Higgs bundles and the real symplectic group

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Abstract. We give an overview of the work of Corlette, Donaldson, Hitchin and Simpson leading to the non-abelian Hodge theory correspondence between representations of the fundamental group of a surface (a *surface group*) and the moduli space of Higgs bundles. We then explain how this can be generalized to a correspondence between character varieties for representations of surface groups in real Lie groups G and the moduli space of G-Higgs bundles. Finally we survey recent joint work with Bradlow, García-Prada and Mundet i Riera on the moduli space of maximal $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles.

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1. INTRODUCTION

Higgs bundles are important in many areas of mathematics and mathematical physics. For example, it was shown by Hitchin that their moduli spaces give examples of Hyper-Kähler manifolds [1] and that they provide important algebraically integrable systems [2]. Recently they have featured for instance in the work of Hausel–Thaddeus [3] on mirror symmetry and in the work of Kapustin–Witten [4] giving a physical derivation of the geometric Langlands correspondence.

In this paper we start by explaining the non-abelian Hodge theory correspondence between representations of the fundamental group of a surface (a *surface group*) and the moduli space of Higgs bundles coming from the work of Corlette [5], Donaldson [6], Hitchin [1] and Simpson [7]. Next we explain how this theory can be generalized in a systematic way to a theory of G-Higgs bundles for real reductive Lie groups G (giving a correspondence with surface group representations in G); this is mainly based on joint work with Bradlow, García-Prada and Mundet i Riera. Finally we focus on the case of the real symplectic group $G = \mathrm{Sp}(2n,\mathbb{R})$ and survey some recent results on the corresponding moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles.

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2. HIGGS BUNDLES AND NON-ABELIAN HODGE THEORY

2.1. Higgs bundles

Let X be a closed Riemann surface of genus g and let $K_X = T^{1,0}X^*$ be the canonical bundle of X, i.e. its holomorphic cotangent bundle.

Definition 2.1. A *Higgs bundle* on X is a pair (E,ϕ), where $E \to X$ is holomorphic vector bundle and the *Higgs field* ϕ is a holomorphic 1-form with values in $\mathrm{End}(E)$, i.e., $\phi \in H^0(X, \mathrm{End}(E) \otimes K_X)$.

Recall that the C^∞ isomorphism class of a complex vector bundle $\mathscr{E}$ on the surface X is given by its first Chern class $c_1(\mathscr{E}) \in H^2(X,\mathbb{Z})$ or, equivalently, by its *degree*,

$$\deg(\mathscr{E}) = \int_X c_1(\mathscr{E}) \in \mathbb{Z}.$$

Recall also that $c_1(\mathscr{E})$ can be represented in de Rham cohomology by $\frac{i}{2\pi}\mathrm{tr}(F(B))$, for any connection B on $\mathscr{E}$. Thus a Higgs bundle (E,ϕ) has as discrete invariants its degree $\deg(E)$ and its rank $\mathrm{rk}(E)$.

As a first example, a rank one Higgs bundle is a pair (L,ϕ), where $L \to X$ is a line bundle and $\phi \in H^0(X,K_X)$ is a holomorphic 1-form (see Goldman and Xia [8] for a careful study of this case). Let $\mathrm{Jac}(X) = \mathrm{Pic}^0(X)$ be the Jacobian of X which parametrizes holomorphic line bundles on X of degree zero. The tangent space to $\mathrm{Jac}(X)$ at any L is just $H^1(X,\mathscr{O})$, which is isomorphic to $H^0(X,K_X)^*$ by Serre duality. Hence a rank 1 degree zero Higgs bundle (L,ϕ) corresponds to a point in the cotangent space $T^*_L\mathrm{Jac}(X)$ and the set of isomorphism classes of all such (L,ϕ) (the *moduli space*) can be identified with the cotangent bundle to the Jacobian $T^*\mathrm{Jac}(X)$.

We now describe abelian Hodge theory correspondence for first cohomology in a manner that points the way to the non-abelian generalization provided by Higgs bundle theory. Let Y be a Kähler manifold and let $H^{p,q}(Y)$ denote its Dolbeault cohomology groups, defined as the $\bar{\partial}$-closed differential forms of type (p,q) modulo the $\bar{\partial}$-exact ones. The abelian Hodge Theorem (see e.g. [9]) says that there is a decomposition of the cohomology with complex coefficients $H^k(Y;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Y)$ by using harmonic representatives of cohomology classes. In particular, for the Riemann surface X we have

$$H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X).$$

In fact, this is the infinitesimal version of an isomorphism

$$H^1(X,\mathbb{C}^*) \cong T^*\mathrm{Jac}(X). \tag{2.1}$$

To see how this comes about, note that an element of $H^1(X,\mathbb{C}^*)$ corresponds to a complex line bundle $\mathscr{L}$ with a flat connection. The corresponding connection form is just a closed 1-form $B \in A^1(X,\mathbb{C})$. If we fix a Hermitian metric on $\mathscr{L}$ and use the decomposition $\mathbb{C} = \mathfrak{gl}(1,\mathbb{C}) = i\mathbb{R} \oplus \mathbb{R}$, we can write

$$B = A + \psi, \qquad A \in A^1(X, i\mathbb{R}) \quad \text{and} \quad \psi \in A^1(X,\mathbb{R}).$$

Then the flatness condition $dB = 0$ is equivalent to saying that A and ψ are both closed 1-forms. Thus A is a flat unitary connection on $\mathscr{L}$. Moreover, the Hodge Theorem says that the cohomology class of ψ has a harmonic representative $\psi + df$ for some smooth f, meaning that

$$d^*(\psi + df) = 0.$$

Note that $\psi + df$ is obtained by applying to ψ the gauge transformation $\chi = \exp(f)$ of $\mathscr{L}$. Any gauge transformation preserves the flatness condition $dB = 0$ and hence the pair (A, ψ) obtained applying the above procedure to the gauge transformed $\chi \cdot B$ satisfies the equations

$$dA = 0, \tag{2.2}$$

$$d\psi = 0, \tag{2.3}$$

$$d^*\psi = 0. \tag{2.4}$$

Now the connection A defines a holomorphic line bundle $L_A \to X$ by taking the holomorphic structure on $\mathscr{L}$ given by the associated $\bar{\partial}$-operator $\bar{\partial}_A$, which is just the $(0,1)$-part of the covariant derivative d_A. We also define

$$\phi = \psi^{1,0} \in \Omega^{1,0}(X).$$

The important point to note is now that the conditions (2.3) and (2.4) together are equivalent to the holomorphicity condition $\bar{\partial}_A \phi = 0$ and hence (2.2)–(2.4) are equivalent to the pair of equations

$$\begin{aligned} dA &= 0, \\ \bar{\partial}_A \phi &= 0. \end{aligned} \tag{2.5}$$

Thus $\phi \in H^0(X, K_X)$ is holomorphic and (L_A, ϕ) is a rank 1 Higgs bundle. This, as we have seen, corresponds to a point in $T^*\mathrm{Jac}(X)$ (the fact that $\deg(L_A) = 0$ follows from $F(A) = dA = 0$).

An analogous argument shows how to recover a flat line bundle from a rank 1 Higgs bundle (L, ϕ), completing the proof of (2.1).

Remark 2.2. It follows that the moduli space $T^*\mathrm{Jac}(X)$ of degree zero Higgs line bundles can be identified with the variety of complex characters of $\pi_1(X)$ via the isomorphism $H^1(X, \mathbb{C}^*) \cong \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$. However the natural complex structures on these spaces do not correspond under this identification.

2.2. Non-abelian Hodge Theory

In this section we explain how the abelian Hodge Theorem for first cohomology generalizes to higher rank. Let $\mathscr{E} \to X$ be a fixed C^∞ complex vector bundle of rank n and degree d. Let B be a connection in $\mathscr{E}$ with constant central curvature:

$$F(B) = -i\mu \,\mathrm{Id}\, \omega, \tag{2.6}$$

where ω is the Kähler class of X, normalized so that $\int_X \omega = 2\pi$. Taking the trace and integrating in this formula, we see that the constant μ is given by

$$\mu = \frac{\deg(\mathscr{E})}{\mathrm{rk}(\mathscr{E})} = \frac{n}{d};$$

the ratio $\mu(E) = \frac{\deg(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}$ is called the *slope* of $\mathscr{E}$. In particular, if $\deg(\mathscr{E}) = 0$, the connection B has zero curvature, i.e., it is flat.

Let h be a Hermitian metric in $\mathscr{E}$. We shall write $\mathscr{E}_h$ for the bundle $\mathscr{E}$ equipped with Hermitian metric h. Splitting the associated Lie algebra valued 1-form in its skew-Hermitian and Hermitian parts, we can write

$$B = A + \psi,$$

where A is a unitary connection on $\mathscr{E}$ and ψ takes values in the bundle $\mathrm{End}^{\mathrm{herm}}(\mathscr{E}_h)$ of Hermitian endomorphisms of $\mathscr{E}$, in other words, ψ descends to a 1-form

$$\psi \in \Omega^1(X, \mathrm{End}^{\mathrm{herm}}(\mathscr{E}_h)).$$

The condition (2.6) can be expressed in terms of (A, ψ) as follows:

$$F(A) + \tfrac{1}{2}[\psi, \psi] = -i\mu \, \mathrm{Id} \, \omega, \tag{2.7}$$

$$d_A \psi = 0. \tag{2.8}$$

One may ask whether there is a preferred choice of Hermitian metric in the flat bundle $\mathscr{E}$. Since the space of metrics in $\mathbb{C}^n$ is $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$, a Hermitian metric h may be viewed as a $\pi_1(X)$-equivariant map

$$\tilde{h} \colon \tilde{X} \to \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n).$$

Because the symmetric space $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$ is a (negatively curved) Riemannian manifold and we have a conformal class of metrics on X defined by the complex structure, it makes sense to ask for $\tilde{h}$ to be an equivariant harmonic[2] map. The derivative of $\tilde{h}$ is a section

$$d\tilde{h} \in \Omega^1(\tilde{X}, \tilde{h}^* T(\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n))).$$

The harmonic map equation (cf. [10]) for $\tilde{h}$ is

$$d_\nabla^*(d\tilde{h}) = 0, \tag{2.9}$$

where d_∇ is the pull-back by h of the Levi-Civita connection on $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n))$ and d_∇^* is its adjoint (constructed again using the conformal class of metrics on X and the Riemannian metric on $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n))$). Moreover, it can be shown that (2.9) is

[2] Recall that harmonicity only depends on the conformal class of the source manifold when this is 2-dimensional.

equivalent to the equation $d_A^* \psi = 0$ so that the pair (A, ψ) obtained from B via a harmonic metric satisfies the equations

$$F(A) + \tfrac{1}{2}[\psi, \psi] = -i\mu \operatorname{Id} \omega, \tag{2.10}$$
$$d_A \psi = 0, \tag{2.11}$$
$$d_A^* \psi = 0. \tag{2.12}$$

Before we can state the main existence result for harmonic metrics we need to introduce the notion of a reductive flat connection. A connection B on $\mathscr{E}$ corresponds to a covariant derivative $d_B\colon \Omega^0(X, \mathscr{E}) \to \Omega^1(X, \mathscr{E})$. We say that a subbundle $\mathscr{F} \subset \mathscr{E}$ is preserved by B if it satisfies $d_B(\Omega^0(X, \mathscr{F})) \subset \Omega^1(X, \mathscr{F})$.

Definition 2.3. A flat connection B on $\mathscr{E}$ is *reductive* if for any subbundle $\mathscr{F} \subset \mathscr{E}$ preserved by B, there is a subbundle $\mathscr{F}'$ which is also preserved by B and such that $\mathscr{E} = \mathscr{F} \oplus \mathscr{F}'$.

It is not hard to show that if a bundle $\mathscr{E}$ with a flat connection B admits a harmonic metric, then B is reductive. The following theorem says that the converse holds. It is due to Donaldson [6] (in the case of rank 2 bundles on Riemann surface) and Corlette [5] (for base manifolds of higher dimension and more general structure groups).

Theorem 2.4. *Let B be a flat reductive connection in $\mathscr{E}$. Then there exists a unique harmonic metric in $\mathscr{E}$.*

In order to get a global statement, we introduce the *moduli space of reductive connections of constant central curvature* on a vector bundle $\mathscr{E}$ of degree d and rank n:

$$M_d^{\mathrm{dR}}(X, \mathrm{GL}(n, \mathbb{C})) = \{B \mid F(B) = -i\mu \operatorname{Id} \omega \text{ and } B \text{ is reductive}\}/\mathscr{G},$$

where $\mathscr{G} = \mathrm{Aut}(\mathscr{E})$ is the complex gauge group of $\mathscr{E}$.

We remark that connections of constant central curvature are related to representations of the fundamental group of X as follows (see Atiyah–Bott [11]). There is a central extension

$$0 \to \mathbb{R} \to \Gamma_{\mathbb{R}} \to \pi_1 X \to 1$$

defined by $\Gamma_{\mathbb{R}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, where the universal central extension Γ is defined by

$$\Gamma = \langle a_1, b_1, \dots, a_g, b_g, J \mid J \text{ is central and } \prod [a_i, b_i] = 1 \rangle.$$

Define the *character variety* for representations of $\Gamma_{\mathbb{R}}$ in $\mathrm{GL}(n, \mathbb{C})$ by

$$\begin{aligned} &M_d^B(X, \mathrm{GL}(n, \mathbb{C})) \\ &\quad = \{\rho\colon \Gamma_{\mathbb{R}} \to \mathrm{GL}(n, \mathbb{C}) \mid \rho(J) = \exp(2\pi i d/n) \operatorname{Id} \text{ and } \rho \text{ is semisimple}\}/\mathrm{GL}(n, \mathbb{C}), \end{aligned}$$

where $\mathrm{GL}(n, \mathbb{C})$ acts by overall conjugation. Note that $M_0^B(X, \mathrm{GL}(n, \mathbb{C}))$ can be naturally identified with the character variety for representations of $\pi_1 X$ in $\mathrm{GL}(n, \mathbb{C})$. It is now a standard fact that there is an (analytic) isomorphism

$$M_d^{\mathrm{dR}}(X, \mathrm{GL}(n, \mathbb{C})) \cong M_d^B(X, \mathrm{GL}(n, \mathbb{C})) \tag{2.13}$$

obtained by taking a connection B to its holonomy representation.

Now fix a reference metric h_0 in $\mathscr{E}$. For a reductive connection B with constant central curvature, let h be the harmonic metric given by the Theorem, so that $(\tilde{A}, \tilde{\psi})$ obtained from B using h satisfies (2.12). Let $g \in \mathscr{G}$ be an isometry between unitary bundles $g\colon \mathscr{E}_{h_0} \to \mathscr{E}_h$. Then the connection g^*B also has constant central curvature and the pair $(A, \psi) = g^*(\tilde{A}, \tilde{\psi})$ obtained from g^*B using the metric h_0, will also solve the equation (2.12). In other words, the metric h_0 is harmonic for g^*B. Any two choices of g differ by a unitary gauge transformation of $(\mathscr{E}, h_0)$. Hence, letting $\mathscr{U}$ denote the unitary gauge group, the theorem can be reformulated as follows:

Theorem 2.5. *There is a bijective correspondence*

$$M_d^{\mathrm{dR}}(X, \mathrm{GL}(n, \mathbb{C})) \cong \{(A, \psi) \mid \textit{satisfying (2.10)–(2.12)}\}/\mathscr{U}$$

where A is a unitary connection on $(\mathscr{E}, h_0)$ and $\psi \in \mathrm{End}^{\mathrm{herm}}(\mathscr{E}_{h_0})$.

In order to obtain a Higgs bundle from a solution (A, ψ) to (2.10)–(2.12), we decompose the covariant derivative associated to A as

$$d_A = \partial_A + \bar{\partial}_A,$$

and denote by E_A the holomorphic vector bundle defined by $\bar{\partial}_A$. Similarly, one can write

$$\psi = \phi + \phi^*$$

for a unique $\phi \in \Omega^{1,0}(X, \mathrm{End}(\mathscr{E}))$ (here ϕ^* denotes the $(0,1)$-form obtained from ϕ by combining the adjoint with respect to h_0 with conjugation on the form component). Then one easily checks that (2.10)–(2.12) are equivalent to *Hitchin's equations*

$$\begin{aligned} F(A) + [\phi, \phi^*] &= -i\mu \,\mathrm{Id}\, \omega, \\ \bar{\partial}_A \phi &= 0. \end{aligned} \tag{2.14}$$

Note that the second equation says that ϕ is an endomorphism valued holomorphic 1-form so (E_A, ϕ) is a Higgs bundle. Now recall that a holomorphic vector bundle which admits a flat unitary connection is the direct sum of stable degree zero vector bundles. To make sense of the analogous statement for Higgs bundles, we define the following stability notions.

Definition 2.6. A Higgs bundle (E, ϕ) is

- *semistable* if $\mu(F) \leq \mu(E)$ for all subbundles $F \subset E$ such that $\phi(F) \subset F \otimes K_X$.
- *stable* if $\mu(F) < \mu(E)$ for all proper subbundles $F \subsetneq E$ such that $\phi(F) \subset F \otimes K_X$.
- *polystable* if $(E, \phi) = (E_1, \phi_1) \oplus \cdots \oplus (E_r, \phi_r)$, where each (E_i, ϕ_i) is stable with $\mu(E_i) = \mu(E)$.

A subbundle $F \subset E$ such that $\phi(F) \subset F \otimes K_X$ is said to be a Φ-*invariant subbundle*. Note that semistability is a weaker condition than polystability, which in turn is weaker than stability.

It is easy to check that a Higgs bundle obtained from a solution to Hitchin's equations (2.14) is polystable. The converse is given by the following theorem, which gives a Hitchin–Kobayashi correspondence for Higgs bundles. It is due to Hitchin [1] (for Higgs bundles on Riemann surfaces) and Simpson [7] (for Higgs bundles over higher dimensional manifolds).

Theorem 2.7. *If (E,ϕ) is polystable then there exists a unique Hermitian metric in E such that (A,ϕ) satisfies Hitchin's equations (2.14), where A is the unique unitary connection compatible with the holomorphic structure (i.e. the Chern connection).*

In order to get the corresponding global statement, we introduce the *moduli space*

$$M_d^{\mathrm{Dol}}(X,\mathrm{GL}(n,\mathbb{C}))$$

of rank n, degree d polystable Higgs bundles. As a set, this is the set of isomorphism classes of polystable Higgs bundles. It can be given the structure of a complex (algebraic) variety using standard gauge theory methods (Hitchin [1]) or using Geometric Invariant Theory[3] (Nitsure [12]). Then Theorem 2.7 implies that $M_d^{\mathrm{Dol}}(X,\mathrm{GL}(n,\mathbb{C}))$ is in bijective correspondence with the space of unitary gauge equivalence classes of solutions to Hitchin's equations (2.14). Putting this together with Theorem 2.5 and the identification (2.13), we finally obtain the non-abelian Hodge Theorem.

Theorem 2.8. *There is a homeomorphism $M_d^{\mathrm{B}}(X,\mathrm{GL}(n,\mathbb{C})) \cong M_d^{\mathrm{Dol}}(X,\mathrm{GL}(n,\mathbb{C}))$.*

3. *G*-HIGGS BUNDLES

We have seen that Higgs bundles correspond to representations $\rho\colon \pi_1 X \to \mathrm{GL}(n,\mathbb{C})$ The use of Higgs bundle methods for studying representations

$$\rho\colon \pi_1 X \to G$$

for more general Lie groups G was pioneered by Hitchin [1, 13] and also, using Tannakian considerations, by Simpson [14]. Subsequently a theory of G-Higgs bundles, appropriate for studying representations of $\pi_1 X$ in real a reductive Lie group G, has been developed in a systematic way. In this section we briefly outline this theory. For more details the reader may consult, for example, [15, 16, 17, 18].

Let G be a real reductive Lie group in the sense of Knapp [19, p. 384]. In particular this means that we are given a maximal compact subgroup $H \subset G$. Also, there is a *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{h}$ is the Lie algebra of H. Moreover, restriction to H of the adjoint action of G on its Lie algebra gives a representation

$$\begin{aligned} \iota\colon H &\to \mathrm{GL}(\mathfrak{m}) \\ g &\mapsto \big(x \mapsto \mathrm{Ad}(g)(x)\big). \end{aligned}$$

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[3] From this point of view it is better to consider $M_d^{\mathrm{Dol}}(X,\mathrm{GL}(n,\mathbb{C}))$ as the space of S-equivalence classes of semistable Higgs bundles.

This representation is called the *isotropy representation*. We shall denote by the same symbol its complexification $\iota\colon H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$ defined on the complexification $H^{\mathbb{C}}$ of H. If E is a principal $H^{\mathbb{C}}$-bundle, we thus have an associated bundle

$$E(\mathfrak{m}^{\mathbb{C}}) = E \times_{\iota} \mathfrak{m}^{\mathbb{C}}$$

with fibres $\mathfrak{m}^{\mathbb{C}}$. Note that if G is itself a complex group, the Cartan decomposition is $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, and hence $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{g}^{\mathbb{C}})$, the adjoint bundle of the G-bundle E.

Definition 3.1. A *G-Higgs bundle* is a pair (E,ϕ), where $E \to X$ is a holomorphic principal $H^{\mathbb{C}}$-bundle, $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K_X)$.

Example 3.2. Let $G = \mathrm{GL}(n,\mathbb{C})$. Then a G-Higgs bundle gives rise to a Higgs bundle $(E \times_{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}^n, \phi)$ as previously defined.

Example 3.3. Let $G = \mathrm{Sp}(2n,\mathbb{R})$. In this case: $H = \mathrm{U}(n)$ and $\mathfrak{m}^{\mathbb{C}} = S^2\mathbb{C}^n \oplus S^2(\mathbb{C}^n)^*$, where $\mathbb{C}^n$ denotes the fundamental representation of $H^{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C})$. Hence a $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle is equivalent to a triple (V,β,γ), where V is a rank n vector bundle and

$$\beta \in H^0(X, S^2V \otimes K_X), \qquad \gamma \in H^0(X, S^2V^* \otimes K_X).$$

The usual Higgs bundle given by the inclusion $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{Sp}(2n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{C})$ is:

$$(E = V \oplus V^*, \phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}).$$

Example 3.4. A $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle is given by $((W,Q),\eta)$, where (W,Q) is an orthogonal bundle and $\eta\colon W \to W \otimes K_X$ is symmetric with respect to Q.

Stability of G-Higgs bundles is in general a complicated notion, and we shall not state it here, since we shall have no explicit need for it. The interested reader is referred to [17]. But it is worth remarking that in all cases of interest to us here, the complexification $G^{\mathbb{C}} \subset \mathrm{GL}(n,\mathbb{C})$ is a linear group and semi- and polystability of (E,ϕ) is equivalent to semi- and polystability of the induced rank n usual Higgs bundle, respectively (cf. [15]). On the other hand, the stability conditions are, in general, different.

When G is connected, the topological classification of G-bundles is given by a characteristic class in $H^2(X,\pi_1 G) \cong \pi_1 G \cong \pi_1 H$. For a fixed topological class $d \in \pi_1 H$ we have can introduce the analogues of the moduli spaces defined above. Thus $M_d^{\mathrm{B}}(X,G)$ denotes the character variety for representations of a suitable central extension of $\pi_1 X$ and $M_d^{\mathrm{Dol}}(X,G)$ denotes the moduli space of polystable G-Higgs bundles. In order to construct the latter space, the general theory of moduli of decorated bundles of Schmitt [20] is required.

We have the following generalization of the non-abelian Hodge Theorem, proved via an intermediate moduli space of solutions to an appropriate version of Hitchin's equations (2.14) for G-Higgs bundles. While Theorem 2.5 essentially applies unchanged in this situation, the generalization of Theorem 2.7 to principal pairs of [21, 22, 17] is required for proving this result.

Theorem 3.5. *There is a homeomorphism* $M_d^{\mathrm{B}}(X,G) \cong M_d^{\mathrm{Dol}}(X,G)$.

4. THE REAL SYMPLECTIC GROUP

In this section we mainly focus on the moduli space of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles, describing some properties of their moduli spaces. Details of these results can be found in [23, 24, 25, 26].

4.1. Stability and the Milnor–Wood inequality

Let (V,β,γ) be a $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle as in Example 3.3. The topological invariant of (V,β,γ) is the degree $d=\deg(V)$. If (V,β,γ) is polystable and $\rho\colon \pi_1 X\to \mathrm{Sp}(2,\mathbb{R})$ is the corresponding representation, it can be seen that d is the so-called *Toledo invariant* of ρ, usually denoted by $\tau(\rho)$. The Toledo invariant is bounded by the inequality

$$|\tau(\rho)|\leq n(g-1), \tag{4.1}$$

usually known as the *Milnor–Wood inequality*. This inequality is due to Milnor [27] in the case $n=1$ and to Dupont [28] and Turaev [29] in the general case, the latter giving the sharp bound. This inequality can be proved easily using $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles as follows.

For definiteness, assume that $\deg(V)>0$. Define subbundles $N\subset V$ and $I\subset V^*$ using the subsheaves $\ker(\gamma)$ and $\mathrm{im}(\gamma)\otimes K_X^{-1}$. Then

$$N\oplus 0\subset V\oplus V^* \qquad \text{and} \qquad V\oplus I\subset V\oplus V^*$$

are Φ-invariant subbundles of the $\mathrm{GL}(2n,\mathbb{R})$-Higgs bundle $(E=V\oplus V^*,\Phi=\left(\begin{smallmatrix}\beta & 0\\ 0 & \gamma\end{smallmatrix}\right))$. If (V,β,γ) is semistable, then so is (E,Φ). It follows that

$$\deg(N)\leq 0, \tag{4.2}$$
$$\deg(V)+\deg(I)\leq 0. \tag{4.3}$$

Moreover, $\gamma\neq 0$ since otherwise $V\subset E$ would be Φ-invariant and thus violate semistability. Hence it induces a non-zero section

$$\bar{\gamma}\in H^0(X,\det(V/N)^*\otimes\det(I)\otimes K^{\mathrm{rk}(\gamma)}), \tag{4.4}$$

so that this line bundle must have positive degree. Together with (4.2) this implies that

$$\deg(V)\leq \mathrm{rk}(\gamma)(g-1),$$

thus demonstrating that $\deg(V)\leq n(g-1)$, which proves the Milnor–Wood inequality (4.1) for $\deg(V)=\tau(\rho)>0$. An analogous argument using β proves the case $\deg(V)<0$.

Note that when the equality $\deg(V)=\mathrm{rk}(\gamma)(g-1)$ holds, the line bundle in (4.4) is forced to have degree zero. Thus the map $\bar{\gamma}$ in (4.4) is an isomorphism. In particular, we have the following important consequence.

Proposition 4.1. *Let* (V,β,γ) *be a semistable* $\mathrm{Sp}(2n,\mathbb{R})$*-Higgs bundle such that* $\deg(V) = n(g-1)$*. Then* γ *induces an isomorphism* $\gamma\colon V \xrightarrow{\cong} V \otimes K$*. If* $\deg(V) = -n(g-1)$ *the analogous statement holds for* β*.*

Definition 4.2. An $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle (V,β,γ) is said to be *maximal* if $|\deg(V)| = n(g-1)$. Similarly, a representation $\rho\colon \pi_1 X \to \mathrm{Sp}(2n,\mathbb{R})$ is *maximal* if $|\tau(\rho)| = n(g-1)$.

The geometric importance of maximal representations is underlined by the following theorem, due to Goldman [30].

Theorem 4.3. $\rho\colon \pi_1 X \to \mathrm{Sp}(2,\mathbb{R})$ *is maximal if and only if it is Fuchsian.*

Recall that a representation $\rho\colon \pi_1 X \to \mathrm{SL}(2,\mathbb{R})$ is called Fuchsian if it is discrete and faithful. More generally, it was shown by Burger–Iozzi–Wienhard [31, 32] that maximal representations are discrete, faithful and reductive.

4.2. The moduli space of maximal $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles

For brevity, denote by $M_{\max}(n)$ the moduli space $M^{\mathrm{Dol}}_{n(g-1)}(X,\mathrm{Sp}(2n,\mathbb{R}))$ of maximal $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles. In this section we shall see how Proposition 4.1 leads to the existence of new invariants of maximal $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles. In particular, this implies that $M_{\max}(n)$ is disconnected and we shall also give a complete count of its connected components.

Choose a square root $K_X^{1/2}$ of K_X. Using the isomorphism $\gamma\colon V \xrightarrow{\cong} V^* \otimes K_X$ and the fact that γ is symmetric, we define an orthogonal holomorphic bundle (W,Q) on X as follows:

$$W = V \otimes K_X^{-1/2} \qquad \text{and} \qquad Q = \gamma \otimes 1_{K^{-1/2}}\colon W \xrightarrow{\cong} W^*.$$

The Stiefel–Whitney classes of (W,Q) define new invariants of the $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle (V,β,γ):

$$w_1(V,\beta,\gamma) \in H^1(X,\mathbb{Z}/2) \qquad \text{and} \qquad w_2(V,\beta,\gamma) \in H^2(X,\mathbb{Z}/2).$$

Remark 4.4. In the case $n=1$, the invariant $w_2(V,\beta,\gamma)$ always vanishes for obvious reasons. The case $n=2$ is also special: when $w_1(V,\beta,\gamma)=0$, there is a lift of $w_2(V,\beta,\gamma)$ to an invariant $c(V,\beta,\gamma) \in H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, coming from a reduction of structure group to $\mathrm{SO}(2) \subset \mathrm{O}(2)$. This invariant satisfies $|c(V,\beta,\gamma)| \leq 2g-2$.

Remark 4.5. We can define $\eta = (\beta \otimes 1) \circ (\gamma \otimes 1)\colon W \to W \otimes K_X^2$. Then $((W,Q),\eta)$ is a *twisted* $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle (cf. Example 3.4), the difference to a usual $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle being the twisting by K_X^2 rather than K_X.

This is an instance of a general phenomenon occurring for maximal G-Higgs bundles when G is isogenous to the isometry group of Hermitian symmetric space of non-compact type. We refer to [18] for this theory.

The connected components of $M_{\max}(1) = M^{\mathrm{B}}_{g-1}(X, \mathrm{SL}(2,\mathbb{R}))$ were determined by Goldman [33] working directly with representations of $\pi_1 X$:

Theorem 4.6. *The connected components of $M_{\max}(1)$ are the 2^{2g} subspaces $M_{w_1} \subset M_{\max}(1)$ of Higgs bundles having invariant $w_1 \in H^1(X, \mathbb{Z}/2)$.*

Remark 4.7. These components are all homeomorphic to the Teichmüller space of X, as also follows from results of Goldman. In particular each component M_{w_1} is homeomorphic to $\mathbb{R}^{6g-6}$. This can bee easily seen from the Higgs bundle point of view (see [1]) by noting that a maximal $\mathrm{Sp}(2,\mathbb{R})$-Higgs bundle is isomorphic to one of the form (L, β, γ), where $L^2 = K_X$ and $\gamma = 1$. The choice of L given by w_1 and thus the choice of $\beta \in H^0(X, K_X^2)$ gives a an identification $M_{w_1} \cong H^0(X, K_X^2)$.

Generalizing the parametrization by quadratic differentials of M_{w_1} of the preceding remark, Hitchin [13] showed the existence of special connected *Hitchin components* $M^H \subset M^{\mathrm{Dol}}(X, G)$ whenever G is a split real form of a simple complex group (the classical examples are $G = \mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n)$, $\mathrm{SO}(n+1,n)$). The Hitchin components are vector spaces of the form $M^H \cong \bigoplus H^0(X, K_X^{m_i+1})$. In the case $G = \mathrm{Sp}(2n,\mathbb{R})$ Hitchin components are maximal and there are 2^{2g} such components

$$M^H_L \subset M_{\max}(n)$$

indexed by square roots L of K_X, just as in the case $n = 1$.

Denote by $M_{w_1,w_2} \subset M_{\max}(n)$ the subspace of **non**-Hitchin $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with invariants $w_i \in H^i(X, \mathbb{Z}/2)$ for $i = 1,2$. In the case $n = 2$, we additionally write $M_{0,c}$ for the subspace of non-Hitchin $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles with invariants $w_1 = 0$ and $c \in H^2(X,\mathbb{Z}) \cong \mathbb{Z}$.

The connected components of $M_{\max}(n)$ were determined in [23] for $n = 2$ and in [25] for $n \geq 3$. We refer the interested reader to these papers for the proof of the following theorem. For more information on maximal $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles and the corresponding representations see for instance [26] and [34].

Theorem 4.8. *For $n = 2$, the decomposition in connected components of $M_{\max}$ is*

$$M_{\max} = \bigcup_{w_1 \neq 0, w_2} M_{w_1,w_2} \cup \bigcup_{0 \leq c < 2g-2} M_{0,c} \cup \bigcup_{L^2 = K_X} M^H_L .$$

For $n \geq 3$, the decomposition in connected components of $M^{\max}$ is

$$M_{\max} = \bigcup_{w_1, w_2} M_{w_1,w_2} \cup \bigcup_{L^2 = K_X} M^H_L .$$

Finally, we mention that the maximal connected components has been carried our in many cases for many non-compact groups G of Hermitian type; see [18] for a survey of such results. On the other hand, the determination of non-maximal components is in general a difficult problem, which in the case of $G = \mathrm{Sp}(2n,\mathbb{R})$) has only been carried out for $n = 2$ [24].

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