On the adjugate of a matrix

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Let $|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ be the characteristic polynomial of an n-by-n matrix A over a given field $\mathbb{K}$. The elegant proof of the Cayley-Hamilton theorem of [1, p. 50] can be easily modified to prove that (see [2, p. 40]):

$$(-1)^{n-1} A^{\mathrm{adj}} = A^{n-1} + c_{n-1} A^{n-2} + \cdots + c_1 I \tag{1}$$

where A^{adj} stands for the *adjugate* of A (or *classical adjoint* — the transpose of the cofactor matrix of A) and I for the identity matrix of order n. More generally, it can be easily modified to prove that (see [3, p. 38]):

$$(\lambda I - A)^{\mathrm{adj}} = A^{n-1} + (\lambda + c_{n-1}) A^{n-2} + \cdots + (\lambda^{n-1} + c_{n-1}\lambda^{n-2} + c_{n-2}\lambda^{n-3} + \cdots + c_1) I \tag{2}$$

Proof. We start from the basic fact that:

$$(\lambda I - A) \cdot (\lambda I - A)^{\mathrm{adj}} = (\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0) I \tag{3}$$

and by noting that, by definition of adjugate, $(\lambda I - A)^{\mathrm{adj}}$ is a polynomial in λ of degree $n-1$ with coefficients in the space of the n-by-n matrices over $\mathbb{K}$, say,

$$(\lambda I - A)^{\mathrm{adj}} = D_{n-1}\lambda^{n-1} + D_{n-2}\lambda^{n-2} + \cdots + D_1\lambda + D_0$$

By equating the terms in λ of the same order in both sides of equation (3), we obtain:

$$\begin{aligned} D_{n-1} &= I \\ -A \cdot D_{n-1} + D_{n-2} &= c_{n-1} I \\ -A \cdot D_{n-2} + D_{n-3} &= c_{n-2} I \\ \vdots \qquad \vdots\ &\ \vdots \\ -A \cdot D_1 + D_0 &= c_1 I \\ -A \cdot D_0 &= c_0 I \end{aligned}$$

Finally, by multiplying the first equation by A^n, the second by A^{n-1}, and so on, up to the last equation, and by adding the new equations up, we obtain the Cayley-Hamilton theorem. This is the proof in [1, p. 50]. If, instead, we multiply the first equation by A^{n-1}, the second by A^{n-2}, etc., and stop precisely before the last equation, by summing up we obtain at the left-hand side D_0, which is $(-1)^{n-1}A^{\mathrm{adj}}$, say, by putting $\lambda = 0$. Hence, we get (1). (2) can be proven by obtaining the coefficients of $(\lambda I - A)^{\mathrm{adj}}$ step by step through the same procedure.

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References

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