RETURN TIME STATISTICS FOR INVARIANT MEASURES FOR INTERVAL MAPS WITH POSITIVE LYAPUNOV EXPONENT

HENK BRUIN AND MIKE TODD

Abstract. We prove that multimodal maps with an absolutely continuous invariant measure have exponential return time statistics around a.e. point. We also show a 'polynomial Gibbs property' for these systems, and that the convergence to the entropy in the Ornstein-Weiss formula has normal fluctuations. These results are also proved for equilibrium states of some Hölder potentials.

1. Introduction

Return time statistics refers to the distribution of return times to (usually small) sets U in the phase space of a measure preserving dynamical system. There have been various approaches to estimate these distributions in the literature. The earlier methods pertain to hyperbolic dynamical systems (such as Anosov diffeomorphisms [H]) as these benefit most directly from the techniques of i.i.d. stochastic processes, the area in which return time statistics was studied first. Gradually methods were developed to treat non-uniformly hyperbolic systems and in [BSTV] it was pointed out that the return time statistics of a dynamical system coincides with the return time statistics of a (first return) induced map. If this first return map itself is hyperbolic, then the above theory can be applied immediately, but the existence of a hyperbolic first return map is a serious restriction on general dynamical systems, especially when (recurrent) critical points are present.

In [BV] this problem was overcome in the context of unimodal interval maps satisfying a summability condition on the derivatives along the critical orbit. Instead of a first return map, a hyperbolic inducing scheme was used, where the inducing time is a suitable, rather than a first, return to a specific subset Y of the interval. The method was to use the so-called 'Hofbauer tower' see [K, B], on which the inducing scheme corresponds to a first return map to a suitable subset $\hat{Y}$ of the Hofbauer tower.

In this paper we will generalise these ideas in two ways:

- f can be any non-flat C^3 multimodal map,

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- μ can be an arbitrary invariant probability measure with positive Lyapunov exponent: $\lambda(\mu) := \int \log |Df| \ d\mu > 0$.

In these cases, exponential return time statistics on balls is obtained for absolutely continuous invariant measures (acips). In addition, we obtain a 'polynomial Gibbs property' and fluctuation results in the Ornstein-Weiss formula, for both acips and equilibrium states of certain Hölder potentials, provided a very weak growth condition of derivatives along critical orbits is satisfied.

Let us start by introducing the concept of return time statistics in more detail. Let (I, f, μ) be a measure preserving ergodic dynamical system. For a measurable set $U_z \subset I$ containing some $z \in I$, let $\mu_{U_z} = \frac{1}{\mu(U_z)}\mu|_{U_z}$ be the conditional measure on U_z and $r_{U_z}(x)$ be the first return time of a point $x \in U_z$ to U_z. Whenever z is not a periodic point, the return time $r_{U_z}(x) \to \infty$ as $\mu(U_z) \to 0$, but Kac's Lemma states that $\int r_{U_z}(x) d\mu_{U_z} = 1$. Therefore, when r_{U_z} is scaled by $\mu(U_z)$, we can hope for a well-defined distribution $g : [0, \infty) \to \mathbb{R}$ such that, for $t \in [0, \infty)$

$$\mu_{U_z}(\{x \in U_z \ : \ r_{U_z}(x)\mu(U_z) > t\}) \to g(t)$$

as $\mu(U_z) \to 0$. We refer to this as the *return time statistics for* (f, μ). For many mixing systems it is known that the return time statistics are exponential, i.e. $g(t) = e^{-t}$, see [A] for a survey of some well behaved systems. This is what we find for systems considered in the latter sections of this paper. For multiple return time statistics for these cases we expect to find Poissonian laws, see [HSV].

The natural choice for the sets U_z are balls or cylinder sets, but results on balls are in general harder to prove because of the lack of (Hölder) regularity of indicator functions χ_{U_n}. Also the Gibbs property only gives information on cylinder sets. Therefore, in dimension greater than 1, most results known pertain to cylinder sets, and not (yet) to balls. See [Sau] for more information on this issue.

However, the literature contains examples of behaviour far from exponential in different settings. For example Coelho and de Faria [CF] find examples of continuous and discontinuous distributions other than exponential, for circle diffeomorphisms, see [DM] for further results in this direction. Moreover, if we do not assume that the sequence of shrinking sets U_n are balls/cylinders then a large class of continuous distributions can be obtained, see Lacroix [La]. Thus it is important to emphasise that in this paper we will focus on balls/intervals.

We next explain the result of [BSTV] which allows us to from return time statistics of a first return map to the return time statistics of the original system. Consider an open set $Y \subset X$ and let $R_Y : Y \to Y$ be the first return map. We will denote by μ_Y the conditional measure on Y, which must be F-invariant and ergodic. For $z \in Y$ and $\alpha > 0$, let $U_\alpha = U_\alpha(z)$ be the α-ball around z. Let $r_{U_\alpha}(x)$ (resp. $r_{R_Y, U_\alpha}(x)$) be the first return time into U_α for f (resp. R_Y). We suppose that (Y, R_Y, μ_Y) has return time statistics $G(t)$, *i.e.,* for μ_Y-a.e. $z \in Y$, there exists $\varepsilon_z(n) \geqslant 0$ with $\varepsilon_z(n) \to 0$ as $\alpha \to 0$ such that

$$\sup_{t \geqslant 0} \left| \mu_Y \left(x \in U_\alpha \ : \ r_{R_Y, U_\alpha}(x) > \frac{t}{\mu(U_\alpha)} \right) - G(t) \right| < \varepsilon_z(n). \tag{1}$$

The key result of [BSTV] is that (Y, R_Y, μ_Y) enjoys the same distribution as (I, f, μ):

Theorem 1. *Suppose that the function G in* (1) *is continuous on* $[0, \infty)$. *Then for μ-a.e. $z \in Y$, there exists $\delta_z(\alpha) > 0$ with $\delta_z(\alpha) \to 0$ as $\alpha \to 0$ such that:*

$$\left|\mu_{U_\alpha}\left(x \in U_\alpha \ : \ r_{U_\alpha}(x) > \frac{t}{\mu(U_\alpha)}\right) - G(t)\right| < \delta_z(\alpha).$$

Note that the theorem can also be applied to cylinders rather than balls.

This theorem requires a first return map, rather than an arbitrary induced map, and we will use the Hofbauer tower to bridge that gap. The requirement that μ has a positive Lyapunov exponent is needed to 'lift' μ to this Hofbauer tower. *Liftability* is an abstract convergence property (in the vague topology) of Cesaro means of a measure μ imposed on the Hofbauer tower. It was introduced by Keller [K]. He showed in the context of one-dimensional maps, that μ having positive entropy ($h_\mu > 0$) or positive Lyapunov exponent both imply liftability. (In fact, for non-atomic measures, $\lambda(\mu) > 0$ is equivalent to liftability, see [BK].)

Let us now explain which type of induced systems we will consider. We fix $\delta > 0$ and some interval Y. We say that the interval Y' is a *δ-scaled neighbourhood of Y* if, denoting the left and right components of $Y' \setminus Y$ by L and R respectively, we have $|L|, |R| = \delta|Y|$. Next define an inducing scheme (Y, F) as follows. Let Y' be a δ-scaled neighbourhood of Y and define $\tau_{Y,\delta}(y)$ to be

$$\min\left\{i \geqslant 1 : f^i(y) \in Y \text{ and } \exists H \ni y \text{ with } f^i|_H : H \to Y' \text{ is a homeomorphism}\right\}$$

We call this the *first δ-extendible return time to Y*. For $y \in Y$ we let $F(y) := f^{\tau_{Y,\delta}(y)}(y)$ and let μ_Y be the conditional measure on Y. Given a point $z \in I$ we will take a sequence of nested intervals $\{J_n\}_n$ such that $\bigcap_n J_n = \{z\}$, we will denote F_n to be the map given above, with $\tau_n = \tau_{J_n,\delta}$. In fact we choose J_n to be so that if $f^k : U \to J_n$ is a diffeomorphism and $U \cap J_n \neq \emptyset$ then $U \subset J_n$, see Section 2.

We denote the, finite, set of critical points by Crit. We say that $c \in \text{Crit}$ is *non-flat* if there exists a diffeomorphism $g_c : \mathbb{R} \to \mathbb{R}$ with $g_c(0) = 0$ and $1 < \ell_c < \infty$ such that for x close to c, $f(x) = f(c) \pm |\varphi_c(x - c)|^{\ell_c}$. The value of ℓ_c is known as the *critical order* of c. We write $\ell_{\max} := \max_{c \in \text{Crit}} \ell_c$. Let

$$NF^k := \left\{f : I \to I : f \text{ is } C^k, \text{ each } c \in \text{Crit is non-flat and } \inf_{f^n(p)=p} |Df^n(p)| > 1\right\}.$$

Maps in NF^2 have no wandering intervals, see [MS], and therefore $\sup_x |Z_n[x]| \to 0$ as $n \to \infty$. By [SV], if $f \in NF^3$ we can use the Koebe Lemma (see [MS]) to say that the first δ-extendible return map F has bounded distortion. For some of the results below we need an *expansion* condition on critical orbits. Therefore we use a result from [BRSS] which states that a map $f \in NF^3$ with $\min_{c \in \text{Crit}} \liminf_n |Df(f(c))| \geqslant L$ has an acip and also satisfies a backward contraction property called $BC(2)$ in [BRSS, Proposition 1]. The number L depends only on the cardinality of the critical set and the maximal critical order $\ell_{\max}$ of f. With this in mind we define

$$ENF^k := \left\{f \in NK^k : \min_{c \in \text{Crit}} \liminf_n |Df(f(c))| \geqslant L(\ \#\text{Crit}(f), \ell_{\max}(f)\ )\right\}.$$

Any map in this class can not be infinitely renormalisable map.

The following is our first main theorem. This theorem also holds for return time statistics to cylinders.

Theorem 2. *Let $f \in NF^3$ and (I, f, μ) be liftable. Suppose that for μ-a.e. $z \in I$ there exists $\delta > 0$ and a nested sequence $\{J_n\}_n$ as above such that $\cap_n J_n = \{z\}$ and for all n, the system (J_n, F_n, μ_{J_n}) defined as above has return time statistics given by a continuous function $G : [0, \infty) \to [0, 1]$. Then (I, f, μ) also has return time statistics given by G.*

Using the ideas of [BSTV], our next main theorem follows almost immediately.

Theorem 3. *Take $f \in NF^3$. If μ is an acip then (I, f, μ) has exponential return time statistics to balls.*

The following results focus on specific measures, namely equilibrium states of some Hölder potentials as well as acips. We say that μ is an *equilibrium state* of *potential* φ if its *free energy* $h_\mu + \int \varphi d\mu$ is equal to the *pressure*

$$P(\varphi) := \sup_{\nu \in \mathcal{M}_{erg}} h_\nu + \int \varphi d\nu,$$

where $\mathcal{M}_{erg}$ denotes the set of all ergodic invariant probability measures. Note that for the specific choice $\varphi = -\log |Df|$, any equilibrium state must be an acip, see [Le, Ru]. We also consider the class of potentials

$$\mathcal{H} := \{\varphi : I \to \mathbb{R} : \varphi \text{ is Hölder and } \sup \varphi - \inf \varphi < h_{top}\},$$

where h_{top} is the topological entropy of f. It is proved in [BT] that $f \in ENF^3$ and $\varphi \in \mathcal{H}$ implies there exists a unique equilibrium state μ_φ. (To trace the origins of this result, [BT, Lemma 2] shows how to use the property $BC(2)$ from [BRSS, Proposition 1] to ensure that induced maps used for the construction of the equilibrium state are indeed sufficiently regular.) Moreover, $\lambda(\mu_\varphi) > 0$ and μ_φ has 'exponential tails', see Section 5.

Theorem 3 also follows for equilibrium states provided certain conditions on the variations of potentials are satisfied. For example the conditions given by Paccaut in [P] are sufficient. There exponential return time statistics to cylinders is proved, but in fact using the method of proof of [BSTV, Theorem 3.2], it can also be proved for balls. We have the following proposition for equilibrium states of our class of potentials. Below its proof we explain why this result is not extended to balls.

Proposition 4. *Take $f \in ENF^3$. If μ is an equilibrium state for a potential $\varphi \in \mathcal{H}$, then (I, f, μ) has exponential return time statistics to cylinders.*

Our next result concerns a weak version of the Gibbs property. Let $\mathcal{P}_1$ be the partition of I into maximal (open) intervals such that $f : Z \to f(Z)$ is a homeomorphism for each $Z \in \mathcal{P}_1$. Refine the partition $\mathcal{P}_n = \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}_1$ and by convention let $\mathcal{P}_0 = \{I\}$. We refer to the elements of $\mathcal{P}_n$ as cylinder sets, and we write $Z_n[x]$ to indicate the cylinder set in $\mathcal{P}_n$ containing x. If $x \in \partial Z_n$ then $Z_n[x]$ is not unique, but this applies only to countably many points.

We say that μ satisfies the *polynomial Gibbs property* with exponent κ if for μ-a.e. x, there is $n_0(x)$ such that

$$\frac{1}{n^\kappa} \leqslant \frac{\mu(Z_n[x])}{e^{\varphi_n(x)-nP(\varphi)}} \leqslant n^\kappa, \tag{2}$$

for all $n \geqslant n_0$. If μ is an acip, and hence an equilibrium state for the potential $\varphi = -\log|Df|$, then the pressure $P(\varphi) = 0$ and the quantity to estimate in (2) simplifies to $\mu(Z_n[x])|Df^n(x)|$. Formula (2) was used in [BV] and can be compared with the 'weak Gibbs property' given by Yuri [Yu], for which the Gibbs constants depend only on n, and the 'non-lacunary measures' of [OV] where the constants depend on x and n, but can grow at any subexponential rate.

Theorem 5. *For any $f \in ENF^3$, the following hold:*

(a) *There is an acip μ which is polynomially Gibbs. More precisely, if $\beta > 2$ and $\alpha > 4\ell_{max}^2$, then for μ-a.e. x there exists n_0 such that $n \geqslant n_0$ such that*
$$\frac{1}{n^{2\alpha}} \leqslant \frac{|f^n(Z_n[x])|}{n^\alpha} \leqslant \mu(Z_n[x])|Df^n(x)| \leqslant n^\beta.$$
(b) *If μ is an equilibrium state of a potential $\varphi \in \mathcal{H}$, then μ is polynomially Gibbs.*

The precise exponent κ of the polynomially Gibbs property in condition (b) is given in the proof of Proposition 11. This depends on the 'rate of decay of the tails' for μ.

The final results of this paper concern the normal fluctuation in the Ornstein-Weiss formula of return times. The Ornstein-Weiss formula says in this context that the first return time to $Z_n[x] \in \mathcal{P}_n$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log r_{Z_n[x]}(x) = h_\mu \text{ for } \mu\text{-a.e. } x. \tag{3}$$

If μ is an invariant probability measure, its variance σ_μ^2 is defined by

$$\sigma_\mu^2 = \sigma_\mu(\varphi)^2 := \int \varphi^2 \, d\mu - \left(\int \varphi \, d\mu\right)^2 + 2\sum_{n=1}^{\infty} \left[\int \varphi \circ f^n \cdot \varphi \, d\mu - \left(\int \varphi \, d\mu\right)^2\right],$$

where in case of the acip, $\varphi := -\log|Df|$. We have $\sigma_\mu > 0$, except when φ is a coboundary, i.e. $\varphi = \psi \circ f - \psi$ for some measurable function ψ. Potentials are unlikely to have zero variance. For example for $\varphi = -\log|Df|$ and $f(x) = ax(1-x)$, the only parameter for which φ is a coboundary is believed to be $a = 4$, cf. Corollary 3 in [BHN]. This is a special case of the broader notion of Livšic regularity.

Theorem 6. *Let $f \in ENF^3$ and define conditions on the system as follows.*

(a) *Suppose that all critical points have the same order. Moreover assume that for some $\beta > 4\ell_{max} - 3$,*
$$|Df^n(f(c))| \geqslant Cn^\beta \qquad \text{for all } c \in Crit \text{ and } n \geqslant 1, \tag{4}$$
and μ is the acip.

(b) *Let $\varphi \in \mathcal{H}$ and μ be the equilibrium state for φ.*

If either (a) or (b) holds and $\sigma_\mu^2 > 0$, then

$$\mu\left\{x \in X : \frac{\log r_{Z_n(x)}(x) - nh_\mu}{\sigma_\mu\sqrt{n}} > u\right\} \to \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-\frac{x^2}{2}}\,dx$$

where the convergence is uniform in x.

For condition (b), see Paccaut [P], where a similar result is proved for another class of equilibrium states.

This paper is organised as follows. In Section 2 we give the basic definitions for interval maps, and we discuss the Hofbauer tower and its lifting properties. Theorem 1 is proved in Section 3. In Section 4 we focus on the exponential return time statistics of acips and equilibrium states of potentials in $\mathcal{H}$. Next, in Section 5 we present our results on the polynomial Gibbs property. The fluctuation results for the Ornstein-Weiss formula (Theorem 6) is given in Section 6.

Throughout calculations, C will be a constant depending only on the map f.

2. Lifting Measures to the Hofbauer Tower

The *Hofbauer tower* (or *canonical Markov extension*) is defined as

$$\hat{I} := \left(\bigsqcup_{n\geqslant 0}\ \bigsqcup_{Z_n\in\mathcal{P}_n} f^n(Z_n)\right)/\sim$$

where $f^n(Z_n) \sim f^k(Z_k)$ if $f^n(Z_n) = f^k(Z_k)$. We denote the domains of $\hat{I}$ by $D = D(Z_n)$ and the collection of all such domains by $\mathcal{D}$. Points in $\hat{I}$ are written as $\hat{x} = (x, D)$, where $D = D_{\hat{x}}$ is element of $\mathcal{D}$ containing $\hat{x}$.

We write $D \to D'$ for $D, D' \in \mathcal{D}$ if there exist $Z_n \in \mathcal{P}_n$ and $Z_{n+1} \in \mathcal{P}_{n+1}$ such that $Z_{n+1} \subset Z_n$, $D = D(Z_n)$ and $D' = D(Z_{n+1})$. This gives $\mathcal{D}$ the a graph structure with domains D as vertices. For each $D = D(Z_n) \in \mathcal{D}$ has at least one and at most $\#\mathcal{P}_1$ outgoing arrows. The map $\hat{f} : \hat{I} \to \hat{I}$ is defined as

$$\hat{f}(x, D) = (f(x), D'),$$

where $D' = f^{n+1}(Z_{n+1})$ for that particular element $Z_{n+1} \in \mathcal{P}_{n+1}$ such that $x \in f^n(Z_{n+1})$ and $D \to D'$. Again $\hat{f}(x, D)$ is uniquely defined for $x \notin f^n(\partial Z_{n+1})$; otherwise $\hat{f}$ is multivalued at $\hat{x} = (x, D)$. By definition we have the following property: The system $(\hat{I}, \hat{f})$ is a Markov map with Markov partition $\mathcal{D}$. The *natural projection* $\pi : \hat{I} \to I$ is the (countable to one) inclusion map from $\hat{I}$ to I, and

$$\pi \circ \hat{f} = f \circ \pi.$$

Let i be the trivial bijection mapping (inclusion) I to $\hat{I}_0$ (note that $i^{-1} = \pi|_{\hat{I}_0}$) and let $\hat{\mu}_0 := \mu \circ i^{-1}$ and

$$\hat{\mu}_n := \frac{1}{n}\sum_{k=0}^{n-1} \hat{\mu}_0 \circ \hat{f}^{-k}. \tag{5}$$

We wish to find some limit $\hat{\mu}$ of a subsequence of $\{\hat{\mu}_n\}_n$.

Note that, as $\hat{I}$ is generally noncompact, the sequence $\{\hat{\mu}_n\}$ may not have a convergent subsequence in the weak topology. Instead we use the vague topology (see e.g [Bi]): Given a topological space, a sequence of measures σ_n is said to converge to a measure σ in the vague topology if for any function $\varphi \in \mathcal{C}_0(\hat{I})$ (where $\mathcal{C}_0(\hat{I})$ is the set of continuous functions with compact support in $\hat{I}$), we have $\lim_{n\to\infty} \sigma_n(\varphi) = \sigma(\varphi)$.

A measure μ on I is *liftable* if a vague limit $\hat{\mu}$ obtained in (5) is not identically 0. We define the *Lyapunov exponent* of μ to be $\int \log|Df|\,d\mu$. In the following theorem we provide assumptions which ensure $\hat{\mu} \not\equiv 0$.

Theorem 7. *Any ergodic invariant measure with positive Lyapunov exponent for a C^1 interval map is liftable to a measure $\hat{\mu}$ where $\hat{\mu}\circ\pi^{-1} = \mu$.*

Proof. For the proof of this see [K], see also [BK]. □

3. Return Statistics via the Hofbauer tower

In [BSTV] it was shown that dynamical systems (X, f, μ) and first return maps (Y, F, μ_Y) to fixed subsets $Y \subset X$ have the same return time statistics. If (Y, F) is hyperbolic, then it is commonly expected (and in many cases proved) that return time statistics will be exponential on balls, or at least on cylinder sets. However, typically no hyperbolic return maps can be found on sets with $\mu(Y) > 0$. The idea from [BV], which we will extend here, is that there frequently are sets Y with induced (rather than first return) maps F such that Y can be lifted to a set $\hat{Y} \subset \pi^{-1}(Y)$ in the Hofbauer tower such that F lifts to a first return map on $\hat{Y}$. As the set Y decreases in size, F will be closer to the true first return map, and hence we can approximate the return time statistics on the original system. That is, we prove Theorem 2.

We first explain the inducing schemes we consider. An inducing scheme (Y, F, τ) for $Y \subset I$ is a generalisation of a first return map. It consists of a collection $\{Y_i\}_i$ such that $F|_{Y_i} = f^{\tau_i}|_{Y_i} : Y_i \to Y$ is monotone onto for some $\tau_i \in \{1, 2, \dots\}$. The function $\tau : \cup_i Y_i \to \mathbb{N}$ with $\tau(x) = \tau_i$ if $x \in Y_i$ is called the *inducing time*. It is well-known that if μ_F is an F-invariant measure, with

$$\Lambda := \sum_i \tau_i\, \mu_F(Y_i) < \infty,$$

then μ defined by

$$\mu(A) = \frac{1}{\Lambda} \sum_i \sum_{n=0}^{\tau_i - 1} \mu_F(f^{-k}(A) \cap Y_i)$$

is f-invariant.

We next explain the relation between a first extendible return map and a first return map on the Hofbauer tower. We fix $\delta > 0$ and let z be a typical point of μ. Let $J_n := Z_n[z]$ and I_n be a δ-neighbourhood of J_n.

Let $\hat{R}_U : \pi^{-1}(U) \to \pi^{-1}(U)$ be the first return map to $\pi^{-1}(U)$ by $\hat{f}$, and denote the return time function by r_U. Define $\hat{I}_n \subset \pi^{-1}(I_n)$ to be the maximal set such that $\hat{I}_n \cap D \neq \emptyset$ for $D \in \mathcal{D}$ implies that $\pi^{-1}(I_n) \cap D$ is compactly contained in D. Now let $\hat{J}_n := \pi^{-1}(J_n) \cap \hat{I}_n$ and denote the first return map by $\hat{f}$ to $\hat{J}_n$ by $R_{\hat{J}_n}$. Note that $R_{\hat{J}_n}$ is extendible to $\hat{I}_n$. Define $\tilde{F}_n(y) := \pi \circ R_{\hat{J}_n} \circ \pi|_{\hat{J}_n}^{-1}(y)$. [B] implies that $\tilde{F}_n$ is well defined. As in the introduction, we consider $\tau_{J_n} = \tau_{J_n,\delta}$.

Lemma 8. $\tau_{J_n} = r_{\hat{J}_n} \circ \pi_{\hat{J}_n}^{-1}$.

This implies that $\tilde{F}_n$ is the same as F_n defined in the introduction.

Proof. This was shown in [B, Lemma 2], □

We say that r_U is (n,δ)-*extendible at* x if $f^{r_U(x)}$ can be extended homeomorphically locally around x to I_n.

Lemma 9. *For any z as above we have*

$$\lim_{n\to\infty} \sup_{z \in U \subset J_n} \mu_U\{x \in U : r_U(x) \text{ is not } (n,\delta)\text{-extendible at } x\} = 0.$$

Proof. Let $\hat{U}_n := \pi^{-1}(U) \cap \hat{I}_n$. By Theorem 7, the construction of $\hat{R}_U$ and Lemma 8, we have

$$\mu\{z \in U : r_U \text{ is not } (n,\delta)\text{-extendible}\} = \hat{\mu}\left(\hat{R}_U^{-1}(\pi^{-1}(U) \cap \hat{U}_n)\right) = \hat{\mu}(\pi^{-1}(U) \setminus \hat{U}_n)$$

by the $\hat{R}_U$-invariance of $\hat{\mu}$. To prove our lemma we must estimate this quantity relative to $\mu(U)$.

Fix $0 < \varepsilon < 1$, and let $D \in \mathcal{D}$ be any domain in the Hofbauer tower. Let $\rho_D(z)$ be given by $\frac{d\hat{\mu}_D \circ \pi^{-1}}{d\mu}(z)$; obviously $\rho_D(z) = 0$ if $z \notin D$. Recalling from Theorem 7 that $\mu = \hat{\mu} \circ \pi^{-1}$, we have that $\sum_{D \in \mathcal{D}} \rho_D(z) = 1$ for μ-a.e. z. Clearly, there exists some finite subcollection $\mathcal{D}'$ of $\mathcal{D}$ such that $\sum_{D \in \mathcal{D}'} \rho_D(z) \geqslant (1-\varepsilon)$. For each D we say that $(*)_D$ holds for n if

(1) $\pi^{-1}(I_n) \cap D$ compactly contained in D; and
(2) for any $U \subset J_n$, $\frac{\hat{\mu}(\pi^{-1}(U) \cap D)}{\mu(U)} \geqslant (1-\varepsilon)\rho_D(z)$.

The first condition trivially holds for any large n. We claim that the second condition holds for a.e. z, when n is sufficiently large. To prove this claim, note that we have $0 \leqslant \rho_D \leqslant 1$. We divide $[0,1]$ into pieces $\{\eta_i\}_i$ of size $\frac{\varepsilon}{2}$. Choose $\beta_i := \rho_D^{-1}(\eta_i)$ so that z is a density point of β_i. Note that for $y \in \beta_i$, $|\rho_D(y) - \rho_D(z)| \leqslant \frac{\varepsilon}{2}$. Then we claim that for $U \ni z$ a small enough neighbourhood of z, denoting $\mu_D := \mu \circ \pi|_D^{-1}$, we have

$$\frac{\mu_D(U)}{\mu(U)} \geqslant (1-\varepsilon)\rho_D(z).$$

To prove the claim, we have

$$\frac{\mu_D(U)}{\mu(U)} = \frac{1}{\mu(U)} \int_U \rho_D \, d\mu = \frac{1}{\mu(U)} \left(\int_{U\cap\beta_i} \rho_D \, d\mu + \int_{U\setminus\beta_i} \rho_D \, d\mu \right)$$
$$\geqslant \frac{\mu(U\cap\beta_i)}{\mu(U)} \left(1 - \frac{\varepsilon}{2}\right) \rho_D(z) - \frac{\mu(U\setminus\beta_i)}{\mu(U)}.$$

Since z is a density point of β_i, we have

$$\frac{\mu(U\cap\beta_i)}{\mu(U)} \to 1 \text{ and } \frac{\mu(U\setminus\beta_i)}{\mu(U)} \to 0$$

as $U \to z$. Thus for large enough n, the second condition must hold for z.

There exists N such that $(*)_D$ holds for all $n \geqslant N$ and $D \in \mathcal{D}'$. Therefore, if $n \geqslant N$ then

$$\frac{\hat\mu(\pi^{-1}(U)\cap \hat U_n)}{\mu(U)} = \sum_{D\in\mathcal{D}} \frac{\hat\mu(\pi^{-1}(U)\cap \hat U_n \cap D)}{\mu(U)}$$
$$= \sum_{D\in\mathcal{D}'} \frac{\hat\mu(\pi^{-1}(U)\cap \hat U_n \cap D)}{\mu(U)} + \sum_{D\in\mathcal{D}\setminus\mathcal{D}'} \frac{\hat\mu(\pi^{-1}(U)\cap \hat U_n \cap D)}{\mu(U)}$$
$$\geqslant (1-\varepsilon) \sum_{D\in\mathcal{D}'} \rho_D(z) \geqslant (1-\varepsilon)^2.$$

Therefore, for all $n \geqslant N$,

$$\frac{\hat\mu(\pi^{-1}(U)\setminus \hat U_n)}{\mu(U)} \leqslant 1 - (1-\varepsilon)^2 < 2\varepsilon.$$

As $\varepsilon > 0$ can be taken arbitrarily small, the proof is complete. □

Proof of Theorem 2. Let $\alpha_n = \sup_{z\in U\subset J_n} \frac{\hat\mu(\hat U_n)}{\mu(U)}$. As we have seen in Lemma 9, $\lim_{n\to\infty} \alpha_n = 1$. Because $f\circ\pi = \pi\circ\hat f$ we have

$$\mu_U\left(\left\{y \ : \ r_U(y) > \frac{t}{\mu(U)}\right\}\right) = \hat\mu_{\pi^{-1}(U)}\left(\left\{\hat y \ : \ r_{\pi^{-1}(U)}(\hat y) > \frac{t}{\mu(U)}\right\}\right).$$

The right hand side is majorised by a sum of three terms:

$$\begin{aligned}
\text{r.h.s.} \quad \leqslant \quad & \hat\mu_{\pi^{-1}(U)}(\pi^{-1}(U)\setminus \hat U_n) \\
& + \hat\mu_{\pi^{-1}(U)}\left(\left\{\hat y \in \hat U_n \ : \ r_{\hat U_n}(\hat y) > \frac{t}{\mu(U)}\right\}\right) \\
& + \hat\mu_{\pi^{-1}(U)}(\{\hat y \in \hat U_n \ : \ r_{\hat U_n}(\hat y) > r_{\pi^{-1}(U)}(\hat y)\}) \\
= \quad & I + II + III.
\end{aligned}$$

We have the estimates

$$I = \frac{\hat\mu(\pi^{-1}(U)\setminus \hat U_n)}{\hat\mu(\pi^{-1}(U))} = \frac{\hat\mu(\pi^{-1}(U)\setminus \hat U_n)}{\mu(U)} \leqslant 1 - \alpha_n \to 0.$$

Next

$$II = \alpha_n \hat\mu_{\hat U_n}\left(\left\{\hat y \ : \ r_{\hat U_n}(\hat y) > \frac{t}{\mu(U)}\right\}\right) = \alpha_n \hat\mu_{\hat U_n}\left(\left\{\hat y \ : \ r_{\hat U_n}(\hat y) > \frac{\tilde t}{\hat\mu(\hat U_n)}\right\}\right)$$

for $\tilde{t} = t\alpha_n$. Theorem 1 says that the return time statistics of a first return map coincides with the return time statistics of the original system. In this case, it means that the system $(\hat{I}, \hat{f}, \hat{\mu})$ has the same return time statistics on $\hat{U}_n$ as the induced system $(\hat{J}_n, \hat{F}_n, \hat{\mu}_{\hat{J}_n})$. By Lemma 9, tends to the same return time statistics as (J_n, F_n, μ_{J_n}). Hence II tends to $\alpha_n G(\tilde{t})$ as $\mu(U) \to 0$, and then, by continuity of G, to $G(t)$ as $n \to \infty$. The third term

$$\begin{aligned} III &= \hat{\mu}_{\pi^{-1}(U)} \left[\hat{R}_U^{-1}(\pi^{-1}(U) \setminus \hat{U}_n) \cap \hat{U}_n \right] \\ &\leqslant \hat{\mu}_{\pi^{-1}(U)}(\pi^{-1}(U) \setminus \hat{U}_n) = I \to 0, \end{aligned}$$

as $n \to \infty$. This gives the required upper bound for $\mu_U(\{y \ : \ r_U(y) > \frac{t}{\mu(U)}\})$. Now for the lower bound

$$\begin{aligned} \text{r.h.s.} &\geqslant \hat{\mu}_{\pi^{-1}(U)} \left(\left\{ \hat{y} \in \hat{U}_n \ : \ r_{\hat{U}_n}(\hat{y}) > \frac{t}{\mu(U)} \right\} \right) \\ &\quad - \hat{\mu}_{\pi^{-1}(U)}(\{\hat{y} \in \hat{U}_n \ : \ r_{\hat{U}_n}(\hat{y}) > r_{\pi^{-1}(U)}(\hat{y})\}) \\ &= II - III. \end{aligned}$$

The above arguments show that this also tends to $G(t)$ as $\mu(U) \to 0$ and $n \to \infty$. This finishes the proof. □

4. Exponential return time statistics

Definition 1 (Rychlik map). *Let $F : \cup_{i \in \mathbb{N}} Y_i \to Y$ be continuous on each Y_i, with $m(\cup_i Y) = m(Y) = 1$ for a given reference measure m. We call F a* Rychlik map *if:*

1. *there exists a neighbourhood $Z \supset Y$ and for each i a neighbourhood $Z_i \supset Y_i$ such that $F|_{Y_i}$ can be extended to a diffeomorphism $F : Z_i \xrightarrow{onto} Z$.*
2. *there exists a function $g : Y \to [0, \infty)$, with* Var $g < +\infty$, $g = 0$ *on* $Y \setminus \cup_i Y_i$, *such that the operator $P : L^1(m) \to L^1(m)$ defined by*
$$Pf(x) = \sum_{y \in F^{-1}(x)} g(y) f(y)$$
preserves m. In other words, $m(Pf) = m(f)$ for each $f \in L^1(m)$ (or equivalently: m is g^{-1}-conformal);
3. *F is expanding: $\sup_{x \in X} g(x) < 1$.*

The following is Theorem 3.2 of [BSTV]. It also applies to cylinders.

Theorem 10. *Suppose (Y, F) is a Rychlik map with conformal measure m and invariant mixing measure $\mu \ll m$. Then (Y, F) has exponential return time statistics to balls.*

Proof of Theorem 3. As in [BSTV], F_n is a Rychlik map, with $g = \frac{1}{|DF_n|}$. So by Theorem 10, each F_n has exponential return time statistics (i.e. $G(t) = e^{-t}$). Since Theorem 2 implies that f has the same return time statistics, we are finished. □

Proof of Proposition 4. The existence of the equilibrium state is proved in [BT]. To give some more details, it is shown that the induced map (J_n, F_n) possesses an induced potential $\Phi(x) = \sum_{k=0}^{\tau(x)-1} \varphi \circ f^k(x)$ which has summable variations. Therefore we can apply theory of Gibbs-Markov systems (see [Sar]) to construct a Φ-conformal measure m_{J_n} and an F_n-invariant Gibbs measure μ_Φ.

We sketch the proof that (J_n, F_n, μ_Φ) has exponential return time statistics to cylinders. Theorem 2 will then complete the proof of the proposition. It is well known, see for example [Yo], that this system has exponential decay of correlations for a class of observables which includes characteristic functions on cylinders. Moreover, the density $\frac{d\mu_\Phi}{dm_{J_n}}$ is uniformly bounded away from 0. By the proof of [BSTV, Theorem 3.2], which uses ideas of [HSV], these facts are sufficient to give exponential return time statistics to cylinders. $\square$

Note that in the above proof, if we had exponential decay of correlations for a class of observables which includes characteristic functions on balls, then we would also have exponential return time statistics to balls. This will follow under Paccaut's conditions, but our conditions are not strong enough.

5. The Polynomial Gibbs Property

We prove Theorem 5 in two parts. The case of equilibrium states for potentials in $\mathcal{H}$ is treated in Proposition 11, and then the upper and lower bounds for the acip is separated into two lemmas.

Proposition 11 relies on the fact that the equilibrium states $\mu = \mu_\varphi$ obtained in [BT] have exponential tails for the induced system, and also that $\varphi \in \mathcal{H}$ are bounded.

Proposition 11. *There exists $\kappa \in (0, \infty)$ such that for μ-a.e. x there exists $n_0 = n_0(x) \in \mathbb{N}$ such that $n \geqslant n_0$ implies*

$$\frac{1}{n^\kappa} \leqslant \frac{\mu(Z_n[x])}{e^{\varphi_n(x) - nP(\varphi)}} \leqslant n^\kappa.$$

This result can be compared with Lemmas 3.2 and 3.3 of [P].

Proof. Firstly we fix an inducing scheme (Y, F) with inducing time τ as in Lemma 9. We define $\varphi_n(x) := \varphi \circ f^{n-1}(x) + \cdots + \varphi(x)$, and for $x \in Y$ let $\Phi(x) := \varphi_{\tau(x)}$. We define Φ_n similarly. We denote the measure on the inducing scheme by μ_Φ.

Let T_φ and T_Φ denote the set of typical points of μ_φ and μ_Φ respectively. Let $f^{k_1}(x) = y_1$ be the first time that x maps into T_Φ, and let $k_2 \in \mathbb{N}$ be minimal such that there exists $y_2 \in T_\Phi$ with $f^{k_2}(y_2) = x$. For $n \geqslant \max\{k_1 + \tau(x), k_2 + \tau(y_2)\}$,

$$\mu_\varphi(Z_n[x]) \leqslant \mu_\varphi(f^{-k_1}(Z_{n-k_1}[y_1])) = \mu_\varphi(Z_{n-k_1}[y_1])$$
$$\mu_\varphi(Z_n[x]) = \mu_\varphi(f^{-k_2}(Z_n[x])) \geqslant \mu_\varphi(Z_{n-k_2}[y_2]).$$

Therefore, we may assume that $x \in T_\Phi$.

By the Gibbs property for (Y, F, μ_Φ), there exists $K > 0$ such that

$$\frac{1}{K} \leqslant \frac{\mu_\Phi(Z_{\tau^n(x)}[x])}{e^{\Phi_n(x)}} \leqslant K.$$

We will use the fact that there exists $\rho(x) \in (0, \infty)$ such that for a nested sequence of open sets $\{U_n\}_n$ such that $\cap U_n\{x\}$ as $n \to \infty$, we have $\frac{\mu_\Phi(U_n(x))}{\mu_\varphi(U_n(x))} \to \rho(x)$. Thus, for large enough n, the estimates we need for $\mu_\varphi(Z_n[x])$ follow immediately from those for $\mu_\Phi(Z_n[x])$.

For each large n, there exists k such that $\tau^{k-1}(x) < n \leqslant \tau^k(x)$. We get

$$\frac{\mu_\Phi(Z_{\tau^k(x)}[x])}{e^{\varphi_n(x)}} \leqslant \frac{\mu_\Phi(Z_n[x])}{e^{\varphi_n(x)}} \leqslant \frac{\mu_\Phi(Z_{\tau^{k-1}(x)}[x])}{e^{\varphi_n(x)}}.$$

So the Gibbs property implies

$$\frac{e^{\varphi_{\tau^k(x)-n}(f^n(x))}}{K} \leqslant \frac{\mu_\Phi(Z_n[x])}{e^{\varphi_n(x)}} \leqslant K e^{-\varphi_{n-\tau^{k-1}(x)}(F^{k-1}(x))}.$$

Since $|\varphi_{\tau^k(x)-n}(f^n(x))| \leqslant \sup_{x\in I} |\varphi(x)||\tau^k(x) - \tau^{k-1}(x)|$, it is sufficient for the lower bound to show that $\tau(F^k(x)) \leqslant \kappa \log n$ for all large n.

Claim. *There exists $\kappa \in (0, \infty)$ such that for μ_Φ-a.e. $x \in Y$ there exists $k_0 = k_0(x) \in \mathbb{N}$ such that $k \geqslant k_0$ implies $\tau(F^k(x)) \leqslant \kappa \log k$.*

Proof. We use the fact that (Y, F, μ_Φ) has exponential tails: in [BT] it is shown that there exists $\alpha > 0$ such that $\mu_\Phi\{\tau \geqslant k\} \leqslant Ce^{-\alpha k}$. We fix $\kappa > \frac{1}{\alpha}$. Let $V_k := \{x \in Y : \tau(F^k(x)) > \kappa \log k\}$. Since μ_Φ is F-invariant and $V_k = F^{-k}\{\tau \geqslant \kappa \log k\}$, we have $\mu_\Phi(V_k) \leqslant Cn^{-\alpha\kappa}$. So by the Borel-Cantelli Lemma we know that for μ_Φ-a.e. x there exists $k_0 = k_0(x)$ such that $k \geqslant k_0$ implies $x \notin V_k$. □

From this claim it follows that $\tau(F^k(x)) \leqslant \kappa \log k$ for all large k. Hence $\tau(F^k(x)) < \kappa \log n$ for all large n. The upper bound follows similarly. □

We now show the polynomial Gibbs property for acips. For $N \in \mathbb{N}$, $\ell > 1$ and $K > 0$, let $\mathcal{A}(N, \ell, K)$ be the set of maps in NF^3 with $\#\mathrm{Crit} = N$ and with each critical point $c \in \mathrm{Crit}$ having order $\ell_c < \ell$ and satisfying

$$|Df^n(f(c))| \geqslant K \text{ for all sufficiently large } n.$$

Clearly, whenever $\min_{c\in\mathrm{Crit}} |Df^n(f(c))| \to \infty$ as $n \to \infty$, f it must lie in $\mathcal{A}(N, \ell, K)$ for some N, ℓ and any $K > 0$.

We let m denote Lebesgue measure on the interval I. The following is proved in [BRSS].

Proposition 12. *Let $\ell > 1$ and $N \in \mathbb{N}$. There exists $K > 0$ such that if $f \in \mathcal{A}(N, \ell, K)$ then there is $C > 0$ such that for any Borel set A and any $n \geqslant 0$,*

$$m[f^{-n}(A)] \leqslant Cm[f(A)]^{\frac{1}{2\ell_{max}}}.$$

We can construct the invariant measure μ by taking a limit of the Cesaro means $\frac{1}{n}\sum_{k=0}^{n-1} m\circ f^{-k}$. From Proposition 12, it is easy to see that for an m-measurable set A, we have

$$\mu(A) \leqslant Cm(f(A))^{\frac{1}{2\ell_{max}}} \leqslant Cm(A)^{\frac{1}{2\ell_{max}^2}}. \tag{6}$$

In particular, $\mu \ll m$.

We prove Theorem 5 for acips in two lemmas. First the upper and then the lower bound.

Lemma 13 (Upper bound)**.** *Fix $\beta > 2$. For μ-a.e. x there is n_0 such that for all $n \geqslant n_0$*

$$\mu(Z_n[x])|Df^n(x)| \leqslant n^\beta.$$

Proof. Our proof follows [BV, Lemma 5]. We will use the Borel-Cantelli Lemma applied to m repeatedly here. Let $\beta' = \frac{\beta}{2} > 1$. Let $W_n := \{Z_n \in \mathcal{P}_n : \mu(Z_n) > n^{\beta'} m(Z_n)\}$, and $A_n := \bigcup_{Z_n\in W_n} Z_n$. Since μ is a probability measure, we have

$$1 \geqslant \mu(A_n) = \sum_{Z_n\in W_n} \mu(Z_n) \geqslant n^{\beta'} \sum_{Z_n\in W_n} m(Z_n) = n^{\beta'} m(A_n).$$

Whence $m(A_n) \leqslant n^{-\beta'}$. The Borel-Cantelli Lemma implies that m-a.e. $x \in I$ belongs to A_n for only finitely many n.

Now for any $Z_n \in \mathcal{P}_n$, let

$$U(Z_n) = \left\{ x \in Z_n : |Df^n(x)| > \frac{n^{\beta'}}{m(Z_n[x])} \right\}.$$

Then for $Z_n \in \mathcal{P}_n$,

$$1 \geqslant m(f^n(Z_n)) \geqslant \int_{U(Z_n)} |Df^n(x)|dm \geqslant \frac{n^{\beta'}}{m(Z_n)} m(U(Z_n)),$$

so $m(Z_n) \geqslant n^{\beta'} m(U(Z_n[x]))$. Letting $B_n := \bigcup_{Z_n\in\mathcal{P}_n} U(Z_n)$, we have

$$m(B_n) = \sum_{Z_n\in\mathcal{P}_n} m(U(Z_n)) \leqslant n^{-\beta'} \sum_{Z_n\in\mathcal{P}_n} m(Z_n) \leqslant n^{-\beta'}.$$

So again the Borel-Cantelli Lemma implies that m-a.e. x belongs to B_n for only finitely many n.

Therefore since $\mu \ll m$, for μ-a.e. $x \in I$ there exists some $n_0 = n_0(x)$ such that $x \notin A_n \cup B_n$ for all $n \geqslant n_0$. Thus $n \geqslant n_0$ implies

$$\mu(Z_n[x])|Df^n(x)| \leqslant n^{\beta'} m(Z_n[x]) \left(\frac{n^{\beta'}}{m(Z_n[x])} \right) \leqslant n^\beta$$

and we have the required upper bound. □

Notice that, unlike the following lemma, the proof of the above lemma did not require Proposition 12.

Lemma 14 (Lower bounds)**.** *For μ-a.e. x there is n_0 such that for all $n \geqslant n_0$, and μ an acip,*

$$\frac{1}{n^{2\alpha}} \leqslant \frac{|f^n(Z_n[x])|}{n^\alpha} \leqslant \mu(Z_n[x])|Df^n(x)|.$$

Proof. Let

$$V_n := \{x \in I : |f^n(x) - \partial f^n(Z_n[x])| < n^{-\alpha}|f^n(Z_n[x])|\}.$$

For $x \in I$, denote the part of $f^n(Z_n[x])$ which lies within $n^{-\alpha}|f^n(Z_n[x])|$ of the boundary of $f^n(Z_n[x])$ by $E_n[x]$. We will estimate the Lebesgue measure of the pullback $f^{-n}(E_n[x])$. Note that this set consists of more than just the pair of connected components $Z_n[x] \cap V_n$.

Clearly, $m(E_n[x]) \leqslant 2n^{-\alpha}m(f^n(Z_n[x]))$. Hence from (6), which follows from Proposition 12, we have

$$m(V_n \cap f^{-n}(E_n[x])) \leqslant K_0(2n^{-\alpha}m(f^n(Z_n[x])))^{\frac{1}{2\ell_{max}^2}} \leqslant 2K_0 n^{-\frac{\alpha}{2\ell_{max}^2}}.$$

There are at most $2n\#\text{Crit}$ domains $f^n(Z_n[x])$, hence

$$m(V_n) \leqslant Cn^{1-\frac{\alpha}{2\ell_{max}^2}}.$$

For $\alpha > 4\ell_{max}^2$ we have $\sum_n m(V_n) < \infty$. So by the Borel-Cantelli Lemma for m-a.e. x there exists n_0 such that $x \notin V_n$ for $n \geqslant n_0$.

We fix $0 < \delta < 1$ and may assume that $n_0^{-\alpha} < \delta$. Let $\tilde{Z}_n[x] \subset Z_n[x]$ be the maximal interval for which $d(f^n(\tilde{Z}_n[x]), \partial f^n(Z_n[x])) = \frac{\delta}{2}|f^n(Z_n[x])|$. Then for x as above, by the Koebe Lemma we obtain for $n \geqslant n_0$,

$$|Df^n(x)| \geqslant \left(\frac{n^{-\alpha}}{1+n^{-\alpha}}\right)^2 \frac{|f^n(\tilde{Z}_n[x])|}{|\tilde{Z}_n[x]|} \geqslant \left(\frac{1-\delta}{2n^\alpha}\right) \frac{|f^n(Z_n[x])|}{|Z_n[x]|}.$$

Letting $b := \inf_{x \in \text{supp}(\mu)} \frac{d\mu}{dm}(x)$, we have

$$\mu(Z_n[x])|Df^n(x)| \geqslant b\left(\frac{1-\delta}{2n^\alpha}\right)|f^n(Z_n[x])|$$

and the first part of the proof is finished if we can show that $b > 0$. Notice that since this works for m-a.e. x, it must also work for μ-a.e. x. To understand why $b > 0$, first note that by the Folklore Theorem, see [MS], the invariant measure μ_F for the induced system (Y, F) has $b' > 0$ so that $\frac{d\mu_F}{dm} \geqslant b'$ on $\text{supp}(\mu_F)$. Also, there exists N such that $\text{supp}(\mu) \subset \overline{\cup_{k=0}^N f^k(Y)}$. Given $y \in \text{supp}(\mu)$ and a set $U \subset Y$ so that $y \in f^k(U)$ for $k \leqslant N$, we have

$$\frac{\mu(f^k(U))}{m(f^k(U))} \geqslant \frac{\mu(f^k(U))}{\int_U |Df^k| \, dm} \geqslant \frac{\mu(U)}{m(U)(\sup|Df|)^k} \geqslant \frac{b'}{(\sup|Df|)^k}.$$

Then shrinking U we see that $\frac{d\mu}{dm}(y) > b$ where $b := \frac{b'}{(\sup|Df|)^N}$.

Let $W_n := \{x \in I : |f^n(Z_n[x])| \leqslant n^{-\alpha}\}$. For each domain Z_n of W_n, we choose a point $x_k \in Z_n$, so that $W_n = \cup_{k=1}^{p_n} Z_n[x_k]$. We have

$$m(W_n) = m\left(\bigcup_{k=1}^{p_n} Z_n[x_k]\right) \leqslant m\left(\bigcup_{k=1}^{p_n} f^{-n}[f^n(Z_n[x_k])]\right) \leqslant (2n\#\mathrm{Crit})n^{-\frac{\alpha}{2\ell_{max}^2}}.$$

Since $\alpha > 4\ell_{max}^2$, the Borel-Cantelli Lemma implies that for μ-a.e. $x \in I$ there is some $n_0 \geqslant 1$ such that $n \geqslant n_0$ implies $m(f^n(Z_n[x])) \geqslant n^{-\alpha}$. Combining this lower bound with the one above, we are finished. □

6. Entropy fluctuations

In this section we prove Theorem 6. This follows the same path as the proof of Theorem 3 in [BV]. For μ and acip, a sketch of the proof is as follows:

Step 1: The log-normal fluctuations in the Ornstein-Weiss Theorem follow (using [Sau]) from
(i) exponential return time statistics to cylinders (which is true for our equilibrium states by Proposition 4, and for acips by Theorem 3 applied to cylinders); and
(ii) log-normal fluctuations in the Shannon-McMillan-Breimann Theorem.

Step 2: Condition (ii) reduces to the regular Central Limit Theorem for observable $\varphi = \log|Df| - \int \log|Df|\, d\mu$, provided there is $\alpha < \frac{1}{2}$ such that

$$\frac{1}{n^\alpha} \leqslant \left|\log(\ \mu(Z_n[x])|Df^n(x)|\ )\right| \leqslant n^\alpha$$

for μ-a.e. x and n sufficiently large. Our polynomial Gibbs property clearly implies this.

Step 3: To prove the CLT for φ, we need Gordin's Theorem (see [BV, Theorem 6]), for which we need to verify that $\varphi \in L^2(\mu)$. Let us do that here.

Lemma 15. *The potential $\varphi := \log|Df| - \int \log|Df|\, d\mu$ belongs to $L^2(\mu)$.*

Proof. Clearly it is enough to show that $\log|Df| \in L^2(\mu)$. Clearly, there exists some $C > 0$ such that $\log|Df(x)| \leqslant C\ell_{max} \log|x - \mathrm{Crit}|$. Also by construction of μ and Proposition 12, we have $\mu(B_\varepsilon(c)) \leqslant C\varepsilon^{\frac{1}{2\ell_{max}^2}}$ for any $c \in \mathrm{Crit}$. For a given $c \in \mathrm{Crit}$, let U be a neighbourhood of c which is away from any other element of Crit. We have

$$\begin{aligned}\int_U (\log|Df(x)|)^2\, d\mu &\leqslant 2\sum_n \int_{(c+2^{-(n+1)},c+2^{-n})} (\log|Df|)^2\, d\mu \\ &\leqslant 2C\sum_n 2^{\frac{1-n}{2\ell_{max}^2}} |C\ell_{max}\log 2^{-n}|^2 \\ &\leqslant 4C^3\ell_{max}^2(\log 2)^2 \sum_n n^2 2^{-\frac{n}{2\ell_{max}^2}} < \infty.\end{aligned}$$

Since we can perform such a calculation at every critical point, the lemma is proved. □

Step 4: Finally, to apply Gordin's Theorem, we follow pages 91-93 of [BV] verbatim, except that neighbourhoods $B(c, L^{-n})$ and $B(c, n^{-5})$ of the critical point c need to be replaced by neighbourhoods $B(\mathrm{Crit}, L^{-n})$ and $B(\mathrm{Crit}, n^{-5})$ of Crit. Equation (11) of [BV] follows from [BLS], also in the multimodal case, whenever the critical orders are all the same.

For μ an equilibrium state for a potential φ, the proof is simplified. Step 1 is the same, so we only need to show that φ satisfies the CLT for (I, f, μ). This follows from [Yo, Theorem 4]. We need to check that for an inducing scheme (Y, F, Φ), the Jacobian of μ is sufficiently smooth and that φ has some Hölder properties. In particular we need to show that for $x, y \in Z_n^F$, there exists $\beta < 1$ such that

(a) $|e^{|\Phi(x)-\Phi(y)|} - 1| \leqslant C\beta^n$; and
(b) $|\Phi(x) - \Phi(y)| \leqslant C\beta^n$.

Here Z_n^F denotes an n-cylinder with respect the map F. In fact, (b) implies (a). Condition (b) follows from the proof of Lemma 2 in [BT].

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Department of Mathematics
University of Surrey
Guildford, Surrey, GU2 7XH
UK
h.bruin@surrey.ac.uk
http://www.maths.surrey.ac.uk/

Departamento de Matemática Pura
Rua do Campo Alegre, 687
4169-007 Porto
Portugal
mtodd@fc.up.pt
http://www.fc.up.pt/pessoas/mtodd