Software *PSDF*
Perturbed Second Degree Forms

A symbolic implementation for deriving some semi-classical properties of perturbed of second degree forms: the case of Chebyshev forms

Version 1.0 - Wolfram *Mathematica*®

TUTORIAL

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2014

Introduction

The software *PSDF*, written in *Mathematica*® [25, 26], includes a set of symbolic tools, of main commands, for dealing with orthogonal polynomials and their characterizations concerning semi–classical, second degree and Laguerre-Hahn forms. In particular, these tools allow the implementation of a general method presented in this work intended to explicit some semi-classical properties of perturbed of second degree forms. The case of Chebyshev form of second kind is taken as study example and we give results for its perturbed of orders zero, one and two. Data for the other forms of Chebyshev is also provided and some results are available in *PSDF*.

Perturbation is a fundamental transformation translated by a modification on the first coefficients of the recurrence relation of order two satisfied by orthogonal polynomial sequences [12]. This transformation can promote a deep change of proprieties, namely in symmetry and positive definite character. Nevertheless, there is a large set of forms that are preserved by perturbation: the second degree forms [15]. In other words, the perturbed of a second degree form still is a second degree form. Moreover, a second degree form is also a semi–classical one [12]. This crucial fact is the basic principle of the aforementioned method, whose aim is to explicit some semi–classical proprieties, in special the second order differential equation. We remark that, in general, the perturbed of a semi–classical form is not semi–classical, but a Laguerre–Hahn [12]. Laguerre–Hahn forms [1, 2] generalize semi–classical and second degree forms, a fact that is evident from the Stieltjes equation or the functional equation satisfied by them. Briefly, the perturbation process appears as a method of constructing new semi–classical sequences from second degree sequences.

The advantage of the cited method is its generality: it is intended to work for any perturbation and any second degree form. Of course, in practice, the computations can be accomplished, even with the help of an automatic manipulator as *Mathematica*®, only in quite simple cases. The four Chebyshev forms are among the most well known second degree forms [15, 16], besides the fact that they belong to the set of second degree classical forms [3]; obviously they are semi–classical. In this work, we will focus our attention only on the form of second kind because the other three can be derived from it by perturbation.

The semi-classical character of perturbed Chebyshev forms is a subject that has already interested several authors. We would like to refer the recent article [17] about the perturbed of order one by dilatation of the second kind form and [16] concerning the four Chebyshev forms; see also [24].

In this work, we give a symbolic implementation intended to treat perturbations of higher order or more complicated cases. The results for the complete perturbed of order one, with three free parameters, and the perturbed of order two by dilatation, with two parameters, are examples of the potential of the software and the general method furnish here.

This tutorial presents the software *PSDF*, explains how one can use its commands, providing also brief information about them, gives one example of application and results for several cases. The reader interested knowing more about implementation of this kind of orthogonality matters in *Mathematica*® can found in the tutorial [21] of the software *CCOP - Connection Coefficients for Orthogonal Polynomials* [20] a source of inspiration (see also the corresponding article [19]).

PSDF is able to compute perturbed orthogonal polynomials of any degree corresponding to any perturbation. Then it is very easy to obtain results and do experiences, in particular, make graphical representations and numerical computations in order to study empirically several properties of polynomials: symmetries, zeros, intersection points, etc. This point is exploited at last.

Let us summarizes the content of this text.

In the first chapter, we present a brief description of the software *PSDF*.

In Chapter 1 , we establish the theoretical framework, we recall the mathematical background necessary to understand the subject of perturbed of second degree forms. In particular, we have collected the formulas and procedures that are implemented in *PSDF*.

Chapter 2 introduces a general method, a procedure of algebraic computations, to explicit some semi–classical properties of perturbed second degree forms, namely: the functional equation, the class of the form, the Stieltjes equation, a structure relation and the second order differential equation. Also, it gives closed formulas for the Stieltjes function. In the first section, we present the method and in the next section, we summarize the flow of computations in a scheme as it is implemented in the software. Tools available in main commands of *PSDF* allow to explicit some additional properties of perturbed of second degree forms according to their Laguerre–Hahn and second degree character, namely the Stieltjes equation and the structure relation.

In Chapter 3 , we detail the content of *PSDF*. In the first two sections, we furnish a description of main commands and of commands for Chebyshev forms. In the last section, we apply the main commands in the implementation of the general method and we perform the computations step by step with a simple example: the perturbed of order zero of the second kind Chebyshev form.

In Chapter 4 , we consider also all perturbed of order one and the perturbed of order two by dilatation and we explicit the announced semi-classical proprieties, listing their characteristics elements, as well as the cited additional proprieties. All results have been obtained with *PSDF*.

In Chapter 5 , we present graphics for some perturbed by dilatation and by translation of the second kind Chebyshev polynomials corresponding to some values of parameters of perturbations and we study, in particular, the behavior of their zeros. We employ some connection relations, derived with the software *CCOP*, to prove the properties exhibit by the intersection points of perturbed polynomials.

We finish this tutorial with some comments concerning the limitations of the method and the software. At last, we remark that we have achieved to explicit new semi–classical proprieties of some perturbed of Chebyshev forms.

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