Novel results of asymmetrically coupled fractional neurons

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Abstract: We consider a fractional-order model of two asymmetrically coupled spiking neurons. The dynamical behaviour of the two neurons is modeled by the fractional-order Hodgkin-Huxley equations. Simulations of the model for distinct values of the order of the fractional derivative, α, and of the coupling constants, k_1, k_2, show interesting features, such as relaxation oscillations, mixed-mode oscillations, small oscillations, and localized solutions. Moreover, α adds extra complexity to the dynamics of the model. These differences may explain certain differences in processing similar tasks in the human brain.

Keywords: asymmetric; coupled; neurons; fractional Hodgkin-Huxley equations

1 Introduction

In 1952, Hodgkin and Huxley [1] conducted experiments, in the squid axon, aimed at a better understanding of the mechanisms and rules governing the flow of the electric current in a nerve cell, during an action potential. The derived equations, known as the Hodgkin-Huxley (HH) equations, have had a decisive influence on the understanding of the neuronal function since then [2], [3], [4], [5]. Phenomena such as in-phase synchronization, anti-phase synchronization, bursting, localization, small oscillations, mixed-mode oscilations, have been modeled by the HH equations.

Synchronization is observed in specific areas of the brain in patients suffering from epilepsy and Parkinson's disease [6]. On the other hand, tasks such as processing sensory information, only occur in synchronized neurons. Localized solutions in oscillatory systems are associated with a partition of the oscillators in two distinct sets. One set is described by oscillators with high amplitudes and the other by oscillators with small amplitudes [7]. This type of patterns may be good

approximations of the dynamics of working memory, in a biologically reasonable parameter region [8]. Relaxation oscillations are solutions defined by long periods of quasi-static behavior interspersed with short periods of rapid transition. They are analysed in the context of the canard phenomenon [9]. A solution showing a combination of traits of relaxation oscillations and small oscillations is defined as a mixed mode oscillation. The later may also be generated by the canard mechanism.

In [2] the authors simulate an integer-order asymmetrically coupled system of two HH equations. They find localized solutions, small oscillations, relaxation oscillations, and mixed-mode oscillations, for certain values of the coupling constants. Localized solutions are seen for negative values of the two coupling constants and when the ratio $\frac{k_1}{k_2}$ between the two constants is far from to -1. Relaxation oscillations occur when this ratio is close to -1, and mixed mode oscillations are the states in between.

Bearing the aforementioned ideas in mind, in this paper we propose a fractional order model for the dynamics of two asymmetrically coupled HH equations, for variation of the order of the fractional derivative and various temperatures. We consider that the coupling is diffusive and is only done in the voltage term. This is, to our best knowledge, not been the issue of any research so far. In Section 2, we introduce the FO model of asymmetrically coupled HH equations. In Section 3 we show and discuss the outcomes of the simulations of the model. In the last section, we conclude our work.

1.1   Non-integer order differentiation

Non-integer order, aka fractional order (FO), differentiation and integration are generalizations of the well known differentiation and integration of integer order. FO systems have been widely applied, for a couple of decades now, to solve problems in engineering, biology, physics, to name a few [10,11,12,13,14].

There are several definitions of FO derivatives. The most commonly used are the Caputo, the Grünwald-LetnikoL), and the Riemann-Liouville derivatives [10]. The GL derivative is given by the equation (1).

$$^{GL}_{a}D^{\alpha}_{t}f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh), t > a, \alpha > 0 \tag{1}$$

Last year, Caputo and Fabrizio (CF) [15] proposed a newly definition for the fractional order derivative. The update with respect to previous definitions is the new non-singular kernel operator. This novel derivative has been used in groundwater and thermal problems. Moreover, Atangana *et al* apply the CF

derivative to find the solutions of the Fisher's reaction-diffusion equation and of the Baggs-Freedman model [16,17].

2 The fractional-order asymmetrically coupled system of two HH equations

The Hodgkin-Huxley equations (1) are a system of 4×4 ordinary differential equations (ODEs). They were derived by Hodgkin and Huxley in 1952 [1] to model the electrical behavior of the squid axon. The first equation refers to the trans-membrane potential dynamics, $v(t)$, for a single neuron, in response to an external stimulus I, and as a function of the ion currents. The ion currents are mostly three, one for the sodium (Na^+), one for the potassium (K^+), and one for a leakage current, associated with other ions, where calcium is included. The ions' conductances are described by the other three equations of the model.

$$\frac{C_m dv}{dt} = f(v, m, n, h) - I$$

$$\frac{dm}{dt} = \Phi(\alpha_m(v)(1 - m) - \beta_m(v)m) \qquad (2)$$

$$\frac{dn}{dt} = \Phi(\alpha_n(v)(1 - n) - \beta_n(v)n)$$

$$\frac{dh}{dt} = \Phi(\alpha_h(v)(1 - h) - \beta_h(v)h)$$

where C_m is the membrane capacitance, $\Phi = 3^{\frac{T-6.3}{10}}$, is the temperature compensating factor. The function f is defined as:

$$f(v, m, n, h) = -g_L(v - V_L) - g_{Na}m^3 h(v - V_{Na}) - g_K n^4(v - V_k)$$

The functions $\alpha_i(v)$ and $\beta_i(v), i = m, n, h$, are given in [1] as:

$$\alpha_m(v) = \psi\left(\frac{v+25}{10}\right), \ \alpha_n(v) = 0.1\psi\left(\frac{v+10}{10}\right), \ \alpha_h(v) = 0.07\psi\left(\frac{v}{20}\right)$$

$$\beta_m(v) = 4e^{\left(\frac{v}{18}\right)}, \beta_n(v) = \frac{1}{8}e^{\left(\frac{v}{80}\right)}, \beta_h(v) = \left(1 + e^{\left(\frac{v+30}{10}\right)}\right)^{(-1)}$$

and $\psi = \begin{cases} \frac{x}{e^x - 1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$.

The asymmetrically coupled system of two fractional-order HH equations is thus given by:

$$\frac{C_m dv_1}{dt} = f(v_1, m_1, n_1, h_1) - I - k_1(v_1 - v_2)$$

$$\frac{dm_1}{dt} = \Phi\big(\alpha_{m_1}(v_1)(1 - m_1) - \beta_{m_1}(v_1)m_1\big)$$

$$\frac{dn_1}{dt} = \Phi\big(\alpha_{n_1}(v_1)(1 - n_1) - \beta_{n_1}(v_1)n_1\big)$$

$$\frac{dh_1}{dt} = \Phi\big(\alpha_{h_1}(v_1)(1 - h_1) - \beta_{h_1}(v_1)h_1\big) \qquad (3)$$

$$\frac{C_m dv_2}{dt} = f(v_2, m_2, n_2, h_2) - I - k_2(v_2 - v_1)$$

$$\frac{dm_2}{dt} = \Phi\big(\alpha_{m_2}(v_2)(1 - m_2) - \beta_{m_2}(v_2)m_2\big)$$

$$\frac{dn_2}{dt} = \Phi\big(\alpha_{n_2}(v_2)(1 - n_2) - \beta_{n_2}(v_2)n_2\big)$$

$$\frac{dh_2}{dt} = \Phi\big(\alpha_{h_2}(v_2)(1 - h_2) - \beta_{h_2}(v_2)h_2\big)$$

where k_1, k_2 are the coupling constants.

3   Numerical simulations

In this section we show simulations of the FO asymmetrically coupled HH equations model (4), for several values of the order of the fractional derivative, α, and distinct coupling constants, k_1, k_2. In Table 1, we list the values of the HH parameters fixed in the simulations. In Table 2, can be found the initial conditions and the values for the varied parameters. The symmetrically coupled FO HH equations model is studied in [18].

Table 1

Values of the Hodgkin-Huxley parameters used in the simulations

Parameters	Values	Units
C_m	1.0	$\mu F/cm^2$
T	6.3, 16.0, 26.0	^oC
V_{Na}	-115.0	mV
V_K	12.0	mV
V_L	-10.599	mV
g_{Na}	120.0	mS/cm^2
g_K	36	mS/cm^2
g_L	0.3	mS/cm^2

Table 2

Initial conditions and parameter values used in the simulations

Fig.	Initial conditions	T, I	$k_1, \ k_2$
1, 2, 3	(-22.18,0.43,0.65,0.07, -20.81,0.38,0.63,0.08)	$26^o C, 155$	$-0.2, -2.01$
4, 5, 6	(-10.67,0.17,0.56,0.17, -18.91,0.67,0.74,0.05)	$16^o C, 60$	$0.7, -1.1$
7, 8, 9	(-16.21,0.36,0.60,0.10, 1.39,0.04,0.52,0.28)	$16^o C, 60$	$0.5, -1.1$
10, 11, 12	(-26.01,0.48,0.64,0.07, -7.43,0.12,0.63,0.10)	$20^o C, 155$	$-0.1, -2.0$

Figures 1-3 depict small oscillations of the FO coupled system (3), for $T = 26.0, I = 155$, and $\alpha = 1.0, 0.8, 0.4$, respectively. One can observe a decrease in the amplitude of the periodic orbits with the order of the fractional derivative, α, and a slight increase in the spiking frequency. Faster ransients are observed for smaller α. In Figures 4-6, we show relaxation oscillations of the model (3), for $T = 16.0, I = 60$, and $\alpha = 1.0, 0.8, 0.4$, respectively. As α is decreased the spiking frequency of the two neurons increases. In Figures 7-8, we show mixed-mode oscillations of the model (2), for $T = 16.0, I = 60$, and $\alpha = 1.0, 0.8$, respectively. Fixing all other parameters and decreasing only α to 0.4, the mixed-mode oscillations are lost, and a relaxation oscillation appears, see Figure 9. Thus, α, causes a, non-expected, change in the behaviour of the system (3). Localized solutions are shown in Figures 10-12, for $T = 20.0, I = 155$, and $\alpha = 1.0, 0.8, 0.4$, respectively. Moreover, we note a decrease in the amplitude and an increase in the spiking frequency of the neurons as α decreases from 1.

As it can be observed from the numerical simulations, the asymmetrically coupled FO model of two HH equations has a rich dynamical behaviour. Mixed-mode oscillations, relaxation oscillations, small oscillations, and localized solutions, are common in specific regions of the coupling constants. These regions agree with the results of paper [2], for most values of α. This means that localized solutions are seen for negative values of the two coupling constants and when the ratio $\frac{k_1}{k_2}$ between the two constants is far from to -1. Relaxation oscillations occur when this ratio is close to -1, and mixed mode oscillations are the states in between. Nevertheless, the value of the fractional order derivative, α, adds extra complexity to the behaviour of the model. We saw in Figure 9 an `expected' mixed-mode oscillation tend towards a relaxation oscillation, when all other parameters `suggested' a mixed-mode oscillation. This `complexity' may be associated with differences in the human brain, when processing and storing information [8,7], when responding to certain stimuli, amongst others. Further study is needed in order

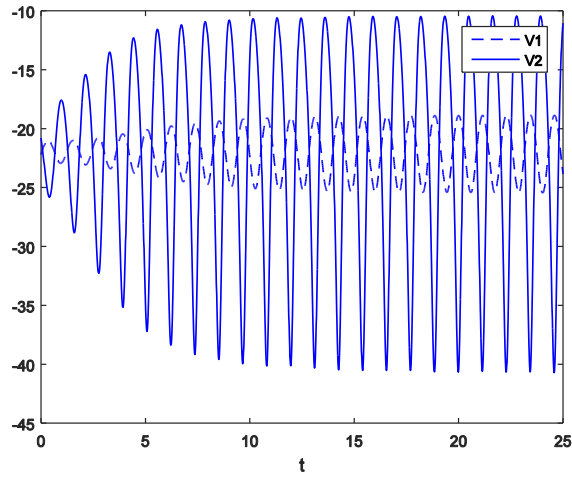to infer the importance of the order of the fractional derivative in these models of spiking neurons.

Figure 1
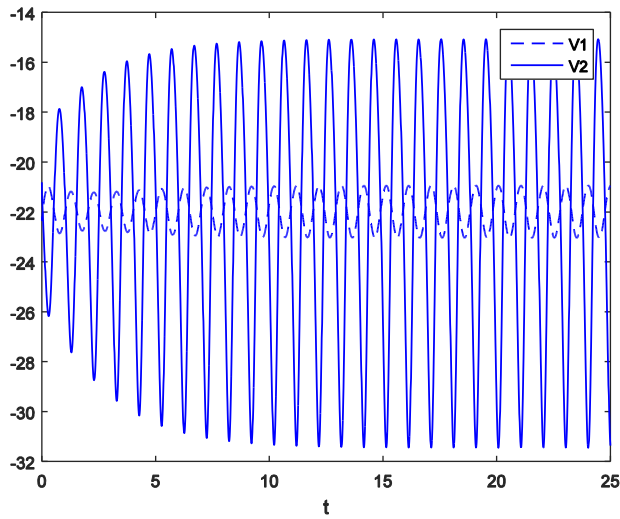Small oscillations of the FO model (2) for $T=26.0$, $I=155$, and $\alpha=1.0$.

Figure 2
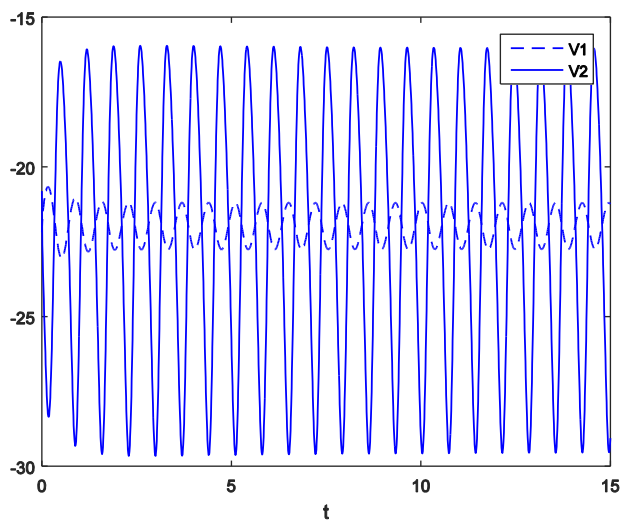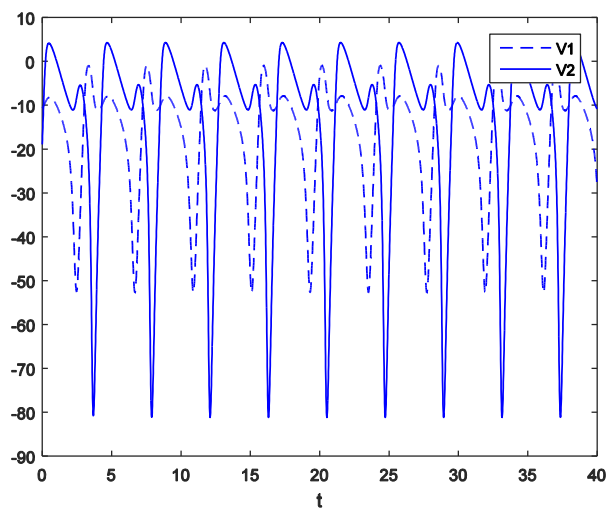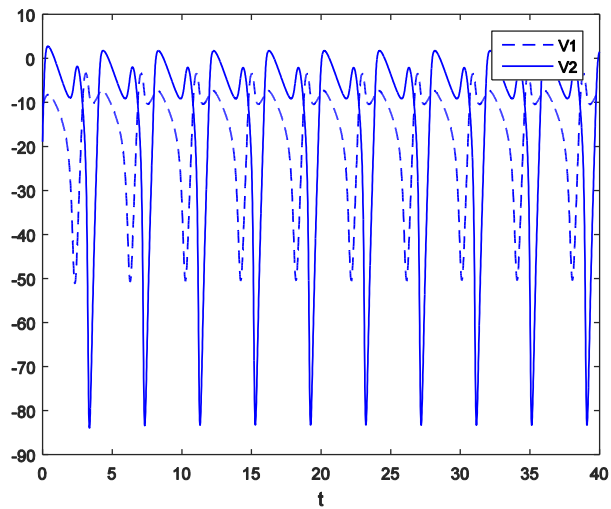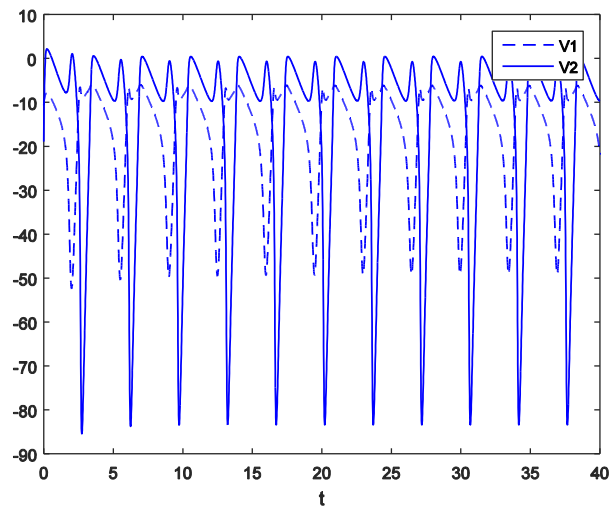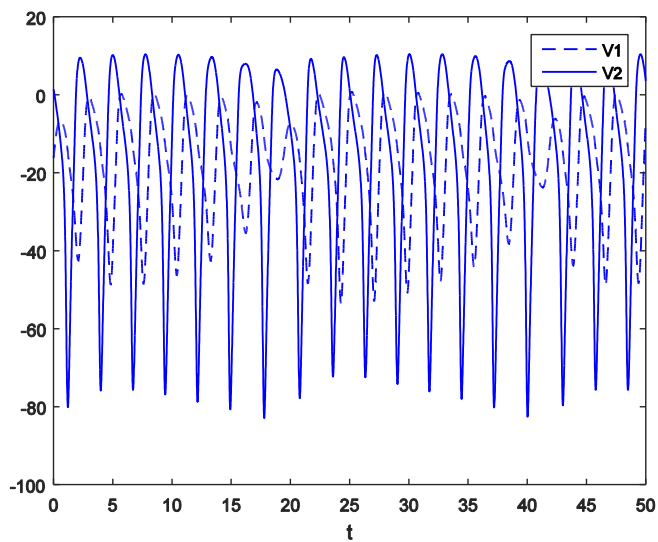Small oscillations of the FO model (2) for $T=26.0$, $I=155$ and $\alpha=0.8$.

Figure 3
Small oscillations of the FO model (2) for *T*=26.0, *I*=155, and α=0.4.

Figure 4
Relaxation oscillations of the FO model (2) for *T*=16.0, *I*=60, and α=1.0.

Figure 5

Relaxation oscillations of the FO model (2) for $T=16.0$, $I=60$, and $\alpha=0.8$.

Figure 6

Relaxation oscillations of the FO model (2) for $T=16.0$, $I=60$, and $\alpha=0.4$.

Figure 7
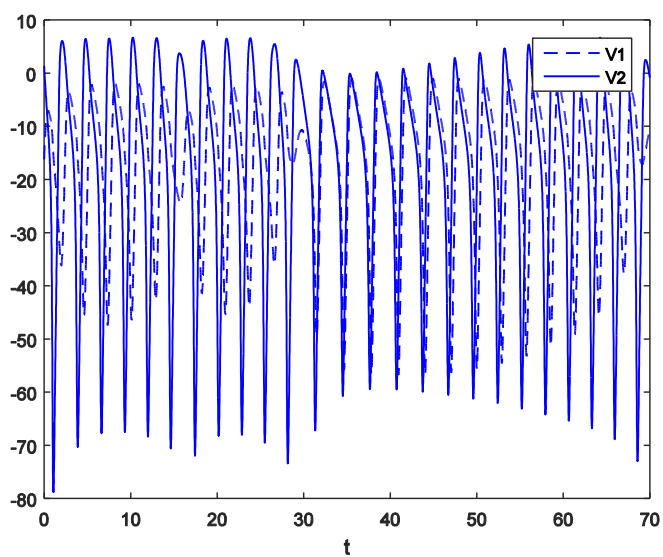Mixed-mode oscillations of the FO model (2) for $T=16.0$, $I=60$, and $\alpha=1.0$.

Figure 8
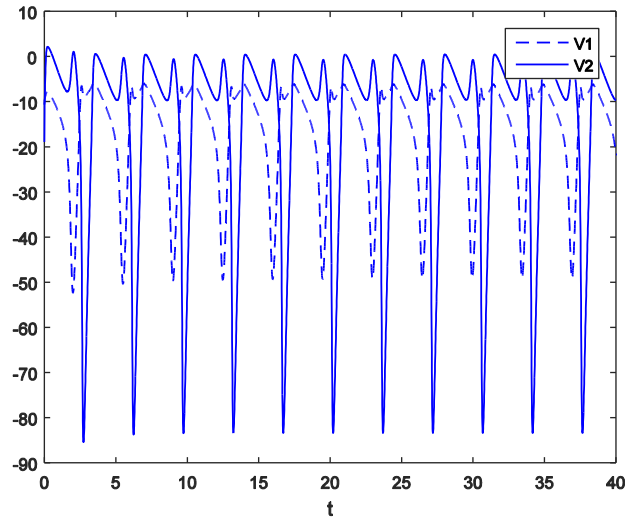Mixed-mode oscillations of the FO model (2) for $T=16.0$, $I=60$, and $\alpha=0.8$.

Figure 9
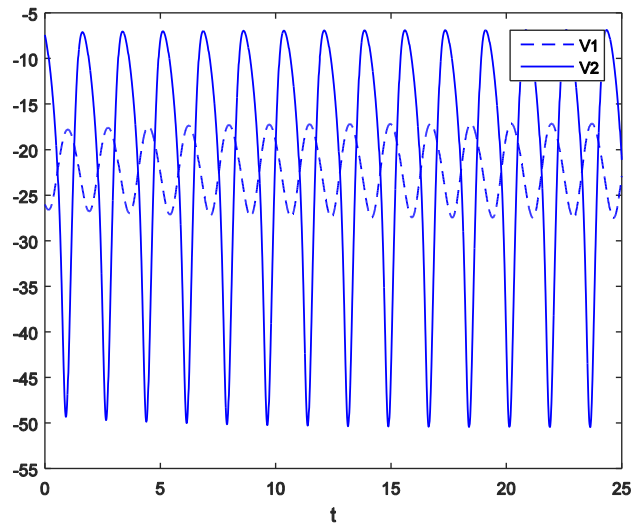Relaxation oscillations of the FO model (2) for $T=16.0$, $I=60$, and $\alpha=0.4$.

Figure 10
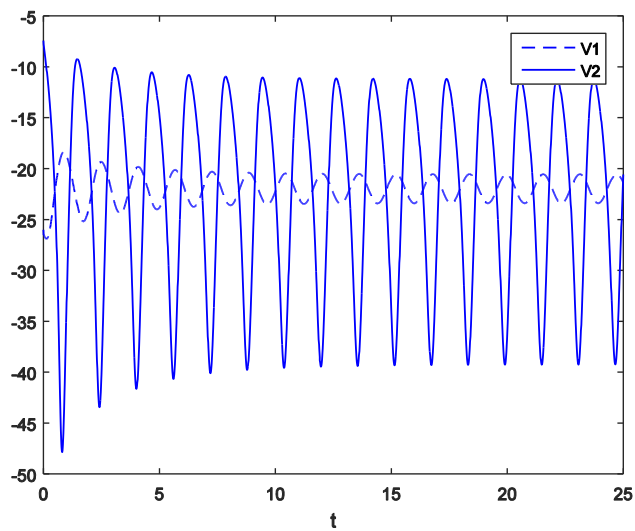Localized solutions of the FO model (2) for $T=20.0$, $I=155$, and $\alpha=1.0$.

Figure 11

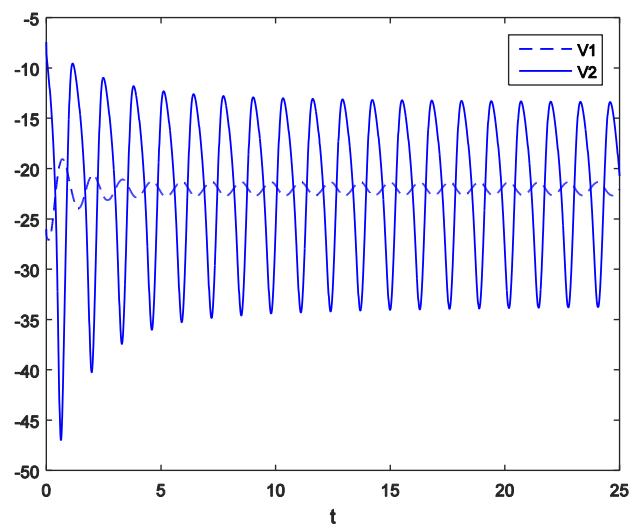Localized solutions of the FO model (2) for T=20.0, *I*=155, and α=0.8.

Figure 12

Localized solutions of the FO model (2) for T=20.0, *I*=155, and α=0.4.

4　Conclusions

We study an asymmetrically coupled model of two FO HH equations. The model is very rich in terms of diversity of dynamical patterns. One can distinguish mixed-mode oscillations, small oscillations, relaxation oscillations, and localized solutions, for certain parameters (coupling constants) regions. Moreover, the value of the order of the fractional derivative comprises more complexity to the coupled FO model. This may be used to explain differences in the human brain when storing and processing memories, or when reacting to the same stimuli. More work is needed in order to fully understand the vast diversity of patterns of the model.

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