprime.

3. λ-connections

In this section, we consider vector bundles with connections, and more generally with λ-connections in the sense of [Si2, p. 87] and [Si1, p. 4]. We denote by

$$\mathcal{M}_{\mathrm{Hod}}(X)$$

the moduli space of λ-connections over X for the group $\mathrm{SL}(r, \mathbb{C})$. Recall that such a λ-connection consists of a number $\lambda \in \mathbb{C}$, a holomorphic vector bundle E over X of rank r together with an isomorphism $\bigwedge^r E \cong \mathcal{O}_X$, and a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : E \longrightarrow E \otimes K_X.$$

This operator ∇ is required to be compatible with the de Rham differential

$$d : \mathcal{O}_X \longrightarrow K_X$$

up to the factor λ, in the sense that it satisfies the following two conditions:

- If f is a locally defined holomorphic function on $\mathcal{O}_X$ and s is a locally defined holomorphic section of E, then
$$\nabla(fs) = f \cdot \nabla(s) + \lambda \cdot s \otimes df.$$
- The operator $\bigwedge^r E \longrightarrow (\bigwedge^r E) \otimes K_X$ induced by ∇ coincides with $\lambda \cdot d$.

The moduli space $\mathcal{M}_{\mathrm{Hod}}(X)$ is a complex algebraic variety of dimension $1 + 2(r^2 - 1)(g - 1)$. It comes with a surjective algebraic morphism

$$\mathrm{pr}_\lambda : \mathcal{M}_{\mathrm{Hod}}(X) \longrightarrow \mathbb{C} \tag{3.1}$$

defined by $(\lambda, E, \nabla) \longmapsto \lambda$.

A λ-connection with $\lambda = 0$ is a Higgs bundle, so

$$\mathcal{M}_{\mathrm{Higgs}}(X) = \mathrm{pr}_\lambda^{-1}(0) \subset \mathcal{M}_{\mathrm{Hod}}(X)$$

is the moduli space of Higgs bundles considered in the previous section. In particular, the embedding (2.1) of $\mathcal{M}_{r,\mathcal{O}_X}$ into $\mathcal{M}_{\mathrm{Higgs}}(X)$ also defines an embedding into $\mathcal{M}_{\mathrm{Hod}}(X)$.

Slightly abusing notation, we denote this embedding again by

$$\iota : \mathcal{M}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\mathrm{Hod}}(X). \tag{3.2}$$

It maps the stable locus $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \subset \mathcal{M}_{r,\mathcal{O}_X}$ into the smooth locus $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Hod}}(X)$.

We let $\mathbb{C}^*$ act on $\mathcal{M}_{\mathrm{Hod}}(X)$ as

$$t \cdot (\lambda, E, \nabla) = (t \cdot \lambda, E, t \cdot \nabla). \tag{3.3}$$

This extends the $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Higgs}}(X)$ introduced above in formula (2.5).

Proposition 3.1. *Let Z be an irreducible component of the fixed point locus*

$$\mathcal{M}_{\mathrm{Hod}}(X)^{\mathbb{C}^*} \subseteq \mathcal{M}_{\mathrm{Hod}}(X).$$

Then $\dim(Z) \leq (r^2 - 1)(g - 1)$*, with equality only for $Z = \iota(\mathcal{M}_{r,\mathcal{O}_X})$.*

Proof. A point $(\lambda, E, \nabla) \in \mathcal{M}_{\mathrm{Hod}}(X)$ can only be fixed by $\mathbb{C}^*$ if $\lambda = 0$. Hence Z is automatically contained in $\mathcal{M}_{\mathrm{Higgs}}(X)$. Now the claim follows from Proposition 2.4 □

A λ-connection with $\lambda = 1$ is a (holomorphic) connection in the usual sense, so

$$\mathcal{M}_{\mathrm{conn}}(X) := \mathrm{pr}_\lambda^{-1}(1) \subset \mathcal{M}_{\mathrm{Hod}}(X) \tag{3.4}$$

is the moduli space of $\mathrm{SL}(r, \mathbb{C})$ connections (E, ∇) over X. We denote by

$$\mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X) \subset \mathcal{M}_{\mathrm{conn}}(X) \qquad \text{and} \qquad \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \subset \mathcal{M}_{\mathrm{Hod}}(X)$$

the open subvarieties where the underlying vector bundle E is stable.

Proposition 3.2. *The forgetful map*

$$\mathrm{pr}_E : \mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X) \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \tag{3.5}$$

defined by $(E, \nabla) \longmapsto E$ admits no holomorphic section.

Proof. This map pr_E is surjective, because a criterion due to Atiyah and Weil states that every stable vector bundle E on X admits a holomorphic connection. In fact E admits a unique unitary holomorphic connection according to the Narasimhan-Seshadri theorem [NS]; this defines a canonical C^∞ section

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X) \tag{3.6}$$

of the map pr_E in question. Because any two holomorphic $\mathrm{SL}(r, \mathbb{C})$-connections on E differ by a Higgs field $\theta : E \longrightarrow E \otimes K_X$ with $\mathrm{trace}(\theta) = 0$, the map pr_E in (3.5) is a holomorphic torsor under the holomorphic cotangent bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$.

Given a complex manifold $\mathcal{M}$, we denote by $T_{\mathbb{R}}\mathcal{M}$ the tangent bundle of the underlying real manifold $\mathcal{M}_{\mathbb{R}}$, and by

$$J_{\mathcal{M}} : T_{\mathbb{R}}\mathcal{M} \longrightarrow T_{\mathbb{R}}\mathcal{M}$$

the almost complex structure of $\mathcal{M}$. Let

$$\pi : \mathcal{X} \longrightarrow \mathcal{M} \tag{3.7}$$

be a holomorphic torsor under a holomorphic vector bundle $\mathcal{V} \longrightarrow \mathcal{M}$. To each C^∞ section $s : \mathcal{M} \longrightarrow \mathcal{X}$ of π, we can associate a $(0,1)$-form

$$\overline{\partial} s \in C^\infty(\mathcal{M}, \Omega^{0,1}\mathcal{M} \otimes \mathcal{V})$$

as follows. The vector bundle homomorphism

$$\widetilde{ds} := ds + J_{\mathcal{X}} \circ ds \circ J_{\mathcal{M}} : T_{\mathbb{R}}\mathcal{M} \longrightarrow s^* T_{\mathbb{R}}\mathcal{X}$$

satisfies

$$J_{\mathcal{X}} \circ \widetilde{ds} + \widetilde{ds} \circ J_{\mathcal{M}} = J_{\mathcal{X}} \circ ds - ds \circ J_{\mathcal{M}} - J_{\mathcal{X}} \circ ds + ds \circ J_{\mathcal{M}} = 0 \tag{3.8}$$

and, since π is holomorphic, also

$$d\pi \circ \widetilde{ds} = d\pi \circ ds + J_{\mathcal{M}} \circ d\pi \circ ds \circ J_{\mathcal{M}} = \mathrm{id} - \mathrm{id} = 0. \tag{3.9}$$

The equation (3.9) means that $\widetilde{ds}$ maps into the subbundle of vertical tangent vectors in $s^* T_{\mathbb{R}}\mathcal{X}$, which is canonically isomorphic to $\mathcal{V}_{\mathbb{R}}$. Thus we can consider $\widetilde{ds}$ as a real 1-form

$$\widetilde{ds} \in C^\infty(\mathcal{M}, T^*_{\mathbb{R}}\mathcal{M} \otimes \mathcal{V}_{\mathbb{R}}).$$

Its complexification $(\widetilde{ds})_{\mathbb{C}}$ is of type $(0,1)$ according to the equation (3.8). We put

$$\overline{\partial} s := \frac{1}{2}(\widetilde{ds})_{\mathbb{C}} \in C^\infty(\mathcal{M}, \Omega^{0,1}\mathcal{M} \otimes \mathcal{V}).$$

Since $\mathcal{V}$ acts on $\pi : \mathcal{X} \longrightarrow \mathcal{M}$, each section $v \in C^\infty(\mathcal{M}, \mathcal{V})$ acts on the sections of π; we denote this action by $s \longmapsto v + s$. The above construction implies

$$\overline{\partial}(v + s) = \overline{\partial} v + \overline{\partial} s$$

and that $\overline{\partial} s$ vanishes if and only if s is holomorphic. Since holomorphic sections of π exist locally, it follows that $\overline{\partial} s$ is always $\overline{\partial}$-closed, that the Dolbeault cohomology class

$$[\pi] := [\overline{\partial} s] \in H^{0,1}_{\overline{\partial}}(\mathcal{M}, \mathcal{V}) \cong H^1(\mathcal{M}, \mathcal{V})$$

does not depend on the choice of the C^∞ section s, and that this class vanishes if and only if the torsor π in (3.7) admits a holomorphic section.

We now take π to be the above torsor pr_E under the cotangent bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, and we take s to be the above C^∞ section (3.6). For this case, the class

$$[\overline{\partial} s] \in H^1(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}, T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \tag{3.10}$$

has been computed in [BR, page 308, Theorem 2.11]; the result is that it is a nonzero multiple of $c_1(\Theta)$, where Θ is the ample generator of $\mathrm{Pic}(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X})$. In particular, the cohomology class (3.10) of the torsor pr_E in question is nonzero. ☐

The forgetful map pr_E in the above Proposition 3.2 extends canonically from $\mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X)$ to $\mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X)$. Slightly abusing notation, we denote this extended map again by

$$\mathrm{pr}_E : \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X};$$

it is defined by $(\lambda, E, \nabla) \longmapsto E$, and also extends the map pr_E in (2.2).

Corollary 3.3. *The only holomorphic map*

$$s : \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X)$$

with $\mathrm{pr}_E \circ s = \mathrm{id}$ *is the restriction*

$$\iota : \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X)$$

of the embedding ι *defined above in* (3.2).

Proof. The composition

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \xrightarrow{s} \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \xrightarrow{\mathrm{pr}_\lambda} \mathbb{C}$$

with the map pr_λ in (3.1) is a holomorphic function on $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, and hence constant. Up to the $\mathbb{C}^*$ action in (3.3), we may assume that this constant is 0 or 1.

If this constant were 1, then s would factor through $\mathrm{pr}_\lambda^{-1}(1) = \mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X)$, which would contradict Proposition 3.2.

Hence this constant is 0, and s factors through $\mathrm{pr}_\lambda^{-1}(0) = \mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X)$. Thus s corresponds, under the isomorphism (2.3), to a holomorphic global section of the vector bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$. But any such section vanishes due to Lemma 2.1; this means that s is indeed the restriction of the canonical embedding ι. □

Corollary 3.4. *The restriction of the holomorphic tangent bundle*

$$T\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$$

to $\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$ *does not admit any nonzero holomorphic section.*

Proof. We denote the holomorphic normal bundle of the restricted embedding

$$\iota : \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$$

by $\mathcal{N}$. Due to Lemma 2.2, it suffices to show that this vector bundle $\mathcal{N}$ over $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ has no nonzero holomorphic sections. One has a canonical isomorphism

$$\mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \xrightarrow{\sim} \mathcal{N}$$

of varieties over $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, defined by sending (λ, E, ∇) to the derivative at $t = 0$ of the map

$$\mathbb{C} \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X), \qquad t \longmapsto (t \cdot \lambda, E, t \cdot \nabla).$$

Using this isomorphism, the claim follows from Corollary 3.3. □

Corollary 3.5. *The isomorphism class of the complex analytic space* $\mathcal{M}_{\mathrm{Hod}}(X)$ *determines uniquely the isomorphism class of the Riemann surface* X.

In other words, if $\mathcal{M}_{\mathrm{Hod}}(X)$ is biholomorphic to $\mathcal{M}_{\mathrm{Hod}}(Y)$ for another compact connected Riemann surface Y of the same genus g, then $Y \cong X$.

Proof. The idea is again that the property in Corollary 3.4 uniquely characterizes the subvariety $\iota(\mathcal{M}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Hod}}(X)$, due to Proposition 3.1. The details of this argument are exactly the same as for $\mathcal{M}_{\mathrm{Higgs}}(X)$ in Corollary 2.5. □

4. The Deligne-Hitchin moduli space

We recall Deligne's construction [De] of the Deligne–Hitchin moduli space $\mathcal{M}_{\mathrm{DH}}(X)$, as described in [Si1, p. 7]. Our target group will always be $\mathrm{SL}(r, \mathbb{C})$.

Let $X_{\mathbb{R}}$ be the C^∞ real surface underlying X, and let

$$\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) := \mathrm{Hom}(\pi_1(X_{\mathbb{R}}), \mathrm{SL}(r, \mathbb{C}))/\mathrm{SL}(r, \mathbb{C})$$

denote the moduli space of representations $\rho : \pi_1(X_{\mathbb{R}}) \longrightarrow \mathrm{SL}(r, \mathbb{C})$. The Riemann–Hilbert correspondence defines a biholomorphic isomorphism

$$\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{conn}}(X). \tag{4.1}$$

It sends each representation $\rho : \pi_1(X_{\mathbb{R}}) \longrightarrow \mathrm{SL}(r, \mathbb{C})$ to the associated holomorphic $\mathrm{SL}(r, \mathbb{C})$-bundle E_ρ^X over X, endowed with the induced connection ∇_ρ^X. The inverse of (4.1) sends each connection to its monodromy representation, which makes sense because any holomorphic connection on a Riemann surface is automatically flat.

Given $\lambda \in \mathbb{C}^*$, we can similarly associate to a representation $\rho : \pi_1(X_{\mathbb{R}}) \longrightarrow \mathrm{SL}(r, \mathbb{C})$ the λ-connection $(E_\rho^X, \lambda \cdot \nabla_\rho^X)$. This defines a holomorphic open embedding

$$\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X)$$

onto the open locus $\mathrm{pr}_\lambda^{-1}(\mathbb{C}^*) \subset \mathcal{M}_{\mathrm{Hod}}(X)$ of all λ-connections (λ, E, ∇) with $\lambda \neq 0$.

Let J_X denote the almost complex structure of the Riemann surface X. Then $-J_X$ is also an almost complex structure on $X_{\mathbb{R}}$; we denote the resulting Riemann surface by $\overline{X}$. Thus we can also consider the moduli space $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ of λ-connections on $\overline{X}$, etc.

Now one defines the Deligne–Hitchin moduli space

$$\mathcal{M}_{\mathrm{DH}}(X) := \mathcal{M}_{\mathrm{Hod}}(X) \cup \mathcal{M}_{\mathrm{Hod}}(\overline{X})$$

by glueing $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ to $\mathcal{M}_{\mathrm{Hod}}(X)$, along the image of $\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$. More precisely, one identifies, for each $\lambda \in \mathbb{C}^*$ and each representation $\rho \in \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$, the two points

$$(\lambda, E_\rho^X, \lambda \cdot \nabla_\rho^X) \in \mathcal{M}_{\mathrm{Hod}}(X) \qquad \text{and} \qquad (\lambda^{-1}, E_\rho^{\overline{X}}, \lambda^{-1} \cdot \nabla_\rho^{\overline{X}}) \in \mathcal{M}_{\mathrm{Hod}}(\overline{X}).$$

This defines a complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ of dimension $2(r^2-1)(g-1)+1$, which has no natural algebraic structure since the Riemann–Hilbert correspondence (4.1) is not algebraic. The forgetful map pr_λ in (3.1) extends to a natural holomorphic morphism

$$\mathcal{M}_{\mathrm{DH}}(X) \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$$

whose fiber over $\lambda \in \mathbb{CP}^1$ is canonically biholomorphic to

- the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$ of $\mathrm{SL}(r, \mathbb{C})$ Higgs bundles on X for $\lambda = 0$,
- the moduli space $\mathcal{M}_{\mathrm{Higgs}}(\overline{X})$ of $\mathrm{SL}(r, \mathbb{C})$ Higgs bundles on $\overline{X}$ for $\lambda = \infty$,
- the moduli space $\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$ of representations $\pi_1(X_{\mathbb{R}}) \longrightarrow \mathrm{SL}(r, \mathbb{C})$ for $\lambda \neq 0, \infty$.

Now we are in a position to prove the main result.

Theorem 4.1. *The isomorphism class of the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines uniquely the isomorphism class of the unordered pair of Riemann surfaces $\{X, \overline{X}\}$.*

Proof. We denote by $\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)$ the smooth locus, and by

$$T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \longrightarrow \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$$

its holomorphic tangent bundle. Since $\mathcal{M}_{\mathrm{Hod}}(X)$ is open in $\mathcal{M}_{\mathrm{DH}}(X)$, Corollary 3.4 implies that the restriction of $T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$ to

$$\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \tag{4.2}$$

does not admit any nonzero holomorphic section. The same argument applies to $\overline{X}$ instead of X. Since $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ is also open in $\mathcal{M}_{\mathrm{DH}}(X)$, the restriction of $T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$ to

$$\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_{\overline{X}}}) \subset \mathcal{M}_{\mathrm{Hod}}(\overline{X})^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \tag{4.3}$$

does not admit any nonzero holomorphic section either. Here $\mathcal{M}_{r,\mathcal{O}_{\overline{X}}}$ is the moduli space of holomorphic $\mathrm{SL}(r,\mathbb{C})$-bundles E on $\overline{X}$, and ι denotes, as in (2.1) and in (3.2), also the canonical embedding $E \longmapsto (E,0)$ of $\mathcal{M}_{r,\mathcal{O}_{\overline{X}}}$ into $\mathcal{M}_{\mathrm{Higgs}}(\overline{X}) \subset \mathcal{M}_{\mathrm{Hod}}(\overline{X})$.

The idea of the proof is again that this property characterizes the two subvarieties (4.2) and (4.3) uniquely, as in Corollary 2.5. To see that, we extend the $\mathbb{C}^*$ action (3.3) from $\mathcal{M}_{\mathrm{Hod}}(X)$ to $\mathcal{M}_{\mathrm{DH}}(X)$, by glueing it with the inverse of the analogous action on $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$. Due to Proposition 3.1, each irreducible component of the fixed point locus

$$\mathcal{M}_{\mathrm{DH}}(X)^{\mathbb{C}^*} \subseteq \mathcal{M}_{\mathrm{DH}}(X)$$

has dimension $\leq (r^2-1)(g-1)$, with equality only for $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ and for $\iota(\mathcal{M}_{r,\mathcal{O}_{\overline{X}}})$.

These observations imply that $\mathcal{M}_{\mathrm{DH}}(X)$ determines the unordered pair of moduli spaces $\{\mathcal{M}_{r,\mathcal{O}_X}, \mathcal{M}_{r,\mathcal{O}_{\overline{X}}}\}$, and hence also the unordered pair of Riemann surfaces $\{X, \overline{X}\}$. The details of this argument are exactly the same as for $\mathcal{M}_{\mathrm{Higgs}}(X)$ in Corollary 2.5. □

Acknowledgements

The first and second authors were supported by the grant MTM2007-63582 of the Spanish Ministerio de Educación y Ciencia. The second author was also supported by the grant 200650M066 of Comunidad Autónoma de Madrid. The third author was supported by the SFB/TR 45 'Periods, moduli spaces and arithmetic of algebraic varieties'.

References

[BBGN] I. Biswas, L. Brambila-Paz, T. L. Gómez and P. E. Newstead, Stability of the Picard bundle, *Bull. London Math. Soc.* **34** (2002), 561–568.

[BG] I. Biswas and T. L. Gómez, A Torelli theorem for the moduli space of Higgs bundles on a curve, *Q. J. Math.* **54** (2003), 159–169.

[BR] I. Biswas and N. Raghavendra, Curvature of the determinant bundle and the Kähler form over the moduli of parabolic bundles for a family of pointed curves, *Asian Jour. Math.* **2** (1998), 303–324.

[De] P. Deligne, Letter to C. T. Simpson (March 20, 1989).

[Hi1] N. J. Hitchin, Stable bundles and integrable systems, *Duke Math. Jour.* **54** (1987), 91–114.

[Hi2] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. Lond. Math. Soc.* **55** (1987), 59–126.

[KP] A. Kouvidakis and T. Pantev, The automorphism group of the moduli space of semistable vector bundles, *Math. Ann.* **302** (1995), 225–268.

[La] G. Laumon, Un analogue global du cône nilpotent, *Duke Math. Jour.* **57** (1988), 647–671.

[NR1] M. S. Narasimhan and S. Ramanan, Moduli of vector bundles on a compact Riemann surface, *Ann. Math.* **89** (1969), 14–51.

[NR2] M. S. Narasimhan and S. Ramanan, Geometry of Hecke cycles. I, *C. P. Ramanujan—a tribute*, pp. 291–345, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin-New York, 1978.

[NS] M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.

[Ni] N. Nitsure, Moduli space of semistable pairs on a curve, *Proc. Lond. Math. Soc.* **62** (1991), 275–300.

[Sc] G. Scheja, Fortsetzungssätze der komplex-analytischen Cohomologie und ihre algebraische Charakterisierung, *Math. Ann.* **157** (1964), 75–94.

[Se] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, (notes written by J.-M. Drézet), *Astérisque* 96, Société Mathématique de France, Paris, 1982.

[Si1] C. T. Simpson, A weight two phenomenon for the moduli of rank one local systems on open varieties, ArXiv:0710.2800.

[Si2] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. I, *Inst. Hautes Études Sci. Publ. Math.* 79 (1994), 47–129.

[Si3] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. II, *Inst. Hautes Études Sci. Publ. Math.* 80 (1994), 5–79.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Serrano 113bis, 28006 Madrid, Spain; and Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address: tomas.gomez@mat.csic.es

Freie Universität Berlin, Institut für Mathematik, Arnimallee 3, 14195 Berlin, Germany

E-mail address: nhoffman@mi.fu-berlin.de

Departamento de Matematica Pura, Facultade de Ciencias, Rua do Campo Alegre 687, 4169-007 Porto Portugal

E-mail address: mlogares@fc.up.pt