Asymptotic and summation formulas related to the Lebedev integrals

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March 30, 2007

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Abstract

We establish asymptotic and summation properties of the Lebedev integrals with respect to an index of the modified Bessel function, which are related to the known Kontorovich-Lebedev transformation. Analogs of the Watson lemma and Poisson summation formulas are proved. As applications certain type series involving Euler's gamma-functions and hyperbolic functions are evaluated.

Keywords: Kontorovich-Lebedev transform, Fourier integrals, Watson lemma, Borel function, Poisson summation formula, modified Bessel functions

2000 Mathematics Subject Classification. Primary: 44A15, 41A60, 44A35, 33C10, 30D20

1 Introduction

Let $z \in \mathbb{C}$ be a complex number and let f be a complex -valued measurable function on $\mathbb{R}_+ = (0, \infty)$. We deal with the following Lebedev integral

$$F(z) \equiv K_z[f] = \int_0^\infty K_z(x) f(x) dx, \tag{1.1}$$

*Work supported by *Fundação para a Ciência e a Tecnologia* (FCT, the project "POCI 2010") through the *Centro de Matemática da Universidade do Porto* (CMUP). Available as a PDF file from http://www.fc.up.pt/cmup.

which is a modification of the Kontorovich-Lebedev transformation (cf. [3], [4], [6], [7]) on general complex index of the modified Bessel function $K_z(x)$ [2], Vol. 2. As it is known, the function $K_z(x)$ satisfies the differential equation

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}-(x^2+z^2)u=0, \tag{1.2}$$

for which it is the solution that remains bounded as x tends to infinity on the real line. The modified Bessel function has the asymptotic behavior [2], Vol. 2

$$K_z(x)=\left(\frac{\pi}{2x}\right)^{1/2}e^{-x}[1+O(1/x)], \qquad x\to\infty, \tag{1.3}$$

and near the origin

$$K_z(x)=O\left(x^{-|\mathrm{Re}z|}\right),\ x\to 0,\ z\neq 0, \tag{1.4}$$

$$K_0(x)=-\log x+O(1),\ x\to 0. \tag{1.5}$$

It can be defined by the following integral representation [6], [7]

$$K_z(x)=\int_0^\infty e^{-x\cosh u}\cosh zu\ du,\ x>0. \tag{1.6}$$

Hence we easily find that $K_z(x)$ is even with respect to z and a real-valued positive function when $z\in\mathbb{R}$. Moreover, it satisfies the following inequality

$$|K_z(x)|\leq K_{\mathrm{Re}z}(x),\ x>0. \tag{1.7}$$

When z is pure imaginary then (1.1) coincides with the Kontorovich-Lebedev transformation introduced by Lebedev in [4], which is different from its original form (cf. [3]).

As it is known, the product of the modified Bessel functions of different arguments can be represented by the Macdonald formula [5], relation (2.16.9.1)

$$K_\nu(x)K_\nu(y)=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\ \left(u\frac{x^2+y^2}{xy}+\frac{xy}{u}\right)}K_\nu(u)\frac{du}{u}. \tag{1.8}$$

This formula generates the following convolution operator

$$(f*g)(x)=\frac{1}{2x}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\ \left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)}f(u)g(y)dudy,\ x>0, \tag{1.9}$$

satisfying under some conditions (see [7], Ch. 4) the factorization property in terms of the Lebedev integrals (1.1). Precisely, we get

$$K_z[f*g]=K_z[f]K_z[g]. \tag{1.10}$$

We will deal in the sequel with a particular case of the Lebedev integral (1.1) when f is an entire function of the exponential type. We will study its analytic properties and prove the Watson lemma, which will give an asymptotic behavior of the Lebedev integral when $z \to \infty$. Then we will establish analogs of the Poisson formulas for the Lebedev integral and its powers, which are known for Fourier integrals (cf. [1]). As applications we will exhibit a few series involving Euler's gamma-functions, which can be evaluated by using these formulas.

2 Watson's lemma

In this section we will study analytic properties of the Lebedev integral $F(z)$ (1.1) when $f(x)$ admits the series representation $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ as an entire function of the exponential type with $\limsup_{n\to\infty} |a_n|^{1/n} = \sigma$, where σ is a type of this function. Then we will prove the Watson lemma for these integrals, which gives the asymptotic behavior of $F(z)$ when $z \to \infty$. Similar questions for pure imaginary z have been studied in [7], Section 2.5.

We start with the following

Lemma 1. *Let f be an entire function of the exponential type with $\sigma < 1$. Then $F(z)$ is analytic in the strip $|\mathrm{Re}\ z| < 1$ and for any $\varepsilon \in (0, 1-\sigma)$ admits there the uniform with respect to* Im z *estimate*

$$|F(z)| < C_\varepsilon \frac{\sin\left(\mathrm{Re} z \arccos(\sigma+\varepsilon)\right)}{\sin\left(\pi \mathrm{Re} z\right)\sqrt{1-(\sigma+\varepsilon)^2}}, \tag{2.1}$$

where $C_\varepsilon > 0$ is a constant depending on ε.

Proof. Indeed, since for any $\varepsilon > 0$

$$|a_n| < C_\varepsilon (\sigma+\varepsilon)^n, \ n = 0, 1, \ldots,$$

we easily have that $|f(x)| < C_\varepsilon e^{(\sigma+\varepsilon)x}, \ x > 0$. Consequently, choosing $\varepsilon \in (0, 1-\sigma)$ we appeal to integral representation (1.6), the estimate (1.7) and evaluating elementary integrals we derive

$$|F(z)| \le \int_0^\infty K_{\mathrm{Re} z}(x)|f(x)|dx < C_\varepsilon \int_0^\infty K_{\mathrm{Re} z}(x) e^{(\sigma+\varepsilon)x} dx$$

$$= C_\varepsilon \int_0^\infty e^{(\sigma+\varepsilon)x} \int_0^\infty e^{-x\cosh u} \cosh \mathrm{Re} z u \ du dx = C_\varepsilon \int_0^\infty \frac{\cosh \mathrm{Re} z u}{\cosh u - \sigma - \varepsilon} du$$

$$= \pi C_\varepsilon \frac{\sin\left(\mathrm{Re} z \arccos(\sigma+\varepsilon)\right)}{\sin\left(\pi \mathrm{Re} z\right)\sqrt{1-(\sigma+\varepsilon)^2}},$$

where the change of the order of integration is due to Fubini's theorem. Thus we arrive at the estimate (2.1). Moreover, it shows via asymptotic formulas (1.3), (1.4), (1.5) for the modified Bessel function that integral (1.1) converges absolutely in the strip $|\mathrm{Re}\ z| < 1$ and uniformly with respect to $\mathrm{Im}\ z$ in each interior substrip of $|\mathrm{Re}\ z| < 1$. Therefore $F(z)$ is analytic in the vertical strip $|\mathrm{Re}\ z| < 1$. Lemma 1 is proved.

This lemma allows us to get another representation of $F(z)$ in the strip $|\mathrm{Re}\ z| < 1$ in terms of the series. Indeed, substituting the series for f into (1.1) we change the order of summation and integration owing to the convergence of the series (see Lemma 1)

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n!} \int_0^{\infty} |K_z(x)| x^n dx.$$

Hence we calculate the inner integral appealing, for instance, to relation (2.140) in [7]. Thus we derive the following equalities

$$F(z) = \int_0^{\infty} K_z(x) \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} K_z(x) x^n dx$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n!} 2^{n-1} \Gamma\left(\frac{n+1+z}{2}\right) \Gamma\left(\frac{n+1-z}{2}\right)$$

$$= \sum_{n=0}^{\infty} 2^{n-1} a_n B\left(\frac{n+1+z}{2}, \frac{n+1-z}{2}\right)$$

$$= \sum_{n=0}^{N} 2^{n-1} a_n B\left(\frac{n+1+z}{2}, \frac{n+1-z}{2}\right) + R_N(z) = S_N(z) + R_N(z), \tag{2.2}$$

where

$$R_N(z) = \sum_{n=N+1}^{\infty} 2^{n-1} a_n B\left(\frac{n+1+z}{2}, \frac{n+1-z}{2}\right)$$

and $\Gamma(w), B(a, b)$ are the Euler gamma- and beta -functions, respectively (cf. [2], Vol. I). With the elementary inequality for beta-functions $|B(a, b)| \le B(\mathrm{Re}\ a, \mathrm{Re}\ b)$, the duplication formula for the gamma-function and via Lemma 1 we have the estimate

$$|R_N(z)| \le \sum_{n=N+1}^{\infty} 2^{n-1} |a_n| B\left(\frac{n+1+\mathrm{Re}\ z}{2}, \frac{n+1-\mathrm{Re}\ z}{2}\right)$$

$$\le C_\varepsilon \sum_{n=N+1}^{\infty} 2^{n-1} (\sigma+\varepsilon)^n B\left(\frac{n+1+\mathrm{Re}\ z}{2}, \frac{n+1-\mathrm{Re}\ z}{2}\right)$$

$$= \frac{\sqrt{\pi} C_\varepsilon}{2} \sum_{n=N+1}^{\infty} (\sigma+\varepsilon)^n \frac{\Gamma\left(\frac{n+1+\mathrm{Re}z}{2}\right)\Gamma\left(\frac{n+1-\mathrm{Re}z}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}, \ \sigma+\varepsilon<1.$$

Hence applying the asymptotic formula for the ratio of gamma- functions (see [2], Vol. I, relation (1.18.4))

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O(z^{-2})\right], \ z\to\infty, \tag{2.3}$$

we get that

$$\frac{\Gamma\left(\frac{n+1+\mathrm{Re}z}{2}\right)\Gamma\left(\frac{n+1-\mathrm{Re}z}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}+1\right)} = O(n^{-1/2}), \ n\to\infty.$$

Therefore the remainder $R_N(z)$ can be estimated uniformly with respect to z in each interior substrip of $|\mathrm{Re}\ z| < 1$ as

$$|R_N(z)| \le D_\varepsilon \sum_{n=N+1}^{\infty} \frac{(\sigma+\varepsilon)^n}{\sqrt{n}} = O\left(\frac{1}{\sqrt{N}}\right), \ N\to\infty.$$

Appealing again to (2.3) we have for any $n = 0,\dots,N$ and $z\to\infty$ the following asymptotic expansions

$$\frac{\Gamma\left(\frac{n+1+z}{2}\right)}{\Gamma\left(\frac{1+z}{2}\right)} = \left(\frac{z}{2}\right)^{n/2}\left[1+\frac{n^2}{4z}+O(z^{-2})\right], \ z\to\infty, \tag{2.4}$$

$$\frac{\Gamma\left(\frac{n+1-z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)} = \left(-\frac{z}{2}\right)^{n/2}\left[1-\frac{n^2}{4z}+O(z^{-2})\right], \ z\to\infty. \tag{2.5}$$

Hence taking into account the elementary identity $\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(\frac{1-z}{2}\right) = \frac{\pi}{\cos(\pi z/2)}$ we return to (2.2) to estimate $S_N(z)$. Invoking (2.4), (2.5) we obtain for each $N\in\mathbb{N}$ and $z\to\infty$, $|\mathrm{Re}\ z|<1$ the representation

$$S_N(z) = \frac{\pi}{2\cos(\pi z/2)} \sum_{n=0}^{N} \frac{a_n}{n!}(iz)^n \left[1+\frac{n^2}{4z}+O(z^{-2})\right]\left[1-\frac{n^2}{4z}+O(z^{-2})\right]$$

$$= \frac{\pi}{2\cos(\pi z/2)}\left[\sum_{n=0}^{N}\frac{a_n}{n!}(iz)^n - \frac{1}{16z^2}\sum_{n=0}^{N}\frac{n^4 a_n}{n!}(iz)^n + O(z^{-2})\right]$$

$$= \frac{\pi}{2\cos(\pi z/2)}\left[\sum_{n=0}^{N}\frac{a_n}{n!}(iz)^n + O(z^{-2})\right].$$

Taking now $N \to \infty$ we derive the following asymptotic expansion of the Lebedev integral (1.1) in the strip $|\text{Re } z| < 1$

$$F(z) = \frac{\pi}{2\cos(\pi z/2)} f(iz)(1 + O\left(z^{-2}\right)), \ z \to \infty. \tag{2.6}$$

Thus we have proved an analog of the Watson lemma for Lebedev's integrals.

Lemma 2. *Let f be an entire function of the exponential type with $\sigma < 1$. Then $F(z)$ admits asymptotic expansion* (2.6) *in the strip* $|\text{Re } z| < 1$ *when $z \to \infty$, i.e.*

$$F(z) \sim \frac{\pi}{2\cos(\pi z/2)} f(iz).$$

3 Summation formulas

We begin to take the known Poisson formula for the cosine Fourier transform [1]

$$\sqrt{\beta}\left[\frac{1}{2}F_c(0) + \sum_{n=1}^{\infty} F_c(n\beta)\right] = \sqrt{\alpha}\left[\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n\alpha)\right], \tag{3.1}$$

where $\alpha\beta = 2\pi, \alpha > 0$ and

$$F_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos xt dt. \tag{3.2}$$

Hence putting in (1.6) $z = i\tau, \ \tau \in \mathbb{R}$ we apply (3.1) to the Lebedev integral and we arrive at the identity

$$K_0(x) + 2\sum_{n=1}^{\infty} K_{in\beta}(x) = \alpha\left[\frac{e^{-x}}{2} + \sum_{n=1}^{\infty} e^{-x\cosh n\alpha}\right], \ x > 0, \ \alpha\beta = 2\pi, \ \alpha > 0. \tag{3.3}$$

This is a key identity we will use to prove an analog of the Poisson summation for the Lebedev integrals (1.1). Indeed, we have

Theorem 1. *Let $f \in L_1(\mathbb{R}_+; K_\mu(\xi x)dx)$, where $\mu, \ \xi \in \mathbb{R}, \ |\mu| > 1/2, \ 0 < \xi < 1$. Then the Poisson type formula is true*

$$K_0[f] + 2\sum_{n=1}^{\infty} K_{in\beta}[f] = \alpha\left[\frac{1}{\pi}\int_0^{\infty} K_{i\tau}[f]d\tau + \sum_{n=1}^{\infty}(Lf)(\cosh n\alpha)\right], \ \alpha\beta = 2\pi, \ \alpha > 0. \tag{3.4}$$

where $(Lf)(x)$ is the Laplace integral

$$(Lf)(x) = \int_0^{\infty} e^{-xt} f(t) dt. \tag{3.5}$$

Proof. The condition $f \in L_1(\mathbb{R}_+; K_\mu(\xi x)dx)$ means that the following integral is finite,

$$||f||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)} = \int_0^\infty K_\mu(\xi t)|f(t)|dt < \infty.$$

Hence we multiply (3.3) on f and integrate with respect to x changing the order of integration and summation in the series. Then invoking (3.5) and the value of the integral (see (1.6), (3.2))

$$e^{-x} = \frac{2}{\pi}\int_0^\infty K_{i\tau}(x)d\tau,\ x > 0,$$

we get (3.4). The change of the order of integration and summation can be motivated by the absolute convergence of the iterated integral in (3.4) and the convergence of the following two series

$$\sum_{n=1}^\infty \int_0^\infty |K_{in\beta}(t)|\ |f(t)|dt,$$

$$\sum_{n=1}^\infty \int_0^\infty e^{-t\cosh n\alpha}|f(t)|dt$$

under conditions of the theorem. To do this we appeal to the inequality (cf. [7, relation (1.100)])

$$|K_{i\tau}(x)| \le e^{-\delta|\tau|}K_0(x\cos\delta),\ \delta \in (0, \pi/2) \tag{3.6}$$

and we choose δ, such that $\cos\delta > \xi$. Hence for the first series we obtain

$$\sum_{n=1}^\infty \int_0^\infty |K_{in\beta}(t)|\ |f(t)|dt \le C_{\mu,\xi,\delta}\int_0^\infty K_\mu(\xi t)\ |f(t)|dt \sum_{n=1}^\infty e^{-\delta n\beta} < \infty,$$

where $C_{\mu,\xi,\delta} > 0$ is a constant since via asymptotic formulas (1.3), (1.4), (1.5) the ratio $\frac{K_0(t\cos\delta)}{K_\mu(\xi t)}$ is bounded, i.e. $\frac{K_0(t\cos\delta)}{K_\mu(\xi t)} < C_{\mu,\xi,\delta}$. The second series can be treated by the estimate

$$\sum_{n=1}^\infty \int_0^\infty e^{-t\cosh n\alpha}|f(t)|dt = \sum_{n=1}^\infty \int_0^\infty e^{-t-2t\sinh^2(n\alpha/2)}|f(t)|dt$$

$$\le \sum_{n=1}^\infty \int_0^\infty \frac{e^{-t}}{1+2t\sinh^2(n\alpha/2)}|f(t)|dt \le \frac{1}{2\sqrt{2}}\sum_{n=1}^\infty \frac{1}{\sinh(n\alpha/2)}$$

$$\times \int_0^\infty \frac{e^{-t}}{\sqrt{t}K_\mu(\xi t)}|f(t)|K_\mu(\xi t)dt < C_{\mu,\xi}\sum_{n=1}^\infty \frac{1}{\sinh(n\alpha/2)}\int_0^\infty |f(t)|K_\mu(\xi t)dt < \infty,$$

where $C_{\mu,\xi} > 0$ is a constant. Finally we observe that the integral in (3.4) is absolutely convergent. Indeed, calling again (3.6) we have

$$\int_0^\infty |K_{i\tau}[f]| d\tau \le \int_0^\infty \int_0^\infty |f(t)||K_{i\tau}(t)| dt d\tau \le \int_0^\infty e^{-\delta\tau} d\tau$$

$$\times \int_0^\infty |f(t)| K_0(t\cos\delta) dt \le C_{\delta,\xi} \int_0^\infty |f(t)| K_\mu(\xi t) dt < \infty,$$

where $C_{\delta,\xi} > 0$ is a constant. Theorem 1 is proved.

Let us exhibit some interesting particular cases of the formula (3.4). Indeed, letting $f(x) \equiv 1$, $\beta = \frac{2\pi}{\alpha}$ we calculate the corresponding Lebedev integral by the relation (2.16.2.1) in [5] and we derive the identity

$$\sum_{n=1}^{\infty} \left[\frac{\pi}{\cosh\left(\frac{n\pi^2}{\alpha}\right)} - \frac{\alpha}{\cosh n\alpha} \right] = \frac{\alpha - \pi}{2}, \ \alpha > 0. \tag{3.7}$$

If $f(x) = x^{\gamma-1}$, $\gamma > 0$ then we appeal to the relation (2.16.2.2) in [5] to obtain the formula

$$2^{\gamma-1} \left[\Gamma^2\left(\frac{\gamma}{2}\right) + 2\sum_{n=1}^{\infty} \left| \Gamma\left(\frac{\gamma}{2} + \frac{\pi i n}{\alpha}\right) \right|^2 \right] = \alpha\Gamma(\gamma) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\cosh^\gamma n\alpha} \right], \ \alpha > 0. \tag{3.8}$$

The value $\gamma = 1$ leads again to (3.7). Let $f(x) = e^{-x}x^{\gamma-1}$, $\gamma > 0$. Then by using relation (2.16.6.4) in [5] the identity (3.4) becomes ($\alpha > 0$)

$$\frac{2^{-\gamma}\sqrt{\pi}}{\Gamma(\gamma + 1/2)} \left[\Gamma^2(\gamma) + 2\sum_{n=1}^{\infty} \left| \Gamma\left(\gamma + \frac{2\pi i n}{\alpha}\right) \right|^2 \right] = \alpha\Gamma(\gamma) \left[2^{-\gamma-1} + \sum_{n=1}^{\infty} \frac{1}{(1 + \cosh n\alpha)^\gamma} \right]. \tag{3.9}$$

Letting $\gamma = 1$ in (3.9) we invoke the reduction and supplement formulas for gamma-functions and we get the identity

$$1 + \frac{2\pi^2}{\alpha} \sum_{n=1}^{\infty} \frac{n}{\sinh\left(\frac{\pi^2 n}{\alpha}\right) \cosh\left(\frac{\pi^2 n}{\alpha}\right)} = \alpha \left[\frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{1 + \cosh n\alpha} \right]. \tag{3.10}$$

In particular, when $\alpha = \pi$ we have

$$\sum_{n=1}^{\infty} \left[\frac{2n}{\sinh \pi n \cosh \pi n} - \frac{1}{1 + \cosh \pi n} \right] = \frac{\pi - 4}{4\pi}.$$

As a consequence of (3.3) and differential equation (1.2) one can get another identity involving series of the modified Bessel functions with respect to an index. In fact, we take into account (see(1.2)) that

$$x^2 \frac{d^2 K_{in\beta}(x)}{dx^2} + x \frac{dK_{in\beta}(x)}{dx} - x^2 K_{in\beta}(x) = -n^2\beta^2 K_{in\beta}(x). \tag{3.11}$$

Hence we differentiate through (3.3) accordingly with respect to x under the series sign by virtue of the uniform convergence at least for $x \in [x_0, \infty),\ x_0 > 0$. The latter fact can be established by using the expressions for the derivatives of the modified Bessel functions (cf. [2, Vol. 2])

$$\frac{dK_{in\beta}(x)}{dx} = -\frac{1}{2}\left[K_{1-in\beta}(x) + K_{1+in\beta}(x)\right],$$

$$\frac{d^2K_{in\beta}(x)}{dx^2} = \frac{1}{4}\left[K_{2-in\beta}(x) + 2K_{in\beta}(x) + K_{2+in\beta}(x)\right]$$

and the inequality (cf. [9]), which generalizes (3.6)

$$|K_{\eta+i\tau}(x)| \le e^{-\delta|\tau|}K_\eta(x\cos\delta), \eta \in \mathbb{R},\ \delta \in (0, \pi/2).$$

Thus combining with (3.11) we finally obtain the equality

$$2\beta^2\sum_{n=1}^{\infty} n^2K_{in\beta}(x) + \alpha x\sum_{n=1}^{\infty} e^{-x\cosh n\alpha}(x\cosh^2 n\alpha - \cosh n\alpha - x)$$
$$= \frac{\alpha x}{2}e^{-x},\ x > 0,\ \alpha\beta = 2\pi, \alpha > 0. \tag{3.12}$$

Let us integrate through (3.12) over $\mathbb{R}_+$ and change the order of integration and summation due to the convergence of the following series (see (3.6))

$$\sum_{n=1}^{\infty} n^2\int_0^\infty |K_{in\beta}(t)|dt \le C_\delta\sum_{n=1}^{\infty} n^2e^{-\delta n\beta} < \infty,$$

$$\sum_{n=1}^{\infty}\cosh^2 n\alpha\int_0^\infty e^{-t\cosh n\alpha}t^2dt = 2\sum_{n=1}^{\infty}\frac{1}{\cosh n\alpha} < \infty,$$

$$\sum_{n=1}^{\infty}\cosh n\alpha\int_0^\infty e^{-t\cosh n\alpha}tdt = \sum_{n=1}^{\infty}\frac{1}{\cosh n\alpha} < \infty,$$

$$\sum_{n=1}^{\infty}\int_0^\infty e^{-t\cosh n\alpha}t^2dt = 2\sum_{n=1}^{\infty}\frac{1}{\cosh^3 n\alpha} < \infty.$$

Calculating elementary integrals and letting $\beta = \frac{2\pi}{\alpha}$ we come out with the identity

$$\frac{4\pi^3}{\alpha^2}\sum_{n=1}^{\infty}\frac{n^2}{\cosh\left(\frac{n\pi^2}{\alpha}\right)} + \alpha\sum_{n=1}^{\infty}\frac{\tanh^2 n\alpha}{\cosh n\alpha} = \frac{\alpha}{2},\ \alpha > 0.$$

In particular, $\alpha = \pi$ gives the value of the series

$$\sum_{n=1}^{\infty}\frac{4n^2 + \tanh^2 \pi n}{\cosh \pi n} = \frac{1}{2}.$$

The Poisson type formula (3.4) can be extended involving convolution operator (1.9). However, first we need to prove the following lemma.

Lemma 3. *Let* $f, g \in L_1(\mathbb{R}_+; K_\mu(\xi x)dx)$, *where* $\mu,\ \xi \in \mathbb{R},\ 0 < \xi < 1$. *Then the convolution* (1.9) $(f * g)(x)$ *exists almost for all* $x > 0$, *belongs to* $L_1(\mathbb{R}_+; K_\mu(\xi x)dx)$ *and satisfies the inequality*

$$||f * g||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)} \le ||f||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)} ||g||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)}. \tag{3.13}$$

Moreover, factorization property (1.10) *holds in the closed strip* $|\text{Re}z| \le |\mu|$.

Proof. Indeed, by the definition of the norm, the Macdonald formula (1.8) via Fubini's theorem and a simple substitution we derive the estimates

$$||f * g||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)} = \int_0^\infty K_\mu(\xi x)|(f * g)(x)|dx$$

$$\le \frac{1}{2}\int_0^\infty \frac{K_\mu(\xi x)}{x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{y^2+u^2}{uy} + \frac{uy}{x}\right)} |f(u)g(y)|dydudx$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty |f(u)g(y)| \int_0^\infty K_\mu(x) e^{-\frac{1}{2}\left(x\frac{y^2+u^2}{\xi uy} + \frac{\xi uy}{x}\right)} \frac{dx}{x} dydu$$

$$\le \frac{1}{2}\int_0^\infty \int_0^\infty |f(u)g(y)| \int_0^\infty K_\mu(x) e^{-\frac{1}{2}\left(x\frac{y^2+u^2}{uy} + \frac{\xi uy}{x}\right)} \frac{dx}{x} dydu$$

$$= \int_0^\infty |f(u)|K_\mu(u\sqrt{\xi})du \int_0^\infty |g(y)|K_\mu(y\sqrt{\xi})dy$$

$$\le \int_0^\infty |f(u)|K_\mu(\xi u)du \int_0^\infty |g(y)|K_\mu(\xi y)dy$$

$$= ||f||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)} ||g||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)}.$$

Thus we arrive at the inequality (3.13) and Fubini's theorem gives us that the double integral (1.9) exists almost for all $x > 0$. Now again by Fubini's theorem we take the Lebedev integral (1.1) and apply to convolution (1.9). After the change of the order of integration we use the Macdonald formula (1.8) to calculate the inner integral. As a result we obtain

$$K_z[f * g] = \frac{1}{2}\int_0^\infty \frac{K_z(x)}{x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{y^2+u^2}{uy} + \frac{uy}{x}\right)} f(u)g(y)dydudx$$

$$= K_z[f]K_z[g].$$

Further, since (see (1.6), (1.7))

$$\int_0^\infty |K_z(x)f(x)|dx \le \int_0^\infty K_{\text{Re}z}(x)|f(x)|dx$$

$$\leq \int_0^\infty K_\mu(x)|f(x)|dx < \infty, \quad |\text{Re}z| \leq |\mu|,$$

and the same integral is finite for $g(x)$, we verify that (1.10) holds in the closed strip $|\text{Re}z| \leq |\mu|$. Lemma 3 is proved.

This lemma drives us to the following extension of the Poisson formula (3.4).

Theorem 2. *Let $f, g \in L_1(\mathbb{R}_+; K_\mu(\xi x)dx)$, where $\mu,\ \xi \in \mathbb{R},\ |\mu| > \frac{1}{4},\ 0 < \xi < 1$. Then the Poisson type formula is true*

$$K_0[f]K_0[g] + 2\sum_{n=1}^\infty K_{in\beta}[f]K_{in\beta}[g] = \alpha\left[\frac{1}{\pi}\int_0^\infty K_{i\tau}[f]K_{i\tau}[g]d\tau\right.$$

$$\left.+\int_0^\infty\int_0^\infty f(u)g(y)\sum_{n=1}^\infty K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right)dudy\right],\ \alpha\beta = 2\pi,\ \alpha > 0. \quad (3.14)$$

In particular, we have the identity

$$|K_0[f]|^2 + 2\sum_{n=1}^\infty |K_{in\beta}[f]|^2 = \alpha\left[\frac{1}{\pi}\int_0^\infty |K_{i\tau}[f]|^2 d\tau\right.$$

$$\left.+\int_0^\infty\int_0^\infty f(u)\overline{f(y)}\sum_{n=1}^\infty K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right)dudy\right],\ \alpha\beta = 2\pi,\ \alpha > 0. \quad (3.15)$$

Proof. The proof is based on Theorem 1, Lemma 3, formula (3.4), factorization property (1.10), Fubini's theorem and the change of the order of summation and integration in the latter term of (3.14). This change is guaranteed by the convergence of the following series

$$\sum_{n=1}^\infty\int_0^\infty\int_0^\infty |f(u)g(y)|K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right)dudy,$$

which we will justify by using the integral representation of the modified Bessel function (cf. (1.6))

$$K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right) = \int_0^\infty e^{-\sqrt{u^2+y^2+2uy\cosh n\alpha}\cosh t}dt.$$

We find the estimate $(n = 1, 2, \dots)$

$$\int_0^\infty e^{-\sqrt{u^2+y^2+2uy\cosh n\alpha}\cosh t}dt = e^{-\sqrt{u^2+y^2+2uy\cosh n\alpha}}$$

$$\times\int_0^\infty e^{-2\sqrt{u^2+y^2+2uy\cosh n\alpha}\sinh^2(t/2)}dt \leq e^{-(u+y)}\int_0^\infty e^{-\frac{t^2}{2}\sqrt{u^2+y^2+2uy\cosh n\alpha}}dt$$

$$= \frac{\sqrt{\pi}e^{-(u+y)}}{\sqrt{2}((u+y)^2+4uy\sinh^2(n\alpha/2))^{1/4}} \le \frac{\sqrt{\pi}e^{-(u+y)}}{2(uy)^{1/4}\sinh^{1/2}(n\alpha/2)}.$$

Consequently,

$$\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}|f(u)g(y)|K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right)dudy$$

$$\le \frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{1}{\sinh^{1/2}(n\alpha/2)}\int_0^{\infty}|f(u)|\frac{e^{-u}}{u^{1/4}}du\int_0^{\infty}|g(y)|\frac{e^{-y}}{y^{1/4}}dy$$

$$\le \frac{\sqrt{\pi}}{2}\left[\sup_{x>0}\frac{e^{-x}}{K_\mu(\xi x)x^{1/4}}\right]^2\sum_{n=1}^{\infty}\frac{1}{\sinh^{1/2}(n\alpha/2)}$$

$$\times||f||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)}||g||_{L_1(\mathbb{R}_+;K_\mu(\xi x)dx)} < \infty,$$

when $\mu,\ \xi \in \mathbb{R},\ |\mu| > \frac{1}{4},\ 0 < \xi < 1$ (see (1.3), (1.4)). Letting $g(x) = \overline{f(x)}$ we get (3.15). Theorem 2 is proved.

An interesting example of the formula (3.15) can be done taking $f(x) \equiv 1$. Appealing to the corresponding values of the Lebedev integrals (see above) we come out with the identity

$$\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{1}{\cosh^2(\pi n\beta/2)} - \alpha\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right)dudy$$

$$= \frac{2\alpha-\pi^2}{4},\ \alpha\beta = 2\pi,\ \alpha > 0.$$

But

$$\int_0^{\infty}\int_0^{\infty}K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right)dudy$$

$$= \int_0^{\pi/2}\int_0^{\infty}rK_0\left(\sqrt{1+2\cos\varphi\sin\varphi\cosh n\alpha}\right)drd\varphi$$

$$= \int_0^{\pi/2}\frac{d\varphi}{1+2\cos\varphi\sin\varphi\cosh n\alpha} = \int_0^{\infty}\frac{du}{u^2+2u\cosh n\alpha+1}$$

$$= \frac{1}{2\sinh n\alpha}\log\frac{\cosh n\alpha+\sinh n\alpha}{\cosh n\alpha-\sinh n\alpha} = \frac{n\alpha}{\sinh n\alpha}.$$

Therefore we finally get

$$\sum_{n=1}^{\infty}\left[\frac{\pi^2}{1+\cosh\pi n\beta} - \frac{n\alpha^2}{\sinh n\alpha}\right] = \frac{2\alpha-\pi^2}{4},\ \alpha\beta = 2\pi,\ \alpha > 0.$$

In particular, letting $\alpha = \pi$ we have

$$\sum_{n=1}^{\infty}\left[\frac{1}{1+\cosh 2\pi n} - \frac{n}{\sinh \pi n}\right] = \frac{2-\pi}{4\pi}.$$

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