KNEADING SEQUENCES FOR DOUBLE STANDARD MAPS

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Dedicated to our common friend, Michał Misiurewicz, on the occasion of his 60th Birthday with an appreciation of all what he has done for us and our field.

Abstract. We investigate the symbolic dynamics for the double standard maps of the circle onto itself, given by $f_{a,b}(x) = 2x + a + (b/\pi)\sin(2\pi x)(\text{mod } 1)$, where $b = 1$ and a is a real parameter $0 \le a < 1$.

1. Introduction

In the family of *double standard maps* of the circle onto itself, given by

$$f_{a,b}(x) = 2x + a + (b/\pi)\sin(2\pi x) \ (\text{mod } 1), \tag{1.1}$$

where the parameters a, b are real and $0 \le b \le 1$, tongues (sets of parameter values for which there is an attracting periodic point) appear (see Misiurewicz-Rodrigues [17, 18, 19]).

The aim of this paper is to study the symbolic dynamics for the double standard family assuming $b = 1$. We define $f_a = f_{a,1}$ for $f_{a,b}$ in (1.1). Moreover, we relate this with the "Real Fatou Conjecture", this is, the density of parameters with attractive periodic orbits.

In the classical theory of Milnor-Thurston the symbolic coding is associated to the two intervals where the restriction of the map to each of them is increasing or decreasing (the basic background for symbolic dynamics and kneading theory may be found in [7]). Consider the *double standard family* (1.1) with $b = 1$ and $a \in [0, 1]$, then for each value of the parameter a the map is increasing for all values of $x \in [0, 1]$ except for the values of the parameter for which there is an attractive periodic orbit of period 1 (see [18]).

The explicit computation of the boundary of the period 1 tongue (see [19]) gives:

$$a = \frac{1}{2} \pm \frac{\sqrt{4b^2 - 1} - \arctan\sqrt{4b^2 - 1}}{2\pi}, \tag{1.2}$$

which allow us to compute the interval $[a_0', a_0]$ for which we have an attractive periodic orbit of period one. We get $a_0 \simeq 0.65$, the bifurcation point for the period 1 tongue for $a > 1/2$, and $a_0' = 1 - a_0$.

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For $a_0 < a < 1$, f_a has a unique fixed point, which we will denote by $p(a)$. This follows from the bifurcation behavior of the fixed point(s). For $a = \frac{1}{2}$, f_a has three fixed points, one at $x = \frac{1}{2}$ and two repelling fixed points. According to the Implicit Function Theorem this behavior persists for $a_0' < a < a_0$. For $a = a_0$, the two rightmost fixed points go through a saddle node bifurcation and disappear. Since the only bifurcation of fixed points for $1/2 < a < 1$ appears at $a = a_0$ it is clear that there is at most one fixed point in this interval, the continuation of the left fixed point that exists for $\frac{1}{2} < a < a_0$. We denote this fixed point by $p(a)$. We will also prove that $p(a)$ for $a_0 < a < 1$ has a unique preimage different from $p(a)$. We denote this preimage by $q(a)$. For $0 < a < a_0'$ the situation is completely symmetric.

We use a symbolic coding related to Yoccoz partitions of the interval [23], but in our case we will apply it to the circle. Let $J_0 = (p(a), q(a))$ for $a > \frac{1}{2}$, where the circle segments are chosen so that they have positive orientation on the circle, and let $J_1 = \mathrm{int}(\mathbb{T} \setminus J_0)$. In the case $0 < a < a_0'$, $J_0 = [0, q(a)) \cup (p(a), 1)$ where the circle is represented by the halfopen fundamental domain $[0, 1)$ and as before J_1 is the interior of its complement.

A more geometric way to express the same is to say that J_0 is the positively oriented (counterclockwise) open arc on the unit circle from $p(a)$ to $q(a)$, and J_1 is the interior of its complement.

For a given initial point x on the circle such that its orbit does not land on p or q let

$$i_n(x) = \begin{cases} 0 \text{ if } f_a^n(x) \in J_0, \\ 1 \text{ if } f_a^n(x) \in J_1. \end{cases} \tag{1.3}$$

For a point that eventually hit p after possibly passing through q, the coding is so far not defined. For these orbits, we define the coding by either $i_0 \dots i_n\, 0\, \overline{1}$ or $i_0 \dots i_n\, 1\, \overline{0}$, and we identify these sequences. Note that this is exactly the same identification as is made of the binary expansions

$$0.i_0 \dots i_n\, 0\, \overline{1} \text{ and } 0.i_0 \dots i_n\, 1\, \overline{0},$$

when they are interpreted as real numbers.

Thus, we associate with each $x \in \mathbb{T}$ a finite or infinite sequence of the symbols $0, 1$ called its *itinerary*. We denote by $I(x)$ the sequence $\{i_n(x)\}_{n=0}^{\infty}$ and this sequence is also naturally identified with a real number in $[0, 1]$.

As usual the kneading sequence will be the itinerary of the critical value $f_a(1/2) = a$ and we denote it by $K(f_a)$.

The paper is organized as follows. In Section 2 we introduce the symbolic coding and we study the monotonicity of the itinerary map and of the kneading sequence. In Section 3 we investigate some properties of the periodic kneading sequences. In Section 4 we investigate some properties of the aperiodic kneading sequences and we show how to connect our work to the "Real Fatou Conjecture", which in the present setting is a result by Levin and van Strien [12], Theorem C. In Section 5 we state some results concerning the abundance of positive Lyapunov exponents and invariant measures. Finally, in Section 6 we state some conjectures.

2. Kneading sequences

Let $I(x), I(y)$ be two sequences of the symbols $\{0,1\}$ such that $I(x) \neq I(y)$. Suppose

$$I(x) = i_0^1 i_1^1 i_2^1 \cdots i_n^1, \quad I(y) = i_0^2 i_1^2 i_2^2 \cdots i_n^2, \tag{2.1}$$

where i_n is the smallest index for which $i_n^1 \neq i_n^2$. We order the sequences (2.1) lexicographically, if $i_n^1 = 0, i_n^2 = 1$ then $I(x) < I(y)$. It is obvious that this order coincides with the order of the real numbers corresponding to the symbol sequences interpreted as binary expansions. It is clear that the map $x \mapsto I(x)$ is continuous, where the topology on the kneading sequences is the topology of real numbers.

We prove the following lemma:

Lemma 2.1. *For $f = f_{a,b}$ as in* (1.1) *with $b = 1$ and for a fixed value of the parameter a, if $x < y$ then $I(x) \leq I(y)$.*

Proof. By the continuity of $x \mapsto I(x)$ it is enough to treat the case when x and y are not binary rationals.

We will prove

$$I(x) > I(y) \implies x > y. \tag{2.2}$$

We argue by induction on the index where $I(x)$ an $I(y)$ differ. For $n = 0$, (2.2) is obvious. Assume it has been proved for $n-1$ and we wish to prove it for n. But $I(x) > I(y) \implies I(f(x)) > I(f(y))$. By (2.2) for $n-1$, we conclude that $f(x) > f(y)$. But by the strict monotonicity of f it now follows that $x > y$. □

We also define an order on the kneading sequences in exactly the same way as we defined an order on the itinereraries.

We have the following lemma:

Lemma 2.2. *Fix $b = 1$ in* (1.1). *Assume that $1/2 < a_* < 1$ does not belong to the period 1 tongue and that f_{a_*} has the property that the critical point $1/2$ is eventually fixed, i.e. there is an integer j so that $f_{a_*}^j(1/2) = p$. Then $a \mapsto K(f_a)$ is strictly monotonic at a_*. More precisely there is a real number $\delta > 0$ so that*

$$a_* - \delta < a_1 < a_* < a_2 < a_* + \delta < 1 \quad \Rightarrow K(f_{a_1}) < K(f_{a_2}).$$

Proof. Outside the period 1 tongue we have $a_0 < a < 1$. We begin by proving that for a in this interval we have $0 < p(a) < 1/6$ and that $p(a)$ decreases as a function of a. Consider the lift F_a of (1.1) to the real line. The fixed point of f_a satisfies

$$F_a(x) = 2x + a + \frac{1}{\pi}\sin(2\pi x) = x + 1.$$

Consider the function $G_a(x) = F_a(x) - x - 1$ and $1/2 < a < 1$. We have $G_{a,b}(0) = a - 1 < 0$ and $G_{a,b}(1/6) = -\frac{5}{6} + a + \frac{\sqrt{3}}{2\pi} > 0$. Thus, there is $0 < c < 1/6$ such that $G_a(c) = F_{a,b}(c) - c - 1 = 0$.

Differentiating $F_a(p(a)) = p(a) + 1$ with respect to a yields

$$[1 + 2\cos(2\pi p(a))]\,\frac{dp(a)}{da} + 1 = 0.$$

Now if $0 \leq p(a) < 1/6$ then $\cos(2\pi p(a)) > -\frac{1}{2}$ and we conclude that $dp(a)/da < 0$. Now we have $p(a) = 0$ only for $a = 1$ so we conclude that $0 < p(a) < 1/6$.

We prove now that $q(a)$ exists and decreases as a function of a and that $5/6 < q(a) < 1$. For $a_0 < a < a_0 + \epsilon, \epsilon$ small we have $q(a) \simeq 0.61$. For $1 - \epsilon < a < 1$, $q(a)$ is small and positive. Since $q(a)$ satisfies

$$F_a(q(a)) = p(a) + 1$$

we obtain

$$[(2 + 2\cos(2\pi q(a))] \frac{dq(a)}{da} = -1 - \frac{dp(a)}{da}.$$

From the Implicit Function Theorem it follows that $q(a)$ exists in the relevant interval $a_0 < a < 1$. Using that $dp(a)/da < 0$ and the behaviour at the end points we conclude that $dq(a)/da < 0$ for $a_0 < a < 1$.

We want to prove that $f_a^n(1/2)$ increases as a function of a. Recall the proof of Lemma 2.6 from [18]: since $f_a^n(x) = f_a^{n-1}(f_a(x))$, we have

$$\frac{\partial f_a^n}{\partial a}(x) = \frac{\partial f_a^{n-1}}{\partial a}(f_a(x)) + (f_a^{n-1})'(f_a(x))\frac{\partial f_a}{\partial a}(x).$$

By (1.1), $\frac{\partial f_a}{\partial a}(y) = 1$ for every y, so by induction we obtain

$$\frac{\partial f_a^n}{\partial a}(x) = \sum_{k=0}^{n-1}(f_a^k)'(f_a^{n-k}(x)).$$

Since f_a' is nonnegative everywhere, so is $(f_a^k)'$. Moreover, if $k = 0$ then $(f_a^k)' \equiv 1$.

Set $\xi_n(a) = f_a^n(1/2)$. We assume that a_0 is such that we do not have a period 1 tongue. Let a be very close to a_0. When a increases, $\xi_n(a) = f_a^n(1/2)$ hits p for $a = a_0$ and there is a change in the n:th symbol of the kneading sequence. But actually ξ_{n-1} must simultaneously hit the preimage q. This means that there are two symbols that changes simultaneously. The monotonicity properties of $\xi_n(a)$, $p(a)$ and $q(a)$ proved above show that I_n changes from 1 to 0 but I_{n-1} changes from 0 to 1. The later change of I_{n-1} has more weight since it is of lower index. Hence the kneading must increase at a_0 when a increases.

□

As a consequence we obtain the following statement on the monotonicity of kneading sequences for the family f_a.

Lemma 2.3. *Let f_{a_1}, f_{a_2} belong to the double standard family. Then*
(i) *$K(f_{a_1}) < K(f_{a_2})$ implies $a_1 < a_2$;*
(ii) *$a_1 < a_2$ implies $K(f_{a_1}) \leq K(f_{a_2})$.*

Proof. We first prove the lemma in the case $a_0 < a_1 < a_2$. The case $0 < a_1 < a_2 < a_0'$ will be postponed until after the proof of Lemma 2.4. Let n be first index where the $K(f_{a_1}) = (k_i^1)_{i=0}^{\infty}$ and $K(f_{a_2}) = (k_i^2)_{i=0}^{\infty}$ differ. The proof of (i) goes by induction on n. To begin the statement is true for $n = 1$ by direct computation. Suppose that (i) is proven for all indices of change $\leq n - 1$ and let $n \geq 2$.

Suppose, arguing by contradiction that $a_1 > a_2$. Let a_1^* be the infimum of the a:s so that $a_1 > a > a_2$ and $K(f_a)_i = k_i^1$, $i = 0, 1, 2, ..., n$. The infimum cannot be

a_2 since the kneading for $0 \leq i \leq n$ is constant near a_2. There must then be a a where the kneading changes. Since $k_n^1 = 0$ actually some $K(f_a)_i$, $0 \leq i \leq n-1$ must change and then we have the monotonicity at that a. We conclude that there is a_1' so that $K(f_{a_1'}) \leq K(f_{a_1}) \leq K(f_{a_2})$, and $a_1' > a_2$ and $K(f_{a_1'})$ and $K(f_{a_2})$ differ at index $\leq n-1$.

We then get a contradiction to the induction assumption that (i) has been proven for change index $\leq n-1$.

Now the proof of (ii) is immediate. We again argue by contradiction. If we assume $K(f_{a_1}) > K(f_{a_2})$ we get by (i) that $a_1 > a_2$ which contradicts the assumption $a_1 < a_2$. □

Lemma 2.4. *Assume that* $1/2 < a_1 < 1,\ a_2 = 1 - a_1$ *and* $K(f_{a_1}), K(f_{a_2})$ *are the corresponding kneading sequences. Then* $K(f_{a_1}) = 1 - K(f_{a_2})$.

Proof. Assume that $1/2 < a_1 < 1$ and $a_2 = 1 - a_1$. We start by proving that if $p(a_1), p(a_2)$ are fixed points of f_{a_1}, f_{a_2}, respectively, then $p(a_2) = 1 - p(a_1)$ and $q(a_2) = 1 - q(a_1)$. Consider the lifting F_a of f_a to the real line. A fixed point of F_a satisfies

$$F_a = 2x + a + \frac{1}{\pi}\sin(2\pi x) = x + 1.$$

Consider the map $G(a,x) = F_a(x) - x - 1$. If $(a_1, p(a_1))$ is a solution of $G(a,x) = 0$ then $(1 - a_1, 1 - p(a_1))$ is also a solution. Assume $G_{a_1}(p(a_1)) = p(a_1) + a_1 + \pi^{-1}\sin(2\pi p(a_1)) - 1 = 0$, then $G_{1-a_1}(1 - p(a_1)) = -G_{a_1}(p(a_1)) = 0$.

We prove now that if $p(a_1) < f_{a_1}^n(1/2) < q(a_1)$, then $q(a_2) < f_{a_2}^n(1/2) < p(a_2)$, for $a_2 = 1 - a_1$. We have that

$$p(a_1) < f_{a_1}^n(1/2) < q(a_1),$$

implies

$$1 - p(a_2) < f_{1-a_2}^n(1/2) < 1 - q(a_2),$$

and so

$$q(a_2) < 1 - f_{1-a_2}^n(1/2) < p(a_2).$$

We will prove that

$$f_{a_2}^n(1/2) = 1 - f_{1-a_2}^n(1/2) \tag{2.3}$$

by induction on n. For $n = 1$ we have $f_{a_2}^n(1/2) = 1 - f_{1-a_2}^n(1/2) = a_2 \pmod 1$. Assume (2.3) holds for n. We show it holds for $n+1$:

$$\begin{aligned} f_{a_2}(f_{a_2}^n(1/2)) = \ & f_{a_2}(1 - f_{1-a_2}^n(1/2)) = f_{a_2}(-f_{a_2}^n(1/2)) = -f_{-a_2}(f_{1-a_2}^n(1/2)) = \\ & -f_{1-a_2}(f_{1-a_2}^n(1/2)) = -f_{1-a_2}^{n+1}(1/2) = 1 - f_{1-a_2}^{n+1}(1/2). \end{aligned}$$

Hence we see that for the n:th item in the kneading sequences if $k_n(f_{a_1}) = 0$ then $k_n(f_{a_2}) = 1$ and vise-versa so $K(f_{a_1}) = 1 - K(f_{a_2})$. □

Proof of Lemma 2.3 for $a < \frac{1}{2}$. We suppose that $0 < a_1 < a_2 < a_0'$. Then $1 - a_2 < 1 - a_1$ and by Lemma 2.3 for $a > \frac{1}{2}$ it follows that $K(1 - a_2) \leq K(1 - a_1)$ and (ii) of Lemma 2.3 follows. The proof of (i) is similar.

The next lemma will be quite useful. Since $K(a)$ for $0 \leq a \leq a_0'$ and $a_0 \leq a < 1$, may be interpreted as real numbers, it will immediately imply an immediate value theorem for kneading sequences:

Lemma 2.5. *The map $a \mapsto K(a)$ is continuous, where the topology on the space of kneading sequences is given by the topology of the real numbers in the interval* $[0,1]$.

Proof. The argument is standard and follows the scheme in [7]. One verifies that $A_0 = \{a|\ K(a) > K_0\}$ and $A_0' = \{a|\ K(a) < K_0\}$ are both open by verifying that each point in these sets is contained in open sets of the type $\{a|\ k_j(a) = k_j^0,\ j = 0, \dots, n\}$, which in its turn is contained in the corresponding set A_0 and A_0' respectively.

□

3. Periodic kneading sequences

In this section we prove that the kneading sequence corresponding to a periodic orbit is a periodic sequence. Furthermore we prove that it corresponds to the binary expansion of the rational number assigning the order of the given tongue as described in [19].

Let W be an open subset of $\mathbb{T}$. We say that $f|_W$ has a *sink* if there is an open interval $K \subset W$ such that $f(K) \subset K$ for some $n \geq 1$ and $f^j(K) \subset W$, $j = 1, \dots, n-1$ (and hence for all j).

We note the following lemma

Lemma 3.1. *If $f = f_a$ is a map from the double standard family and W is an open subset of $\mathbb{T}$ such that $f|_W$ has a sink then $\overline{W}$ contains an attractive periodic orbit.*

Proof. If f has a sink then $g = f^n|_K$ is a homeomorphism of K into itself. We want to show that g has a stable fixed point in $\overline{K}$. It is obvious that there is a point $x \in \overline{K}$ such that $g(x) = x$. If g maps $\overline{K}$ into K the fixed point lies in K and must satisfy $0 \leq g'(x) \leq 1$. If $\leq g'(x) < 1$, we are done. If $g'(x) = 1$ then x is stable from the left.

If $g(x) = x$ for some $x \in \partial\overline{K}$ then we argue as follows. Suppose x is the left endpoint. If $g'(x)$ and $g(y) < y$ for y near x in K then x is one-sided stable. If $g(y) > y$ for y near x or $g'(x) > 1$ then we argue as in the case $g(\overline{K}) \subset K$. The right endpoint is handled in a symmetric fashion.

□

We next turn to a result, which in the unimodal case is due to Guckenheimer [9].

Theorem 3.2. *If f_a belongs to the double standard family then*

(i) *f_a has a stable periodic orbit if and only if $K(f_a)$ is periodic.*

(ii) *If $K(f_{a_1})$ is periodic and $K(f_{a_1}) = K(f_{a_2})$ then f_{a_1} and f_{a_2} are topologically conjugate.*

Proof. We start by proving (i). Assume that f has a stable periodic orbit of period n. Let x be a point on the orbit. Let y be a critical point of f^n so that $0 < |Df^n| \leq 1$ on $(x;y)$ ($(x;y)$ is (x,y) or (y,x), respectively, depending on whether $x < y$ or $x > y$). That such a critical point exists follows since there is always a critical point in the immediate basin of an attractive periodic orbit by Singer's theorem.

Let j be such that $f^j(y) = \frac{1}{2}$. Now $f^{rn}(y)$ converges monotonously to x when $r \to \infty$. Therefore $f^j(f^{rn}(y)) = f^{rn}(\frac{1}{2})$ converges monotonously to $f^j(x)$.

Assume now $p \in (f^k(\frac{1}{2}); f^{k+j}(x))$ for some k. This implies $p \in (f^{k+j}(y); f^{k+j}(x))$; in particular when $rn > k + j$ there is a z in $(y;x)$ such that $f^{rn}(z) = p$. But this contradicts that the entire interval $(y;x)$ is attracted to the attractive periodic orbit.

Hence the case $p \in (f^k(\frac{1}{2}); f^{k+j}(x))$ does not occur. We conclude that $f(\frac{1}{2})$ and $f^{j+1}(x)$ have the same itinerary. Hence $I(f(\frac{1}{2}))$ is periodic (and not only eventually periodic).

If on the other hand $I(f(\frac{1}{2}))$ is periodic then all points in the interval $(f^{m+1}(\frac{1}{2}); f(\frac{1}{2}))$ have the same itineraries. Either $f^{m+1}(\frac{1}{2}) = f(\frac{1}{2})$ or we have a superattractive orbit. If not, let us for convenience assume that $f^{m+1}(\frac{1}{2}) < f(\frac{1}{2})$. Then the interval $(f^{m+1}(\frac{1}{2}), f(\frac{1}{2}))$ is mapped to $(f^{2m+1}(\frac{1}{2}), f^{m+1}(\frac{1}{2}))$. In the same way

$$(f^{jm+1}(\frac{1}{2}), f^{(j-1)m}(\frac{1}{2})) \mapsto (f^{(j+1)m+1}(\frac{1}{2}), f^{jm}(\frac{1}{2})).$$

It follows that the sequence $f^{jm+1}(\frac{1}{2})$ is monotonous as a function of j and it converges to a point. This must be a point of an attractive periodic orbit, which obviously then exists. The case $f^{p+1}(\frac{1}{2}) > f(\frac{1}{2})$ is completely analogous.

Now we turn to the proof of (ii). We follow the proof of Theorem II.6.3 in [7], but the present case is actually easier since $Df(x) > 0$. Let us use the notations $f = f_{a_1}$ and $g = f_{a_2}$. Let U_f be the stable neighborhood of 0 for f. Then $f^n(x) \in U_f$ if and only if $\mathcal{S}^{n+1}(I(x)) = I_f(f(\frac{1}{2})) = K(f)$. We denote

$$E_f = \{x : f^n(x) \text{ does not tend to the stable periodic orbit}\}.$$

Hence we can determine whether $x \in E_f$ from its itinerary. The same considerations apply to g and hence we can define a homeomorphism $h : E_f \mapsto E_g$ by the property $I_f(x) = I_g(h(x))$. We extend h to the set $\bigcup_{i>0} f^{-i}(U_f)$ assuming that we have already constructed a topological equivalence h from $f^n|_{U_f}$ to $f^n|_{U_g}$, where n is the smallest integer for which $f^n(U_f) \subset U_f$ (and hence also $g^n(U_g) \subset U_g$). For each component K of $f^{-i}(U_f)$ other than U_f there is a $j \leq i$ with f^j mapping K homeomorphically into U_f. Then we define h for $x \in K$ by $h(f^j(x)) = g^j(h(x))$ and the requirement that the itineraries $I_f(x)$ and $I_g(h(x)$ are the same (this identifies the component). It is easy to see that h, defined this way is a topological equivalence. It remains to define h on U_f. U_f must always be of the form (ξ, ξ') where ξ and ξ' are either in E_f or on the periodic orbit. It is clear how to define h in this case. □

The order of the tongues for the family (1.1) correspond to the order of rational numbers with denominator given by $2^n - 1$. Consider $F_{a,b}$ the lift of $f_{a,b}$ to the real line. The limit

$$\Phi_{a,b}(x) = \lim_{n\to\infty} \frac{F^n_{a,b}(x)}{2^n}, \tag{3.1}$$

where each $F_{a,b}$ is continuous increasing (as a function of x) and $F_{a,b}(x+k) = F_{a,b}(x)+2k$ for every integer k, exists uniformly in x (see Lemma 3.1 [19]).

The fact that the periodic part of the binary expansion of $\Phi_a(x)$ gives us a periodic coding in 0's and 1's suggests that there is a relation between a periodic kneading and the periodic part of this binary expansion.

Theorem 3.3. *If a is such that f_a has a periodic orbit of period n ($n > 1$) then $K(f_a)$ corresponds to the binary expansion of $k/(2^n - 1)$ where k is a natural number and $k = 1, \ldots, 2^n - 2$.*

Proof. We make a linear conjugacy (or rotation)

$$T(\xi) = \xi + p. \tag{3.2}$$

The map in the new coordinates is

$$\hat{f}(\xi) = T^{-1} \circ f \circ T, \tag{3.3}$$

and it has a fixed point at $\xi = 0$.

We want to compute the limit

$$\lim_{n\to\infty} \frac{F^n_{a,b}(x)}{2^n},$$

in particular for $x = c$ we want to show that the itinerary goes with this limit.

The point q is defined by $F(q) = 1$. We have

$$F(x) = \begin{cases} f(x), \ 0 \le x \le q \\ f(x) + 1, \ q \le x \le 1 \end{cases}$$

and

$$F(x) = \begin{cases} f(x-k) + k, \ k \le x \le q + k \\ f(x-k) + k + 1, \ q + k \le x \le k + 1. \end{cases}$$

For an itinerary $\{i_j\}_{j=0}^{n-1}$ we have the recursion formula

$$\begin{cases} a_{n+1} = 2a_n + i_n, \\ \xi_{n+1} = f(\xi_n), \end{cases}$$

for $n = 0, 1, 2$ which is equivalent to

$$\begin{cases} x_{n+1} = F(x_n), \\ x_n = a_n + \xi_n, \end{cases}$$

for $a_n \in \mathbf{Z}, 0 \le \xi_n < 1$.

From these we get the following recursion formula for a_i:

$$a_0 = 0, \quad a_1 = i_0, \quad a_2 = 2i_0 + i_1, \quad a_3 = 4i_0 + 2i_1 + i_2, \quad \dots,$$

and we have

$$\begin{aligned} a_{kp} &= 2^{kp-1} i_0 + 2^{kp-2} i_1 + \ldots + 2^{k(p-1)} i_{p-1} + 2^{k(p-1)-1} i_0 + \cdots + i_{p-1} \\ &= i_0 2^{p-1} \frac{2^{kp} - 2^{p-1}}{2^p - 1} + \cdots + i_{p-1} \frac{2^{kp} - 1}{2^p - 1}. \end{aligned}$$

Since we know that the limit exists (see [19], Theorem 3.1), we get

$$\lim_{n\to\infty} \frac{F^n_{a,b}(x)}{2^n} = \lim_{k\to\infty} \frac{a_{kp}}{2^{kp}} = \frac{i_0 2^{p-1} + \cdots + i_{p-1}}{2^p - 1},$$

which is the statement of our theorem. □

4. Aperiodic kneading sequences

In this section we study the monotonicity for f_a from the double standard family of maps for which there is an aperiodic kneading sequence. We prove that the itinerary map is strictly increasing in this case. Furthermore, we state how to relate the work in the present paper with the "Real Fatou Conjecture".

Theorem 4.1. *Assume that $K(f_{a_1})$ and $K(f_{a_2})$ are aperiodic kneading sequences. Then the two maps are topologically conjugate and both are conjugate to the doubling map of the circle*

$$x \mapsto 2x \pmod 1.$$

Theorem 4.2. *If $f = f_a$ from the double standard family has an aperiod kneading sequence, then the itinerary map $x \mapsto I(x)$ is strictly increasing, i.e.,*

$$x < y \implies I(x) < I(y).$$

Before we prove Theorem 4.2 we recall the following definition.

Definition 4.3. We say that J is a *wandering interval* if
1. $\{f^j(J)\}_{j=0}^{\infty}$ are disjoint.
2. The ω-limit set of J is not a single period orbit.

Let N be either the interval $[-1, 1]$ or the circle $\mathbb{T}$. We say that a critical point c of a C map $f : N \mapsto N$ is *non-flat* if there is a C local diffeomorphism φ with $\varphi(c) = 0$ such that $f(x) = \pm|\varphi(x)|^{\alpha} + f(c)$ in a neighborhood of c.

The double standard family is obviously a circle map with non flat critical points.

The proof of Theorem 4.2 will follow from the following result obtained by [13]. For an exposition see the book [15], Chapter 4.

Theorem 4.4. *[Theorem A,* [15]*, Chapter 4]. If N is a C map with non-flat critical points, then f has no wandering intervals.*

Proof of Theorem 4.2 Let $x < y$ and $I(x) = I(y)$ and define $J = (x, y)$. We first claim $\{f^j(J)\}_{j=0}^{\infty}$ must be disjoint. Suppose not. Then there are $n \geq 0$ and $k > 0$ so that $f(J)$ and $f^{n+k}(J)$ are not disjoint. Thus $K = \bigcup_{p\geq 0} f^{n+kp}(J)$ is an interval. Form $L = \bigcup_{j=0}^{p-1} f^j(K)$. We know that L is not the entire circle $\mathbb{T}$ since $I(x) = I(y)$, p and q cannot be in $\{f^j(J)\}_{j=0}^{\infty}$. Obviously $f(K) \subset K$ and then $f|_L$ has a sink and hence an attractive periodic orbit. Since the kneading sequence is aperiodic this contradicts Theorem 3.2.

We also have to verify that the ω limit set of J is not a single periodic orbit. We first claim that the critical point $c = \frac{1}{2}$ is in the ω limit set. Suppose not. Then there is a neighborhood U of the critical point so that for all $x \in J$ $f^j(x) \notin U$ for $j \geq 0$. This means that Mañé:s theorem, see e.g. [15], Theorem III.5.1 is applicable. We conclude that there exists constants $C > 0$ and $\lambda > 1$ so that

$$|Df^n(x)| \geq C\lambda^n \qquad \forall n \geq 0.$$

This would mean that $|f^n(J)|$ grows exponentially, a contradiction.

Hence $f^n(J)$ accumulates on c. We have to prove part 2. in the definition of a wandering interval. Suppose that the ω limit set of J is a periodic orbit. Since c is in the ω limit set the periodic orbit must contain c. Hence f_a has a superattractive periodic orbit and c and the kneading sequence is periodic. This is a contradiction. We conclude that $I(x) < I(y)$ and the theorem is proved.

Proof of Theorem 4.1. Suppose f_a has an aperiodic kneading sequence. Let $\varphi(x) = I_f(x)$ be the kneading map and $D(x)$ be the doubling map

$$D(x) = 2x \qquad (\text{mod } 1).$$

It is then clear that φ is a homemorphism and $f = \varphi^{-1} \circ D \circ \varphi$, so the conjugacy is proven.

Theorem 4.5. *Assume that and $K(f_{a_1})$ and $K(f_{a_2})$ are aperiodic kneading sequences such that $K(f_{a_1}) = K(f_{a_2})$. Then $a_1 = a_2$.*

Proof. Since the map $a \mapsto K(a)$ is non-decreasing, i.e.

$$a_1 < a_2 \implies K(a_1) \leq K(a_2),$$

it is enough to prove that there are no intervals where $K(a)$ is constant. But by the density of hyperbolicity [12], in such a parameter interval there must be a point a' with a attractive periodic orbit. But for this parameter a' by Theorem 3.2, $K(a')$ is periodic and this is a contradiction.

□

It is clear that Theorem 4.5 really is equivalent to the density of parameters with attractive periodic orbits ("The Real Fatou Conjecture") in our present case of the double standard map.

We can realize this as follows. Let a^* be a parameter point. We want to prove that a^* can be approximated by a sequence of points a_n so that $K(a_n)$ is periodic and therefore corresponds to an attractive periodic orbit. If $K(a^*)$ itself is periodic there is nothing to prove. If $K(a^*)$ is aperiodic we can approximate it from below by a sequence K_n of increasing periodic kneading sequences. Since $a \mapsto K(a)$ is increasing and continuous by the intermediate value theorem there is an increasing sequence $\{a_n\}_{n=0}^{\infty}$ so that $K(a_n) = K_n$. Obviously a_n tends to a limit which we denote by a'. By the continuity of $a \mapsto K(a)$, $K(a') = K(a^*)$. If $a' < a$, $[a', a^*]$, there would be an interval of parameters with constant aperiodic kneading sequence, which is not possible according to Theorem 4.5.

The following conjecture is in many cases implied by Theorem B of [12].

Conjecture 4.6. *Suppose that $K(a_1) = K(a_2)$ are two aperiodic kneading sequences. Then the corresponding maps f_{a_1} and f_{a_2} are quasisymmetrically conjugate.*

An independent proof of Conjecture 4.6, in particular, if it could be done with purely real methods, would be of interest.

5. Abundance of positive Lyapunov exponents and invariant measures

We first state a theorem which corresponds to a result, which in the case of the quadratic family, is a consequence of results of M. Misiurewicz from his famous paper [16].

Theorem 5.1. *If a is such that $K(a)$ is preperiodic but not periodic, then f_a has an absolutely continuous invariant measure.*

We will not give the proof here since it can be given following the ideas given by Misiurewicz. See also the book [7] for an exposition of this proof.

One would expect that the set of parameters satisfying the condition that $K(a)$ is preperiodic but not periodic is of Lebesgue measure 0. In the case of the quadratic family the corresponding result is due to D. Sands [22]. Because of this fact the following result of of Collet-Eckmann type is of interest:

Theorem 5.2. *There is a subset $A \subset [0,1]$ of positive Lebesgue measure and constants c and $C > 0$ so that for all $a \in A$*

$$|Df_a^n(f_a(1/2))| \geq Ce^{cn}, \qquad \forall\ n \geq 0.$$

We will not give the proof of this result either since it can be given along the same principles as the proof of the corresponding theorem in the case of the quadratic family following e.g. [4] and [5].

Corollary 5.3. *For $a \in A$ the map f_a has an absolutely continuous invariant measure.*

As a direct consequence of Theorem 5.2 it follows immediately that $D_n = |Df_a^n(f(\frac{1}{2}))|$ satisfies

$$\sum_{n=0}^{\infty} D_n^{-1/l} < \infty.$$

In our case $l = 3$, since this is the power-law behaviour of the maps at the critical inflexion point $\frac{1}{2}$.

This indicates that an analogy to a result of Nowicki-van Strien, [21], Main Theorem, proved in the interval case, should be applicable, and if so we conclude that there is an absolutely continuous invariant measure.

It is also clear that a direct proof of Corollary 5.3 can also be given following the ideas of e.g. [4], part II, or [6].

6. Conjectures and further results

It seems reasonable that the analogy to the quadratic family could be carried even further. In particular Nowicki, Martens and Lyubich [14] proved that the parameter space of the quadratic family $x \mapsto q_a(x) = 1 - ax^2,\ 0 < a < 2$ can be written as a union

$$(0,2) = A \cup B \cup S \quad \text{a.e.},$$

where A is the set of parameters for which q_a has an an absolutely continuous invariant measure, B is the set of parameters for which q_a has a attractive periodic orbit and S is the set of parameters for which q_a has a singular non-atomic invariant measure μ_s, which is also the unique *physical measure*, i.e. a.e. point in the dynamic interval are Birkhoff generic:

$$\sum_{k=0}^{n-1} \delta_{f^k(x)} \to \mu_s, \qquad \text{for a.e. } x,$$

in the weak-$*$ topology.

Later Lyubich [11] proved that S is of Lebesgue measure 0. It seems natural to conjecture that the corresponding result is true for the double standard family for $b = 1$.

Conjecture 6.1. *For the double standard family f_a, $0 \leq a < 1$ the parameter space can be subdivided into three sets*

$$[0,1] = A \cup B \cup S \qquad a.e., \tag{6.1}$$

so that

(i) *A, the set of a:s such that f_a has an absolutely continuous invariant measure is of positive Lebesgue measure;*
(ii) *B the set of a:s such that f_a has a stable periodic orbit is open and dense;*
(iii) *S, the set of parameters for which f_a has a unique non-atomic singular physical measure, is of Lebesgue measure 0.*

Let us finish with some remarks about what is known as for this conjecture.

(i) is Theorem 5.2 above. (ii) is the Real Fatou Conjecture in this case and is a result of Levin and van Strien, [12]. The subdivision (6.1) and (iii) are open problems in the case of the double standard family even if these facts are both known for the quadratic family.

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