Bounding the gap between a free group (outer) automorphism and its inverse

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Abstract

Two complexity functions α_r and β_r are defined to measure the maximal possible gap between the norm of an automorphism (respectively outer automorphism) of F_r and the norm of its inverse. The exact complexity of α_2 and β_2 is computed. For rank $r \geqslant 3$, polynomial lower bounds are provided for α_r and β_r, and the existence of a polynomial upper bound is proved for β_r.

1 Introduction

Let $A_r = \{a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}\}$ be an *alphabet* of r symbols together with their formal inverses (a total of $2r$ symbols different from each other). All along the paper we assume $r \geqslant 2$ to avoid trivial cases.

The set of all words on A_r, including the empty one denoted 1, together with the operation of concatenation of words, forms a free monoid denoted A_r^*. For any subset $S \subseteq A_r^*$, the symbol S^* denotes the submonoid generated by S, namely the set of all (arbitrarily long) finite formal products of elements in S. For example, $\{a_1, \ldots, a_r\}^*$ is precisely the set of all *positive* words on the alphabet A_r.

Let $F_r = \langle a_1, \dots, a_r \rangle$ be the free group (of rank r) on the alphabet A_r, i.e. $A_r^*/\sim$ where $\sim$ is the congruence generated by the elementary reductions $a_i a_i^{-1} \sim a_i^{-1} a_i \sim 1$. A word of A_r^* is said to be *(cyclically) reduced* if it contains no (cyclic) factor of the form $a_i^{\epsilon} a_i^{-\epsilon}$, $\epsilon = \pm 1$. Given a word $w \in A_r^*$, we shall denote by $\overline{w}$ its reduction, namely the unique reduced word representing the same element of F_r as w. We shall do the standard abuse of notation consisting on using words, specially reduced ones, to refer to elements of F_r.

The *length* of an element $w \in F_r$, denoted $|w|$, is the total number of letters in $\overline{w}$, understanding $|1| = 0$. It is straightforward to see that $|w^n| \leqslant |n||w|$ and $|vw| \leqslant |v| + |w|$ hold for all $v, w \in F_r$.

We are interested in the automorphism group of F_r, denoted $\operatorname{Aut} F_r$. We let automorphisms act on the right, so we write $\varphi\colon F_r \to F_r$, $w \mapsto w\varphi$. Since every $\varphi \in \operatorname{Aut} F_r$ is determined by the images of $a_1, \dots, a_r$, say $a_1\varphi = u_1, \dots, a_r\varphi = u_r$, we shall adopt the notation $\varphi = \eta_{u_1, \dots, u_r}$ on occasions. When all of the u_i's are positive words, we say that $\eta_{u_1, \dots, u_r}$ is a *positive* automorphism (also known in the literature as *invertible substitutions*, see e.g. [7]). The submonoid of $\operatorname{Aut} F_r$ consisting of all positive automorphisms is denoted by $\operatorname{Aut}^+ F_r$. An automorphism $\eta_{u_1, \dots, u_r}$ is said to be *cyclically reduced* when $u_1, \dots, u_r$ are all cyclically reduced. For every $w \in F_r$, we denote by λ_w the right conjugation by w, namely $x\lambda_w = w^{-1}xw$. Since $\lambda_w \varphi = \varphi \lambda_{w\varphi}$, it follows easily that $\Lambda_r = \{\lambda_w \mid w \in F_r\}$ is a normal subgroup of $\operatorname{Aut} F_r$. Each of the cosets $[\varphi] = \varphi\Lambda_r$ is said to be an *outer automorphism* of F_r. We write $\operatorname{Out} F_r = (\operatorname{Aut} F_r)/\Lambda_r$.

Given $\varphi \in \operatorname{Aut} F_r$, we consider

$$\|\varphi\|_1 = |a_1\varphi| + \cdots + |a_r\varphi|,$$

as a measure of its complexity. Note that there is no $\varphi \in \operatorname{Aut} F_r$ with $\|\varphi\|_1 \leqslant r-1$, and there are exactly $r!2^r$ automorphisms with $\|\varphi\|_1 = r$, namely those of the form $a_1 \mapsto a_{1\pi}^{\epsilon_1}, \dots, a_r \mapsto a_{r\pi}^{\epsilon_r}$, where $\pi \in S_r$ is a permutation of $\{a_1, \dots, a_r\}$ and $\epsilon_i = \pm 1$. These automorphisms are the simplest ones and are called *letter permutation* automorphisms of F_r. They will be useful to reduce the number of cases in our arguments below. Note that, for increasing values of $n \geqslant r$, there is an increasing number of automorphisms φ with $\|\varphi\|_1 \leqslant n$, but only finitely many for every fixed n.

This measure induces a measure on $\operatorname{Out} F_r$ defined as follows. Given $\Phi \in \operatorname{Out} F_r$, let

$$\|\Phi\|_1 = \min\{\|\varphi\|_1 \mid \varphi \in \Phi\}.$$

Once again, for every fixed n, there exists a finite number of outer automorphisms Φ with $\|\Phi\|_1 \leqslant n$.

Very little is known in general about the relation between the complexity of an (outer) automorphism of F_r and the complexity of its inverse. In order to establish a first step into this direction, we define the following complexity functions:

$$\alpha_r(n) = \max\{\|\varphi^{-1}\|_1 \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_1 \leqslant n\},$$

$$\beta_r(n) = \max\{\|\Phi^{-1}\|_1 \mid \Phi \in \operatorname{Out} F_r, \quad \|\Phi\|_1 \leqslant n\}.$$

Clearly, $\alpha_r(n) \leqslant \alpha_r(n+1)$, hence α_r is a non-decreasing function, and so is β_r. It is easy to see that $\beta_r(n) = \max\{\|[\varphi^{-1}]\|_1 \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_1 \leqslant n\}$, hence $\beta_r(n) \leqslant \alpha_r(n)$ for every n. Observe also that the natural inclusion $\operatorname{Aut} F_r \hookrightarrow \operatorname{Aut} F_{r+1}$ defined by fixing the last generator, gives the inequality $\alpha_{r+1}(n+1) \geqslant 1 + \alpha_r(n)$.

Note that $\|\varphi\|_1$ depends on the prefixed basis $\{a_1, \dots, a_r\}$ in which one computes the norm; in other words, given $\psi \in \operatorname{Aut} F_r$, $\|\varphi\|_1$ and $\|\psi^{-1}\varphi\psi\|_1$ are not equal in general, although they differ only by a multiplicative constant, as stated in Corollary 2.3 below. However, the functions α_r and β_r do not depend on the chosen basis and constitute canonical invariants of the group F_r.

The goal of this paper is to investigate the asymptotic behavior of $\alpha_r(n)$ and $\beta_r(n)$. We can complete this project for the rank two case, which is quite special compared with higher ranks. On one hand, we show that, for every $\Phi \in \operatorname{Out} F_2$, $\|\Phi^{-1}\|_1 = \|\Phi\|_1$ and so $\beta_2(n) = n$, while the same equality in higher rank is far from true. On the other hand, we prove that $\alpha_2(n)$ is bounded above and below by quadratic functions, i.e. there is an exact quadratic gap between $\|\varphi\|_1$ and $\|\varphi^{-1}\|_1$ in the rank two case. Collecting Theorems 3.5, 3.6 and 3.7, we have

Theorem.

(i) *For* $n \geqslant 4$, $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$,

(ii) *for* $n \geqslant 10$, $\frac{n^2}{4} - 6n + 42 \leqslant \alpha_2(n)$,

(iii) *for* $n \geqslant 1$, $\beta_2(n) = n$.

For higher rank, the problem is much more tricky and our results are less precise. We show that $\alpha_r(n)$ grows at least polynomially with degree r, and $\beta_r(n)$ grows polynomially with degree at least $r-1$. Collecting Theorem 4.4 and Corollary 4.6, we have

Theorem. *For every* $r \geqslant 3$, *there exist constants* $K_r, K'_r, K''_r, M_r > 0$ *such that, for every* $n \geqslant 1$,

(i) $K_r n^r \leqslant \alpha_r(n)$,

(ii) $K'_r n^{r-1} \leqslant \beta_r(n) \leqslant K''_r n^{M_r}$.

Finally, we write a couple of interesting open questions.

2 Preliminaries

2.1 The p-norm of an automorphism

To prove the main results in the paper, we need to use standard facts about norms on real (or complex) vectors and matrices. Recall that the maps $\|\cdot\|_p\colon \mathbb{R}^k \to \mathbb{R}$, $\|(x_1, \ldots, x_k)\|_p = (|x_1|^p + \cdots + |x_k|^p)^{1/p}$ (for $p \in \mathbb{R}^+$) and $\|\cdot\|_\infty\colon \mathbb{R}^k \to \mathbb{R}$, $\|(x_1, \ldots, x_k)\|_\infty = \max\{|x_1|, \ldots, |x_k|\}$ are *vector norms*, i.e. they satisfy the following axioms: (1) $\|\mathbf{x}\|_p \geqslant 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$; (2) $\|\mu\mathbf{x}\|_p = |\mu|\|\mathbf{x}\|_p$; and (3) $\|\mathbf{x} + \mathbf{y}\|_p \leqslant \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.

Let us extend these notions to the non-abelian context, via the length function. For $p \in \overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$ and $\mathbf{w} = (w_1, \ldots, w_k) \in F_r^k$, we define

$$\|\mathbf{w}\|_p = \|(w_1, \ldots, w_k)\|_p = (|w_1|^p + \cdots + |w_k|^p)^{1/p}$$

for $p \in \mathbb{R}^+$, and

$$\|\mathbf{w}\|_\infty = \|(w_1, \ldots, w_k)\|_\infty = \max\{|w_1|, \ldots, |w_k|\}$$

for $p = \infty$. Note that the notation is coherent with the fact $\|\mathbf{w}\|_\infty = \lim_{p\to\infty} \|\mathbf{w}\|_p$.

Observe that this map $F_r^k \to \mathbb{R}$ can be expressed in terms of the corresponding vector norm, $\|(w_1, \ldots, w_k)\|_p = \|(|w_1|, \ldots, |w_k|)\|_p$. Hence, it satisfies the following properties:

1) (positivity) $\|\mathbf{w}\|_p \geqslant 0$ with equality if and only if $\mathbf{w} = (1, \ldots, 1)$;
2) (powers) $\|(w_1^n, \ldots, w_k^n)\|_p \leqslant |n|\|(w_1, \ldots, w_k)\|_p$;
3) (triangular inequality) $\|(v_1w_1, \ldots, v_kw_k)\|_p \leqslant \|(v_1, \ldots, v_k)\|_p + \|(w_1, \ldots, w_k)\|_p$.

By analogy, we shall emphasize on these three properties by referring to $\|\cdot\|_p$ as the *p-norm* in F_r^k.

Let us move now to morphisms. Thinking endomorphisms of F_r (and, in particular, automorphisms) as r-tuples of elements, $\varphi \leftrightarrow (a_1\varphi, \ldots, a_r\varphi)$, we define the *p-norm* of an endomorphism $\varphi \in \operatorname{End} F_r$, $p \in \overline{\mathbb{R}}^+$, as

$$\|\varphi\|_p = \|(a_1\varphi, \ldots, a_r\varphi)\|_p.$$

Given $\Phi \in \operatorname{Out} F_r$, define also

$$\|\Phi\|_p = \min\{\|\varphi\|_p \mid \varphi \in \Phi\}.$$

Further, we define the corresponding complexity functions α_r and β_r, in an attempt to measure the deviation in size between an automorphism (outer automorphism) and its inverse:

$$\begin{aligned} \alpha_r^p(n) = &\ \max\{\|\varphi^{-1}\|_p \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_p \leqslant n\}, \\ \beta_r^p(n) = &\ \max\{\|\Phi^{-1}\|_p \mid \Phi \in \operatorname{Out} F_r, \quad \|\Phi\|_p \leqslant n\}. \end{aligned}$$

Note that α_r and β_r above are α_r^1 and β_r^1, respectively. However, as expressed in the following proposition, for the different values of $p \in \overline{\mathbb{R}}^+$, all functions α_r^p are closely related to each other, like β_r^p do. For this reason, we shall restrict our attention to the case $p = 1$ (with occasional references to the ∞-norm for some technical arguments).

Proposition 2.1. *For all $p, q \in \overline{\mathbb{R}}^+$ there exists a natural number $C = C_{p,q,r} > 0$ such that*

$$\frac{1}{C}\|\varphi\|_q \leqslant \|\varphi\|_p \leqslant C\|\varphi\|_q \quad \text{and} \quad \frac{1}{C}\|\Phi\|_q \leqslant \|\Phi\|_p \leqslant C\|\Phi\|_q$$

hold for all $\varphi \in \operatorname{End} F_r$ and $\Phi \in \operatorname{Out} F_r$. Furthermore,

$$\frac{1}{C}\alpha_r^p\left(\frac{n}{C}\right) \leqslant \alpha_r^q(n) \leqslant C\alpha_r^p(Cn) \quad \text{and} \quad \frac{1}{C}\beta_r^p\left(\frac{n}{C}\right) \leqslant \beta_r^q(n) \leqslant C\beta_r^p(Cn),$$

for all n multiple of C.

Proof. It is well-known (see [4, Corollary 5.4.5]) that the exact similar fact holds for the corresponding vector norms: there exists a positive constant, and so a natural number $C = C_{p,q,r}$ such that

$$\frac{1}{C}\|\mathbf{x}\|_q \leqslant \|\mathbf{x}\|_p \leqslant C\|\mathbf{x}\|_q$$

for every $\mathbf{x} \in \mathbb{R}^r$. Now $\frac{1}{C}\|\varphi\|_q \leqslant \|\varphi\|_p \leqslant C\|\varphi\|_q$ follows immediately from the equality

$$\|\varphi\|_p = \|(a_1\varphi, \ldots, a_r\varphi)\|_p = \|(|a_1\varphi|, \ldots, |a_r\varphi|)\|_p.$$

On the other hand, since $\|\Phi\|_q = \|\theta\|_q$ for some $\theta \in \Phi$, we get

$$\|\Phi\|_p = \min\{\|\varphi\|_p \mid \varphi \in \Phi\} \leqslant \|\theta\|_p \leqslant C\|\theta\|_q = C\|\Phi\|_q$$

and $\frac{1}{C}\|\Phi\|_q \leqslant \|\Phi\|_p \leqslant C\|\Phi\|_q$ follows by symmetry.

For the second part of the statement, we have

$$\begin{aligned} \alpha_r^q(n) = &\ \max\{\|\varphi^{-1}\|_q \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_q \leqslant n\} \\ \leqslant &\ \max\{\|\varphi^{-1}\|_q \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_p \leqslant Cn\} \\ \leqslant &\ C\max\{\|\varphi^{-1}\|_p \mid \varphi \in \operatorname{Aut} F_r, \quad \|\varphi\|_p \leqslant Cn\} \\ = &\ C\alpha_r^p(Cn) \end{aligned}$$

for all n. Symmetrically, $\alpha_r^p(n) \leqslant C\alpha_r^q(Cn)$ and so $\frac{1}{C}\alpha_r^p(\frac{n}{C}) \leqslant \alpha_r^q(n)$ for all n multiple of C. The same argument gives the corresponding inequalities for the β functions. □

The following lemma states some initial properties of norms of automorphisms, that will be useful later. As a corollary, we deduce the fact that, up to multiplicative constants, $\|\cdot\|_1$ (and so $\|\cdot\|_p$) do not depend on the (prefixed) basis of F_r chosen to compute norms.

Lemma 2.2. *Let φ, θ, ψ_1, $\psi_2 \in \operatorname{Aut} F_r$ with ψ_1 and ψ_2 letter permuting, and let $w \in F_r \setminus \{1\}$. Then:*

(i) $\frac{\|\varphi\|_1}{r} \leqslant \|\varphi\|_\infty < \|\varphi\|_1$,

(ii) $\|\psi_1\varphi\psi_2\|_p = \|\varphi\|_p$ *for all* $p \in \overline{\mathbb{R}}^+$,

(iii) $||\varphi\theta||_1 \leqslant ||\varphi||_1 \cdot ||\theta||_\infty < ||\varphi||_1 \cdot ||\theta||_1$,

(iv) $||\lambda_w\varphi||_1 \leqslant (2r|w| + r - 2)||\varphi||_\infty < (2r|w| + r - 2)||\varphi||_1$.

Proof. (i) and (ii) are clear from the definitions.

(iii) For every $a \in A_r$, we have $|a\varphi\theta| \leqslant |a\varphi| \cdot ||\theta||_\infty$ and so

$$||\varphi\theta||_1 = \sum_{i=1}^{r} |a_i\varphi\theta| \leqslant \sum_{i=1}^{r} |a_i\varphi| \cdot ||\theta||_\infty = ||\varphi||_1 \cdot ||\theta||_\infty < ||\varphi||_1 \cdot ||\theta||_1.$$

(iv) Since $w \neq 1$, exactly one of the words $w^{-1}a_iw$ is non reduced, and so

$$\begin{aligned} ||\lambda_w\varphi||_1 = & \sum_{i=1}^{r} |(\overline{w^{-1}a_iw})\varphi| \leqslant (r-1)(2|w|+1)||\varphi||_\infty + (2|w|-1)||\varphi||_\infty \\ = & \ (2r|w| + r - 2)||\varphi||_\infty < (2r|w| + r - 2)||\varphi||_1. \end{aligned}$$

□

Corollary 2.3. *Let* $\psi \in \operatorname{Aut} F_r$ *(thought as a change of basis), and let* $C = \|\psi\|_1 \cdot \|\psi^{-1}\|_1$. *Then:*

(i) for every $\varphi \in \operatorname{Aut} F_r$, *we have* $\frac{1}{C}\|\varphi\|_1 \leqslant \|\psi^{-1}\varphi\psi\|_1 \leqslant C\|\varphi\|_1$,

(ii) $\alpha_r(n) = \max\{\|\psi^{-1}\varphi^{-1}\psi\|_1 \mid \varphi \in \operatorname{Aut} F_r, \quad \|\psi^{-1}\varphi\psi\|_1 \leqslant n\}$, *i.e. the definition of* $\alpha_r(n)$ *does not depend on the basis chosen to compute norms.*

Proof. (i) By Lemma 2.2(iii), $\|\psi^{-1}\varphi\psi\|_1 \leqslant \|\psi^{-1}\|_1 \cdot \|\varphi\|_1 \cdot \|\psi\|_1 = C\|\varphi\|_1$. Analogously, $\|\varphi\|_1 = \|\psi(\psi^{-1}\varphi\psi)\psi^{-1}\|_1 \leqslant \|\psi\|_1 \cdot \|\psi^{-1}\varphi\psi\|_1 \cdot \|\psi^{-1}\|_1 = C\|\psi^{-1}\varphi\psi\|_1$.

(ii) Considering the change of variable $\nu = \psi^{-1}\varphi\psi$ in $\operatorname{Aut} F_r$, we get $\nu^{-1} = \psi^{-1}\varphi^{-1}\psi$ and so $\max\{\|\psi^{-1}\varphi^{-1}\psi\|_1 \mid \varphi \in \operatorname{Aut} F_r, \ \|\psi^{-1}\varphi\psi\|_1 \leqslant n\} = \{\|\nu^{-1}\|_1 \mid \nu \in \operatorname{Aut} F_r, \ \|\nu\|_1 \leqslant n\} = \alpha_r(n)$. □

Lemma 2.4. *Let* $\Phi, \Theta \in \operatorname{Out} F_r$ *and let* $\psi_1, \psi_2 \in \operatorname{Aut} F_r$ *be letter permuting. Then:*

(i) $\|[\psi_1]\Phi[\psi_2]\|_1 = \|\Phi\|_1$,

(ii) $\|\Phi\Theta\|_1 \leqslant \|\Phi\|_1\|\Theta\|_1$.

Proof. We have $[\psi_1]\Phi[\psi_2] = \psi_1\Lambda_r\Phi\psi_2\Lambda_r = \psi_1\Lambda_r\Phi\Lambda_r\psi_2 = \psi_1\Phi\psi_2$. Now Lemma 2.2(ii) yields

$$\|[\psi_1]\Phi[\psi_2]\|_1 = \min\{\|\psi_1\varphi\psi_2\|_1 \mid \varphi \in \Phi\} = \min\{\|\varphi\|_1 \mid \varphi \in \Phi\} = \|\Phi\|_1$$

and so (i) holds.

For (ii), we use Lemma 2.2(iii) to get

$$\begin{aligned} \|\Phi\Theta\|_1 = & \ \min\{\|\psi\|_1 \mid \psi \in \Phi\Theta\} = \min\{\|\varphi\theta\|_1 \mid \varphi \in \Phi, \ \theta \in \Theta\} \\ \leqslant & \ \min\{\|\varphi\|_1\|\theta\|_1 \mid \varphi \in \Phi, \ \theta \in \Theta\} = (\min\{\|\varphi\|_1 \mid \varphi \in \Phi\})(\min\{\|\theta\|_1 \mid \theta \in \Theta\}) \\ = & \ \|\Phi\|_1\|\Theta\|_1 . \end{aligned}$$

□

Now, similarly to Corollary 2.3, one could prove that the definition of $\beta_r(n)$ does not depend on the basis chosen to compute norms.

The proof of the following lemma is immediate:

Lemma 2.5. *Let* $\varphi \in \operatorname{Aut} F_r$ *be cyclically reduced. Then* $\|[\varphi]\|_1 = \|\varphi\|_1$.

2.2 Abelianizing

Abelianizing will be a valuable tool to derive lower bounds for $\|\varphi\|_1$ and $\|\Phi\|_1$.

The 1-norm for vectors $\|(x_1,\ldots,x_r)\|_1 = |x_1|+\cdots+|x_r|$ gives rise to the 1-norm for matrices, namely

$$\|M\|_1 = \sum_{i,j} |m_{i,j}|,$$

where $M = (m_{i,j}) \in \mathrm{GL}_r(\mathbb{Z})$. It is straightforward to verify that, for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^r$ and $M, N \in \mathrm{GL}_r(\mathbb{Z})$, we have the inequalities $\|\mathbf{x}+\mathbf{y}\|_1 \leqslant \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$, $\|\mathbf{x}M\|_1 \leqslant \|\mathbf{x}\|_1 \cdot \|M\|_1$, $\|M+N\|_1 \leqslant \|M\|_1 + \|N\|_1$, and $\|MN\|_1 \leqslant \|M\|_1\|N\|_1$.

Let us denote the abelianization map by $(\cdot)^{\mathrm{ab}}\colon F_r \twoheadrightarrow \mathbb{Z}^r$, $w \mapsto w^{\mathrm{ab}} = ([w]_{a_1}, \ldots, [w]_{a_r})$. Here, $[w]_{a_i}$ is the total exponent of a_i in w, i.e. the total number of times the letter a_i occurs in $\overline{w}$, taking into account the exponents' signs (for example, $[a_1a_2a_1^{-2}]_{a_1} = -1$ and $[a_1a_1^{-1}a_2]_{a_1} = [a_2]_{a_1} = 0$).

Every automorphism $\varphi \in \mathrm{Aut}\, F_r$ abelianizes to an automorphism φ^{ab} of $\mathbb{Z}^r$ which we shall represent by its $r\times r$ (invertible) matrix over $\mathbb{Z}$. We want automorphisms to act on the right, and so we write matrices by rows, i.e. with the i-th row describing the image of the i-th generator:

$$\varphi^{\mathrm{ab}} = \begin{pmatrix} [a_1\varphi]_{a_1} & \cdots & [a_1\varphi]_{a_r} \\ \cdots & \cdots & \cdots \\ [a_r\varphi]_{a_1} & \cdots & [a_r\varphi]_{a_r} \end{pmatrix} \in \mathrm{GL}_r(\mathbb{Z}).$$

This way, for every $w \in F_r$, $(w\varphi)^{\mathrm{ab}} = w^{\mathrm{ab}}\varphi^{\mathrm{ab}}$. Furthermore, $(\varphi\theta)^{\mathrm{ab}} = \varphi^{\mathrm{ab}}\theta^{\mathrm{ab}}$, and $(\varphi^{-1})^{\mathrm{ab}} = (\varphi^{\mathrm{ab}})^{-1}$.

Observe that, for every $w \in F_r$, $|w| \geqslant \|w^{\mathrm{ab}}\|_1 = |[w]_{a_1}| + \cdots + |[w]_{a_r}|$ with equality if and only if no letter occurs in $\overline{w}$ with the two opposite signs. This can be expressed in the following useful way:

Lemma 2.6. *For every $\varphi \in \mathrm{Aut}\, F_r$, $\|\varphi\|_1 \geqslant \|[\varphi]\|_1 \geqslant \|\varphi^{\mathrm{ab}}\|_1$, with equalities if and only if, for every $i = 1,\ldots,r$, no letter occurs in $\overline{a_i\varphi}$ with the two opposite signs. In particular, $\|\varphi\|_1 = \|\varphi^{\mathrm{ab}}\|_1$ for positive automorphisms.*

Proof. Clearly, $\|\varphi\|_1 \geqslant \|[\varphi]\|_1$. We may write $\|[\varphi]\|_1 = \|\varphi\lambda_w\|_1$ for some $w \in F_r$. Then

$$\begin{array}{rcl} \|\varphi\|_1 & \geqslant & \|[\varphi]\|_1 = \|\varphi\lambda_w\|_1 = \sum_{i=1}^r |a_i\varphi\lambda_w| \geqslant \sum_{i=1}^r \|(a_i\varphi)^{\mathrm{ab}}\|_1 = \sum_{i=1}^r \|a_i^{\mathrm{ab}}\varphi^{\mathrm{ab}}\|_1 \\ & = & \sum_{i=1}^r \sum_{j=1}^r |[a_i\varphi]_{a_j}| = \|\varphi^{\mathrm{ab}}\|_1, \end{array}$$

where a_i^{ab} is the i-th canonical vector and so, $a_i^{\mathrm{ab}}\varphi^{\mathrm{ab}}$ is the i-th row in φ^{ab}. It is immediate that the inequality $\|\varphi\|_1 \geqslant \|\varphi^{\mathrm{ab}}\|_1$ becomes an equality if and only if, for every $i = 1,\ldots,r$, no letter occurs in $\overline{a_i\varphi}$ with the two opposite signs. This is the case when $\varphi \in \mathrm{Aut}^+ F_r$. □

3 The rank two case

In this section we shall deal with the rank 2 case. Along it, we simplify our notation to $A = A_2 = \{a, b, a^{-1}, b^{-1}\}$.

We start by proving that inversion preserves the norm in the case of positive automorphisms. It is known that positive automorphisms of F_2 are generated as a monoid by $\Delta = \{\eta_{b,a},\, \eta_{a,ab},\, \eta_{a,ba}\}$, that is, they all can be obtained as a composition of these elementary ones, i.e. $\mathrm{Aut}^+ F_2 = \Delta^*$ (see [7]).

Lemma 3.1. *Let $\varphi \in \mathrm{Aut}^+ F_2$ and write $\varphi^{-1} = \eta_{u,v}$. Then either $u \in \{a, b^{-1}\}^*$ and $v \in \{a^{-1}, b\}^*$, or $u \in \{a^{-1}, b\}^*$ and $v \in \{a, b^{-1}\}^*$. In particular, φ^{-1} is cyclically reduced.*

Proof. The result is clear for the three elementary positive automorphisms, $\eta_{b,a}^{-1} = \eta_{b,a}$, $\eta_{a,ab}^{-1} = \eta_{a,a^{-1}b}$, $\eta_{a,ba}^{-1} = \eta_{a,ba^{-1}}$. Since all positive automorphisms are compositions of elements in Δ, we are reduced to show that, given a positive automorphism φ and $\theta \in \Delta$, the lemma holds for $\varphi\theta$ whenever it holds for φ. To see this, write $\varphi^{-1} = \eta_{u,v}$ and assume u and v are like in the statement. Then we get

$$(\varphi\eta_{b,a})^{-1} = \eta_{b,a}\eta_{u,v} = \eta_{v,u},$$
$$(\varphi\eta_{a,ab})^{-1} = \eta_{a,a^{-1}b}\eta_{u,v} = \eta_{u,u^{-1}v},$$
$$(\varphi\eta_{a,ba})^{-1} = \eta_{a,ba^{-1}}\eta_{u,v} = \eta_{u,vu^{-1}},$$

completing the proof. □

Proposition 3.2. *Let $\varphi \in \operatorname{Aut}^+ F_2$. Then $\|\varphi^{-1}\|_1 = \|\varphi\|_1$.*

Proof. Abelianizing, we have

$$\varphi^{\mathrm{ab}} = \begin{pmatrix} [a\varphi]_a & [a\varphi]_b \\ [b\varphi]_a & [b\varphi]_b \end{pmatrix} \quad \text{and} \quad (\varphi^{-1})^{\mathrm{ab}} = \pm \begin{pmatrix} [b\varphi]_b & -[a\varphi]_b \\ -[b\varphi]_a & [a\varphi]_a \end{pmatrix};$$

hence, $\|(\varphi^{-1})^{\mathrm{ab}}\|_1 = \|\varphi^{\mathrm{ab}}\|_1$. Also, $\|\varphi^{\mathrm{ab}}\|_1 = \|\varphi\|_1$ since φ is positive (see Lemma 2.6). Now, write $\varphi^{-1} = \eta_{u,v}$. By Lemma 3.1 no letter occurs with both signs in neither u nor v so, again by Lemma 2.6, $\|(\varphi^{-1})^{\mathrm{ab}}\|_1 = \|\varphi^{-1}\|_1$, concluding the proof. □

From positive automorphisms we can gain control of all cyclically reduced ones.

Lemma 3.3. *For every cyclically reduced $\varphi \in \operatorname{Aut} F_2$, there exist two letter permuting automorphisms ψ_1, $\psi_2 \in \operatorname{Aut} F_2$ and $\theta \in \operatorname{Aut}^+ F_2$ such that $\varphi = \psi_1\theta\psi_2$.*

Proof. Write $\varphi = \eta_{u,v}$. Since both u and v are cyclically reduced, the main result in [2] tells us that at most two letters of A occur in u, and at most two of them (not necessarily the same ones) occur in v. Without loss of generality, we may assume that two different letters occur in either u or v, say in u. Inverting all possibly negative letters in u, we can write $\eta_{u,v} = \eta_{u',v'}\eta_{a^\epsilon,b^\delta}$ with ϵ, $\delta = \pm 1$, $u' \in \{a,b\}^*$ and $|u'| = |u|$ and $|v'| = |v|$.

If $v' \in \{a, b\}^*$, i.e. is a positive word, then $\eta_{u',v'} \in \operatorname{Aut}^+ F_2$ and we are done. If $v' \in \{a^{-1}, b^{-1}\}^*$, take $\eta_{u,v} = \eta_{a,b^{-1}}\eta_{u',v'^{-1}}\eta_{a^\epsilon,b^\delta}$ and we are also done. The remaining cases to consider are $v' \in \{a^{-1}, b\}^*$ or $v' \in \{a, b^{-1}\}^*$ with exactly two letters occurring in v'; they will lead us to contradiction. Indeed, abelianizing we get $u'^{\mathrm{ab}} = ([u]_a, [u]_b) = (p,q)$ with $p, q > 0$, and $v'^{\mathrm{ab}} = ([v]_a, [v]_b) = (r,s)$ with $rs < 0$. This contradicts $ps - qr = \pm 1$ coming from the fact that $\eta_{u',v'}$ is an automorphism of F_2. □

And from those, we can reach the general case:

Lemma 3.4. *For every $\varphi \in \operatorname{Aut} F_2$, there exist two letter permuting automorphisms ψ_1, $\psi_2 \in \operatorname{Aut} F_2$, $\theta \in \operatorname{Aut}^+ F_2$, and an element $g \in F_2$ such that $\varphi = \psi_1\theta\psi_2\lambda_g$ and $\|\theta\|_1 + 2|g| \leqslant \|\varphi\|_1$.*

Proof. Note that, by Lemmas 2.2(ii) and 3.3, we are reduced to show that there exists a cyclically reduced $\varphi' \in \operatorname{Aut} F_2$ and $g \in F_2$, such that $\varphi = \varphi'\lambda_g$ and $\|\varphi'\|_1 + 2|g| \leqslant \|\varphi\|_1$. Let us prove this claim by induction on $\|\varphi\|_1$.

If $\|\varphi\|_1 = 2$ the claim is trivial since φ is cyclically reduced. So, suppose $\varphi = \eta_{u,v} \in \operatorname{Aut} F_2$ is given with $\|\eta_{u,v}\|_1 > 2$, and let us assume the claim holds for all automorphisms of smaller 1-norm. Again, if u and v are cyclically reduced the claim is trivial so, by symmetry, we can assume that u is not cyclically reduced, say $\overline{u} = c^{-1}u'c$ for some $c \in A$ and $u' \in F_2$. If $\overline{v}$ neither begins with c^{-1} nor ends with c then it could be easily seen that c would not be contained in $\langle u, v\rangle$ contradicting the fact that $\{u, v\}$ generates F_2. Hence, $v \in c^{-1}A^* \cup A^*c$, and so $|\overline{cvc^{-1}}| \leqslant |v|$. Now, factoring $\eta_{u,v}$ as $\eta_{u,v} = \eta_{u',\overline{cvc^{-1}}}\lambda_c$, we have

$$\|\eta_{u',\overline{cvc^{-1}}}\|_1 = |u'| + |\overline{cvc^{-1}}| \leqslant |u| - 2 + |v| = \|\eta_{u,v}\|_1 - 2,$$

and we can apply the induction hypothesis to get a factorization $\eta_{u',\overline{cvc^{-1}}} = \varphi'\lambda_h$ with φ' cyclically reduced and $\|\varphi'\|_1 + 2|h| \leqslant \|\eta_{u',\overline{cvc^{-1}}}\|_1$. Thus, we have $\eta_{u,v} = \eta_{u',\overline{cvc^{-1}}}\lambda_c = \varphi'\lambda_h\lambda_c = \varphi'\lambda_{hc}$ with

$$\|\varphi'\|_1 + 2|hc| \leqslant \|\varphi'\|_1 + 2|h| + 2 \leqslant \|\eta_{u',\overline{cvc^{-1}}}\|_1 + 2 \leqslant \|\eta_{u,v}\|_1 = \|\varphi\|_1.$$

This completes the proof of the claim and so, of the lemma. □

Theorem 3.5. *For every $n \geqslant 4$, we have $\alpha_2(n) \leqslant \frac{(n-1)^2}{2}$.*

Proof. Let $\varphi \in \operatorname{Aut} F_2$ with $\|\varphi\|_1 \leqslant n$, and let us prove that $\|\varphi^{-1}\|_1 \leqslant \frac{(n-1)^2}{2}$. Consider the decomposition given in Lemma 3.4, $\varphi = \psi_1\theta\psi_2\lambda_g$ for some letter permuting $\psi_1, \psi_2 \in \operatorname{Aut} F_2$, some $\theta \in \operatorname{Aut}^+ F_2$, and some $g \in F_2$ such that $\|\theta\|_1 + 2|g| \leqslant \|\varphi\|_1$.

If $g = 1$ then

$$\|\varphi^{-1}\|_1 = \|\psi_2^{-1}\theta^{-1}\psi_1^{-1}\|_1 = \|\theta^{-1}\| = \|\theta\|_1 = \|\varphi\|_1 \leqslant n \leqslant \frac{(n-1)^2}{2},$$

by Lemma 2.2(ii) and Proposition 3.2 (and using in the last step that $n \geqslant 4$).

So, let us assume $g \neq 1$ in which case we have $\varphi^{-1} = \lambda_{g^{-1}}\psi_2^{-1}\theta^{-1}\psi_1^{-1}$. By Lemma 2.2 and Proposition 3.2,

$$\|\varphi^{-1}\|_1 \leqslant 4|g| \cdot \|\psi_2^{-1}\theta^{-1}\psi_1^{-1}\|_\infty = 4|g| \cdot \|\theta^{-1}\|_\infty \leqslant 4|g|(\|\theta^{-1}\|_1 - 1) = 4|g|(\|\theta\|_1 - 1).$$

Since we also have $\|\theta\|_1 + 2|g| \leqslant \|\varphi\|_1 \leqslant n$, we deduce $|g| \leqslant \frac{n-\|\theta\|_1}{2}$ and so,

$$\|\varphi^{-1}\|_1 \leqslant 2(n - \|\theta\|_1)(\|\theta\|_1 - 1).$$

Finally, since the parabola $f(x) = 2(n-x)(x-1)$ has its absolute maximum in the point $x = \frac{n+1}{2}$, we conclude

$$\|\varphi^{-1}\|_1 \leqslant 2(n - \|\theta\|_1)(\|\theta\|_1 - 1) \leqslant 2\Big(n - \frac{n+1}{2}\Big)\Big(\frac{n+1}{2} - 1\Big) = \frac{(n-1)^2}{2}. \quad \square$$

In order to establish lower bounds for $\alpha_2(n)$, we need to construct explicit automorphisms of F_2 having inverses with big 1-norm compared to that of themselves.

Theorem 3.6. *For $n \geqslant 10$, we have $\alpha_2(n) \geqslant \frac{n^2}{4} - 6n + 42$.*

Proof. For $k \geqslant 0$ consider the automorphisms

$$\psi_k = \eta_{ab^{2k},\, ab^{2k+1}}\lambda_{a^{-k}b} = \eta_{b^{-1}a^{k+1}b^{2k}a^{-k}b,\, b^{-1}a^{k+1}b^{2k+1}a^{-k}b}.$$

We have $\|\psi_k\|_1 = 8k + 7$. For the inverse, we have

$$\psi_k^{-1} = \lambda_{b^{-1}a^k}\eta^{-1}_{ab^{2k},\, ab^{2k+1}} = \lambda_{b^{-1}a^k}\eta_{a(b^{-1}a)^{2k},\, a^{-1}b} = \eta_{u,\, v},$$

where u and v are the two words $u = ((a^{-1}b)^{2k}a^{-1})^k a^{-1}ba(b^{-1}a)^{2k}b^{-1}a(a(b^{-1}a)^{2k})^k$ and $v = ((a^{-1}b)^{2k}a^{-1})^k a^{-1}b(a(b^{-1}a)^{2k})^k$. Hence, $\|\psi_k^{-1}\|_1 = 4(4k+1)k + 4k + 7 = 16k^2 + 8k + 7$.

Writing $n = \|\psi_k\|_1 = 8k + 7$, we have $k = \frac{n-7}{8}$ and then

$$\|\psi_k^{-1}\|_1 = 16\frac{(n-7)^2}{64} + n - 7 + 7 = \frac{n^2 - 10n + 49}{4}.$$

Thus, for $n \equiv 7 \mod 8$, we have $\alpha_2(n) \geqslant \frac{n^2-10n+49}{4}$.

Finally, for every $n \geqslant 7$, let n' be the unique integer congruent with 7 modulo 8 in the set $\{n-7, \ldots, n-1, n\}$. We have

$$\alpha_2(n) \geqslant \alpha_2(n') \geqslant \frac{n'^2 - 10n' + 49}{4} \geqslant \frac{(n-7)^2 - 10(n-7) + 49}{4} = \frac{n^2}{4} - 6n + 42,$$

where the last inequality uses $n \geqslant 10$ since the parabola $f(x) = \frac{x^2-10x+49}{4}$ has its minimum at $x = 5$. □

The outer automorphism case turns out to be simpler:

Theorem 3.7. *For every $\Phi \in \operatorname{Out} F_2$, $\|\Phi^{-1}\|_1 = \|\Phi\|_1$. Consequently, $\beta_2(n) = n$.*

Proof. Take $\varphi \in \Phi$. By Lemma 3.4, $\varphi = \psi_1\theta\psi_2\lambda_g$ for some letter permuting automorphisms ψ_1, $\psi_2 \in \operatorname{Aut} F_2$, some $\theta \in \operatorname{Aut}^+ F_2$ and some element $g \in F_2$. Then Lemmas 2.4(i) and 2.5 yield

$$\|\Phi\|_1 = \|[\varphi]\|_1 = \|[\psi_1\theta\psi_2\lambda_g]\|_1 = \|[\psi_1\theta\psi_2]\|_1 = \|[\theta]\|_1 = \|\theta\|_1.$$

Also by Lemma 2.4(i), we get

$$\|\Phi^{-1}\|_1 = \|[\varphi^{-1}]\|_1 = \|[\lambda_{g^{-1}}\psi_2^{-1}\theta^{-1}\psi_1^{-1}]\|_1 = \|[\psi_2^{-1}\theta^{-1}\psi_1^{-1}]\|_1 = \|[\theta^{-1}]\|_1.$$

Since θ^{-1} is cyclically reduced by Lemma 3.1, we may use Lemma 2.5 to get $\|\Phi^{-1}\|_1 = \|[\theta^{-1}]\|_1 = \|\theta^{-1}\|_1$. Since $\|\theta\|_1 = \|\theta^{-1}\|_1$ by Proposition 3.2, we get $\|\Phi^{-1}\|_1 = \|\Phi\|_1$. Therefore $\beta_2(n) = n$. □

4 Higher rank

In this section, we consider arbitrary rank $r \geqslant 3$, compute polynomial lower bounds for both $\alpha_r(n)$ and $\beta_r(n)$, and show that $\beta_r(n)$ admits a polynomial upper bound.

The polynomial lower bounds for $\alpha_r(n)$ and $\beta_r(n)$ have degrees r and $r-1$, respectively. In particular, this separates the asymptotic behavior of the rank two case from all other ranks, with respect to both complexity functions. That is, $\xi_2(n)$ grows more slowly than $\xi_r(n)$ for all $r \geqslant 3$ and $\xi \in \{\alpha, \beta\}$, which agrees with the intuitive fact that $\operatorname{Aut} F_r$ is a much easier group for $r = 2$ than for higher rank.

Finally, the polinomial upper bound for $\beta_r(n)$ is established with the help of the theory of Outer space.

We assume the rank r fixed throughout the whole section.

4.1 Lower bounds

Our lower bound for $\beta_r(n)$ is obtained by abelianizing positive automorphisms. The extra unit in the degree of the lower bounds from $\beta_r(n)$ to $\alpha_r(n)$ will be achieved by additionally composing the positive automorphisms with a suitable conjugation that increases in size when inverting. We thank Warren Dicks for suggesting us to use the following automorphisms; this significantly simplified our previous proof of the lower bounds for $\alpha_r(n)$ and $\beta_r(n)$.

We start by defining, for every $p \in \mathbb{Z}$, a matrix $M^{(p)} = (m_{i,j}^{(p)}) \in \mathrm{GL}_r(\mathbb{Z}) = \operatorname{Aut} \mathbb{Z}^r$ given by

$$m_{i,j}^{(p)} = \begin{cases} 1, & \text{if } i = j; \\ p, & \text{if } j = i + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\det M^{(p)} = 1$ and so $M^{(p)}$ is indeed invertible.

Lemma 4.1. *For all $r \geqslant 2$ and $p \in \mathbb{Z}$, let $N^{(p)} = (n^{(p)}_{i,j}) \in \mathrm{GL}_r(\mathbb{Z})$ be defined by*

$$n^{(p)}_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ (-p)^{j-i}, & \text{if } i < j; \\ 0, & \text{otherwise.} \end{cases}$$

Then $N^{(p)} = (M^{(p)})^{-1}$.

Proof. It suffices to show that $M^{(p)}N^{(p)}$ is the identity matrix. Indeed, the (i,j)-th entry of the product matrix is $\sum_{k=1}^{r} m^{(p)}_{i,k} n^{(p)}_{k,j} = \sum_{k=i}^{\min\{i+1,j\}} m^{(p)}_{i,k} n^{(p)}_{k,j}$ which is 0 if $j < i$ and 1 if $j = i$. If $j > i$, we get $m^{(p)}_{i,i} n^{(p)}_{i,j} + m^{(p)}_{i,i+1} n^{(p)}_{i+1,j} = (-p)^{j-i} + p(-p)^{j-i-1} = 0$ and the lemma is proved. □

We immediately obtain:

Lemma 4.2. *For all $r \geqslant 2$ and $p \in \mathbb{Z}$, we have $\|M^{(p)}\|_1 = r + (r-1)p$ and $\|(M^{(p)})^{-1}\|_1 \geq p^{r-1}$.* □

For every integer $p \geqslant 2$, define $\varphi_p \in \mathrm{Aut}^+ F_r$ by

$$a_i\varphi_p = \begin{cases} a_i a_{i+1}^p, & \text{if } 1 \leq i < r; \\ a_r, & \text{if } i = r. \end{cases}$$

Note that φ_p is clearly onto and therefore an automorphism since free groups of finite rank are hopfian [5].

Lemma 4.3. *For all $r \geqslant 2$ and $p \geqslant 2$:*

(i) $\varphi_p^{\mathrm{ab}} = M^{(p)}$,

(ii) $a_r\varphi_p^{-1} = a_r$ *and* $a_i\varphi_p^{-1} = a_i(a_{i+1}\varphi_p^{-1})^{-p}$ *for* $i = 1, \ldots, r-1$,

(iii) $\overline{a_i\varphi_p^{-1}} \in a_i A_r^* a_{i+1}^{-1}$ *for* $i = 1, \ldots, r-1$,

(iv) $||\varphi_p^{-1}||_1 < 2|a_1\varphi_p^{-1}|$.

Proof. (i) is clear.

To get (ii), it suffices to compute $(a_i(a_{i+1}\varphi_p^{-1})^{-p})\varphi_p = (a_i\varphi_p)a_{i+1}^{-p} = a_i$ for $i < r$. Then (iii) follows from (ii) by reverse induction.

Finally, to see (iv) observe that by (iii) the product $a_i(a_{i+1}\varphi_p^{-1})^{-p}$ is reduced and so $|a_i\varphi_p^{-1}| > p|a_{i+1}\varphi_p^{-1}|$ for every $i < r$. Hence $|a_i\varphi_p^{-1}| < \frac{1}{p^{i-1}}|a_1\varphi_p^{-1}|$ for $i = 2, \ldots, r$ and so

$$||\varphi_p^{-1}||_1 = \sum_{i=1}^{r} |a_i\varphi_p^{-1}| < (1 + \frac{1}{p} + \cdots + \frac{1}{p^{r-1}})|a_1\varphi_p^{-1}| < 2|a_1\varphi_p^{-1}|. \quad \square$$

Now we are ready to state and prove the lower bounds for our complexity functions.

Theorem 4.4. *For every $r \geqslant 2$, there exists constants $K_r, K'_r > 0$ such that, for every $n \geqslant 1$:*

(i) $K_r n^r \leqslant \alpha_r(n)$,

(ii) $K'_r n^{r-1} \leqslant \beta_r(n)$.

Proof. Let $p \geqslant r$. By Lemmas 2.6, 4.2 and 4.3(i), we have

$$\|\varphi_p\|_1 = \|[\varphi_p]\|_1 = \|\varphi_p^{\mathrm{ab}}\|_1 = \|M^{(p)}\|_1 = r + (r-1)p \leqslant rp. \tag{1}$$

On the other hand, the same results yield

$$\|\varphi_p^{-1}\|_1 \geqslant \|[\varphi_p^{-1}]\|_1 \geqslant \|(\varphi_p^{-1})^{\mathrm{ab}}\|_1 = \|(\varphi_p^{\mathrm{ab}})^{-1}\|_1 = \|(M^{(p)})^{-1}\|_1 \geqslant p^{r-1}. \tag{2}$$

Let $n_0 = \max\left\{r^2, \frac{(r-1)2^{\frac{1}{r-1}}}{2^{\frac{1}{r-1}}-1}\right\}$ and consider $n \geqslant n_0$. Take the integer $p = \lfloor \frac{n}{r} \rfloor \geqslant r$, which satisfies $\frac{n-(r-1)}{r} \leqslant p \leqslant \frac{n}{r}$ and so $rp \in \{n-(r-1), \ldots, n\}$. The outer automorphism $[\varphi_p] \in \mathrm{Out}(F_r)$ satisfies $\|[\varphi_p]\|_1 \leqslant rp \leqslant n$; and, on the other hand, $\|[\varphi_p^{-1}]\|_1 \geqslant p^{r-1} \geqslant (\frac{n-(r-1)}{r})^{r-1} = \frac{(n-(r-1))^{r-1}}{r^{r-1}}$. Now it is straightforward to check that

$$(n-a)^s \geqslant \frac{n^s}{2} \iff n \geqslant \frac{a2^{\frac{1}{s}}}{2^{\frac{1}{s}}-1}$$

holds for all positive integers s, a, n. Hence, we deduce that

$$\|[\varphi_p^{-1}]\|_1 \geqslant \frac{1}{2r^{r-1}} n^{r-1}$$

(using that $n \geqslant \frac{(r-1)2^{\frac{1}{r-1}}}{2^{\frac{1}{r-1}}-1}$). We conclude that $\beta_r(n) \geqslant \frac{1}{2r^{r-1}} n^{r-1}$ for $n \geqslant n_0$. Adjusting the value of the constant $\frac{1}{2r^{r-1}}$ to cover the finitely many missing values of n, (ii) holds.

To prove (i) let us restrict ourselves to the case $r \geqslant 3$ (Theorem 3.6 already deals with the case $r = 2$). Fix $p \geqslant r$ and let $\psi_p = \varphi_p \lambda_{a_1^p}$. Then (1) yields

$$\|\psi_p\|_1 = \sum_{i=1}^{r} |a_1^{-p}(a_i\varphi_p)a_1^p| \leqslant 2rp + \|\varphi_p\|_1 \leqslant 3rp.$$

On the other hand,

$$\|\psi_p^{-1}\|_1 = \|\lambda_{a_1^{-p}}\varphi_p^{-1}\|_1 > \sum_{i=3}^{r} |(a_1^p a_i a_1^{-p})\varphi_p^{-1}|.$$

Since the products $(a_1\varphi_p^{-1})^p(a_i\varphi_p^{-1})(a_1^{-1}\varphi_p^{-1})^p$ are reduced by Lemma 4.3(iii), it follows that $\|\psi_p^{-1}\|_1 > 2(r-2)p|a_1\varphi_p^{-1}| > (r-2)p\|\varphi_p^{-1}\|_1 \geqslant (r-2)p^r$, by Lemma 4.3(iv) and (2).

This shows that, for $n = 3rp$ and $p \geqslant r$, we have $\alpha_r(n) > (r-2)p^r = \frac{r-2}{(3r)^r} n^r$ i.e., (i) is proven for all such values of n. Finally, the extension of this inequality to all values of n (after adjusting properly the multiplicative constant) proceeds similarly to part (ii). □

As a final remark for this section, it seems clear that this exhausts the potential of abelianization techniques to provide lower bounds. If the growths of our complexity functions are strictly bigger than what we have proven here, this will have to be obtained by more intricate counting techniques working above the abelian level.

4.2 Upper bounds

We can present a polynomial upper bound for $\beta_r(n)$ using Outer space techniques. We thank M. Bestvina for suggesting a simplification of our initial arguments, which leads to a very easy and elegant proof of such a polynomial upper bound, now essentially a corollary of a recent result about the asymmetry of the Lipschitz metric in Outer space.

Let us briefly recall what Outer space $\mathcal{X}_r$ is, $r \geqslant 2$, following the notation from [1] (see [6] for more details).

With the term *graph* we mean a finite graph Γ of rank r, all whose vertices have degree at least three. A *metric* on Γ is a function $\ell\colon E\Gamma \to [0,1]$ defined on the set of edges of Γ such that $\sum_{e\in E\Gamma} \ell(e) = 1$ and the set of length zero edges forms a forest. Let us denote by Σ_Γ the space of all such metrics ℓ on Γ, viewed as a "simplex with missing faces" (corresponding to degenerate metrics that vanish on a subgraph which is not a forest). If Γ' is obtained from Γ by collapsing a forest, then we will naturally consider $\Sigma_{\Gamma'}$ as a subset of Σ_Γ along the inclusion given by assigning length zero to the collapsed edges.

Fix the rose graph R_r with one vertex and r edges, and identify the free group F_r with the fundamental group $\pi_1(R_r)$ in such a way that each generator a_i corresponds to a single oriented edge of R_r. Under this identification, each reduced word in F_r corresponds to a reduced edge-path loop starting and ending at the basepoint of R_r.

A *marked graph* is a pair (Γ, f) where f is a *marking*, i.e. a homotopy equivalence from the rose R_r to Γ. It is standard to consider the set of marked graphs modulo the following equivalence relation: $(\Gamma, f) \sim (\Gamma', f')$ if and only if there is a homeomorphism $\mu\colon \Gamma \to \Gamma'$ such that $f\mu$ is homotopic to f'. Denote it by $\mathcal{MG}/\sim$.

Noting that all representatives of a given class $[(\Gamma, f)] \in \mathcal{MG}/\sim$ share a common underlying graph, we can consider the space of metrics on Γ and denote it $\Sigma_{[(\Gamma,f)]}$. Now, the *Outer Space* $\mathcal{X}_r$ is obtained from the disjoint union

$$\bigsqcup_{[(\Gamma,f)]\in\mathcal{MG}/\sim} \Sigma_{[(\Gamma,f)]}$$

by identifying the faces of the simplices along the above natural inclusions. Thus, a point in $\mathcal{X}_r$ is represented by a triple of the form (Γ, f, ℓ).

There is a natural action of $\operatorname{Aut} F_r$ on $\mathcal{X}_r$. Given $\varphi \in \operatorname{Aut} F_r$, realize it on the rose, say $\varphi\colon R_r \to R_r$, and for every point $x = (\Gamma, f, \ell) \in \mathcal{X}_r$ define $\varphi \cdot x$ to be $(\Gamma, \varphi f, \ell)$. It is easy to see that this is well defined and gives an action of $\operatorname{Aut} F_r$ on $\mathcal{X}_r$. Notice that, by construction, inner automorphisms act trivially; so, what we have is in fact an action of $\operatorname{Out} F_r$ on $\mathcal{X}_r$.

Recently, the Lipschitz metric for $\mathcal{X}_r$ has been introduced and initially studied in [3], followed by other authors (see, for example, [1]). This metric can be defined as follows.

Let $x, x' \in \mathcal{X}_r$ be two points in the Outer space; take representatives, say (Γ, f, ℓ) and (Γ', f', ℓ'), respectively. A *difference of markings* is a map $\mu\colon \Gamma \to \Gamma'$ which is linear on edges, and such that $f\mu$ is homotopic to f'. For such a difference of markings one can define $\sigma(\mu)$ to be the largest slope of μ over all edges $e \in E\Gamma$. Then define the distance from x to x' as

$$d(x, x') = \min_{\mu} \{\log \sigma(\mu)\},$$

where the minimum is taken over all possible differences of markings (and achieved by Arzela-Ascoli's Theorem).

The basic properties of this "distance" are the following: (1) $d(x, y) \geqslant 0$, with equality if and only if $x = y$; (2) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}_r$; (3) $\operatorname{Out} F_r$ acts by isometries, i.e. $d([\varphi]\cdot x, [\varphi]\cdot y) = d(\varphi\cdot x, \varphi\cdot y) = d(x, y)$ for all $x, y \in \mathcal{X}_r$ and $\varphi \in \operatorname{Aut} F_r$; but (4) $d(x, y) \neq d(y, x)$ in general. See [3] and [1] for details.

For $\epsilon > 0$, define the *ϵ-thick part* of $\mathcal{X}_r$ as

$$\mathcal{X}_r(\epsilon) = \{(\Gamma, f, \ell) \in \mathcal{X}_r \mid \ell(p) \geqslant \epsilon \ \ \forall p \text{ nontrivial closed path in } \Gamma\}.$$

The following is an interesting result from Y. Algom-Kfir and M. Bestvina (see [1, Theorem 23]):

Theorem 4.5 (Algom-Kfir, Bestvina)**.** *Let $r \geqslant 2$. For any $\epsilon > 0$ there is a constant $M = M(r, \epsilon) > 0$ such that, for all $x, y \in \mathcal{X}_r(\epsilon)$,*

$$d(x, y) \leqslant M \cdot d(y, x).$$

As an easy corollary, we obtain our polynomial upper bound for $\beta_r(n)$:

Corollary 4.6. *For every $r \geqslant 2$, there exist constants $K_r, M_r > 0$ such that $\beta_r(n) \leqslant K_r n^{M_r}$ for every $n \geqslant 1$.*

Proof. Fix an automorphism $\varphi \in \operatorname{Aut} F_r$.

Consider the point of the Outer space $x \in \mathcal{X}_r$ represented by (R_r, id, ℓ_0), i.e. by the identity marking over the balanced rose (here, ℓ_0 assigns constant length $1/r$ to each petal). Now consider the point $[\varphi] \cdot x = (R_r, \varphi, \ell_0) \in \mathcal{X}_r$. It is straightforward to see from the definitions that, given a difference of markings $\mu : R_r \to R_r$, then μ is homotopic to φ if and only if $\mu = \varphi\lambda_w$ for some $w \in F_r$. Moreover, $\sigma(\varphi\lambda_w) = \|\varphi\lambda_w\|_\infty$. It follows that

$$d(x, [\varphi] \cdot x) = \min_{w \in F_r} \{\log(\sigma(\varphi\lambda_w))\} = \log(\min_{w \in F_r} \|\varphi\lambda_w\|_\infty) = \log \|[\varphi]\|_\infty.$$

Similarly,

$$d([\varphi] \cdot x, x) = d(x, [\varphi^{-1}] \cdot x) = \log \|[\varphi^{-1}]\|_\infty.$$

But, since all the involved points belong to the $(1/r)$-thick part $\mathcal{X}_r(\frac{1}{r})$, we can take the constant $M_r = M(r, \frac{1}{r})$ from Theorem 4.5 to get $\log \|[\varphi^{-1}]\|_\infty \leqslant M_r \log \|[\varphi]\|_\infty$ and so $\|[\varphi^{-1}]\|_\infty \leqslant \|[\varphi]\|_\infty^{M_r}$. Bringing in the constant $C_r = C_{\infty,1,r}$ from Proposition 2.1, we obtain

$$\|[\varphi^{-1}]\|_1 \leqslant C_r \|[\varphi^{-1}]\|_\infty \leqslant C_r \|[\varphi]\|_\infty^{M_r} \leqslant C_r^{M_r+1} \|[\varphi]\|_1^{M_r}.$$

Hence $\beta_r(n) \leqslant K_r n^{M_r}$ holds for $K_r = C_r^{M_r+1}$. □

We remark that the polynomial upper bound for $\beta_r(n)$ established above is intuitively far from sharp. The proof of the Algom-Kfir-Bestvina's theorem is indirect and the actual constant provided there is quite big. The provided lower and upper bounds for $\beta_r(n)$, namely n^{r-1} and n^{M_r}, are the only known information about the following open question:

Question 4.7. *What is the exact asymptotic behavior of the function β_r, for $r \geqslant 3$?*

We also remark that getting a polynomial upper bound for $\alpha_r(n)$ seems to be a more tricky problem. On the one hand, the geometric techniques coming from Outer space do not seem to provide control on the length of possible conjugators showing up when computing the pre-image of the generators a_i by an (even cyclically reduced) given automorphism of F_r. Additionally, and oppositely to the much easier case $r = 2$, these conjugators cannot be avoided in general by just composing with an appropriate inner automorphism because they can affect differently to the various generators.

Question 4.8. *Is there a polynomial upper bound for $\alpha_r(n)$? What is the exact asymptotic behavior of the function α_r, for $r \geqslant 3$?*

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References

[1] Y. Algom-Kfir and M. Bestvina, Asymmetry of Outer Space, *Geom. Dedicata* 156 (2012), 81–92.

[2] M. Cohen, W. Metzler and A. Zimmermann, What does a basis of $F(a, b)$ look like?, *Math. Ann.* 257 (1981), 435–445.

[3] S. Francaviglia and A. Martino, The isometry group of outer space, *Adv. Math.* 231 (2012), 1940–1973.

[4] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1990.

[5] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin-New York, 1977.

[6] K. Vogtmann, Automorphisms of free groups and outer space, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000), *Geom. Dedicata* 94 (2002), 1–31.

[7] Z. X. Wen and Z. Y. Wen, Local isomorphisms of invertible substitutions, *C. R. Acad. Sci. Paris Sér. I Math.* 318 (1994), 299–304.