ARE THERE CHAOTIC MAPS IN THE SPHERE?

MÁRIO BESSA

Abstract. Two of the most popular notions of chaoticity are the one due to Robert Devaney and the one that assumes positive Lyapunov exponents. In this note we discuss the coexistence of both definitions for conservative discrete dynamical sistems in the two-sphere and with respect to the C^1-generic point of view.

1. Introduction and basic definitions

As far as we know the first time the word *chaos* appeared in the mathematical literature was in Li and York mid-seventies' paper [7]. The notion has been popularized and now we have available lots of definitions of chaoticity in various different contexts. For a survey on the subject see [8].

We recall two well-known definition of chaos; Devaney's definition [5] and the definition that excludes zero Lyapunov exponents. In this remark we study the coexistence of both definitions in one of the most common two-dimensional manifold. Actually, we will conclude that, in the two-sphere, typically area-preserving diffeomorphisms do not satisfy both aforementioned definitions simultaneously.

Let M^d, where $d \geq 2$, be a closed Riemannian d-dimensional manifold. We center our attention in the particular cases when $M^2 = \mathbb{S}^2$ (the sphere) and when $M^2 = \mathbb{T}^2$ (the torus). To define $\mathbb{T}^2$ we take the quotient of $\mathbb{R}^2$ under the identification $(x, y) \backsim (x', y')$ if $x - x'$ and $y - y'$ are integers. We denote this identification by $(x, y) = (x', y')(\text{mod} 1)$.

Given a volume form ω on M^d, let μ be the probability measure associated to ω, which we call Lebesgue measure. Let $\text{Diff}^1_\mu(M^d)$ denotes the class of C^1 diffeomorphisms $f\colon M^d \to M^d$ that preserves the Lebesgue measure, that is, if you pick any measurable set $\mathcal{A} \subseteq M^d$, then, $\mu(f(\mathcal{A})) = \mu(\mathcal{A})$. Another way to check the Lebesgue preserving property is by computing the determinant of the derivative at all points and see if its modulus is one.

If one wants to estimate distances between two diffeomorphisms we use the C^1-metric. In broad terms, two diffeomorphisms f and g are C^1-close if they are uniformly close as well as their first derivatives

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computed in any $x \in M^d$. To derive the distance between linear maps like $Df(x)$ and $Dg(x)$ we use the uniform norm $\|\cdot\|$ of linear operators.

Given $f\colon M^d \to M^d$, we denote $f^n(x) = f \circ f \circ f ... \circ f(x)$ by composing f n-times. We say that a point $x \in M^d$ is periodic of period n, if $f^n(x) = x$ and $f^i(x) \neq x$ for every $i = \{1, 2, ..., n-1\}$. The forward orbit of x is defined by $\cup_{n \in \mathbb{N}} f^n(x)$ and we say that f has a dense orbit if, for some $x \in M^d$, the manifold M^d is the closure of the forward orbit of x. In this case we say that f is *transitive*.

We recall one of the most celebrated definitions of chaos due to Devaney (see [5, Definition 8.5]): $f\colon M^d \to M^d$ is *chaotic* if:

(a) f is transitive;
(b) the periodic points are dense in M^d and
(c) f is *sensitive to the initial conditions*, i.e., there exists $\delta > 0$ such that for all $x \in M^d$ and all neighborhood of x, V_x, there exists $y \in V_x$ and an integer n where $d(f^n(y), f^n(x)) > \delta$, where $d(\cdot, \cdot)$ is the distance inherit from the Riemannian structure.

In this case we also say that f is *chaotic in the topological sense.*

Example 1: (Arnold's cat map) The map $\alpha\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\alpha(x, y) = (2x+y, x+y)(\mod 1)$ is an area-preserving (two-dimensional case) diffeomorphim on the torus which is chaotic in the sense of Devaney.

It was proved in [2] that (a) and (b) implies (c), and so in order to be chaotic in the sense of Devaney the system only has to satisfy the transitivity property and the density of periodic points.

The other definition of chaotic map that we are going to use is the one that says that there are no zero Lyapunov exponents. The point $x \in M^d$ has a zero Lyapunov exponent if there exists $v \in T_x M^d$ (the tangent space at x) such that

$$\frac{1}{n} \log \|Df^n(x) \cdot v\| \to 0 \text{ as } n \to \pm\infty. \tag{1}$$

In the area-preserving setting if f has a zero Lyapunov exponent, then for all $v \in T_x M^2$ (1) holds.

At this time you ask yourself if this limit always exists and, moreover, if it is the same when we iterate forward and backward? Well, Oseledets' theorem (see [9]) assure the existence of this limits at least for almost every point in M^d with respect to any f-invariant measure. For a proof of the Oseledets theorem on surfaces see [11]. When, in our area-preserving setting, we have two non-zero (thus symmetric) Lyapunov exponents we say that f is *chaotic in the measurable sense.* The Arnold cat map is also chaotic in this sense.

2. Lots of zero Lyapunov exponents in $\mathbb{S}^2$

We say that a diffeomorphism is *stable* if any other sufficiently close, for the C^1-metric, has equivalent behavior, i.e., it is possible to find a change of coordinates conjugating the two dynamics (for more details see [10]). The notion of hyperbolicity goes a long way if one wants to establish the stability of dynamical systems. We say that $f \in \mathrm{Diff}^1_\mu(M^d)$ is *hyperbolic* (or *Anosov*, see [14]) if, at each point $x \in M^d$, there exists a splitting $T_xM^d = E^s_x \oplus E^u_x$ such that both E^s_x and E^u_x are invariant by the derivative and we have

$$\|Df(x) \cdot s\| < 1/2 \text{ and } \|Df^{-1}(x) \cdot u\| < 1/2,$$

where $s \in E^s_x$ and $u \in E^u_x$ are unitary vectors.

The definition of hyperbolic diffeomorphisms for the dissipative case is analog. Notice that Arnold's cat map is hyperbolic.

This definition is very rigid and imposes certain topological constraints on M^d. Actually, in the late sixties, John Franks proved the following result (see [6]).

Theorem 2.1. *The only surfaces that support hyperbolic diffeomorphisms are the tori.*

It is well-known that the set of area-preserving diffeomorphism in M^d endowed with the C^1-metric is complete (see [10] and the references therein). Moreover, a subset of it is called a *residual subset* if it contains a countable intersection of open and dense subsets in the C^1-metric. A classical result from general topology asserts that in a complete space any residual subset is also dense (see e.g. [12, Theorem 7.2]). We say that a property is *generic* if it holds for a residual subset, that is, is typical from the topological viewpoint. Notice that an intersection of two residual subsets is also residual.

Recently, Jairo Bochi proved in [3], the following result.

Theorem 2.2. *There exists a residual subset $\mathcal{R}$ of the set of area-preserving C^1 diffeomorphisms on M^2 such that if $f \in \mathcal{R}$, then f is* ***hyperbolic*** *or else f has zero Lyapunov exponents at almost every point in M^2.*

Therefore, previous theorem together with Theorem 2.1 yields the following corollary.

Corollary 2.3. *There exists a residual subset $\mathcal{R}$ of the set of area-preserving C^1 diffeomorphisms on $\mathbb{S}^2$ such that if $f \in \mathcal{R}$, then f has zero Lyapunov exponents at almost every point in $\mathbb{S}^2$.*

We conclude that, from the topological point of view, typical area-preserving diffeomorphisms in the two-sphere satisfies the following: if you pick randomly a point in the sphere, then, with probability one, its Lyapunov exponent is zero.

3. Lots of chaos in the sense of Devaney

One of the most important outcome of the remarkable Bonatti and Crovisier [4] recent work on recurrence is the following result.

Theorem 3.1. *There exists a residual subset* $\mathcal{R}_1$ *of* $\text{Diff}^1_\mu(M^d)$ *such that if* $f \in \mathcal{R}_1$*, then* f *is transitive.*

Moreover, by Pugh-Robinson's general density theorem [13] we get:

Theorem 3.2. *There exists a residual subset* $\mathcal{R}_2$ *of* $\text{Diff}^1_\mu(M^d)$ *such that if* $f \in \mathcal{R}_2$*, then the periodic points of* f *are dense in* M^d*.*

Therefore, defining $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ and recalling [2] we conclude that:

Corollary 3.3. *There exists a residual subset* $\mathcal{R}$ *of* $\text{Diff}^1_\mu(M^d)$ *such that if* $f \in \mathcal{R}$*, then* f *is chaotic in the sense of Devaney.*

4. Conclusion

It is also interesting to recall the definition of chaos in the sense of Auslander-Yorke [1]: the diffeomorphism must be transitive and sensitive to the initial conditions. We easily obtain the following result.

Theorem 1. *There exists a residual subset* $\mathcal{R}$ *of* $\text{Diff}^1_\mu(M^d)$ *where the definitions of chaotic in the sense of Auslander-Yorke and in the sense of Devaney coincide.*

In previous sections we saw that the Arnold cat map is both chaotic in the sense of Devaney and also has non-zero Lyapunov exponents. Nevertheless, as we will see in Theorem 2, in the two-sphere the two definitions are not equivalent, in fact, there are examples of diffeomorphisms satisfying only the firstmentioned definition of chaoticity and not the second. Moreover, these examples are generic.

Our main conclusion is that examples which satisfies both definitions are, at most, a countable union of nowhere dense sets. Actually, intersecting the residuals of both Corollaries 2.3 and 3.3 we obtain:

Theorem 2. *There exists a residual subset* $\mathcal{R}$ *of the set of area-preserving* C^1 *diffeomorphisms on* $\mathbb{S}^2$ *such that if* $f \in \mathcal{R}$*, then* f *is chaotic in the sense of Devaney and* f *has zero Lyapunov exponents at almost every point in* $\mathbb{S}^2$*.*

Going back to the title of this note and considering the C^1-generic point of view, the answer is *yes* in the topological sense and *no* in the measurable sense.

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Mário Bessa (bessa@fc.up.pt) FCUP, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal *and* ESTGOH - IPC, Rua General Santos Costa, 3400-124, Oliveira do Hospital, Portugal.