An index integral and convolution operator related to the Kontorovich-Lebedev and Mehler-Fock transforms

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Abstract

We deal with an index integral involving the product of the modified Bessel functions and associated Legendre functions. It was discovered by Ferrell [2] while comparing solutions of the Laplace equation in different coordinate systems in his study of the so-called surface plasmons in various condensed matter samples. This integral is quite interesting from the pure mathematical point of view and it is absent in famous reference books for series and integrals. We give a rigorous proof of this formula and discuss its particular cases. We also construct a convolution operator associated with this integral, which is related to the classical Kontorovich-Lebedev and Mehler-Fock transforms. Mapping properties and the norm estimates in weighted L_p-spaces, $1 \le p \le 2$ are investigated. An application to a class of convolution integral equations is considered. Necessary and sufficient conditions are found for the solvability of these equations in L_2.

Keywords: *Kontorovich-Lebedev transform, Mehler-Fock transform, Modified Bessel function, Associated Legendre functions, Convolution integral equations, Parseval equality, Index integrals*

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1 Introduction and preliminary results

In this paper we investigate the following integral with respect to an index or a parameter of the modified Bessel function and associated Legendre functions [1], Vols. 1-2

$$e^{-x\mu\eta} J_0\left(x\sqrt{(\eta^2-1)(1-\mu^2)}\right) = \sqrt{\frac{2}{\pi x}} \int_0^\infty \tau \tanh(\pi\tau) K_{i\tau}(x) P_{-1/2+i\tau}(\mu) P_{-1/2+i\tau}(\eta) d\tau, \tag{1.1}$$

where $x > 0, \mu, \eta > -1$, $J_0(z)$, $K_\nu(z)$ are Bessel and modified Bessel functions and $P_{-1/2+i\tau}(z)$ is the associated Legendre or conical function. It was discovered by Ferrell [2] while comparing solutions of the Laplace equation in different coordinate systems in his study of the so-called surface plasmons in various condensed matter samples. However, integral (1.1) is quite interesting from the pure mathematical point of view. As far as the author aware, there is no a rigorous proof of this formula and it is absent in the corresponding reference book [8]. We will prove this formula in the sequel, will discuss its particular cases and represent new index integrals as a consequence of a relationship of the integral (1.1) with the Kontorovich-Lebedev and Mehler-Fock transforms (see [9], [10], [11], [12]). Important and recent applications of the Ferrell integral (1.1) to index integral representations for connection between different coordinate systems see in [5]. Moreover, our goal is to construct a new convolution operator related to (1.1)

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for the Kontorovich-Lebedev and Mehler-Fock transforms. We will also prove the factorization property for this convolution in the weighted L_p-spaces, $1 \leq p \leq 2$ and discuss its algebraic properties. Finally we will apply it to a class of the corresponding convolution integral equations, finding necessary and sufficient conditions for the solvability of these equations in L_2.

As it is known [9], [1], Vol.2, the modified Bessel function $K_{i\tau}(x)$ can be represented by the Fourier integral

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u} \cos xudu, \ x > 0. \tag{1.2}$$

Hence, when $\tau \in \mathbb{R}$, it is real-valued and even with respect to the pure imaginary index $i\tau$. Furthermore, this integral can be extended to the strip $\delta \in [0, \pi/2)$ in the upper half-plane, i.e.

$$K_{i\tau}(x) = \frac{1}{2} \int_{i\delta-\infty}^{i\delta+\infty} e^{-t\cosh z + i\tau z} dz, \tag{1.3}$$

and leads for each $x > 0$ to a uniform estimate

$$|K_{i\tau}(x)| \leq e^{-|\tau| \arccos \beta} K_0(\beta x), \quad 0 < \beta \leq 1, \tag{1.4}$$

which will be used in the sequel. We note also its asymptotic behaviour [1] at infinity

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \quad z \to \infty, \tag{1.5}$$

and hear the origin

$$z^\nu K_\nu(z) = 2^{\nu-1}\Gamma(\nu) + o(1), \quad z \to 0, \tag{1.6}$$

$$K_0(z) = -\log z + O(1), \quad z \to 0. \tag{1.7}$$

When x is fixed we have the following behavior of the modified Bessel function $K_{i\tau}(x)$ with respect to the index $\tau \to +\infty$

$$K_{i\tau}(x) = \sqrt{\frac{2\pi}{\tau}} e^{-\frac{\pi}{2}\tau} \sin\left(\frac{\pi}{4} + \tau \log\left(\frac{2\tau}{x}\right) - \tau\right) \left[1 + O\left(\frac{1}{\tau}\right)\right]. \tag{1.8}$$

By $L_p(\Omega; w(x)dx), \ 1 < p < \infty$ we denote the weighted L_p- space with the norm

$$||f||_{L_p(\Omega;w(x)dx)} = \left(\int_\Omega |f(x)|^p w(x)dx\right)^{1/p}$$

$$||f||_{L_\infty(\Omega;w(x)dx)} = \text{ess sup}_{x\in\Omega}|f(x)|.$$

As it is known [9], [11] the modified Bessel function $K_{i\tau}(x)$ is the kernel of the following operator of the Kontorovich-Lebedev transformation

$$K_{i\tau}[f] = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^\infty K_{i\tau}(x) f(x) \frac{dx}{\sqrt{x}}, \tag{1.9}$$

which is an isometric isomorphism (see [12])

$$K_{i\tau} : L_2(\mathbb{R}_+; dx) \to L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau),$$

and the convergence of the integral (1.9) is in the mean-square sense with respect to the norm of the space $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$. Moreover, the Parseval identity

$$\frac{2}{\pi^2} \int\limits_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau = \int\limits_0^\infty |f(x)|^2 dx \tag{1.10}$$

holds and the inverse operator is defined by the formula

$$f(x) = \operatorname*{l.i.m.}_{N\to\infty} \frac{2}{\pi^2} \int\limits_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{\sqrt{x}} K_{i\tau}[f] d\tau, \tag{1.11}$$

where the convergence is in mean-square with respect to the norm of $L_2(\mathbb{R}_+; dx)$.

Formula (1) involves the product of the associated Legendre functions of different parameters [1], Vol. 1 and [4]. The function $P_\nu(z)$ is the associated Legendre function of the first kind, which is analytic in the half-plane $\mathrm{Re\, z} > -1$ and entire with respect to ν. The following integral representations will be useful in the sequel (see [4])

$$P_{-1/2+i\tau}(\mu) = \frac{2}{\pi} \cosh(\pi\tau) \int_0^\infty J_0\left(y\sqrt{\frac{\mu-1}{2}} \right) K_{2i\tau}(y) dy,\ \mu \geq 1, \tag{1.12}$$

$$P_{-1/2+i\tau}(\cos\beta) = \frac{2}{\pi} \int_0^\beta \frac{\cosh \theta\tau}{\sqrt{2(\cos\theta - \cos\beta)}} d\theta,\ 0 \leq \theta \leq \pi, \tag{1.13}$$

$$P_{-1/2+i\tau}(\cosh\alpha) = \sqrt{\frac{2}{\pi}} \frac{\cosh(\pi\tau)}{\pi} \int_0^\infty e^{-y\cosh\alpha} K_{i\tau}(y) \frac{dy}{\sqrt{y}},\ \alpha \geq 0. \tag{1.14}$$

$$P_{-1/2+i\tau}(\cosh\alpha) = \frac{1}{\pi} \int_0^\pi \frac{d\theta}{(\cosh\alpha + \sinh\alpha\ \cos\theta)^{1/2+i\tau}}. \tag{1.15}$$

We note the important values $P_\nu(1) = 1,\ \nu \in \mathbb{C}$,

$$P_\nu(0) = \frac{\sqrt{\pi}}{\Gamma((1-\nu)/2)\Gamma(1+\nu/2)}$$

and uniform asymptotic expansions with respect to τ at infinity

$$P_{-1/2+i\tau}(\cos\theta) = O\left(\frac{e^{\theta\tau}}{\sqrt{2\pi\tau \sin\theta}} \right),\ \delta \leq \theta \leq \pi - \delta,\ \delta \in (0, \pi),\ \tau \to +\infty, \tag{1.16}$$

$$P_{-1/2+i\tau}(\cosh\alpha) = O\left(\sqrt{\frac{2}{\pi\tau \sinh\alpha}} \sin\left(\alpha\tau + \frac{\pi}{4} \right) \right),\ \delta \leq \alpha \leq A < \infty,\ \delta > 0,\ \tau \to +\infty. \tag{1.17}$$

The associated Legendre function of the second kind is denoted by $Q_\nu(z)$ and it is analytic in the half-plane $\mathrm{Re}\ z > 1$. It has the following uniform asymptotic behavior at infinity [1], Vol. 1

$$Q_\nu(z) = O\left(\frac{\sqrt{\pi}}{2^{\nu+1}} \frac{\Gamma(1+\nu)}{\Gamma(\nu+3/2)} z^{-\nu-1} \right),\ z \to \infty, \tag{1.18}$$

which can be easily obtained from its representation in terms of the Gauss hypergeometric function (see [4]).

We will appeal below to the following integral representation

$$Q_{\nu-1/2}(\cosh\alpha) = \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-y\cosh\alpha} I_\nu(y)\frac{dy}{\sqrt{y}},\ \mathrm{Re}\,\nu > -\frac{1}{2},\ \alpha > 0, \tag{1.19}$$

where $I_\nu(z)$ is the modified Bessel function of the third kind [1], Vol. 2.

The classical Mehler-Fock transform in the space $L_2((1,\infty);dx)$ we define in the form [9], [10]

$$MF[f](\tau) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_1^N P_{-1/2+i\tau}(x)f(x)dx, \tag{1.20}$$

where integral (1.20) is convergent in the mean square sense with respect to the norm in $L_2(\mathbb{R}_+;\tau\tanh\pi\tau d\tau)$. It is known [3] that MF is an isometric isomorphism

$$MF : L_2((1,\infty);dx) \to L_2(\mathbb{R}_+;\tau\tanh\pi\tau d\tau)$$

with the inverse operator

$$f(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_0^N \tau\tanh\pi\tau\ P_{-1/2+i\tau}(x)MF[f](\tau)d\tau, \tag{1.21}$$

where the convergence is with respect to the norm in $L_2((1,\infty);dx)$, and the generalized Parseval equality

$$\int_0^\infty \tau\tanh\pi\tau MF[f_1](\tau)MF[f_2](\tau)d\tau = \int_1^\infty f_1(x)f_2(x)dx \tag{1.22}$$

for any $f_1, f_2 \in L_2((1,\infty);dx)$.

2 Convergence properties and the validity of (1.1) under various parameters

We begin this section with the following

Theorem 1. *Let $x > 0$ and $\mu,\eta > -1$. Formula (1.1) is valid and the corresponding integral converges absolutely if*

$$\mu,\eta > 0, \quad \mu^2+\eta^2 > 1. \tag{2.1}$$

The convergence in (1.1) is conditional if:

$$\mu = 0, \quad \eta \geq 1 \tag{2.2}$$

or vice versa, or

$$(\mu,\eta) \in (0,1)\times(0,1),\ \mu^2+\eta^2 = 1. \tag{2.3}$$

Finally, when at least one of the parameters μ or η belongs to the interval $(-1,0)$ or $(\mu,\eta)\in[0,1)\times[0,1)$ such that $\mu^2+\eta^2 < 1$ the integral (1.1) is divergent.

Proof. Denoting by

$$I_{\mu,\eta}(x) = \sqrt{\frac{2}{\pi x}}\int_0^\infty \tau\tanh(\pi\tau)K_{i\tau}(x)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta)d\tau, \tag{2.4}$$

the right-hand side of the integral (1.1) we first consider the case $(\mu, \eta) \in [1, \infty) \times [1, \infty)$. Taking into account asymptotic behavior by the index of the modified Bessel function and associated Legendre functions (see formulas (1.8), (1.16), (1.17)) it is not difficult to observe the absolute convergence of the integral (2.4) in this case. Moreover, multiplying both sides of (2.4) by $\sqrt{x}e^{-x}$, we appeal to inequality (1.4) for the modified Bessel function in order to motivate the use of the Mellin transform [8], [10]

$$f^*(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{2.5}$$

with respect to x through the obtained equality. Changing the order of integration by Fubini's theorem, we employ relation (8.4.23.3) in [8]

$$\int_0^\infty e^{-x}K_{i\tau}(x)x^{s-1}dx = 2^{-s}\sqrt{\pi}\frac{\Gamma(s+i\tau)\Gamma(s-i\tau)}{\Gamma(s+1/2)}, \ \mathrm{Re}\, s > 0 \tag{2.6}$$

and come out with the equality

$$I^*_{\mu,\eta}(s) = \frac{2^{1/2-s}}{\Gamma(s+1/2)}\int_0^\infty \tau \tanh(\pi\tau)\Gamma(s+i\tau)\Gamma(s-i\tau)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta)d\tau, \tag{2.7}$$

where

$$I^*_{\mu,\eta}(s) = \int_0^\infty I_{\mu,\eta}(x)e^{-x}x^{s-1/2}dx. \tag{2.8}$$

Further, appealing to (1.12) we substitute it in (2.7). Then denoting by $a = \sqrt{\frac{\mu-1}{2}}, \ b = \sqrt{\frac{\eta-1}{2}}$ we consider the triple integral

$$I^*_{\mu,\eta}(s) = \frac{2^{3/2-s}}{\pi^2\Gamma(s+1/2)}\int_0^\infty\int_0^\infty\int_0^\infty \tau\sinh(2\pi\tau)\Gamma(s+i\tau)\Gamma(s-i\tau)$$
$$\times J_0(ay)J_0(bu)K_{2i\tau}(y)K_{2i\tau}(u)dudyd\tau. \tag{2.9}$$

By virtue of the Stirling asymptotic formula for gamma-functions [1], Vol. 1 we have

$$|\Gamma(s+i\tau)| = O\left(e^{-\pi\tau/2}\tau^{\mathrm{Re} s-1/2}\right), \ \tau \to +\infty.$$

Therefore taking into account asymptotic properties of the Bessel functions (see (1.3), (1.4), (1.5), (1.7)), we verify the absolute convergence of the integral (2.8) and the possibility to change the order of integration via Fubini's theorem. Since the inner index integral with respect to τ is calculated by relation (2.16.53.1) in [7] we arrive at the equality

$$I^*_{\mu,\eta}(s) = \frac{2^{1/2-3s}}{\Gamma(s+1/2)}\int_0^\infty\int_0^\infty J_0(ay)J_0(bu)\left(\frac{y^2u^2}{y^2+u^2}\right)^s K_{2s}\left(\sqrt{u^2+y^2}\right)dudy. \tag{2.10}$$

In the meantime (see relation (2.3.16.1) in [6]),

$$\left(\frac{y^2u^2}{y^2+u^2}\right)^s K_{2s}\left(\sqrt{u^2+y^2}\right) = \frac{1}{2}\int_0^\infty t^{2s-1}e^{-t\frac{y^2+u^2}{2uy}-\frac{uy}{2t}}dt.$$

Consequently, after simple substitutions (2.9) yields

$$I^*_{\mu,\eta}(s)\Gamma\left(\frac{1}{2}+s\right) = 2^{-3/2}\int_0^\infty\int_0^\infty\int_0^\infty J_0(ay)J_0(bu)$$

$$\times \exp\left(-\left[\sqrt{8t}\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right]\right) t^{s-1} dudydt. \tag{2.11}$$

Now we are going to cancel the Mellin transform (2.5) in the latter equality (2.11) by using its uniqueness for integrable functions [10]. Taking into account factorization properties of the Mellin type convolution transforms (see [8], [10]) (2.11) implies for $t > 0$

$$\int_0^\infty I_{\mu,\eta}(x) e^{-x-\frac{t}{x}} \frac{dx}{x} = \frac{2^{-3/2}}{\sqrt{t}} \int_0^\infty \int_0^\infty J_0(ay) J_0(bu)$$

$$\times \exp\left(-\left[\sqrt{8t}\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right]\right) dudy, \tag{2.12}$$

where the left-hand side represents a modified Laplace transform [10] of the function $e^{-x} I_{\mu,\eta}(x)$. Meanwhile, the double integral in the right-hand side of (2.12) can be treated by polar coordinates. This drives at the form

$$\frac{2^{-3/2}}{\sqrt{t}} \int_0^\infty \int_0^\infty J_0(ay) J_0(bu) \exp\left(-\left[\sqrt{8t}\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right]\right) dudy$$

$$= \frac{2^{-3/2}}{\sqrt{t}} \int_0^\infty \int_0^{\pi/2} J_0(ar\sin\varphi) J_0(br\cos\varphi) \exp\left(-\left[\frac{\sqrt{8t}}{\sin 2\varphi}+\frac{r^2 \sin 2\varphi}{4\sqrt{8t}}\right]\right) r dr d\varphi.$$

The latter integral by r is calculated via relation (2.12.39.3) in [7]. Substituting the result in (2.11) we obtain

$$\int_0^\infty I_{\mu,\eta}(k) e^{-x-\frac{t}{x}} \frac{dx}{x}$$

$$= 2I_0(ab\sqrt{8t}) \int_0^{\pi/2} \exp\left(-\sqrt{8t}\ \frac{1+a^2\sin^2\varphi+b^2\cos^2\varphi}{\sin 2\varphi}\right) d\varphi, \tag{2.12}$$

where $I_0(z)$ is the modified Bessel function [4]. Hence calculating the integral with respect to φ in (2.12) by using an elementary substitutions and calling relation (2.3.16.1) in [6], we get finally the equality

$$\int_0^\infty I_{\mu,\eta}(x) e^{-x-\frac{t^2}{8x}} \frac{dx}{x} = 2I_0(abt) K_0\left(t\sqrt{(1+a^2)(1+b^2)}\right). \tag{2.13}$$

Taking into account definitions of the involved parameters, the property of Bessel functions $J_0(iz) = I_0(z)$ and relation (2.12.10.1) in [7], we write (2.13) in the form

$$\int_0^\infty I_{\mu,\eta}(x) e^{-x-\frac{t^2}{8x}} \frac{dx}{x} = \int_0^\infty e^{-x(z+1)-\frac{t^2}{8x}} J_0(xR) \frac{dx}{x},\ t > 0, \tag{2.14}$$

where we denote by $z = \mu\eta$, and $R = \sqrt{(\eta^2-1)(1-\mu^2)}$. Consequently, the final equality gives the value of the integral (1.1) via the uniqueness theorem for the modified Laplace transform of integrable functions [9], [10]. Canceling this transform in (2.14) we find the value of (1.1), namely

$$I_{\mu,\eta}(x) = e^{-xz} J_0(xR),\ \mu,\nu \in [1,\infty) \times [1,\infty). \tag{2.15}$$

We will prove now the validity of integral (1) under conditions (2.1) and its absolute convergence. Indeed, we could see already that (1.1) converges absolutely when $\mu, \nu \in [1,\infty)$. Suppose that $\mu \in (0,1), \eta \geq 1$ or vice versa. In this case formula (1.1) is true because it can be prolonged analytically by μ or η. In fact, $P_{-1/2+i\tau}(z)$ is analytic in the right half-plane $\mathrm{Re}\, z > -1$ and integral (1.1) converges absolutely and uniformly with respect to $\theta = \arccos\mu \in \left[\varepsilon,\ \frac{\pi}{2}-\varepsilon\right],\ \eta \geq 1$ or $\xi = \arccos\eta \in \left[\varepsilon,\ \frac{\pi}{2}-\varepsilon\right],\ \mu \geq 1$ for any

$k > 0$ owing to uniform estimates (1.16), (1.17). When $\mu,\ \eta \in (0,1)\times(0,1)$ we employ again (1.8), (1.16) and an elementary equality [6]

$$\arccos\mu + \arccos\eta = \arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right)$$

to obtain for sufficiently big $A > 0$

$$\int_A^\infty \tau\tanh(\pi\tau)\left|K_{i\tau}(x)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta)\right| d\tau$$

$$\leq C\int_A^\infty \frac{\tanh(\pi\tau)}{\sqrt{\tau}}\exp\left(-\tau\left[\frac{\pi}{2} - \arccos\mu - \arccos\eta\right]\right) d\tau$$

$$= C\int_A^\infty \frac{\tanh(\pi\tau)}{\sqrt{\tau}}\exp\left(-\tau\left[\frac{\pi}{2} - \arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right)\right]\right) d\tau, \tag{2.16}$$

where $C > 0$ is an absolute constant. Therefore, the latter integral in (2.16) converges uniformly if

$$\arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right) \leq \frac{\pi}{2} - \varepsilon,\ \varepsilon > 0.$$

This means the condition $\mu^2 + \eta^2 > 1$ and proves (2.1).

Further, when $\mu,\ \eta \in [0,1)\times[0,1)$ such that $\mu^2+\eta^2 < 1$, then

$$\int_A^\infty \tau\tanh(\pi\tau)K_{i\tau}(x)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta)d\tau$$

$$= O\left(\int_A^\infty \exp\left(\tau\left[\arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right) - \frac{\pi}{2}\right]\right) d\tau\right) \to \infty,\ A\to\infty$$

since $\arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right) > \frac{\pi}{2}$. Thus integral (1.1) diverges under these conditions. Moreover, since the uniform asymptotic formula (1.16) keeps true for $\theta \in \left(\frac{\pi}{2},\ \pi-\varepsilon\right],\ \varepsilon > 0$ (see [4]), we easily get assuming $\mu \in (-1,0),\ \eta > -1$ or vice versa, that integral (1.1) is divergent, namely

$$\int_A^\infty \tau\tanh(\pi\tau)K_{i\tau}(x)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta)dq$$

$$= O\left(\int_A^\infty P_{-1/2+i\tau}(\eta)\exp\left(q\left[\theta - \frac{\pi}{2}\right]\right) dq\right) \to \infty,\ A\to\infty.$$

Finally, we will establish the validity of the integral (1.1) under more delicate conditions (2.2), (2.3). To do this we recall (1.8), (1.16), (1.17) to verify the uniform convergence of the integral

$$\int_A^\infty \tau\tanh(\pi\tau)K_{i\tau}(x)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta)dq$$

$$= O\left(\int_A^\infty \exp\left(\tau\left(\arccos\mu - \frac{\pi}{2}\right)\right)\frac{\tanh(\pi\tau)}{\sqrt{\tau}}\sin\left(\tau\log\left(\frac{2\tau}{x}\right) - \tau + \frac{\pi}{4}\right)\right.$$
$$\left.\times\cos\left(\tau\ \mathrm{arccosh}\ \eta - \frac{\pi}{4}\right) d\tau\right) \tag{2.17}$$

with respect to $\mu \in [0,\varepsilon],\ \varepsilon > 0$ for $\eta > 1,\ k > 0$ and sufficiently big $A > 0$. But the latter fact follows immediately from the Abel test of the uniform convergence. This means that we can put $\mu = 0$ in (1.1) and condition (2.2) holds for any $\eta > 1$. Moreover, taking into account values $P_q(0)$ (see above) we derive the formula

$$J_0\left(x\sqrt{\eta^2-1}\right) = \sqrt{\frac{2}{x}} \int_0^\infty \frac{\tau\tanh(\pi\tau)}{\left|\Gamma\left(\frac{3}{4}+\frac{i\tau}{2}\right)\right|^2} K_{i\tau}(x) P_{-1/2+i\tau}(\eta) d\tau, \tag{2.18}$$

which coincides with a particular case of the relation (2.17.27.21) in [8]. We will show that formula (2.18) is true also for $\eta = 1$, which gives a new index integral ($P_{-1/2+i\tau}(1) = 1$)

$$\int_0^\infty \frac{\tau\tanh(\pi\tau)}{\left|\Gamma\left(\frac{3}{4}+\frac{i\tau}{2}\right)\right|^2} K_{i\tau}(x) d\tau = \sqrt{\frac{x}{2}}.$$

In fact, integral (2.18) converges uniformly by $\eta \in [1, 1+\varepsilon],\ \varepsilon > 0$ since (see above) with the integration by parts

$$\int_A^\infty \frac{\tau\tanh(\pi\tau)}{\left|\Gamma\left(\frac{3}{4}+\frac{i\tau}{2}\right)\right|^2} K_{i\tau}(x) P_{-1/2+i\tau}(\eta) dq$$

$$= O\left(\int_A^\infty \sin\left(\tau\log\left(\frac{2\tau}{x}\right) - \tau + \frac{\pi}{4}\right)\cos\left(\tau \text{ arccosh } \eta - \frac{\pi}{4}\right) d\tau\right)$$

$$= O\left(\text{arccosh } \eta \int_A^\infty \cos\left(\tau\log\left(\frac{2\tau}{x}\right) - \tau(1 - \text{arccosh } \eta) + \frac{\pi}{4}\right) \frac{d\tau}{\log(2\tau/x)}\right)$$

and the latter integral converges uniformly by $\eta \in [1, 1+\varepsilon],\ \varepsilon > 0$ via the Dirichlet test.

Analogously, if $\mu,\ \eta \in (0,1)\times(0,1)$ such that $\mu^2+\eta^2 = 1$, then (1.1) converges because

$$\int_A^\infty \tau\tanh(\pi\tau) K_{i\tau}(x) P_{-1/2+i\tau}(\mu) P_{-1/2+i\tau}(\eta) d\tau$$

$$= O\left(\int_A^\infty \exp\left(\tau\left(\text{arccos}\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right) - \frac{\pi}{2}\right)\right)\right.$$
$$\left.\times \frac{\tanh(\pi\tau)}{\sqrt{\tau}} \sin\left(\tau\log\left(\frac{2\tau}{x}\right) - \tau + \frac{\pi}{4}\right) d\tau\right)$$
$$= O\left(\int_A^\infty \frac{\tanh(\pi\tau)}{\sqrt{\tau}} \sin\left(\tau\log\left(\frac{2\tau}{x}\right) - \tau + \frac{\pi}{4}\right) d\tau\right) < \infty$$

due to the Dirichlet test. Thus we get condition (2.3) and complete the proof of Theorem 1.

Since $P_{-1/2+i\tau}(1) = 1$ we get the following value of the index integral

$$\int_0^\infty \tau\tanh(\pi\tau) K_{i\tau}(x) P_{-1/2+i\tau}(\mu) d\tau = \sqrt{\frac{\pi x}{2}} e^{-x\mu},\ x, \mu > 0,$$

which coincides with relation (2.17.26.15) in [8]. For $\mu = 1$ it gives the value

$$\int_0^\infty \tau\tanh(\pi\tau) K_{i\tau}(x) d\tau = \sqrt{\frac{\pi x}{2}} e^{-x},\ x > 0, \tag{2.19}$$

which is the limit case of the relation (2.16.48.15) in [7]. When $\mu = \eta \geq \frac{1}{\sqrt{2}}$ formula (1.1) becomes

$$\int_0^\infty \tau\tanh(\pi\tau) K_{i\tau}(x) [P_{-1/2+i\tau}(\mu)]^2 d\tau = \sqrt{\frac{\pi x}{2}} e^{-x\mu^2} I_0(x(\mu^2-1)),$$

which represents a slightly corrected relation (2.17.29.4) in [8].

Corollary 1. *Let $x > 0$ and $\mu \geq 0,\ \eta \geq 0,\ z = \mu\eta$, and $R = \sqrt{(\eta^2-1)(1-\mu^2)}$. Then*

$$\frac{\sqrt{2}}{\pi\sqrt{\pi}}\int_0^\infty e^{-xz}J_0(xR)K_{i\tau}(x)\frac{dx}{\sqrt{x}} = \textit{sech}(\pi\tau)P_{-1/2+i\tau}(\mu)P_{-1/2+i\tau}(\eta), \tag{2.20}$$

where the integral converges absolutely. Moreover, for $\mu_j, \eta_j,\ \ j = 1,2$ such that $\eta_j > 1,\ 0 \leq \mu_j < 1,\ \ j = 1,2$, or vice versa, or $\eta_j > 1,\ \mu_j > 1,\ j = 1,2$, or $0 < \eta_j < 1,\ 0 < \mu_j < 1,\ \eta_j^2 + \mu_j^2 > 1,\ j = 1,2$ we get the value of a new index integral

$$\int_0^\infty \tau\frac{\tanh(\pi\tau)}{\cosh(\pi\tau)}P_{-1/2+i\tau}(\mu_1)P_{-1/2+i\tau}(\eta_1)P_{-1/2+i\tau}(\mu_2)P_{-1/2+i\tau}(\eta_2)d\tau$$

$$= \frac{1}{\pi^2\sqrt{R_1R_2}}Q_{-1/2}\left(\frac{(z_1+z_2)^2+R_1^2+R_2^2}{2R_1R_2}\right), \tag{2.21}$$

where $z_j = \mu_j\eta_j,\ R_j = \sqrt{(\eta_j^2-1)(1-\mu_j^2)}, j = 1,2$. If one of the parameters $\mu_j, \eta_j,\ j = 1,2$ is equal to 1, say $\mu_1 = 1$, then

$$\int_0^\infty \tau\frac{\tanh(\pi\tau)}{\cosh(\pi\tau)}P_{-1/2+i\tau}(\eta_1)P_{-1/2+i\tau}(\mu_2)P_{-1/2+i\tau}(\eta_2)d\tau = \frac{1}{\pi}\frac{1}{\sqrt{(\eta_1+z_2)^2+R_2^2}}. \tag{2.22}$$

Finally, when $\mu_1 = \mu_2 = \eta_1 = \eta_2 = a \in \left(\frac{1}{\sqrt{2}}, \infty\right)\setminus\{1\}$ we have in particular, a new index integral

$$\int_0^\infty \tau\frac{\tanh(\pi\tau)}{\cosh(\pi\tau)}\left[P_{-1/2+i\tau}(a)\right]^4 d\tau = \frac{1}{\pi^2|a^2-1|}Q_{-1/2}\left(\frac{a^4+2a^2-1}{(a^2-1)^2}\right). \tag{2.23}$$

The limit case $a = 1$ (see (1.18)) coincides with the known value

$$\int_0^\infty \tau\frac{\tanh(\pi\tau)}{\cosh(\pi\tau)}d\tau = \frac{1}{2\pi}.$$

Proof. The proof is based on the Plancherel theorem and Parseval's equality (1.10) for the Kontorovich -Lebedev transform (1.9). In fact, via the parallelogram equality for the inner product, (1.10) yields

$$\frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}[f_1]K_{i\tau}[f_2]d\tau = \int_0^\infty f_1(x)f_2(x)dx. \tag{2.24}$$

Taking $0 < \mu \leq 1,\ \eta \geq 1$ we easily check via the asymptotic behavior of Bessel functions that $e^{-xz}J_0(xR) \in L_2(\mathbb{R}_+; dx)$. Therefore (2.20) holds and the integral converges absolutely to the same limit under these conditions. However the absolute and uniform convergence keeps true for any $\mu \geq 0,\ \eta \geq 0$. Therefore the equality (2.20) is still valid for any nonnegative parameters $\mu,\ \nu$. Further, we immediately get (2.21) and its particular cases (2.22), (2.23) as a consequence of the Parseval identity (26) and relations (2.12.38.1), (2.12.8.2) in [7]. The corresponding conditions on parameters guarantee the absolute and uniform convergence of the integral (2.21). Corollary 1 is proved.

3 A convolution operator and its mapping properties

We begin with

Definition 1. Let f, g be functions from $(1,\infty)$ into $\mathbb{C}$. Then the function $f * g$ defined on $\mathbb{R}_+$ by

$$(f*g)(x) = \frac{\sqrt{2}}{\pi\sqrt{\pi}}\int_1^\infty\int_1^\infty e^{-xuv}I_0\left(x\sqrt{(u^2-1)(v^2-1)}\right)f(u)g(v)dudv \tag{3.1}$$

is called the convolution related to the Kontorovich-Lebedev and Mehler-Fock transforms (1.9) and (1.20), respectively (provided that it exists).

Theorem 2. $f, g \in L_p((1,\infty); dx),\ 1 < p < 2$. *Then the convolution* $(f * g)(x)$ *exists for almost all* $x > 0$ *and belongs to* $L_2(\mathbb{R}_+; dx)$. *The convolution is commutative and*

$$||f * g||_{L_2(\mathbb{R}_+;dx)} \leq C||f||_{L_p((1,\infty);dx)}||g||_{L_p((1,\infty);dx)}, \tag{3.2}$$

where $C > 0$ *is an absolute constant*.

Proof. Indeed, from Definition 1 it follows that $f * g$ is a commutative operation. Further, by virtue of Fubini's theorem with the use of the generalized Minkowski inequality there exists

$$||f * g||_{L_2(\mathbb{R}_+;dx)} \leq \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_1^\infty \int_1^\infty \left(\int_0^\infty e^{-2xuv} I_0^2 \left(x\sqrt{(u^2-1)(v^2-1)} \right) dx \right)^{1/2} |f(u)g(v)| dudv. \tag{3.3}$$

The integral with respect to x

$$I = \int_0^\infty e^{-2xuv} I_0^2 \left(x\sqrt{(u^2-1)(v^2-1)} \right) dx$$

is calculated by relation (2.15.20.1) in [7]. Consequently, we obtain

$$I = \frac{1}{\pi\sqrt{(u^2-1)(v^2-1)}} Q_{-1/2} \left(\frac{2u^2v^2}{(u^2-1)(v^2-1)} - 1 \right).$$

Substituting this value in (3.3) and using the Hölder inequality for double integrals it becomes

$$||f * g||_{L_2(\mathbb{R}_+;dx)} \leq \frac{\sqrt{2}}{\pi^2} \left(\int_1^\infty \int_1^\infty [(u^2-1)(v^2-1)]^{-q/4} Q_{-1/2}^{q/2} \left(\frac{2u^2v^2}{(u^2-1)(v^2-1)} - 1 \right) dudv \right)^{1/q}$$

$$\times ||f||_{L_p((1,\infty);dx)}||g||_{L_p((1,\infty);dx)}, \qquad q = \frac{p}{p-1}. \tag{3.4}$$

Meanwhile, calling representation (1.19) of the associated Legendre function $Q_{\nu-1/2}(\cosh\alpha)$, we put $\nu = 0$ and use relation (8.4.22.3) in [8]

$$e^{-x} I_0(x) = \frac{1}{2\pi i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1/2-s)}{\Gamma(1-s)} (2x)^{-s} ds,\ 0 < \gamma < \frac{1}{2}.$$

Substituting the latter integral into (1.19) and changing the order of integration via Fubini's theorem, we have

$$Q_{-1/2} \left(\frac{2u^2v^2}{(u^2-1)(v^2-1)} - 1 \right) = \frac{1}{4\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma^2(1/2-s)}{\Gamma(1-s)}$$

$$\times \left(\frac{u^2v^2}{(u^2-1)(v^2-1)} - 1 \right)^{s-1/2} ds.$$

Hence

$$Q_{-1/2}^{1/2} \left(\frac{2u^2v^2}{(u^2-1)(v^2-1)} - 1 \right) \leq \frac{1}{2\sqrt{\pi}} \left(\frac{u^2v^2}{(u^2-1)(v^2-1)} - 1 \right)^{\frac{\gamma}{2}-\frac{1}{4}}$$

$$\times \left(\int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Gamma(s)\Gamma^2(1/2-s)}{\Gamma(1-s)} ds \right| \right)^{1/2} = C_\gamma \left(\frac{u^2v^2}{(u^2-1)(v^2-1)} - 1 \right)^{\frac{\gamma}{2}-\frac{1}{4}},$$

where

$$C_\gamma = \frac{1}{2\sqrt{\pi}} \left(\int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Gamma(s)\Gamma^2(1/2-s)}{\Gamma(1-s)} ds \right| \right)^{1/2},\ 0 < \gamma < \frac{1}{2}.$$

Therefore with elementary substitutions

$$\frac{\sqrt{2}}{\pi^2}\left(\int_1^\infty\int_1^\infty [(u^2-1)(v^2-1)]^{-q/4} Q^{q/2}_{-1/2}\left(\frac{2u^2v^2}{(u^2-1)(v^2-1)}-1\right)dudv\right)^{1/q}$$

$$\leq \frac{C_\gamma\sqrt{2}}{\pi^2}\left(\int_1^\infty\int_1^\infty [(u^2-1)(v^2-1)]^{-q\gamma/2}\left(u^2v^2-(u^2-1)(v^2-1)\right)^{q(\gamma-1/2)/2}dudv\right)^{1/q}$$

$$= \frac{C_\gamma\sqrt{2}}{\pi^2}\left(\int_0^\infty\int_0^\infty [\sinh u\sinh v]^{1-q\gamma}\left(\sinh^2 u+\sinh^2 v+1\right)^{q(\gamma-1/2)/2}dudv\right)^{1/q}$$

$$\leq \frac{C_\gamma\sqrt{2}}{\pi^2}\left(\int_0^\infty \sinh^{1-q\gamma} u\, du\right)^{2/q},$$

where the latter integral is evidently convergent for $q \in \left(\frac{1}{\gamma}, \frac{2}{\gamma}\right) \subset (2,\infty)$ because $0 < \gamma < \frac{1}{2}$. Since γ is arbitrary from this interval, inequality (3.5) is true for any $2 < q = \frac{p}{p-1}$. Hence appealing to relation (2.4.4.7) in [6] and putting

$$C = \frac{[\Gamma(1-q\gamma/2)\Gamma((q\gamma-1)/2)]^{2/q}}{2^{(4+q)/2q}\pi^{(2+5q)/2q}}\left(\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma(s)\Gamma^2(1/2-s)}{\Gamma(1-s)}ds\right|\right)^{1/2},\ 0<\gamma<\frac{1}{2}$$

we get (3.2) and complete the proof of Theorem 2.

In order to study mapping properties of the Mehler-Fock transform (1.20) we will need a uniform estimate by $\tau \in \mathbb{R}_+$ of the associated Legendre function. But this easily follows from representation (1.15). Precisely, we have $|P_{-1/2+i\tau}(x)| \leq P_{-1/2}(x),\ x \geq 1$.

Lemma 1. *Operator of the Mehler-Fock transform* (1.20)

$$MF[f](\tau) = \int_1^\infty P_{-1/2+i\tau}(x)f(x)dx, \tag{3.5}$$

is bounded as the operator from $L_p((1,\infty);dx),\ 1 \leq p < 2$ *into the space* $C_0(\mathbb{R}_+)$ *of bounded continuous functions on* $\mathbb{R}_+$ *and the integral* (3.5) *converges absolutely and uniformly by* $\tau \geq 0$. *Moreover,*

$$||MF[f]||_{C_0(\mathbb{R}_+)} \leq (2/q)^{1/q}\frac{\Gamma^2(2^{-1}-q^{-1})}{\pi\Gamma(1-q^{-1})}||f||_{L_p((1,\infty);dx)},\ q=\frac{p}{p-1}. \tag{3.6}$$

When $p = 1$ *it gives, correspondingly,*

$$||MF[f]||_{C_0(\mathbb{R}_+)} \leq ||f||_{L_1((1,\infty);dx)}. \tag{3.7}$$

Proof. Indeed, taking into account the previous uniform estimate of the associated Legendre function, we use the Hölder inequality for $1 < p < 2$ to find

$$||MF[f]||_{C_0(\mathbb{R}_+)} = \sup_{\tau\geq 0}|MF[f](\tau)| \leq \int_1^\infty P_{-1/2}(x)|f(x)|dx$$

$$\leq \left(\int_1^\infty P^q_{-1/2}(x)dx\right)^{1/q} ||f||_{L_p((1,\infty);dx)},\ q=\frac{p}{p-1}. \tag{3.8}$$

Meanwhile, representation (1.14) and the generalized Minkowski inequality yield

$$\left(\int\limits_1^\infty P^q_{-1/2}(x)dx\right)^{1/q} \le \sqrt{\frac{2}{\pi^3}}\int_0^\infty \left(\int_1^\infty e^{-qxy}dx\right)^{1/q} K_0(y)\frac{dy}{\sqrt{y}}$$

$$= q^{-1/q}\sqrt{\frac{2}{\pi^3}}\int_0^\infty e^{-y}K_0(y)y^{-(q^{-1}+1/2)}dy.$$

The latter integral is calculated by relation (2.16.6.4) in [7]. Therefore,

$$\left(\int\limits_1^\infty P^q_{-1/2}(x)dx\right)^{1/q} \le (2/q)^{1/q}\frac{\Gamma^2(2^{-1}-q^{-1})}{\pi\Gamma(1-q^{-1})} \tag{3.9}$$

and combining with (3.8) we establish (3.6). For the limit case $p = 1$ we deduce (see (3.9))

$$||MF[f]||_{C_0(\mathbb{R}_+)} \le ||f||_{L_1((1,\infty);dx)} \sup_{x\ge 1}[P_{-1/2}(x)] \le ||f||_{L_1((1,\infty);dx)}\sqrt{\frac{2}{\pi^3}}\int_0^\infty e^{-y}K_0(y)\frac{dy}{\sqrt{y}}$$

$$= ||f||_{L_1((1,\infty);dx)}.$$

Thus (3.7) holds and Lemma 1 is proved.

Corollary 2. *The norm of the operator* $MF : L_p((1,\infty);dx) \to C_0(\mathbb{R}_+),\ 1 < p < 2$ *satisfies the estimate*

$$||MF|| \le (2/q)^{1/q}\frac{\Gamma^2(2^{-1}-q^{-1})}{\pi\Gamma(1-q^{-1})}.$$

When $p = 1$*, we have* $||MF|| \le 1$*.*

Theorem 3. *Let* $f, g \in L_p((1,\infty);dx),\ 1 \le p < 2$*. Then for all* $x > 0$ *the following generalized Parseval equality takes place*

$$(f * g)(x) = \frac{2}{\pi^2}\int_0^\infty \tau\tanh\pi\tau\frac{K_{i\tau}(x)}{\sqrt{x}}MF[f](\tau)MF[g](\tau)d\tau, \tag{3.10}$$

where the integral is absolutely convergent.

Proof. In fact, we employ Ferrell's integral (1.1) and substitute it in (3.1). The change of the order of integration is guaranteed by Theorem 2 and Fubini's theorem. Finally the definition of the Mehler-Fock transform (3.5) leads to (3.10). Theorem 3 is proved.

Corollary 3. *Under conditions of Theorem 2 the product*

$$MF[f](\tau)MF[g](\tau) \in L_2\left(\mathbb{R}_+; \tau\frac{\tanh\pi\tau}{\cosh\pi\tau}d\tau\right).$$

Moreover, the factorization identity (see (1.9)*)*

$$K_{i\tau}[f * g] = \frac{1}{\cosh\pi\tau}MF[f](\tau)MF[g](\tau) \tag{3.11}$$

and the Parseval equality hold

$$\int_0^\infty |(f * g)(x)|^2dx = \frac{2}{\pi^2}\int_0^\infty \tau\frac{\tanh\pi\tau}{\cosh\pi\tau}\left|MF[f](\tau)MF[g](\tau)\right|^2 d\tau. \tag{3.12}$$

Proof. Since via Theorem 2 $f * g \in L_2(\mathbb{R}_+; dx)$ the statement is an immediate consequence of the L_2-theory for the Kontorovich-Lebedev transform (1.9) by virtue of equalities (1.10), (1.11). Corollary 3 is proved.

Theorem 4. *Let* $f \in L_p((1,\infty); dx),\ 1 \le p < 2$. *The Mehler-Fock transform* (3.5) *is the composition of the Kontorovich-Lebedev transform* (1.9) *and the following Laplace transform*

$$(Lf)(x) = \int_1^\infty e^{-xt} f(t)dt,\ x > 0. \tag{3.13}$$

Namely, we have the equality

$$MF[f](\tau) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}} \cosh \pi\tau K_{i\tau}\left[Lf\right],\ \tau \ge 0, \tag{3.14}$$

where all involved integrals are absolutely convergent.

Proof. In fact, (3.14) takes place due to (1.9), (1.14), (3.5) and Fubini's theorem. The latter fact can be verified employing the estimate

$$\int\limits_1^\infty |P_{-1/2+i\tau}(x)f(x)|dx \le \sqrt{\frac{2}{\pi^3}} \int_0^\infty \left(\int_1^\infty e^{-qxy}dx\right)^{1/q} K_0(y)\frac{dy}{\sqrt{y}} \left(\int\limits_1^\infty |f(x)|^p dx\right)^{1/p}$$

$$= q^{-1/q}\sqrt{\frac{2}{\pi^3}}||f||_{L_p((1,\infty);dx)} \int_0^\infty e^{-y} K_0(y) y^{-(q^{-1}+1/2)} dy < \infty,\ q = \frac{p}{p-1}.$$

Theorem 4 is proved.

4 Convolution integral equations

This section will be devoted to a class of integral equations of the first kind related to the convolution operator (3.1). Namely, we will examine a solvability of the following integral equations

$$\int_1^\infty K(x,y)f(y)dy = g(x),\ x > 0, \tag{4.1}$$

$$\int_1^\infty \left[\lambda e^{-xy} + K(x,y)\right] f(y)dy = g(x),\ \lambda \in \mathbb{C},\ x > 0, \tag{4.2}$$

where the kernel $K(x,y)$ is defined by the integral

$$K(x,y) \equiv K_h(x,y) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_1^\infty e^{-xyu} I_0\left(x\sqrt{(y^2-1)(u^2-1)}\right) h(u)du, \tag{4.3}$$

h, g are given functions and f is to be determined.

Definition 2. Let $1 < p < 2$. We call by

$$\mathcal{M}F_{p,2} \equiv \{\psi(\tau) \in L_2\left(\mathbb{R}_+; \tau \tanh \pi\tau d\tau\right);\ \psi(\tau) = MF[f](\tau),\ f \in L_2((1,\infty); dx) \cap L_p((1,\infty); dx)\}$$

a class of images of $f \in L_2((1,\infty); dx) \cap L_p((1,\infty); dx)$ under the Mehler-Fock transform (1.20), considering a restriction of this map to

$$MF : L_2((1,\infty); dx) \cap L_p((1,\infty); dx) \to \mathcal{M}F_{p,2}.$$

We note that $\mathcal{M}F_{p,2}$ is a subspace of $L_2(\mathbb{R}_+;\tau\tanh\pi\tau d\tau)$ and by virtue of Lemma 1 we have $\mathcal{M}F_{p,2}\subset C_0(\mathbb{R}_+)$.

Theorem 5. *Let* $1<p<2,\ g\in L_2(\mathbb{R}_+;dx)$ *and* $h(x)\in L_p((1,\infty);dx)$. *Then for the solvability of the equation* (4.1) *in* $L_2((1,\infty);dx)\ \cap L_p((1,\infty);dx)$ *it is necessary and sufficient that* $\frac{\cosh\pi\tau K_{i\tau}[g]}{MF[h](\tau)}\in\mathcal{M}F_{p,2}$. *Moreover, the corresponding solution* $f(x)$ *is unique and given by the formula*

$$f(x)=\int\limits_0^\infty\tau\sinh\pi\tau\ P_{-1/2+i\tau}(x)\frac{K_{i\tau}[g]}{MF[h](\tau)}d\tau,\ x>1, \tag{4.4}$$

where the convergence is with respect to the norm in $L_2((1,\infty);dx)$.

Proof. *Necessity.* Indeed, if under conditions of the theorem equations (4.1) is satisfied, then convolution (3.1) exists and by (3.11)

$$K_{i\tau}[g]=\frac{1}{\cosh\pi\tau}MF[f](\tau)MF[h](\tau).$$

However, $MF[f]\in\mathcal{M}F_{p,2}$. Hence $\frac{\cosh\pi\tau K_{i\tau}[g]}{MF[h](\tau)}\in\mathcal{M}F_{p,2}$ and the corresponding solution in $L_2(\mathbb{R}_+;dx)$ is given by (4.4) via inversion formula (1.21) for the Mehler-Fock transform (1.20).

Sufficiency. Now assuming $\frac{\cosh\pi\tau K_{i\tau}[g]}{MF[h](\tau)}\in\mathcal{M}F_{p,2}$ we get correspondingly via (4.4) and Definition 2 that $f(x)\in L_2((1,\infty);dx)\ \cap L_p((1,\infty);dx)$. Further, owing to conditions of the theorem the left-hand side of (4.1) is the convolution like (3.1) $(f*h)(x)$, which belongs to $L_2(\mathbb{R}_+;dx)$. Therefore due to the factorization identity (3.1) we obtain

$$K_{i\tau}[f*h]=\frac{1}{\cosh\pi\tau}MF[f](\tau)MF[h](\tau). \tag{4.5}$$

But (see (1.21) and (4.4)) $MF[f]=\frac{\cosh\pi\tau K_{i\tau}[g]}{MF[h](\tau)}$. Substituting this expression into (4.5) we find

$$K_{i\tau}[f*h]=\frac{1}{\cosh\pi\tau}MF[h](\tau)\frac{\cosh\pi\tau K_{i\tau}[g]}{MF[h](\tau)}=K_{i\tau}[g].$$

So by the uniqueness property for the Kontorovich-Lebedev transform equation (4.1) is satisfied, and (4.4) its unique solution from the class $L_2((1,\infty);dx)\ \cap L_p((1,\infty);dx)$. Theorem 5 is proved.

Equation (4.2) can be treated similarly by using composition representation (3.14). Indeed, under conditions of Theorem 5 after applying the Kontorovich-Lebedev transform to both sides of (4.2) and taking into account factorization identity (4.5) we get the following algebraic equality

$$\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}\,MF[f](\tau)+MF[f](\tau)MF[h](\tau)=\cosh\pi\tau K_{i\tau}[g], \tag{4.6}$$

which can be solved with respect to $MF[f](\tau)$ if

$$\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\neq 0,\ \tau\in\mathbb{R}_+.$$

Hence

$$MF[f](\tau)=\cosh\pi\tau K_{i\tau}[g]\left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\right]^{-1} \tag{4.7}$$

and we come out with the following result.

Theorem 6. *Under conditions of Theorem* 5 *for the solvability of equation* (4.2) *in* $L_2((1,\infty);dx)\ \cap$ $L_p((1,\infty);dx),\ 1<p<2$ *it is necessary and sufficient that the right-hand side of* (4.7) *belongs to* $\mathcal{M}F_{p,2}$. *Then the corresponding solution* $f(x)$ *is unique and given by the formula*

$$f(x)=\int\limits_0^\infty \tau\sinh\pi\tau\ P_{-1/2+i\tau}(x)K_{i\tau}[g]\left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\right]^{-1}d\tau,\ x>1, \tag{4.8}$$

where the convergence is with respect to the norm in $L_2((1,\infty);dx)$.

Let us indicate a special case of the equation (4.2) when its solution (4.8) can be represented in the resolvent form. Suppose that $g(x)$ is the modified Laplace transform (3.13) of some function $\varphi(t)\in L_2((1,\infty);dt)\cap L_p((1,\infty);dt),\ 1<p<2$, i.e.

$$g(x)=\int_1^\infty e^{-xt}\varphi(t)dt. \tag{4.9}$$

A class of such functions g belongs to $L_2(\mathbb{R}_+;dx)$. In fact, by virtue of the generalized Minkowski and Hölder inequalities we have the estimate

$$||g||_{L_2(\mathbb{R}_+;dx)}=\left(\int_0^\infty|g(x)|^2dx\right)^{1/2}\le\int_1^\infty\left(\int_0^\infty e^{-2xt}dx\right)^{1/2}|\varphi(t)|dt$$
$$=\frac{1}{\sqrt{2}}\int_1^\infty|\varphi(t)|\frac{dt}{\sqrt{t}}\le\frac{1}{\sqrt{2}}||\varphi||_{L_p((1,\infty);dt)}\left(\int_1^\infty\frac{dt}{t^{q/2}}\right)^{1/q}$$
$$=\frac{2^{q^{-1}-2^{-1}}}{(q-2)^{1/q}}||\varphi||_{L_p((1,\infty);dt)}<\infty,\ q=\frac{p}{p-1}.$$

Therefore by composition representation (3.14) and inversion formula (1.21) for the Mehler-Fock transform solution (4.8) becomes in the form

$$f(x)=\int\limits_0^\infty\tau\sinh\pi\tau\ P_{-1/2+i\tau}(x)K_{i\tau}[g]\left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\right]^{-1}d\tau$$
$$=\frac{\pi\sqrt{\pi}}{\sqrt{2}}\int\limits_0^\infty\tau\tanh\pi\tau\ P_{-1/2+i\tau}(x)MF[\varphi]\left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\right]^{-1}d\tau$$
$$\frac{\pi^2}{2\lambda}\int\limits_0^\infty\tau\tanh\pi\tau\ P_{-1/2+i\tau}(x)MF[\varphi]d\tau$$
$$-\frac{\pi^2}{2\lambda}\int\limits_0^\infty\tau\tanh\pi\tau\ P_{-1/2+i\tau}(x)MF[\varphi]MF[h](\tau)\left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\right]^{-1}d\tau$$
$$=\frac{\pi^2}{2\lambda}\left[\varphi(x)-\int\limits_0^\infty\tau\tanh\pi\tau P_{-1/2+i\tau}(x)MF[\varphi]MF[h](\tau)\right.$$
$$\left.\times\left[\frac{\lambda}{\pi}\sqrt{\frac{2}{\pi}}+MF[h](\tau)\right]^{-1}d\tau\right],\ \lambda\neq 0. \tag{4.10}$$

Finally, let us consider an example of the equation (4.2), letting $MF[h](\tau) = [\cosh \pi\tau]^{-1}$. In order to find an original we use the inversion formula (1.21) and integral representations (1.14), (2.19). So we have, correspondingly, changing the order of integration and calculating elementary integrals

$$h(u) = \int_0^\infty \tau \frac{\tanh \pi\tau}{\cosh \pi\tau} P_{-1/2+i\tau}(u) d\tau = \sqrt{\frac{2}{\pi^3}} \int_0^\infty \tau \tanh \pi\tau \int_0^\infty e^{-yu} K_{i\tau}(y) \frac{dy}{\sqrt{y}} d\tau$$

$$= \frac{1}{\pi} \int_0^\infty e^{-y(u+1)} dy = \frac{1}{\pi(u+1)}.$$

Equation (4.2) in this case can be written in the form (see (4.9))

$$\int_1^\infty e^{-xy}(\lambda f(y) - \varphi(y))dy + \int_1^\infty K_h(x,y) f(y) dy = 0, \tag{4.11}$$

where the kernel $K_h(x,y)$ can be calculated explicitly by using relation (2.16.51.10) in [7]. Precisely we obtain

$$K_h(x,y) = \frac{2\sqrt{2}}{\pi^3 \sqrt{\pi x}} \int_0^\infty \tau \tanh(\pi\tau) K_{i\tau}(x) \int_0^\infty e^{-ty} K_{i\tau}(t) \frac{dt}{\sqrt{t}} d\tau$$

$$= \sqrt{\frac{2}{\pi^5}} e^{xy} \int_{x(y+1)}^\infty e^{-u} \frac{du}{u} = -\sqrt{\frac{2}{\pi^5}} e^{xy} Ei(-x(y+1)),$$

where $Ei(z)$ is the integral exponential function [7]. Now taking $\lambda = \pi\sqrt{\pi/2}$ in (4.11), consider the integral equation

$$\int_1^\infty e^{-xy} \left(\pi \sqrt{\frac{\pi}{2}} f(y) - \varphi(y) \right) dy - \sqrt{\frac{2}{\pi^5}} \int_1^\infty e^{xy} Ei(-x(y+1)) f(y) dy = 0, \; x > 0. \tag{4.12}$$

According to (4.10) under the corresponding conditions its unique solution is given by the formula

$$f(x) = \sqrt{\frac{\pi}{2}} \left[\varphi(x) - \frac{1}{2} \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2(\pi\tau/2)} P_{-1/2+i\tau}(x) MF[\varphi](\tau) d\tau \right]. \tag{4.13}$$

Meanwhile, substituting the value of $MF[\varphi]$ by formula (3.5) into (4.13) and changing the order of integration, we will have the inner integral of the form

$$\int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2(\pi\tau/2)} P_{-1/2+i\tau}(x) P_{-1/2+i\tau}(y) d\tau.$$

Its value is still unknown. However, one can represent it in another form employing integral (1.1) and relation (2.16.33.2) in [7]. Indeed, by straightforward calculations this gives the result

$$\int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2(\pi\tau/2)} P_{-1/2+i\tau}(x) P_{-1/2+i\tau}(y) d\tau = \left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty e^{-txy} I_0\left(t\sqrt{(x^2-1)(y^2-1)}\right) K_0(t) \sqrt{t} dt.$$

Hence taking into account (4.3) the unique solution (4.13) of the integral equation (4.12) can be written in the equivalent form

$$f(x) = \sqrt{\frac{\pi}{2}} \left[\varphi(x) - \int_0^\infty K_\varphi(x,t) K_0(t) \sqrt{t} dt \right], \; x > 1,$$

where

$$K_\varphi(x,t) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^\infty e^{-txy} I_0\left(t\sqrt{(x^2-1)(y^2-1)}\right) \varphi(y)dy.$$

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