Non-Noetherian generalized Heisenberg algebras

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Abstract

In this note we classify the non-Noetherian generalized Heisenberg algebras $\mathcal{H}(f)$ introduced in [8]. In case $\deg f > 1$, we determine all locally finite and also all locally nilpotent derivations of $\mathcal{H}(f)$ and describe the automorphism group of these algebras.

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1 Introduction

Fix a polynomial $f \in \mathbb{C}[h]$. The *generalized Heisenberg algebra* $\mathcal{H}(f)$ is the unital associative $\mathbb{C}$-algebra with generators x, y, h satisfying the relations:

$$hx = xf(h), \quad yh = f(h)y, \quad yx - xy = f(h) - h. \tag{1.1}$$

See [8] and the references therein for information on how these algebras first appeared and on their applications to theoretical physics.

Ambiskew polynomial rings were introduced by Jordan over a series of papers (see the references in [5]), but for our purposes the best suited definition is the one found in [5], which we briefly recall. Let σ be an endomorphism of a commutative $\mathbb{C}$-algebra B, $c \in B$ and $p \in \mathbb{C}$. The ambiskew polynomial ring $R(B, \sigma, c, p)$ is the $\mathbb{C}$-algebra generated by B and two indeterminates, x and y, subject to the relations

$$bx = x\sigma(b), \quad yb = \sigma(b)y, \quad yx - pxy = c, \quad \text{for all } b \in B.$$

On comparing these relations with those in (1.1), one immediately sees that

$$\mathcal{H}(f) \cong R(\mathbb{C}[h], \sigma, f(h) - h, 1), \tag{1.2}$$

where $\sigma : \mathbb{C}[h] \to \mathbb{C}[h]$ is the algebra endomorphism given by $\sigma(h) = f(h)$. In particular, one can see that there is an overlap between the generalized Heisenberg algebras defined above and (generalized) down-up algebras (see Corollary 2.7 below).

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The algebras $\mathcal{H}(f)$ can also be seen as weak generalized Weyl algebras over a polynomial algebra in two variables, in the sense of [7], a construction which includes the generalized Weyl algebras introduced by V.V. Bavula in [1]. In [8] the authors determine a basis for $\mathcal{H}(f)$ over $\mathbb{C}$, compute the center of $\mathcal{H}(f)$, solve the isomorphism problem for this family of algebras and classify all the finite-dimensional irreducible representations of $\mathcal{H}(f)$.

In this note we show that $\mathcal{H}(f)$ is (right or left) Noetherian if and only if $\mathsf{deg}\, f = 1$ and that $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra if and only if $\mathsf{deg}\, f \leq 1$. For this reason, we then concentrate on the case where $\mathsf{deg}\, f > 1$ and determine the locally nilpotent and the locally finite derivations of $\mathcal{H}(f)$, all $\mathbb{Z}$-gradings of $\mathcal{H}(f)$ and describe the automorphism group of $\mathcal{H}(f)$. In particular, we obtain the following results in case $\mathsf{deg}\, f > 1$:

(i) $\mathcal{H}(f)$ in neither right nor left Noetherian (Proposition 2.4);

(ii) $\mathcal{H}(f)$ admits a unique (up to an integer multiple) nontrivial $\mathbb{Z}$-grading, in which x has degree 1, y has degree -1 and h has degree 0 (Corollary 4.10);

(iii) the automorphism group of $\mathcal{H}(f)$ is abelian: it is isomorphic to $\mathbb{C}^* \times \mathsf{C}$, where C is a finite cyclic group whose order divides $(\mathsf{deg}\, f) - 1$ (Theorem 5.5).

In Section 2 of the paper we review some properties of $\mathcal{H}(f)$ which have been established in [8], determine when $\mathcal{H}(f)$ is Noetherian and when it is isomorphic to a generalized down-up algebra, while in Section 3 we introduce a useful commutative subalgebra of $\mathcal{H}(f)$, which is a maximal commutative subalgebra if $\mathsf{deg}\, f > 1$. Assuming that $\mathsf{deg}\, f > 1$, we then investigate the locally finite and the locally nilpotent derivations of $\mathcal{H}(f)$ and also its $\mathbb{Z}$-gradings in Section 4, and in the final section, Section 5, we describe the automorphism group of $\mathcal{H}(f)$ and show that it is always an abelian group generated by the automorphisms which fix h and the automorphisms which fix x.

We make use of the commutator notation $[a, b] = ab - ba$. The sets of integers, nonnegative integers and positive integers are denoted by $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$, respectively. The field of complex numbers is denoted by $\mathbb{C}$, and the multiplicative group of nonzero complex numbers is denoted by $\mathbb{C}^*$. For a polynomial $g \in \mathbb{C}[h]$, $\mathsf{deg}\, g$ will always denote the degree of g as a polynomial in h.

Throughout the paper, $\sigma : \mathbb{C}[h] \to \mathbb{C}[h]$ is the algebra endomorphism given by $\sigma(h) = f(h)$. For any function $\phi : X \to X$, we will use the notation ϕ^k to mean the k-th power of ϕ with respect to composition. In particular, ϕ^0 denotes the identity on the set X.

2 The Noetherian property

Below we record a few results from [8] which will be useful in the course of this paper. As usual, $Z(\mathcal{H}(f))$ denotes the center of $\mathcal{H}(f)$.

Lemma 2.1 ([8, Lemma 1, Lemma 2, Theorem 4])**.** *Let $f \in \mathbb{C}[h]$. Then:*

(a) *The set* $\{x^i h^j y^k \mid i,j,k \in \mathbb{Z}_{\geq 0}\}$ *is a basis of* $\mathcal{H}(f)$.

(b) *The algebra* $\mathcal{H}(f)$ *is a domain if and only if* $\deg f \geq 1$.

(c) *The center of* $\mathcal{H}(f)$ *contains the polynomial algebra* $\mathbb{C}[z]$*, where* $z = xy - h = yx - f(h)$*. If* $\deg f \neq 1$*, then* $Z(\mathcal{H}(f)) = \mathbb{C}[z]$.

Remarks 2.2.

1. Identifying $\mathcal{H}(f)$ with the ambiskew polynomial ring $R(\mathbb{C}[h], \sigma, f(h) - h, 1)$ as in (1.2), it follows that $\mathcal{H}(f)$ is conformal, as defined in [5, Section 2.3], and the corresponding Casimir element is precisely the central element $z = xy - h$ defined above.

2. Suppose $f \in \mathbb{C}$. Then by considering the generators $-x$, y and $h - f$, we see that $\mathcal{H}(f) \cong R(\mathbb{C}[h], \sigma, h, 1)$, with $\sigma = 0$, and from [5, Theorem 7.10] we conclude that $\mathcal{H}(f)$ is a prime ring. Thus by Lemma 2.1(b), $\mathcal{H}(f)$ is a prime ring for any $f \in \mathbb{C}[h]$.

3. Since the center of $\mathcal{H}(f)$ contains the polynomial algebra $\mathbb{C}[z]$ and $\mathcal{H}(f)$ has countable dimension over $\mathbb{C}$, it follows from Dixmier's version of Schur's Lemma that $\mathcal{H}(f)$ is never primitive.

There is an order two anti-automorphism of $\mathcal{H}(f)$, denoted by ι, that fixes h and interchanges x and y:

$$\iota : \mathcal{H}(f) \to \mathcal{H}(f), \qquad x \mapsto y, \quad y \mapsto x, \quad h \mapsto h. \tag{2.3}$$

Hence $\mathcal{H}(f)$ is isomorphic to its opposite algebra $\mathcal{H}(f)^{\mathrm{op}}$.

Proposition 2.4. *The algebra* $\mathcal{H}(f)$ *is right (respectively, left) Noetherian if and only if* $\deg f = 1$.

Proof. If $\deg f = 1$ then $\mathcal{H}(f)$ is a generalized Weyl algebra over a polynomial ring in two variables, and thus it is right and left Noetherian. So assume that $\deg f \neq 1$. In particular, $f(h) - h$ has some root $\alpha \in \mathbb{C}$. Let $F(h) = f(h + \alpha) - \alpha$. Then $\deg F = \deg f$ (here we assume the zero polynomial has degree 0) and $F(h) \in h\mathbb{C}[h]$. Moreover, $F(h - \alpha) = f(h) - \alpha$ and then $\mathcal{H}(f) \cong \mathcal{H}(F)$ by [8, Lemma 3]. So there is no loss in assuming that $f(h) \in h\mathbb{C}[h]$. By the isomorphism $\mathcal{H}(f) \cong \mathcal{H}(f)^{\mathrm{op}}$ it will be enough to show that $\mathcal{H}(f)$ is not left Noetherian.

For each $n \in \mathbb{Z}_{\geq 0}$ define the left ideal

$$I_n = \sum_{i=0}^{n} \mathcal{H}(f) h y^i.$$

Then $I_n \subseteq I_{n+1}$ for all $n \geq 0$ and we finish the proof by showing that these inclusions are strict. Note that by Lemma 2.1(a),

$$\mathcal{H}(f) = \bigoplus_{j,k \geq 0} x^j \mathbb{C}[h] y^k.$$

Given $j, k \geq 0$ and $g(h) \in \mathbb{C}[h]$, we have $x^j g(h) y^k h y^i = x^j g(h) \sigma^k(h) y^{k+i}$. Assume, by way of contradiction, that $hy^{n+1} \in I_n$. Then there exist $g_i(h) \in \mathbb{C}[h]$, $i = 0, \ldots, n$, such that $hy^{n+1} = \sum_{i=0}^{n} g_i(h) \sigma^{n+1-i}(h) y^{n+1}$. It follows by Lemma 2.1(a) that

$$h = \sum_{i=0}^{n} g_i(h) \sigma^{n+1-i}(h). \tag{2.5}$$

As by hypothesis $\sigma(h) = f(h) \in h\mathbb{C}[h]$, one can deduce that $\sigma^{n+1-i}(h) \in f(h)\mathbb{C}[h]$ for all $0 \leq i \leq n$ and (2.5) then implies that $h \in f(h)\mathbb{C}[h]$, which is a contradiction since under our hypothesis either $f(h) = 0$ or $\deg f > 1$. This proves that $hy^{n+1} \notin I_n$ for any $n \geq 0$ and hence $\{I_n\}_{n \geq 0}$ is a strict ascending chain of left ideals of $\mathcal{H}(f)$. □

Remark 2.6. The case $f \in \mathbb{C}$ of Proposition 2.4 follows also from [5, Corollary 7.3], which applies when σ is not injective. In terms of the endomorphism σ, Proposition 2.4 could be restated as: The algebra $\mathcal{H}(f)$ is right (respectively, left) Noetherian if and only if σ is an automorphism.

We recall that a generalized down-up algebra $L(g, r, s, \gamma)$, given by the parameters $g \in \mathbb{C}[H]$ and $r, s, \gamma \in \mathbb{C}$, is defined as the unital associative $\mathbb{C}$-algebra generated by d, u and H, subject to the relations:

$$dH - rHd + \gamma d = 0, \quad Hu - ruH + \gamma u = 0, \quad du - sud + g(H) = 0.$$

Generalized down-up algebras were defined in [4] as generalizations of the down-up algebras introduced by Benkart and Roby in [2]. Generalized down-up algebras include all down-up algebras, the algebras similar to the enveloping algebra of $\mathfrak{sl}_2$ defined by Smith [11], Le Bruyn's conformal $\mathfrak{sl}_2$ enveloping algebras [6] and Rueda's algebras similar to the enveloping algebra of $\mathfrak{sl}_2$ [10].

Corollary 2.7. *The algebra $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra if and only if $\deg f \leq 1$.*

Proof. Suppose first that $\deg f \leq 1$, say $f(h) = ah + b$ for $a, b \in \mathbb{C}$. Then it is straightforward to verify that $\mathcal{H}(f) \cong L(H - f(H), a, 1, -b)$, under an isomorphism that sends x, y and h to u, d and H, respectively. Conversely, suppose that $\deg f > 1$. Then by Proposition 2.4 and Lemma 2.1(b), $\mathcal{H}(f)$ is a non-Noetherian domain. Hence $\mathcal{H}(f)$ cannot be isomorphic to a generalized down-up algebra, as a generalized down-up algebra is a domain if and only if it is Noetherian, by Propositions 2.5 and 2.6 of [4]. □

In view of this result, we will henceforth focus most of our attention on the generalized Heisenberg algebras $\mathcal{H}(f)$ with $f \in \mathbb{C}[h]$ such that $\deg f > 1$.

3 The commutative algebra $\mathcal{H}(f)_0$

In this short section we record a few useful formulas for computing in $\mathcal{H}(f)$ and then explore an interesting commutative subalgebra of $\mathcal{H}(f)$.

Lemma 3.1. *Let $k \in \mathbb{Z}_{\geq 0}$ and $g \in \mathbb{C}[h]$. Then the following hold:*

(a) $[y, x^k] = x^{k-1}(\sigma^k(h) - h)$;

(b) $[y^k, x] = (\sigma^k(h) - h)y^{k-1}$;

(c) $(x^k g y^k)x = x(x^k \sigma(g) y^k + x^{k-1}(\sigma^k(h) - h) g y^{k-1})$;

(d) $y(x^k g y^k) = (x^k \sigma(g) y^k + x^{k-1}(\sigma^k(h) - h) g y^{k-1})y$;

(e) *$x^k g y^k$ commutes with $x^j \tilde{g} y^j$ for all $\tilde{g} \in \mathbb{C}[h]$ and all $j \in \mathbb{Z}_{\geq 0}$.*

Proof. Parts (a) and (b) have been established in [5], formulas (6a)–(6b). We prove part (c) using (b):

$$\begin{aligned}(x^k g y^k)x &= x^k g x y^k + x^k g[y^k, x] \\ &= x^{k+1}\sigma(g) y^k + x^k g(\sigma^k(h) - h)y^{k-1} \\ &= x(x^k \sigma(g) y^k + x^{k-1}(\sigma^k(h) - h) g y^{k-1}).\end{aligned}$$

Formula (d) follows from applying the anti-automorphism ι of (2.3) to (c).

Finally, we prove (e) by induction on k, the case $k = 0$ being trivial:

$$g(x^j \tilde{g} y^j) = x^j \sigma^j(g) \tilde{g} y^j = x^j \tilde{g} \sigma^j(g) y^j = (x^j \tilde{g} y^j)g.$$

Now suppose (e) holds for a certain $k \geq 0$. Thus we have:

$$\begin{aligned}(x^{k+1} g y^{k+1})(x^j \tilde{g} y^j) &= (x^{k+1} g y^k) y (x^j \tilde{g} y^j) \\ &= x(x^k g y^k)(x^j \sigma(\tilde{g}) y^j + x^{j-1}(\sigma^j(h) - h)\tilde{g} y^{j-1})y \qquad \text{by (d)} \\ &= x(x^j \sigma(\tilde{g}) y^j + x^{j-1}(\sigma^j(h) - h)\tilde{g} y^{j-1})(x^k g y^k)y \qquad (*) \\ &= (x^j \tilde{g} y^j) x (x^k g y^k) y \qquad \text{by (c)} \\ &= (x^j \tilde{g} y^j)(x^{k+1} g y^{k+1}),\end{aligned}$$

where $(*)$ follows from the induction hypothesis. So (e) holds for all $k \in \mathbb{Z}_{\geq 0}$. □

There is an obvious grading of $\mathcal{H}(f)$ relative to which x has degree 1, y has degree -1 and h has degree 0. We denote the corresponding homogeneous subspaces by $\mathcal{H}(f)_\ell$, for $\ell \in \mathbb{Z}$, so that

$$\mathcal{H}(f) = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}(f)_\ell, \qquad \text{with} \qquad \mathcal{H}(f)_\ell = \bigoplus_{i-k=\ell} \mathbb{C}\, x^i \mathbb{C}[h] y^k. \tag{3.2}$$

We call this the *standard grading* of $\mathcal{H}(f)$, and, whenever we mention a homogeneous component or element of $\mathcal{H}(f)$, we will always be referring to this standard grading.

The subalgebra $\mathcal{H}(f)_0$ has basis $\{x^k h^j y^k \mid k, j \geq 0\}$ and $\mathcal{H}(f)_\ell = x^\ell\,\mathcal{H}(f)_0$ if $\ell \geq 0$; $\mathcal{H}(f)_\ell = \mathcal{H}(f)_0\, y^{-\ell}$ if $\ell \leq 0$. Thus we have the decomposition

$$\mathcal{H}(f) = \bigoplus_{\ell>0} x^\ell\,\mathcal{H}(f)_0 \oplus \mathcal{H}(f)_0 \oplus \bigoplus_{\ell>0} \mathcal{H}(f)_0\, y^\ell.$$

Proposition 3.3. *The subalgebra $\mathcal{H}(f)_0$ is commutative. If $\mathsf{deg}\, f > 1$, then $\mathcal{H}(f)_0$ is a maximal commutative subalgebra of $\mathcal{H}(f)$ which strictly contains $\mathbb{C}[z, h]$, the polynomial subalgebra of $\mathcal{H}(f)$ generated by h and the central element $z = xy - h$.*

Proof. The first statement is a direct consequence of Lemma 3.1(e). Assume now that $\mathsf{deg}\, f > 1$. Then σ is injective and has infinite order. For any $i, k \in \mathbb{Z}_{\geq 0}$ and $g \in \mathbb{C}[h]$, $[h, x^i g y^k] = x^i(\sigma^i(h) - \sigma^k(h))g y^k$. Hence, if $g \neq 0$, we deduce that $[h, x^i g y^k] = 0 \iff i = k$, and from this it is straightforward to conclude that $\mathcal{H}(f)_0$ is the centralizer of h, hence a maximal commutative subalgebra of $\mathcal{H}(f)$.

The commuting elements h and z are homogeneous of degree 0 and are easily seen to be algebraically independent, as $z^k - x^k y^k$ is in the span of $\{x^i g y^i \mid i < k,\ g \in \mathbb{C}[h]\}$. Suppose, by contradiction, that there exist $g_k \in \mathbb{C}[h]$ such that $xhy = \sum_{k\geq 0} g_k z^k$. Then by the argument above we must have $g_k = 0$ for all $k > 1$ and $\sigma(g_1) = h$, which is possible only if $\mathsf{deg}\, f = 1$. Therefore $xhy \in \mathcal{H}(f)_0 \setminus \mathbb{C}[z, h]$. □

By Lemma 3.1(c)–(d), it is possible to extend σ to a $\mathbb{C}$-linear endomorphism $\tilde{\sigma}$ of $\mathcal{H}(f)_0$ so that $\tilde{\sigma}(x^k g y^k) = x^k \sigma(g) y^k + x^{k-1}(\sigma^k(h) - h) g y^{k-1}$, for all $k \in \mathbb{Z}_{\geq 0}$ and $g \in \mathbb{C}[h]$. For simplicity, we still denote this endomorphism by σ instead of $\tilde{\sigma}$. By Lemma 3.1(c)–(d) and Lemma 2.1(a), σ is defined by the relations:

$$\theta x = x\sigma(\theta), \qquad y\theta = \sigma(\theta) y, \qquad \text{for all } \theta \in \mathcal{H}(f)_0. \tag{3.4}$$

In particular, (3.4) implies that σ is an algebra endomorphism of $\mathcal{H}(f)_0$.

4 locally finite derivations of $\mathcal{H}(f)$ when $\mathsf{deg}\, f > 1$

Henceforth we will assume that $\mathsf{deg}\, f > 1$. By Corollary 2.7 we are assuming that $\mathcal{H}(f)$ is not a generalized down-up algebra. Most of our subsequent results do not hold if $\mathsf{deg}\, f \leq 1$.

Our goal in this section is to determine all locally finite derivations of $\mathcal{H}(f)$. In particular, we will classify all $\mathbb{Z}$-gradings of $\mathcal{H}(f)$ and show that $\mathcal{H}(f)$ has no nontrivial locally nilpotent derivations. Our methods are akin to those used in [12].

Let δ be a $\mathbb{C}$-linear endomorphism of $\mathcal{H}(f)$. We recall the following standard definitions:

- δ is a derivation of $\mathcal{H}(f)$ if $\delta(ab) = \delta(a)b + a\delta(b)$;
- δ is locally finite if for every $a \in \mathcal{H}(f)$ the $\mathbb{C}$-linear span of $\{\delta^k(a) \mid k \in \mathbb{Z}_{\geq 0}\}$ is finite dimensional;
- δ is locally nilpotent if for every $a \in \mathcal{H}(f)$ there is $k \in \mathbb{Z}_{\geq 0}$ such that $\delta^k(a) = 0$;

- δ is homogeneous of degree $r \in \mathbb{Z}$ if $\delta(\mathcal{H}(f)_\ell) \subseteq \mathcal{H}(f)_{\ell+r}$ for all $\ell \in \mathbb{Z}$.

Assume δ is any derivation of $\mathcal{H}(f)$. Since $\mathcal{H}(f)$ is finitely generated, there exist homogeneous derivations $\delta_1, \ldots, \delta_k$ of strictly increasing degrees such that $\delta = \delta_1 + \cdots + \delta_k$. Moreover, as seen in [12, Lemma 1.1], if δ is locally finite, then so are δ_1 and δ_k, and if δ_1 (respectively, δ_k) is of nonzero degree, then it must be locally nilpotent.

We need one final definition. Given a locally nilpotent derivation δ and $a \in \mathcal{H}(f)$, define

$$\deg_\delta(a) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \delta^k(a) \neq 0\} \quad \text{if } a \neq 0;$$

define also $\deg_\delta(0) = -\infty$. It can be easily checked (see for example [9]) that for $a, b \in \mathcal{H}(f)$, $\deg_\delta(a+b) \leq \max\{\deg_\delta(a), \deg_\delta(b)\}$, with equality if $\deg_\delta(a) \neq \deg_\delta(b)$. Since $\mathcal{H}(f)$ is a domain and $\mathbb{C}$ has characteristic 0, we also have, from the Leibniz rule, $\deg_\delta(ab) = \deg_\delta(a) + \deg_\delta(b)$. In particular, $\mathsf{ker}\delta$ is factorially closed: if $\delta(ab) = 0$ for some nonzero $a, b \in \mathcal{H}(f)$, then $\delta(a) = 0 = \delta(b)$.

Lemma 4.1. *Assume that* $\mathsf{deg}\, f > 1$. *Then all locally finite derivations of* $\mathcal{H}(f)$ *are homogeneous of degree* 0.

Proof. Let δ be a locally finite derivation of $\mathcal{H}(f)$. By the above decomposition $\delta = \delta_1 + \cdots + \delta_k$ of δ into homogeneous derivations of strictly increasing degrees, it will be enough to show that there are no nonzero homogeneous locally nilpotent derivations of $\mathcal{H}(f)$ of degree $r \neq 0$.

So assume δ is a homogeneous locally nilpotent derivation of $\mathcal{H}(f)$ of degree $r \neq 0$. Let $d = \deg_\delta(h)$ and suppose that $d > 0$. Then $\deg_\delta(f(h)) = d\,\mathsf{deg}\, f$ and the relation $hx = xf(h)$ yields

$$d + \deg_\delta(x) = \deg_\delta(x) + d\,\mathsf{deg}\, f,$$

so $\mathsf{deg}\, f = 1$, which contradicts our hypothesis. Hence $d = 0$ and $\delta(h) = 0$.

By replacing δ with $\iota\delta\iota^{-1}$, where ι is the anti-automorphism defined in (2.3), we can assume that $r > 0$. Then $\mathsf{ker}\delta$ contains some nonzero homogeneous element of positive degree. Since elements of $\mathcal{H}(f)$ of positive degree lie in $x\mathcal{H}(f)$ and $\mathsf{ker}\delta$ is factorially closed, we deduce that $\delta(x) = 0$.

Any derivation maps the center of an algebra into itself, so δ restricts to a locally nilpotent derivation of $\mathbb{C}[z]$, by Lemma 2.1(c), and thus $\delta(z) \in \mathbb{C}$. On the other hand, since $z = xy - h$ is homogeneous of degree 0 and δ has positive degree, it must be that $\delta(z) = 0$, and from $0 = \delta(z) = x\delta(y)$, we conclude that $\delta(y) = 0$. Then $\delta = 0$ and the lemma is proved. □

The next theorem, our main result on derivations of $\mathcal{H}(f)$ when $\mathsf{deg}\, f > 1$, shows that the space of locally finite derivations of $\mathcal{H}(f)$ is one-dimensional over $\mathbb{C}$, spanned by the derivation ∂ defined by

$$\partial(x^i h^j y^k) = (i-k)x^i h^j y^k, \quad \text{for all } i, j, k \in \mathbb{Z}_{\geq 0}. \tag{4.2}$$

Theorem 4.3. *Assume that* $\mathsf{deg}\, f > 1$. *If* δ *is a locally finite derivation of* $\mathcal{H}(f)$, *then there is* $\lambda \in \mathbb{C}$ *such that* $\delta(x) = \lambda x$, $\delta(y) = -\lambda y$ *and* $\delta(h) = 0$.

Proof. Let δ be a locally finite derivation of $\mathcal{H}(f)$. By Lemma 4.1, we know that δ is homogeneous of degree 0, so there are $\theta_x, \theta_h, \theta_y \in \mathcal{H}(f)_0$ so that

$$\delta(x) = x\theta_x, \quad \delta(h) = \theta_h, \quad \text{and} \quad \delta(y) = \theta_y y.$$

In particular, since h commutes with θ_h, we have $\delta(g(h)) = g'(h)\theta_h$ for all $g(h) \in \mathbb{C}[h]$, where $g'(h)$ denotes the derivative of $g(h)$ with respect to h.

Claim 1: $\theta_h = 0$ and $\theta_x + \theta_y = 0$.

Proof of Claim 1: Write

$$\theta_h = \sum_{k\geq 0} x^k g_k(h) y^k, \tag{4.4}$$

with $g_k(h) \in \mathbb{C}[h]$ and $g_k(h) = 0$ except for finitely many indices k.

As observed in the proof of Lemma 4.1, δ restricts to a locally finite derivation of $\mathbb{C}[z]$, the center of $\mathcal{H}(f)$, and thus $\delta(z) \in \mathbb{C} \oplus \mathbb{C}z$, say $\delta(z) = \mu z - \lambda$, with $\lambda, \mu \in \mathbb{C}$. Since $\mu z - \lambda = \delta(xy - h) = x(\theta_x + \theta_y)y - \theta_h$, we have

$$\theta_h = x(\theta_x + \theta_y)y - \mu z + \lambda = x(\theta_x + \theta_y - \mu)y + \mu h + \lambda. \tag{4.5}$$

In particular, $g_0(h) = \mu h + \lambda$.

We now apply δ to the relation $yh = f(h)y$ and get $\theta_y yh + y\theta_h = f'(h)\theta_h y + f(h)\theta_y y$. As h and θ_y commute, and $y\theta_h = \sigma(\theta_h)y$, by (3.4), we obtain

$$\sigma(\theta_h) = f'(h)\theta_h. \tag{4.6}$$

Now combining (4.4) and (4.6) we deduce that, for every $k \geq 0$:

$$\sigma^k(f'(h))g_k(h) = \sigma(g_k(h)) + (\sigma^{k+1}(h) - h)g_{k+1}(h). \tag{4.7}$$

Setting $k = 0$ in (4.7) we obtain $(f(h) - h)g_1(h) = f'(h)g_0(h) - \sigma(g_0(h))$. Since we have already established that $\deg g_0 \leq 1$, we deduce now from the latter equation that $\deg (f(h) - h)g_1(h) \leq \deg f$, and thus $g_1 \in \mathbb{C}$. Combining this with the $k = 1$ case of (4.7), $\sigma(f'(h))g_1(h) = \sigma(g_1(h)) + (\sigma(f(h)) - h)g_2(h)$, yields $g_2 = 0$, and in turn the latter gives $g_k = 0$ for all $k \geq 2$. Using again the relation $\sigma(f'(h))g_1(h) = \sigma(g_1(h)) + (\sigma(f(h)) - h)g_2(h)$ with $g_2 = 0$ and $g_1 \in \mathbb{C}$ gives $g_1 = 0$. Therefore we have

$$\sigma(g_0) = f'(h)g_0. \tag{4.8}$$

Suppose $g_0 \neq 0$, and let a be the leading coefficient of $f(h)$. Then $\mu \neq 0$ and comparing leading coefficients in (4.8) yields $\mu a = a(\deg f)\mu$, whence $\deg f = 1$, which is a contradiction. Thus $g_0 = 0$.

From the above we conclude that $\theta_h = \sum_{k\geq 0} x^k g_k(h) y^k = 0$ and finally by (4.5) we get $\theta_x + \theta_y = 0$, establishing Claim 1.

So far we have shown that

$$\delta(x) = x\theta_x, \quad \delta(h) = 0, \quad \text{and} \quad \delta(y) = -\theta_x y,$$

so it remains to be inferred that $\theta_x \in \mathbb{C}$.

Claim 2: $\delta(\theta) = 0$, *for all* $\theta \in \mathcal{H}(f)_0$.

Proof of Claim 2: Since $\delta(g) = 0$ for all $g \in \mathbb{C}[h]$, it suffices to show that if $\theta \in \mathcal{H}(f)_0$ and $\delta(\theta) = 0$, then also $\delta(x\theta y) = 0$. This follows easily using the fact that $\mathcal{H}(f)_0$ is commutative, as proved in Proposition 3.3.

From Claim 2 it follows that, for all $k \geq 0$, $\delta(\theta_x^k) = 0$, which implies that $\delta^k(x) = x\theta_x^k$. As δ is locally finite, the span of $\{\theta_x^k \mid k \in \mathbb{Z}_{\geq 0}\}$ must then be finite dimensional. This is possible only if $\theta_x \in \mathbb{C}$, thus finishing the proof of the theorem. □

Since locally nilotent derivations are locally finite, we derive the following corollary.

Corollary 4.9. *Assume that* $\deg f > 1$. *Then* $\mathcal{H}(f)$ *has no nonzero locally nilpotent derivations.*

Suppose that $\mathcal{H}(f) = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$ is a grading. Define the $\mathbb{C}$-linear endomorphism δ of $\mathcal{H}(f)$ by $\delta(v_\alpha) = \alpha v_\alpha$ for all $v_\alpha \in V_\alpha$ and all $\alpha \in \mathbb{C}$. It is immediate to check that δ is a diagonalizable derivation of $\mathcal{H}(f)$ whose eigenvalues are those $\alpha \in \mathbb{C}$ such that $V_\alpha \neq (0)$. Conversely, if δ is a diagonalizable derivation, then δ determines a grading where V_α is the α-eigenspace of δ. Furthermore, diagonalizable derivations are clearly locally finite.

Thus, we deduce from Theorem 4.3 that, except for the trivial grading in which every element of $\mathcal{H}(f)$ has degree 0, $\mathcal{H}(f)$ only admits the standard grading defined in (3.2), up to scaling by some integer. More precisely, we have:

Corollary 4.10. *Assume that* $\deg f > 1$. *Then for any* $\mathbb{Z}$*-grading of* $\mathcal{H}(f)$, *there is an integer* $\ell \in \mathbb{Z}$ *such that, relative to that grading,* x *has degree* ℓ, y *has degree* $-\ell$ *and* h *has degree* 0.

5 Automorphisms of $\mathcal{H}(f)$ when $\deg f > 1$

When $\deg f = 1$ the algebra $\mathcal{H}(f)$ is a Noetherian generalized down-up algebra, by Corollary 2.7, and the automorphisms of the latter have been investigated in [3]. We continue to assume that $\deg f > 1$ and note again that our results do not generalize to the cases with $\deg f \leq 1$.

Since $\mathcal{H}(f)$ has no nonzero locally nilpotent derivations, it seems natural to conjecture that the automorphism group of $\mathcal{H}(f)$ is somewhat small. However, over $\mathbb{C}$ we can consider also the exponential of a diagonalizable derivation. Specifically, let $c \in \mathbb{C}$ and let ∂ be the derivation of $\mathcal{H}(f)$ defined in (4.2). Then the expression

$$\exp(c\partial) := \sum_{k=0}^{\infty} \frac{(c\partial)^k}{k!}$$

defines an automorphism of $\mathcal{H}(f)$ satisfying

$$\exp(c\partial)(x) = \sum_{k=0}^{\infty} \frac{c^k}{k!} x = \exp(c)x, \quad \exp(c\partial)(y) = \exp(-c)y, \quad \exp(c\partial)(h) = h,$$

with inverse $\exp(-c\partial)$.

The above motivates the following definition. For each $\lambda \in \mathbb{C}^*$, let ϕ_λ be the automorphism of $\mathcal{H}(f)$ defined by

$$\phi_\lambda(x) = \lambda x, \quad \phi_\lambda(y) = \lambda^{-1} y, \quad \phi_\lambda(h) = h. \tag{5.1}$$

The group of algebra automorphisms of $\mathcal{H}(f)$ will be denoted by $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$. We have a first description of $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$ below.

Proposition 5.2. *Assume* $\deg f > 1$. *Then the following hold:*

(a) *Any automorphism of* $\mathcal{H}(f)$ *restricts to an automorphisms of* $\mathbb{C}[h]$, *and* x *and* y *are eigenvectors.*

(b) $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(h) = h\} = \{\phi_\lambda \mid \lambda \in \mathbb{C}^*\} \cong \mathbb{C}^*$, *and this is a central subgroup of* $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$.

(c) $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ *is a finite cyclic subgroup whose order divides* $(\deg f) - 1$.

Proof. Let ϕ be an automorphism of $\mathcal{H}(f)$. Then as argued in Claim 4 of the proof of [8, Theorem 5], the relation $\phi(h)\phi(x) = \phi(x)f(\phi(h))$ with $\deg f > 1$ implies that $\phi(h) \in \mathbb{C}[h]$; applying this result to ϕ^{-1} gives that $\phi(h) = ah + b$, for some $a, b \in \mathbb{C}$ with $a \neq 0$.

Now writing $\phi(x)$ as a sum of terms of the form $x^i g_{i,j} y^j$ with $i, j \in \mathbb{Z}_{\geq 0}$ and $g_{i,j} \in \mathbb{C}[h]$, and comparing the corresponding expressions for $\phi(h)\phi(x)$ and $\phi(x)f(\phi(h))$, we obtain $\phi(x) \in \mathcal{H}(f)_1$. Similarly, $\phi(y) \in \mathcal{H}(f)_{-1}$, so ϕ is homogeneous of degree 0. Thus, there exist $\theta_x, \theta_y \in \mathcal{H}(f)_0$ such that $\phi(x) = x\theta_x$ and $\phi(y) = \theta_y y$. Applying the same reasoning to ϕ^{-1}, we deduce that $\theta_x, \theta_y \in \mathbb{C}^*$, which proves (a).

Now assume $\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$ and $\phi(h) = h$. By (a) there exist $\lambda, \mu \in \mathbb{C}^*$ such that $\phi(x) = \lambda x$ and $\phi(y) = \mu y$. Applying ϕ to the relation $[y, x] = f(h) - h$ yields $\lambda\mu = 1$, so $\phi = \phi_\lambda$. This proves the equality in (b), and the isomorphism $\{\phi_\lambda \mid \lambda \in \mathbb{C}^*\} \cong \mathbb{C}^*$ is clear, as $\phi_\lambda \circ \phi_\mu = \phi_{\lambda\mu}$ for all $\lambda, \mu \in \mathbb{C}^*$.

Next, we show that the subgroup $\{\phi_\lambda \mid \lambda \in \mathbb{C}^*\}$ is central in $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$. Let $\lambda \in \mathbb{C}^*$, and suppose $\psi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$ is arbitrary. By (a) we know that $\psi(h) \in \mathbb{C}[h]$, which implies that $\phi_\lambda \circ \psi(h) = \psi \circ \phi_\lambda(h)$. But as x and y are eigenvalues for any automorphism of $\mathcal{H}(f)$, $\phi_\lambda \circ \psi$ and $\psi \circ \phi_\lambda$ also agree on these generators, and thus $\phi_\lambda \circ \psi = \psi \circ \phi_\lambda$.

To prove part (c), suppose that $\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$ and $\phi(x) = x$. We know already that $\phi(h) = ah + b$ and $\phi(y) = cy$, for some $a, b, c \in \mathbb{C}$ with $a, c \neq 0$. Then $xf(ah + b) = x\phi(f(h)) = \phi(h)x = (ah + b)x = x(af(h) + b)$, and we obtain

$$f(ah + b) = af(h) + b. \tag{5.3}$$

Therefore,

$$\begin{aligned} c(f(h) - h) = c[y, x] = \phi([y, x]) = \phi(f(h) - h) = af(h) + b - (ah + b) \\ = a(f(h) - h), \end{aligned}$$

and we conclude that $c = a$.

Write $f(h) = \sum_{k=0}^{n} a_k h^k$, where $n = \mathsf{deg}\, f$ and $a_k \in \mathbb{C}$. Applying the derivation $\frac{d}{dh}$ to (5.3) $n-1$ times yields $a^{n-1} f^{(n-1)}(ah+b) = af^{(n-1)}(h)$, as $n-1 \geq 1$. As $f^{(n-1)}(h) = (n-1)!(na_n h + a_{n-1})$, we obtain

$$a^{n-1} = 1 \quad \text{and} \quad b = \frac{(a-1)a_{n-1}}{na_n}. \tag{5.4}$$

Let $\mathsf{U}_{n-1} = \{\zeta \in \mathbb{C}^* \mid \zeta^{n-1} = 1\}$ be the cyclic group of order $n-1$, and define a map

$$\Psi : \{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\} \longrightarrow \mathsf{U}_{n-1}, \quad \phi \mapsto a, \quad \text{where } \phi(h) = ah + b.$$

Then Ψ is well defined by (5.4), and it is a group homomorphism. If $\Psi(\phi) = 1$ for some $\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$ with $\phi(x) = x$, then the above shows that $\phi(y) = y$ and $\phi(h) = h + b$. Again by (5.4) we deduce that $b = 0$, so ϕ is the identity on $\mathcal{H}(f)$. This shows that Ψ is an injective group homomorphism and thus $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ is isomorphic to a subgroup of U_{n-1}; hence it is a finite cyclic group whose order divides $n-1$. □

It is now an easy matter to determine the structure of $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$. The symbol $\dot{\times}$ used below denotes the internal direct product of subgroups of a group.

Theorem 5.5. *Assume* $\mathsf{deg}\, f > 1$. *Then*

$$\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) = \{\phi_\lambda \mid \lambda \in \mathbb{C}^*\} \dot{\times} \{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\} \tag{5.6}$$

is an abelian group, where:

- $\{\phi_\lambda \mid \lambda \in \mathbb{C}^*\} \cong \mathbb{C}^*$ *and* ϕ_λ *is defined in* (5.1);
- $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ *is a finite cyclic group whose order divides* $(\mathsf{deg}\, f) - 1$ *and which, as a set, can be identified with* $\{(a,b) \in \mathbb{C}^* \times \mathbb{C} \mid f(ah+b) = af(h) + b\}$ *via the correspondence* $\phi \mapsto (a,b)$, *where* $\phi(h) = ah + b$.

Proof. Since we have already seen in Proposition 5.2 that $\{\phi_\lambda \mid \lambda \in \mathbb{C}^*\}$ is central, in order to prove the direct product decomposition in (5.6), it remains to show that the two subgroups have trivial intersection, which is clear, and generate $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$. Let $\psi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$. Then there is $\lambda \in \mathbb{C}^*$ such that $\psi(x) = \lambda x$, whence $\phi_\lambda^{-1} \circ \psi \in \{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$, and this shows the latter claim. Moreover, since $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ is abelian, by Proposition 5.2(c), the group $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f))$ must also be abelian.

The remaining parts of the theorem have already been proved, except for the observation that $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ can be identified with the set $\{(a,b) \in \mathbb{C}^* \times \mathbb{C} \mid f(ah+b) = af(h) + b\}$. Indeed, if $\phi(x) = x$, then we have seen in the proof of Proposition 5.2 that $\phi(h) = ah + b$ and $\phi(y) = ay$, for some $a, b \in \mathbb{C}$ with $a \neq 0$, and (5.3) must hold. This shows that the correspondence $\phi \mapsto (a,b)$ is well defined and one-to-one. Conversely, given $(a,b) \in \mathbb{C}^* \times \mathbb{C}$ satisfying $f(ah+b) = af(h) + b$, it is routine to check that there is an automorphism of $\mathcal{H}(f)$ defined by the conditions $\phi(x) = x$, $\phi(y) = ay$, $\phi(h) = ah + b$, and this shows the correspondence is onto. □

Remark 5.7. Any pair $(a,b) \in \mathbb{C}^* \times \mathbb{C}$ satisfying $f(ah+b) = af(h)+b$ must also satisfy (5.4), where $n = \deg f$, although the conditions in (5.4) are not sufficient (see the examples below). Thus, for each $(n-1)$-th root of unity a, the corresponding scalar b is determined by (5.4), but one still needs to check the relation $f(ah+b) = af(h)+b$ for the pair (a,b).

Examples 5.8.

(a) If $\deg f = 2$, then $n = 2$ in (5.4), so $a = 1$ and $b = 0$, and the pair $(1,0)$ corresponds to the identity map. It follows that the group $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ is trivial and $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \cong \mathbb{C}^*$.

(b) Let $f(h) = h^3 + h$. Then $n = 3$ in (5.4), so $a = \pm 1$. If $a = 1$, then $b = 0$, and the corresponding automorphism is the identity. If $a = -1$, then $b = 0$, and in fact $f(-h) = -f(h)$. Therefore there is an automorphism ϕ of $\mathcal{H}(f)$ such that $\phi(x) = x$, $\phi(y) = -y$, $\phi(h) = -h$, and $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \cong \mathbb{C}^* \times \mathbb{Z}_2$.

(c) Let $f(h) = h^3 + h + 1$. Then $n = 3$ in (5.4), so $a = \pm 1$. If $a = -1$, then (5.4) yields $b = 0$, but $f(-h) \neq -f(h)$, so the group $\{\phi \in \mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \mid \phi(x) = x\}$ is trivial and $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \cong \mathbb{C}^*$.

(d) Let $f(h) = h^n$, for $n > 1$. Then (5.4) says that a is a $(n-1)$-th root of unity and $b = 0$. Moreover, $f(ah) = a^n f(h) = af(h)$ for any $(n-1)$-th root of unity a. Hence $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \cong \mathbb{C}^* \times \mathbb{Z}_{n-1}$.

(e) Let $f(h) = h^n + 1$, for $n > 1$. Then, as before, a is a $(n-1)$-th root of unity and $b = 0$. However, in this case, $f(ah) = ah^n + 1$ whereas $af(h) = ah^n + a$, so equality holds if and only if $a = 1$. Hence $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \cong \mathbb{C}^*$.

(f) Let $f(h) = h^n + h^{k+1}$, for some $n \geq 4$, and take any $1 \leq k < n-1$ such that $k \mid n-1$. Then a is a $(n-1)$-th root of unity and $b = 0$. In this case $f(ah) = ah^n + a^{k+1}h^{k+1}$ whereas $af(h) = ah^n + ah^{k+1}$, so equality holds if and only if $a^k = 1$. By the hypothesis that $k \mid n-1$, we deduce that $\mathsf{Aut}_{\mathbb{C}}(\mathcal{H}(f)) \cong \mathbb{C}^* \times \mathbb{Z}_k$.

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