# Persistent Switching near a Heteroclinic Model for the Geodynamo Problem

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December 30, 2012

**Keywords:**
Georeversals, Heteroclinic Networks, Switching, Symmetry Breaking

**AMS Subject Classifications:**
Primary: 34C28; Secondary: 34E10, 37C20, 37C27, 37C29, 76W05

**Abstract**

Modelling chaotic and intermittent behaviour, namely the excursions and reversals of the geomagnetic field, is a big problem far from being solved. Armbruster *et al* [5] considered that structurally stable heteroclinic networks associated to invariant saddles may be the mathematical object responsible for the aperiodic reversals in spherical dynamos. In this paper, invoking the notion of heteroclinic switching near a network of rotating nodes, we present analytical evidences that the mathematical model given by Melbourne, Proctor and Rucklidge [22] contributes to the study of the georeversals. We also present numerical plots of solutions of the model, showing the intermittent behaviour of trajectories near the heteroclinic network under consideration.

## 1 Introduction

The geomagnetic field can be roughly described as a dipole aligned with the axis of rotation of the Earth as in figure 1(a), but the polarity of the dipole did not always coincide with the one observed nowadays. Paleomagnetic studies show that the Earth's magnetic field flips from time to time, reversing poles. Although we know that in the past the average time between such reversals was about $2 \times 10^5$ years (source: Stefani *et al* [30]), the time of the next reversal is unpredictable, since the reversals occur irregularly. The reasons of this irregularity is an important question of modern geophysics.

The geodynamo model, based on equations of magnetohydrodynamics, is at present the only plausible model of the mechanism, by which the Earth generates and maintains a magnetic field. Several phenomenological models were proposed to explain the reversals of the geomagnetic field. According to Pétrélis *et al* [23], one of these mechanisms is the use of normal forms, based on the assumption that magnetic eigenmodes are competing. In this context, heteroclinic structures provide a framework to describe georeversals connecting quasi-steady states with a given polarity.

The dynamics near heteroclinic cycles and networks has many properties that appeal for a model of aperiodic georeversals. Armbruster *et al* [5] proposed a dynamo model considering convection in a spherical shell without rotation where symmetric heteroclinic cycles associated to seven equilibria appear in the associated flow. The heteroclinic cycles are structurally stable due to the symmetries of the system.

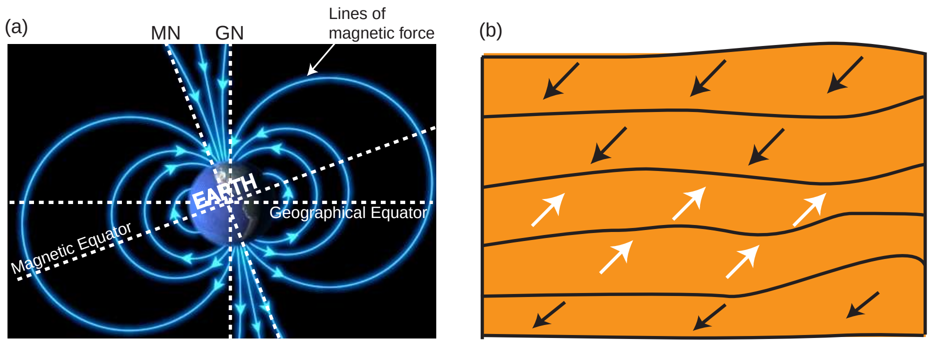


Figure 1: (a): Scheme of the Earth's magnetic field, approximated as a magnetic axial dipole. Note that the locus of points with the same intensity of magnetic field is a closed curve; MN – Magnetic North; GN – Geographical North; (b): Scheme of the lave flow layers; the magnetic alignment preserved after cooling records reversion of the geomagnetic field. Each layer maintains the original magnetic field at its time of cooling.

Introduction of a magnetic field transformed the hydrodynamic equilibria into others with a non-vanishing magnetic field. The dynamics near the network explains the changes of orientation of the magnetic dipole and then the georeversals.

Later, Chossat and Armbruster [6] gave a rigorous proof of the existence of structurally stable heteroclinic cycles involving dipolar magnetic fields generated by convection in a spherical shell. The authors demonstrated the existence of heteroclinic cycles only in the case of non-rotating spherical dynamo; they have also presented numerical simulations showing the existence of heteroclinic cycles for order–1 rotation rates. More recently, in Podvigina [24], reversals may be seen as a interaction of two distinct attractors, apparently via heteroclinic connections due to the intersection of stable and unstable manifolds of periodic orbits. The interactions depend on the Reynolds number.

In 2001, Melbourne, Proctor and Rucklidge [22] designed a simple model involving nine parameters with a heteroclinic network in which the associated geometry resembled the Earth's magnetic field reversals. The system of [22] has four ordinary differential equations compatible with the symmetries of the velocity fields in a rotation sphericall shell. It is based on the fact that the dominant modes in the geomagnetic field are dipoles and quadrupoles with some symmetry features. The authors assumed that the flow has $(\mathbf{Z}_3 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$–symmetry because the flow is invariant under rotations of 120 degrees[1]. Since the $\mathbf{Z}_3$–symmetry acts non-absolutely irreducibly, slight changes of the magnetic diffusivity behave like a Hopf bifurcation and the velocity acquires an $\mathbf{SO(2)}$–symmetry. The amplitude of the corresponding normal form, $(y_1, y_2, x_3, x_4)$, may be seen as coordinates of a continuous dynamical system whose flow contains an asymptotically stable heteroclinic network associated to two pairs of saddle-foci and a non trivial closed trajectory with real Floquet multipliers, embedded in a four dimensional manifold.

Since the network is asymptotically stable, trajectories whose initial condition lies outside the invariant subspaces will approach one of the cycles in the heteroclinic network. The fixed point subspaces work as barriers and the time spent near consecutive saddles increases geometrically with each visit. The absence of geomagnetic reversals on numerics is one of the reasons why the constructed model was unsatisfactory. This fact led the authors in [22] to consider breaking the original symmetries to kick the system away from the invariant saddles and to generate random reversals with a finite mean period. Observe that the dynamo model must have $\mathbf{Z}_2$–symmetry which is never broken: $B \mapsto -B$ (this means that if a magnetic field $B$ is a solution for the differential equation, then $-B$ is another solution).

Recently, Aguiar *et al* [4] defined and proved a strong form of *switching* near generic heteroclinic networks of rotating nodes embedded in a three dimensional smooth manifold. This phenomenon is characterized by the following property: close to the heteroclinic network there are trajectories that visit

[1]The evidence from magnetic field measurements at the surface, extrapolated down to the core-mantle boundary suggest that in fact the flow field is neither $\mathbf{Z}_2$ (180 degrees rotation) nor $\mathbf{Z}_3$ (120 degrees rotation) symmetric, but somewhere in between – see Jackson *et al* [16].

neighbourhoods of the saddles following any prescribed and admissible set of heteroclinic connections of the network. More recently, Homburg and Knobloch [15] gave an equivalent definition of forward switching for a heteroclinic network, using the notion of connectivity matrix (which characterizes the admissible sequences) and symbolic dynamics.

The perturbed dynamo model presented in [22] has a flow-invariant plane whose invariance is not broken and the nearby dynamics have not been studied rigorously yet. In particular, the considered perturbation does not break the heteroclinic network. The retention of this flow-invariant set has some effects on the switching properties. This is the interesting bridge between the works [4] and that of [22].

The main goal of this paper is to frame the model of Melbourne, Proctor and Rucklidge [22] and its numerics, in the context of the works of Aguiar *et al* [2, 3], Aguiar *et al* [4] and Rodrigues *et al* [26], to conclude analytically that all the properties for infinite switching hold. We show that the flow associated to the amplitude equations of the model in [22] has a heteroclinic network involving a periodic solution and four equilibria, embedded in an flow-invariant three-dimensional manifold. Near the heteroclinic network of the perturbed flow, there are trajectories making excursions around the whole network in an irregular way.

### Framework of the paper

This paper is organized as follows. Section 3 sets up the construction of the heteroclinic model given by Melbourne *et al* [22] as a convection dynamo problem with symmetry, preceded by definitions and preliminary results in section 2. Using the equivariance of the system under a Lie group, section 4 gives an analytical description of the associated equivariant Birkhoff normal form of degree 3 and we discuss the geometry of the flow of the corresponding lift. Section 5 contains a proof of switching near the network which appears near the perturbation constructed in section 4. This phenomenon enables the occurrence of chaotic intermittency. Section 6 is a short explanation of the physics underlying the system, in the context of georeversals. After the analysis of the geometry of some model's solutions and the corresponding time series, we include a discussion and a conclusion about the main result.

# 2 Preliminaries

This section provides some preliminary results and notation required. Let $f$ be a smooth vector field on $\mathbf{R}^n$ with flow given by the unique solution $x(t) = \varphi(t, x_0) \in \mathbf{R}^n$ of

$$\dot{x} = f(x), \qquad x(0) = x_0. \tag{2.1}$$

## 2.1 Heteroclinic cycles and networks

Following Field [9], if $A \subset \mathbf{R}^n$ is a compact flow-invariant set for the flow of $f$, we say that $A$ is an *invariant saddle* if

$$A \subseteq \overline{W^s(A)\backslash A} \ \text{ and } \ A \subseteq \overline{W^u(A)\backslash A},$$

where $W^s(A)$ and $W^u(A)$ is the stable and unstable manifold of $A$, respectively. In this paper, the saddles consist of either saddle-foci or non-trivial closed trajectories – as in [4], these are what we call hereafter *rotating nodes*. We assume that all the saddles are hyperbolic.

Given two invariant saddles $A$ and $B$, an $m$–dimensional *heteroclinic connection* from $A$ to $B$, denoted $[A \to B]$, is an $m$–dimensional connected invariant manifold contained in $W^u(A) \cap W^s(B)$.

Let $\mathcal{S} = \{A_j : j \in \{1, \ldots, k\}\}$ be a finite ordered set of mutually disjoint invariant saddles. Following Field [9], we say that there is a *heteroclinic cycle* associated to $\mathcal{S}$ if

$$\forall j \in \{1, \ldots, k\}, W^u(A_j) \cap W^s(A_{j+1}) \neq \emptyset \pmod{k}.$$

Sometimes, we refer to the saddles defining the heteroclinic cycle as *nodes*. In this paper, a *heteroclinic network* is simply a finite connected union of heteroclinic cycles such that given any two saddles in the network, there is a sequence of heteroclinic connections taking one to the other.

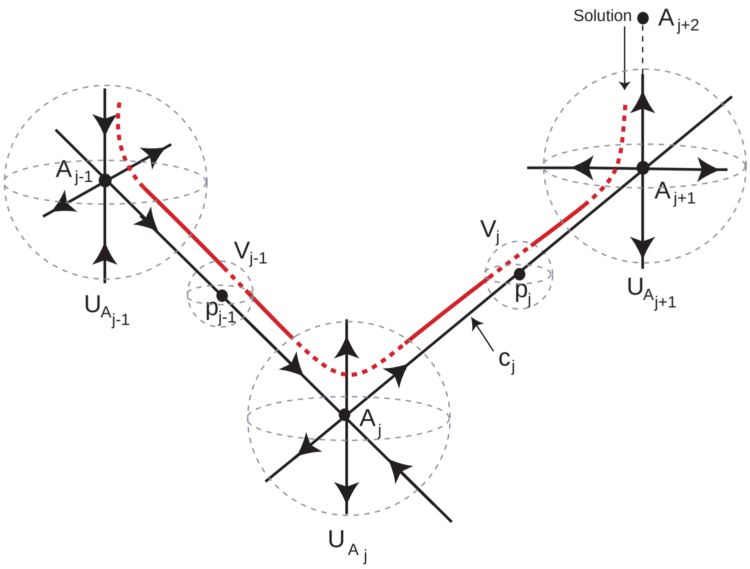


Figure 2: Solution shadowing a path of order 3, in a three–dimensional manifold: given the three heteroclinic connections $[A_{j-1} \to A_j]$, $[A_j \to A_{j+1}]$ and $[A_{j+1} \to A_{j+2}]$, there is a solution following them.

## 2.2 Symmetry and Heteroclinic Switching

Here, we introduce some background on group theory and equivariant dynamics. We refer the reader to Golubitsky *et al* [10] for other concepts and results about differential equations with symmetry. Let $\Gamma$ be a compact Lie group acting linearly on $\mathbf{R}^n$. The vector field $f$ is $\Gamma-$equivariant if for all $\gamma \in \Gamma$ and $x \in \mathbf{R}^n$, we have $f(\gamma x) = \gamma f(x)$. In this case $\gamma \in \Gamma$ is said to be a symmetry of $f$.

The $\Gamma$*–orbit* of $x_0 \in \mathbf{R}^n$ is the set $\Gamma(x_0) = \{\gamma x_0, \gamma \in \Gamma\}$. If $x_0$ is an equilibrium of (2.1), so are the elements in its $\Gamma$-orbit. More generically, if $\xi$ is a flow-invariant set, then so are the sets $\gamma\xi$, with $\gamma \in \Gamma$. A *relative equilibrium* is a group orbit of an equilibrium.

The *isotropy subgroup* of $x_0 \in \mathbf{R}^n$ is $\Gamma_{x_0} = \{\gamma \in \Gamma, \quad \gamma x_0 = x_0\}$. For an isotropy subgroup $\Sigma$ of $\Gamma$, its *fixed-point subspace* is

$$Fix(\Sigma) = \{x \in \mathbf{R}^n : \forall \gamma \in \Sigma, \gamma x = x\}.$$

We say that the action $\Gamma$ on $\mathbf{R}^n$ is *absolutely irreducible* when the only linear mappings $\mathbf{R}^n \to \mathbf{R}^n$ that commute with the action of $\Gamma$ on $\mathbf{R}^n$ are the scalar real multiples of *Id*.

The existence of heteroclinic networks is a common phenomenon in problems where there exist invariant spaces forced by symmetry. Symmetry forces the existence of fixed point subspaces, which are not destroyed under symmetric perturbations.

We follow the set-up in Aguiar *et al* [4] to define the concept of heteroclinic switching. Roughly speaking, switching near a heteroclinic network means that any infinite sequence of pseudo-orbits (defined by heteroclinic connections) can be shadowed. For a heteroclinic network $\Sigma$ with node set $\mathcal{A}$, a *path of order* $k$, on $\Sigma$ is a finite sequence $s^k = (c_j)_{j\in\{1,\ldots,k\}}$ of heteroclinic connections $c_j = [A_j \to B_j]$ in $\Sigma$ such that $A_j, B_j \in \mathcal{A}$ and $B_j = A_{j+1}$ *i.e.* $c_j = [A_j \to A_{j+1}]$. An infinite path, corresponds to an infinite sequence of connections in $\Sigma$.

Let $N_\Sigma$ be a neighbourhood of the network $\Sigma$ and let $U_A$ be a neighbourhood of each node $A$ in $\Sigma$. For each heteroclinic connection in $\Sigma$, consider a point $p_j$ on it and a small neighbourhood $V_j$ of $p_j$. We assume that the neighbourhoods of the nodes are pairwise disjoint, as well for those of points in connections as shown in figure 2.

Given neighbourhoods as before, the trajectory $\varphi(t, q)$, *follows* the finite path $s^k = (c_j)_{j\in\{1,\ldots,k\}}$ of order $k$, if there exist two monotonically increasing sequences of times $(t_i)_{i\in\{1,\ldots,k+1\}}$ and $(z_i)_{i\in\{1,\ldots,k\}}$ such that for all $i \in \{1, \ldots, k\}$, we have $t_i < z_i < t_{i+1}$ and:

- $\varphi(t, q) \subset N_\Sigma$ for all $t \in (t_1, t_{k+1})$;
- $\varphi(t_i, q) \in U_{A_i}$ and $\varphi(z_i, q) \in V_i$ and
- for all $t \in (z_i, z_{i+1})$, $\varphi(t, q)$ does not visit the neighbourhood of any other node except that of $A_{i+1}$.

There is *finite switching* near $\Sigma$ if for each finite path there is a trajectory that follows it as depicted in scheme 2. Analogously, we define *infinite* switching near $\Sigma$ by requiring that each infinite path is followed by a trajectory. We refer to the type of switching described as *persistent.* In Aguiar *et al* [4], the authors proved that under generic hypothesis on the heteroclinic network, persistent switching arises.

In a three–dimensional manifold this seems to be the only possible mechanism for the existence of infinite switching. This mechanism also works in higher dimensions if we can capture the heteroclinic network inside a three-dimensional manifold. This may be achieved using either the *center manifold of heteroclinic cycles* [28, 29] or the *normal hyperbolicity* of a three-dimensional flow–invariant set.

# 3 Construction and Description of the Model

In this section, we give a short description of the model designed by Melbourne *et al* [22]. It describes the evolution of a magnetic field $B$, through the magnetic induction equation (3.2), obtained combining Maxwell's equations and Ohm's Law, and the equation (3.3). The latter means that $B$ is a *divergence-free* vector field. Here $R_m = \frac{UL}{\eta}$ is the dimensionless magnetic Reynolds number, where $U$ is the characteristic flow velocity, $L$ is the typical length scale of the flow and $\eta$ is magnetic diffusivity.

$$\frac{\partial B}{\partial t} = R_m \nabla \times (v \times B) + \nabla^2 B \tag{3.2}$$

$$divB = 0. \tag{3.3}$$

Since (3.2) is linear in $B$, it is known that solutions of (3.2) with different symmetries are independent – see Gubbins [11]. The velocity field assumed by Melbourne *et al* [22] in (3.2) can be regarded as the system of rolls which has the so-called *cartridge belt* structure, with three pairs of columnar cells aligned with the rotation axis. Then $v$ is invariant under the equatorial reflection and the rotation by $\frac{2\pi}{3}$ about the axis of the Earth. Because of the symmetries of the velocity field, the solution of (3.2) can be expressed as a linear combination of the symmetric modes $D_e$, $Q_a$, $D_a$ and $Q_e$ (notation of Holme [14]), as depicted in figure 3, where:

- $D_a$ is a dipolar solution of (3.2) antisymmetric with respect to the equatorial plane and symmetric with respect to rotations by angle $\pi$ around the polar axis;
- $Q_a$ is a quadrupolar solution of (3.2) symmetric with respect to the equatorial plane and symmetric with respect to rotations by angle $\pi$ around the polar axis;
- $D_e$ is a dipolar solution of (3.2) symmetric with respect to the equatorial plane and antisymmetric with respect to rotations by angle $\pi$ around the polar axis;
- $Q_e$ is a quadrupolar solution of (3.2) antisymmetric with respect to the equatorial plane and antisymmetric with respect to rotations by angle $\pi$ around the polar axis

For the dynamo model involving the Kumar-Roberts flow, the modes $D_a$, $D_e$ and $Q_a$ become unstable for close values of magnetic diffusivity, which hinted to the authors of [22] to use the following ansatz for the magnetic field $B(r, t)$:

$$B(r, t) = y_1(t)D_e^1(r) + y_2(t)D_e^2(r) + x_2(t)Q_a(r) + x_3(t)D_a(r), \tag{3.4}$$

where $r = (||(y_1, y_2)||, x_2, x_3)$ and $D_e = (D_e^1, D_e^2)$ is an oscillatory mode. Note that the symmetric modes may be seen as vector fields in $\mathbf{R}^3$ where $r$ is a general position in the phase space. We are interested in the geometry of $(y_1, y_2, x_2, x_3)$, the coefficients of the symmetric modes, hereafter called the *amplitude* of the symmetric modes.

In the next section, we concentrate our attention in the dynamics of the amplitude equations of $B(r, t)$.

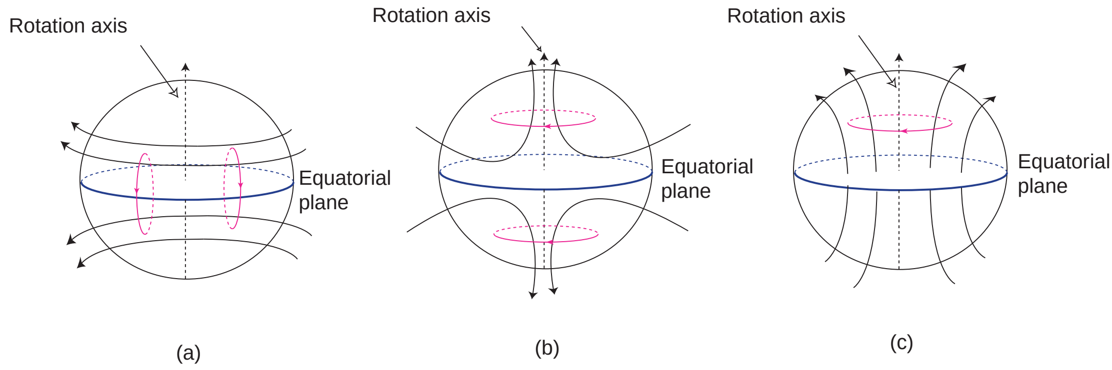


Figure 3: Symmetric Modes. (a): Equatorial Dipole $D_e$; (b): Axial Quadripole $Q_a$; (c): Axial Dipole $D_a$.

# 4 Truncated Birkhoff Normal Form

Before adding perturbing terms, note that the model constructed in [22] has the particularity that the velocity of the flow which generates the dynamo action is equivariant under the action of compact Lie group $\Gamma = \mathbf{SO}(2) \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Recall that the authors of [22] assumed that the flow had $(\mathbf{Z}_3 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$–symmetry – the flow is invariant under 120 degree rotations. Since the $\mathbf{Z}_3$–symmetry acts non-absolutely irreducibly, slight changes of the magnetic diffusivity $\eta$ give rise generically to a Hopf Bifurcation and the system acquires a $\mathbf{SO(2)}$–symmetry.

## 4.1 Heteroclinic Network in $\mathbf{R}^3$

In $\mathbf{R}^4$, assuming the usual representation of $\mathbf{SO}(2)$ in the first two coordinates and the usual representation of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ generated by the linear maps $\zeta_1$ and $\zeta_2$:

$$\zeta_1(y_1, y_2, x_2, x_3) = (y_1, y_2, -x_2, x_3) \qquad \text{and} \qquad \zeta_2(y_1, y_2, x_2, x_3) = (y_1, y_2, x_2, -x_3),$$

the truncated Birkhoff normal form of degree 3 in real coordinates, acting as organizing center, is given by:

$$\begin{cases} \dot{y}_1 = y_1(\mu_1 - (y_1^2 + y_2^2) + A_{12}x_2^2 + A_{13}x_3^2) - \omega_1 y_2 \\ \dot{y}_2 = y_2(\mu_1 - (y_1^2 + y_2^2) + A_{12}x_2^2 + A_{13}x_3^2) + \omega_1 y_1 \\ \dot{x}_2 = x_2(\mu_2 + A_{21}(y_1^2 + y_2^2) - x_2^2 + A_{23}x_3^2) \\ \dot{x}_3 = x_3(\mu_3 + A_{31}(y_1^2 + y_2^2) + A_{32}x_2^2 - x_3^2) \end{cases} \tag{4.5}$$

The procedure of reduction to the normal form is rather straightforward involving step by step elimination of the non resonant terms. Following Aguiar *et al* [3] and Rodrigues *et al* [26], the vector field $X_4$ associated to system (4.5) can be seen as the *lifted by rotation* associated to the pair $(x_1, \omega_1)$ applied to the vector field $X_3$ where

$$X_3(x_1, x_2, x_3) = \begin{pmatrix} x_1(\mu_1 - x_1^2 + A_{12}x_2^2 + A_{13}x_3^2), \\ x_2(\mu_2 + A_{21}x_1^2 - x_2^2 + A_{23}x_3^2), \\ x_3(\mu_3 + A_{31}x_1^2 + A_{32}x_2^2 - x_3^2) \end{pmatrix}. \tag{4.6}$$

Passing from (4.5) to (4.6) can be understood as a standard technique of phase-amplitude equations: (4.6) corresponds to the amplitude equations. We refer the technique *lifting by rotation* presented in the papers of Aguiar *et al* [3] and Rodrigues *et al* [26], because their results about lifted vector fields will be relevant for our further conclusions. The vector field $X_3$ is equivariant under the action of the finite Lie group $\Theta = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ where

$$\gamma_1(x_1, x_2, x_3) = (-x_1, x_2, x_3), \quad \gamma_2(x_1, x_2, x_3) = (x_1, -x_2, x_3) \quad \text{and} \quad \gamma_3(x_1, x_2, x_3) = (x_1, x_2, -x_3),$$

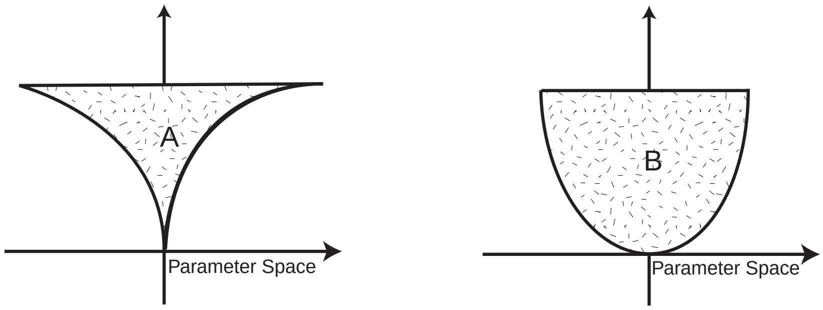


Figure 4: The set $A$ is not a large set (bounded by tangent lines) and $B$ is a large set (bounded by tangent lines). The origin represents the bifurcation point.

whose action is isomorphic to the usual action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ in $\mathbf{R}^3$. This implies that the coordinate planes and axes are flow-invariant; they correspond to $Fix\mathbf{Z}_2(\gamma_i)$ and $Fix(\mathbf{Z}_2(\gamma_i) \oplus \mathbf{Z}_2(\gamma_j))$, respectively ($i \neq j \in \{1, 2, 3\}$).

**Definition 1** *If $A$ is a set of parameter space such that $A$ is not bounded by surfaces with tangencies at the bifurcation point, we say that $A$ is* large *(see figure 4).*

It has already been proved by Melbourne [21] that:

**Theorem 1 (Melbourne [21], 1989)** *For $i \in \{1, 2, 3\}$, if the following conditions hold (if $i = 2$, then $i + 2 = 1$; if $i = 3$, then $i + 1 = 1$):*

*(1)* $\mu_i > 0$ ;

*(2)* $\mu_i + A_{i,i+2}\mu_{i+2} > 0$;

*(3)* $\mu_i + A_{i,i+1}\mu_{i+1} < 0$;

*(4)* $\mu_i + A_{i,i+1}\mu_{i+1} + \mu_{i+1} + A_{i+1,i}\mu_i < 4\min\{\mu_i, \mu_{i+1}\}$;

*(5)* $-\prod_{i=1}^{3}(\mu_i + A_{i,i+1}\mu_{i+1}) > \prod_{i=1}^{3}(\mu_i + A_{i,i+2}\mu_{i+2})$,

*then the flow associated to equation (4.6) has an asymptotically stable heteroclinic cycle $\Sigma_2$ associated to six equilibria on the three–coordinate axes. Moreover, there is a* large *set of values of $A_{ij}$ for which there is a large set of choices of parameters $(\mu_1, \mu_2, \mu_3)$ where conditions (1)–(5) hold.*

The network $\Sigma_2$ is the union of eight heteroclinic cycles (related by the symmetry group $\Theta$), each one lying on the boundary of each octant (see figure 5). Next table contains the information about the eigenvalues and eigenvectors for the $\Theta$–orbit of the equilibria contained in $\Sigma_2$.

| Equilibrium | Radial | | Contracting | | Expanding | |
|---|---|---|---|---|---|---|
| | Eigenvalue | Direction | Eigenvalue | Direction | Eigenvalue | Direction |
| $\pm P_1 \mapsto (\pm\sqrt{\mu_1}, 0, 0)$ | $-2\mu_1$ | $(1, 0, 0)$ | $\mu_3 + A_{31}\mu_1$ | $(0, 0, 1)$ | $\mu_2 + A_{21}\mu_1$ | $(0, 1, 0)$ |
| $\pm P_2 \mapsto (0, \pm\sqrt{\mu_2}, 0)$ | $-2\mu_2$ | $(0, 1, 0)$ | $\mu_1 + A_{12}\mu_2$ | $(1, 0, 0)$ | $\mu_3 + A_{32}\mu_2$ | $(0, 0, 1)$ |
| $\pm P_3 \mapsto (0, 0, \pm\sqrt{\mu_3})$ | $-2\mu_3$ | $(0, 0, 1)$ | $\mu_2 + A_{23}\mu_3$ | $(0, 1, 0)$ | $\mu_1 + A_{13}\mu_3$ | $(1, 0, 0)$ |

Since the network is asymptotically stable, solutions starting near the heteroclinic cycle will approach closer and closer the heteroclinic cycle and they remain near the equilibria for increasing periods of time; moreover, near the network, solutions make fast transitions from one node to the next. The existence of a radial component near each equilibrium stresses the existence of a two-dimensional flow-invariant region $\mathcal{S}_2^\star \supset \Sigma_2$ contained in an ellipsoid (the origin is repelling). In the restriction to its first octant, the unstable manifolds of the saddles are given by:

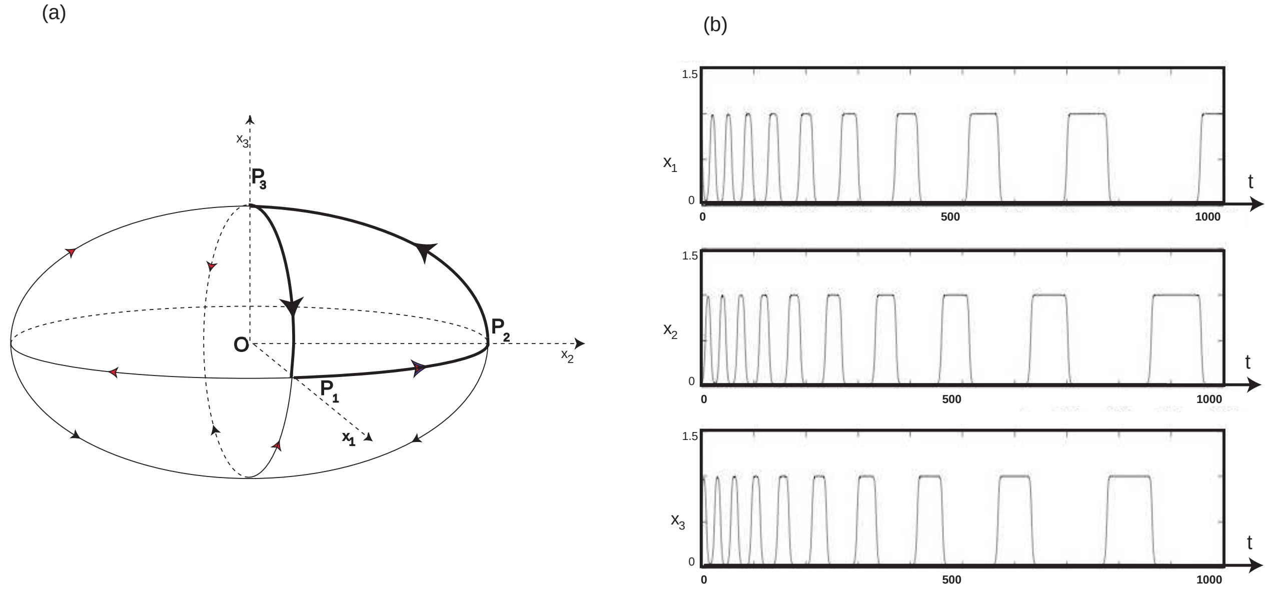


Figure 5: (a): The network $\Sigma_2$ can be seen as the union of eight heteroclinic cycles (related by symmetry), each one lying on the boundary of each octant. The network $\Sigma_2$ is the $\Theta$–orbit of the cycle depicted on the first octant. (b): Time series for a trajectory starting in the first octant – it starts near the heteroclinic cycle approaching closer and closer the heteroclinic cycle, remaining near consecutive equilibria for increasing periods of time.

1. $W^u(\{+P_1\}) \subset \{(x_1, x_2, x_3) \in (\mathbf{R}_0^+)^3 : x_3 = 0 \wedge x_1 \geq 0\}$;
2. $W^u(\{+P_2\}) \subset \{(x_1, x_2, x_3) \in (\mathbf{R}_0^+)^3 : x_1 = 0 \wedge x_3 \geq 0\}$;
3. $W^u(\{+P_3\}) \subset \{(x_1, x_2, x_3) \in (\mathbf{R}_0^+)^3 : x_2 = 0 \wedge x_2 \geq 0\}$.

This network is very similar to that of Dos Reis [25], who observed that it is possible to have robust cycles of non-transverse saddle-connections for a $\mathbf{Z}_2^3$–equivariant vector field. This kind of networks has also been considered by Guckenheimer and Holmes [12] – their case is more restrict since the differential equation has $\mathbf{O}_3$–symmetry.

### 4.2 Lifting by rotation

Due to the $\mathbf{Z}_2(\gamma_1)$–equivariance, the vector field $X_3$ is lifted by the rotation associated to the pair $(x_1, \omega_1)$, to the $(\mathbf{SO(2)} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$–equivariant vector field $\dot{X} = X_4(X)$ where $X_4$ is the vector field defined in (4.5).

Using the notation of Rodrigues *et al* [26] and according to the proposition 15 of [26], the asymptotically stable heteroclinic network $\Sigma_2 \subset \mathcal{S}_2^\star$ gives rise to an asymptotically stable heteroclinic network $\mathcal{L}(\Sigma_2) = \Sigma_3 \subset \mathcal{S}_3^\star$, associated to a relative equilibrium $c = \mathcal{L}(\pm P_1)$, with real Floquet multipliers and to the set of four equilibria

$$\mathcal{B} = \{\mathcal{Q} := \pm i_3(\pm P_2), \mathcal{D} := \pm i_3(\pm P_3)\},$$

two by two $\mathbf{Z}_2$–symmetric. The Floquet multipliers are precisely the eigenvalues whose eigendirections are transverse to the rotation. Here, it is important to note that $c$ lies in the plane $Fix(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ defined by $x_2 = x_3 = 0$ and that the one-dimensional heteroclinic connections from $\pm\mathcal{Q}$ to $\pm\mathcal{D}$ lie in the plane $Fix(\mathbf{SO(2)})$ defined by the equations $y_1 = y_2 = 0$ (see figure 6).

Any one-dimensional heteroclinic connection lying inside $Fix(\mathbf{Z}_2(\gamma_1))$ lifts to a one-dimensional heteroclinic connection; a one-dimensional connection lying outside $Fix(\mathbf{Z}_2(\gamma_1)$ lifts to a two-dimensional connection of the unique relative equilibrium. To be more precise, in the restriction to $\mathcal{S}_3^\star \subset \mathbf{R}^4$, each

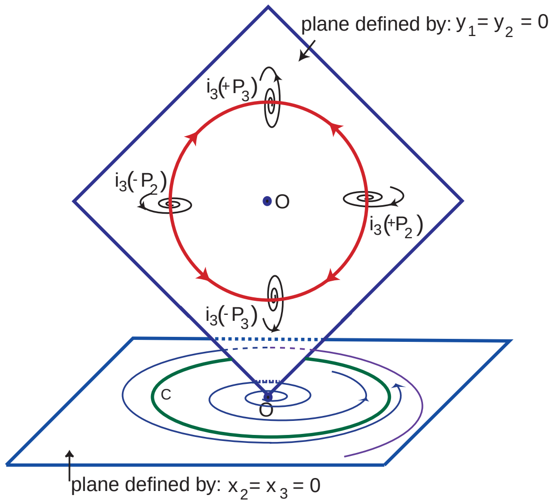


Figure 6: Representation of the two planes $Fix(\mathbf{SO(2)})$ (horizontal) and $Fix(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ (vertical), in which the invariant saddles associated to $\Sigma_3$ lie. In $Fix(\mathbf{SO(2)})$, the non trivial closed trajectory $c$ is a sink; in $Fix(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, we observe the non robust heteroclinic connections of the network which do not involve $c$. Note that $\mathbf{R}^4 = Fix(\mathbf{SO(2)}) \oplus Fix(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$.

heteroclinic connection involving $c$ has dimension 2 and the others have dimension 1. The invariant manifolds $W^s(c)$ and $W^u(\mathcal{Q})$ coincide, as well as $W^s(\mathcal{D})$ and $W^u(c)$. After the lift by rotation, the map $DX_4$, at an equilibrium $i_3(\pm P_i)$, $i = 2, 3$, has four eigenvalues. The eigenvalue of $DX_3(P_i)$ whose associated eigenvector has the same direction as the rotation gives rise to two complex non-real eigenvalues: the real part is equal to that of the original linearization at each equilibrium and the imaginary part is equal to the speed of rotation $\omega_1$ ($\omega_1 \neq 0$); the other eigenvalues and the corresponding eigendirection (transverse to the lift) remain. This is a general property for the eigenvalues of the equilibria for liftings of vector fields. In summary, we have proved that:

**Theorem 2** *Under the conditions of theorem 1, the flow associated to equation (4.5), has an asymptotically stable heteroclinic cycle $\Sigma_3$ associated to four saddle-foci and a non-trivial periodic solution with real Floquet multipliers.*

The heteroclinic network $\Sigma_3$ can be decomposed into four cycles. Due to the symmetry, trajectories whose initial condition starts outside the invariant subspaces will approach in positive time one of the cycles. The hyperplanes defined by $x_2 = 0$ and $x_3 = 0$ prevent switching and the time spent near either each equilibrium or the periodic solution increases geometrically. The limit of the ratio between consecutive times of flight inside the neighbourhoods of the saddles is related to the ratio between the real part of the eigenvalues at the corresponding saddles.

At this point, the existence of the heteroclinic network $\Sigma_3$ "connecting" $\pm B$ cannot describe georeversals because the period goes to infinity as the solutions are attracted to one cycle. In the next section, an arbitrary symmetry breaking term will kick the trajectories away from the nodes and will generate random reversals.

# 5 Analysis of the Perturbed Vector Field in $\mathbf{R}^4$

The flow associated to the differential equation (4.5) has a heteroclinic network but no connection of states with opposite polarities, except when additional perturbing terms that break the symmetries are taken into account. The perturbation (5.7) considered in [22] may be explained by the fact that the Earth does not have exact rotational and reflectional symmetries due to the inhomogeneities of the convection of the fluid motions inside the Earth's core. It corresponds to the equations (6)–(8) of [22].

$$\begin{cases} \dot{y}_1 = y_1(\mu_1 - (y_1^2 + y_2^2) + A_{12}x_2^2 + A_{13}x_3^2) - \omega_1 y_2 + \\ \qquad + \varepsilon_1 y_1(y_1^4 - 10y_1^2y_2^2 + 5y_2^4) + \varepsilon_2 x_2(y_1^4 + 2y_1^2y_2^2 + y_2^4) + \varepsilon_3 y_1 x_2 x_3^3 \\ \dot{y}_2 = y_2(\mu_1 - (y_1^2 + y_2^2) + A_{12}x_2^2 + A_{13}x_3^2) + \omega_1 y_1 + \\ \qquad + \varepsilon_1 y_1(-5y_1^4 + 10y_1^2y_2^2 - y_2^4) + \varepsilon_3 y_2 x_2 x_3^3 \\ \dot{x}_2 = x_2(\mu_2 + A_{21}(y_1^2 + y_2^2) - x_2^2 + A_{23}x_3^2) \\ \qquad + \varepsilon_1 y_1(y_1^2 - 3y_2^2)x_2^2 + \varepsilon_2 y_1^5 + \varepsilon_3 x_3^5 \\ \dot{x}_3 = x_3(\mu_3 + A_{31}(y_1^2 + y_2^2) + A_{32}x_2^2 - x_3^2) \\ \qquad + \varepsilon_1 y_1(3y_1^2 - y_2^2)x_2 x_3 + \varepsilon_2 y_1^3 x_2 x_3 + \varepsilon_3 x_2^5 \end{cases} \tag{5.7}$$

Observe that the vector field associated to (5.7) is equivariant under the action of the symmetry $-Id$. The nodes of $\Sigma_3$ are hyperbolic and persist under the perturbation. It is done in such a way that the invariance of the plane defined by the equations $y_1 = y_2 = 0$ is not broken. This means that the non-robust heteroclinic connections lying in $Fix(\mathbf{SO(2)})$ from $\pm\mathcal{Q}$ to $\pm\mathcal{D}$ persist. Since $\pm\mathcal{Q}$ and $c$ are hyperbolic, $\dim W^u(\pm\mathcal{Q}) = 2$ and $\dim W^s(c) = 3$, then for a non-empty open set of parameters $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, the invariant manifolds meet transversely and their intersection consists of a finite number of trajectories. The same holds for the other intersections involving the saddle $c$. Hence, generically the perturbation (5.7) does not destroy the heteroclinic network, which we denote from now on by $\Sigma_3^\varepsilon$.

Using the *center manifold theorem for heteroclinic cycles* studied by Shaskov *et al* [28] in the context of homoclinic loops, and by Shilnikov *et al* [29] in the general case, provided that the radial eigenvalue is the most contractive at all saddles, generically it is possible to reduce the interesting dynamical behaviour to a three-dimensional manifold $\mathcal{M}^3$ containing the network $\Sigma_3^\varepsilon$. More precisely, we require that:

$$-2\mu_1 < \mu_1 + A_{12}\mu_2 < 0, \quad -2\mu_2 < \mu_2 + A_{23}\mu_3 < 0 \quad \text{and} \quad -2\mu_3 < \mu_3 + A_{31}\mu_1 < 0 \tag{5.8}$$

to assure the *trichotomy condition* (meaning that the radial component must correspond to the strong stable manifold of each node). In particular, we have shown that restricted to the center manifold $\mathcal{M}^3$, the heteroclinic network $\Sigma_3^\varepsilon$ is of the type studied by Aguiar *et al* [4]: the invariant saddles of the network are either periodic solutions with non zero real Floquet exponents or hyperbolic saddle-foci, and all connections that take place in two–dimensional invariant manifolds occur as tranverse intersections. In the following lemma, we provide sufficient conditions for the flow to be $C^1$–linearizable around each saddle. These conditions are related to the eigenvalues of the linearization of $X_4$ at the saddles.

**Lemma 3** *For all* $i \in \{1, 2, 3\}$, *if*

$$|\mu_i + A_{i,i+1}\mu_{i+1}| \neq \mu_{i+2} + A_{i+2,i-1}\mu_{i-1}, \tag{5.9}$$

*then the flow is* $C^1$*–linearizable around each saddle.*

**Proof** See Aguiar *et al* [4] and Rodrigues *et al* [26] – the proof relies on the Samovol's theorem [27]. □

Condition (5.9) for $C^1$–linearization shows that linearization is not possible for subsets defined by lines in the parameter space. These restrictions have zero Lebesgue measure, thus they do not place serious constraint on the following analysis.

## 5.1 Poincaré maps

In this section, we describe the construction of the local and global maps required for modelling the dynamics near the heteroclinic network $\Sigma_3^\varepsilon$, via the composition in the usual way. We give the details for the flow near the nodes $c$, $\pm\mathcal{Q}$ and $\pm\mathcal{D}$.

By lemma 3, the vector field of (5.7) may be linearised around each equilibrium point, up to a set of measure zero. Hence it is possible to obtain cylindrical neighbourhoods near each invariant saddle (hollow cylinder in the case of the non trivial closed trajectory). The boundary of each cylindrical neighbourhood

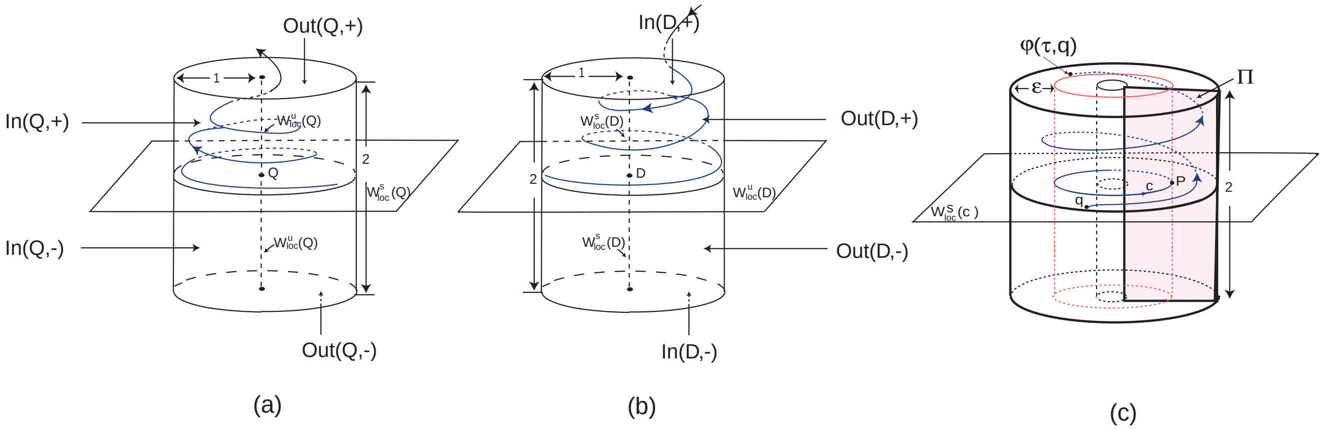


Figure 7: (a) and (b): cylindrical neighbourhoods of the saddle-foci $\mathcal{Q}$, $\mathcal{D}$; (c): hollow cylindrical neighbourhood of the non-trivial closed trajectory $c$.

forms an *isolating block*: the flow is transverse to the cylinder walls, top and bottom. We study the discrete time dinamics obtained by looking at points on the isolating block. Let $Q^+$, $Q^-$, $D^+$, $D^-$ cylindrical neighbourhoods of $+\mathcal{Q}$, $-\mathcal{Q}$, $+\mathcal{D}$ and $-\mathcal{D}$, respectively, where the linearization holds. After a linear rescaling of the local variables, the cylinders may be choosen to have radius 1 and height 2 as shown in figure 7. We omit the symbols $+$ and $-$ since the local dynamics near $+\mathcal{Q}$ and $-\mathcal{Q}$ are the same (the same holds for $+\mathcal{D}$ and $-\mathcal{D}$). In cylindrical coordinates $(\rho, \theta, z) \in \mathbf{R}_0^+ \times \mathbf{R} \pmod{2\pi} \times [-1, 1]$, the linearizations have the form:

$$\mathcal{Q}: \quad \dot{\rho} = C_1\rho \quad \wedge \quad \dot{\theta} = \omega_1 \quad \wedge \quad \dot{z} = E_1 z$$

and

$$\mathcal{D}: \quad \dot{\rho} = E_2\rho \quad \wedge \quad \dot{\theta} = \omega_1 \quad \wedge \quad \dot{z} = C_2 z,$$

where

$$C_1 = \mu_1 + A_{12}\mu_2 < 0, \quad E_1 = \mu_3 + A_{32}\mu_2 > 0, \quad C_2 = \mu_2 + A_{23}\mu_3 < 0 \quad \text{and} \quad E_2 = \mu_1 + A_{13}\mu_3 > 0.$$

We define the cross sections transverse to the cycle:

$$In(\mathcal{Q}) = Out(\mathcal{D}) = \{(\rho, \theta, z) \in \mathbf{R}_0^+ \times \mathbf{R} \pmod{2\pi} \times [-1, 1] : \rho = 1 \wedge |z| \leq 1\}$$

and

$$Out(\mathcal{Q}) = In(\mathcal{D}) = \{(\rho, \theta, z) \in \mathbf{R}_0^+ \times \mathbf{R} \pmod{2\pi} \times [-1, 1] : \rho \leq 1 \wedge |z| = \pm 1\},$$

parametrized by the coverings:

$$(x, y) \mapsto (1, x, y) = (\rho, \theta, z) \quad \text{and} \quad (r, \varphi) \mapsto (\rho, \theta, \pm 1) = (\rho, \theta, z),$$

respectively. Note that $Out(\mathcal{Q})$ and $In(\mathcal{D})$ have two connected components: top and bottom that will be denoted by $Out(\mathcal{Q}, +)$, $Out(\mathcal{Q}, -)$, $Out(\mathcal{D}, +)$ or $Out(\mathcal{D}, -)$, according to the sign of $z$ – see figure 7.

Close to the saddle-foci $\mathcal{Q}$, the flow goes in, at the cylinder walls, and it goes out at the top and bottom. Near $\mathcal{D}$, the flow goes in, at the cylinder top and bottom and it goes out at the wall. Inside the cylinder the vector field is linear, so the transition from the wall to top/bottom and top/bottom to the wall is well understood. The local flows near $\mathcal{Q}$ and $\mathcal{D}$ induce the maps $\phi_1 : In(\mathcal{Q}) \to Out(\mathcal{Q})$ and $\phi_2 : In(\mathcal{D}) \to Out(\mathcal{D})$ which to lowest order is given by:

$$\phi_1(x, y) = \left(y^{\delta_1}, -\frac{\omega_1}{E_1}\ln(y) + x\right) \qquad \text{and} \qquad \phi_2(r, \phi) = \left(-\frac{\omega_1}{E_2}\ln(r) + \phi, r^{\delta_2}\right)$$

where for $i \in \{1,2\}$, we have $\delta_i = -\frac{C_i}{E_i}$. Assuming (5.9), let $\Pi$ the cross section to $p \in c$. The Poincaré first return map defined on $\Pi$ may be linearized around $p$. The linearization takes the form:

$$\dot{\rho} = C_3(\rho - 1) \quad \wedge \quad \dot{\theta} = \omega_1 \quad \wedge \quad \dot{z} = E_3 z$$

where

$$C_3 = \mu_3 + A_{31}\mu_1 < 0, \quad \text{and} \quad E_3 = \mu_2 + A_{21}\mu_1 > 0.$$

Let $\mathcal{C}$ be a hollow three-dimensional cylindrical neighbourhood of $c$. Define the cross sections:

$$In(c) = \{(\rho, \theta, z) \in \mathbf{R}_0^+ \times \mathbf{R} \pmod{2\pi} \times [-1,1] : |\rho - 1| = \varepsilon \wedge |z| \leq 1\}$$

and

$$Out(c) = \{(\rho, \theta, z) \in \mathbf{R}_0^+ \times \mathbf{R} \pmod{2\pi} \times [-1,1] : |\rho - 1| \leq \varepsilon \wedge |z| = \pm 1\},$$

parametrized by the covering:

$$(x, y) \mapsto (1 \pm \varepsilon, x, y) = (\rho, \theta, z) \quad \text{and} \quad (r, \varphi) \mapsto (\rho, \theta, 1 \pm \epsilon) = (\rho, \theta, z),$$

The local flow near c induces the map $\phi_3 : In(c) \to Out(c)$ which to lowest order is given by:

$$\phi_3(x, y) = \left( \pm y^{\delta_3} + 1, -\frac{\omega_1}{E_3} \ln(y) + x \right),$$

where $\delta_3 = -\frac{C_3}{E_3}$.

The heteroclinic trajectories from $Out(\mathcal{Q})$ to $In(\mathcal{D})$, from $Out(\mathcal{D})$ to $In(c)$ and from $Out(c)$ to $In(\mathcal{Q})$ are nonsingular. Applying the *flow box theorem* around each heteroclinic connection, we shall define the connecting diffeomorphism $\Psi_{[\mathcal{D} \to \mathcal{Q}]}$ from a neighbourhood of $Out(\mathcal{Q}) \cap W^u(\mathcal{Q})$ into a neighbourhood of $In(\mathcal{D}) \cap W^s(\mathcal{D})$. We may define similar diffeomorphisms $\Psi_{[\mathcal{Q} \to c]}$ and $\Psi_{[c \to \mathcal{D}]}$, which will be called by *transition maps.*

The transition from one isolating block to the next may be seen, to first order, either a linear map, when the nodes are both saddle foci, or a rotation, when at least one saddle is the periodic solution. The study of the local map near rotating nodes is given in sections 5 and 6 of [4]. Return maps approximating the dynamics near the heteroclinic cycle can now be computed by composing the local and global maps in an appropriate order.

## 5.2 Coarse geometry near the saddles

We recall the results in Aguiar *et al* [4]. A *segment* $\beta$ *on* $In(\mathcal{Q})$ (resp.: $In(c)$) is a smooth regular parametrized curve $\beta : [0,1] \to In(\mathcal{Q})$ (resp.: $\beta : [0,1] \to In(c)$), that meets the stable manifold of $\mathcal{Q}$ (resp. the stable manifold of $c$) transversely at the point $\beta(1)$ only.

Let $U$ be an open set in a plane in $\mathbf{R}^n$ and $p \in U$. A *spiral on* $U$ *around* $p$ is a curve $\alpha : [0,1) \to U$ satisfying $\lim_{s \to 1^-} \alpha(s) = p$ and such that, if $\alpha(s) = (\alpha_1(s), \alpha_2(s))$ are its expressions in polar coordinates $(\rho, \theta)$ around $p$, then $\alpha_1$ and $\alpha_2$ are monotonic, with $\lim_{s \to 1^-} |\alpha_2(s)| = +\infty$.

Let $a, b \in \mathbf{R}$ such that $a < b$ and let $H$ be a surface parametrized by a cover $(\theta, h) \in \mathbf{R} \times [a, b]$ where $\theta$ is periodic. A *helix on* $H$ *accumulating on the circle* $h = h_0$ is a curve $\gamma : [0,1) \to H$ such that its coordinates $(\theta(s), h(s))$ are monotonic functions of $s$ with $\lim_{s \to 1^-} h(s) = h_0$ and $\lim_{s \to 1^-} |\theta(s)| = +\infty$.

**Proposition 4**

1. *A segment* $\beta$ *on* $In(\mathcal{Q})$ *is mapped by* $\phi_1$ *into a spiral on* $Out(\mathcal{Q})$ *around the local unstable manifold of* $\mathcal{Q}$.
2. *A spiral on* $In(\mathcal{D})$ *around the local stable manifold of* $\mathcal{D}$ *is mapped by* $\phi_2$ *into a helix on* $Out(\mathcal{D})$ *accumulating on the circle* $Out(\mathcal{D}) \cap W^u_{loc}(\mathcal{D})$.
3. *A segment* $\beta$ *on* $In(c)$ *is mapped by* $\phi_3$ *into a helix on* $Out(c)$ *accumulating on the circle* $Out(c) \cap W^u_{loc}(c)$.

**Proof:** See Aguiar *et al* [4] (section 6) and Rodrigues *et al* [26] (section 4) – the proof is based on the $\lambda$–lemma for flows. $\square$

On the proof of item 2 of the above lemma, the authors of [4] used implicitly that the orientations in which trajectories turn around the connection $[\mathcal{Q} \to \mathcal{D}]$ are the same (when restricted to the neighbourhoods of $\mathcal{Q}$ and $\mathcal{D}$) – detailed study in Labouriau and Rodrigues [19]. By the way the vector field (4.5) has been constructed, this hypothesis is straightforwardly satisfied.

## 5.3 Main Result

Here, we put together the local behaviour of trajectories near each node, to state the main result of the present paper. We adapt the proofs of [2] and [4] to our purposes.

**Theorem 5** *Under the conditions of Theorem 1, (5.8) and (5.9), near $\Sigma_3^\varepsilon \cap \mathcal{M}^3$, there exists:*

**(a)** *a set of initial conditions with positive Lebesgue measure (on a section transverse to the network) exhibiting finite switching of any order;*

**(b)** infinite switching *which may be realized by infinitely many initial conditions;*

**(c)** *a suspended horseshoe $\mathcal{H}$ containing the heteroclinic network on its closure.*

**Proof:** We suggest that the reader follows the proof observing figure 8.

**(a)** The initial conditions which shadow a given finite heteroclinic path are obtained by a recursive construction. Any segment of initial conditions lying across the stable manifold of $\mathcal{Q}$, say $\beta$, is wrapped around the isolating block. By item 1 of proposition 4, it accumulates as a spiral on the unstable manifold of $\mathcal{Q}$ and then on the stable manifold of the next saddle, $\mathcal{D}$ (see figure 8(a)). This spiral of initial conditions on the top/bottom of the neighbourhood around $\mathcal{D}$ is mapped, by the map $\phi_2$, into points lying on a helix accumulating on its unstable manifold (item 2 of proposition 4), which crosses transversely the stable manifold of $c$ infinitely many times (see figure 8(b)), giving rise to infinitelly many segments lying across $In(c)$.

This is the crucial point in which the equilibria must be saddle-foci. The complex eigenvalues force the spreading of solutions around all the unstable manifold of $\mathcal{D}$, allowing visits to all possible connections starting at $\mathcal{D}$. The transversality enables the existence of solutions that follow heteroclinic connections on the two different connect components of $\mathcal{M}^3 \backslash W^s(c)$, the upper and the lower part on the wall of the hollow cylinder.

Any curve of initial conditions lying across the stable manifold of $c$ winds around the isolating block $\mathcal{C}$ and, by item 3 of proposition 4, it accumulates as a helix on its unstable manifold following all the possible heteroclinic connections starting at $c$. Any heteroclinic connection is followed by a piece of helix and it is mapped, under the transition map $[c \to \mathcal{Q}]$, into a new segment across the stable manifold of $\mathcal{Q}$ (see figure 8(c)), giving rise to a new segment across the stable manifold of $\mathcal{Q}$.

The composition of consecutive local and transition functions, maps the original segment of initial conditions lying across the stable manifold of $\mathcal{Q}$ into infinitely many segments with the same property. Thus, it is possible to construct recursively a nested sequence of intervals accumulating on the stable manifold of $\mathcal{Q}$ that follow a prescribed sequence of heteroclinic connections and thus that exhibit finite heteroclinic switching.

**(b)** On the initial segment $\beta \subset In(\mathcal{Q})$, there are infinitely many (sub)segments of initial conditions following any heteroclinic connection starting at $\mathcal{D}$ (see figure 9). Each (sub)segment contains a point which is mapped by $\Psi_{[\mathcal{Q}\to c]} \circ \phi_2 \circ \Psi_{[\mathcal{D}\to\mathcal{Q}]} \circ \phi_1$ into $W^s_{loc}(c)$. This point divides the segments into two parts (points that are mapped into the lower part of $\mathcal{C}$ and that whose solutions follow the upper part). This construction may be continued *ad infinitum*, giving rise to a nested sequence of subsegments in $\beta$, whose infinite intersection is non-empty. In particular, for each sequence of nested compacts, there is at least a point realizing any given infinite heteroclinic path.

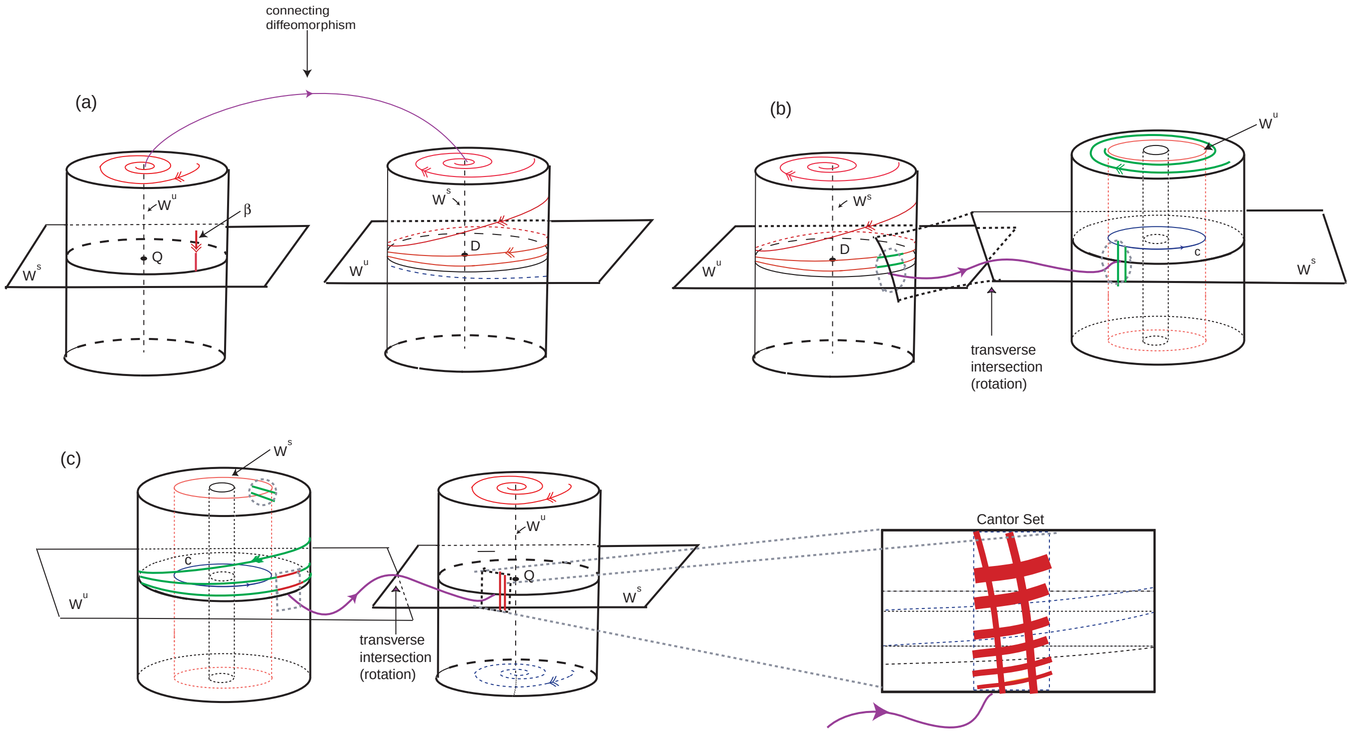


Figure 8: Local and global dynamics near the heteroclinic network. (a): a segment of initial conditions lying across the stable manifold of $\mathcal{Q}$ is mapped, by the local map $\phi_1$, into a spiral accumulating on its unstable manifold and thus on the stable manifold of $\mathcal{D}$; (b): a spiral of initial conditions on the top of the neighbourhood of $\mathcal{D}$ is mapped, by the local map $\phi_2$, into a helix accumulating on the unstable manifold of $\mathcal{D}$, which crosses transversely the stable manifold of $c$ infinitely many times; (c): a segment of initial conditions lying across the stable manifold of $c$ is mapped into a spiral accumulating on its unstable manifold. Each piece of helix is a new segment across the stable manifold of $\mathcal{Q}$. In (c), it is also possible to observe the first step of the construction of the Cantor set on the wall of the cylinder $In(\mathcal{Q})$.

**(c)** Allowing some thickness on the segments, instead of a nested sequence of intervals, we may construct a nested sequence of rectangles $(H_n)_{n\in\mathbf{N}}$ with positive Lebesgue measure[2] of initial conditions which shadow a given finite heteroclinic path (see figure 9). The sequence of heights associated to a sequence of rectangles accumulating on the network is decreasing. By construction, for each $n \in \mathbf{N}$, the intersection of $\phi_3 \circ \Psi_{[\mathcal{D}\to c]} \circ \phi_2 \circ \Psi_{[\mathcal{Q}\to\mathcal{D}]} \circ \phi_1(H_n)$ with $\Psi^{-1}_{[c\to\mathcal{Q}]}(H_n)$ is a topological horseshoe as depicted in figure 8.

□

The first return map $\Psi$ to $In(\mathcal{Q})$ is hyperbolic in all strips. For each $i \in \mathbf{N}$, the set

$$\Lambda = \bigcap_{k\in\mathbf{Z}} \left( \bigcup_{i=1}^{k} \Psi^k(H_i) \right)$$

is a Cantor set of initial conditions where the return map to $\bigcup_{i\in\mathbf{N}} H_i$ is well defined in forward and backward time, for arbitrarily large times. The dynamics of $\Psi$ restricted to invariant set $\Lambda$ is semi-conjugated to a full shift over an infinite alphabet that represents the paths in $\Sigma^3_\varepsilon$ (possibly not unique).

The two main ingredients of the previous proof were the transversality and the existence of rotating nodes which forces the spreading of solutions along the unstable manifolds of the nodes and then along the connections starting there.

A heteroclinic path on the network can be realized by trajectories in $W^u(c)$ since the unstable manifold of $c$ meets $In(\mathcal{Q})$ at a segment. In particular, there are infinitely many transverse *homoclinic connections* associated to $c$. Beside the original transverse connections, there exists infinitely many subsidiary heteroclinic trajectories turning around the original network.

## 5.4 Georeversals as a consequence of heteroclinic switching

The existence of heteroclinic switching near $\Sigma^\varepsilon_3 \cap \mathcal{M}^3$ implies that trajectories will make repeated passes near the network, which may be seen as a consequence of the hyperbolic suspensed horseshoe with the same shape as the whole recorrent set. The suspended horseshoe meets each cross section in a Cantor set – the *first iteration* of this set is depicted in figure 8(c). Next result is the core of this paper: it says that the intermittency of the flow associated to perturbation (5.7), numerically studied in Melbourne *et al* [22], is closely linked to the georeversals.

**Corollary 6** *According to [22], the geomagnetic field $B$ changes its polarity.*

**Proof:** From Theorem 5(b) we can conclude the existence of persistent switching near $\Sigma^\varepsilon_3$, *ie* close to the network, there are trajectories that visit the neighbourhoods of the saddles following all the heteroclinic connections of the network in any given order. These trajectories correspond to the evolution of the amplitude equations of (3.4). In particular, since the coefficient of the axial dipole, $x_3$, changes its sign then the vector field $B$ changes its polarity. □

The proof of theorem 5(c) also shows that switching is realized by infinitely many initial conditions, which are coded by the number of revolutions near each saddle with respect to a Poincaré section. The arguments of Rodrigues *et al* [26] used for non trivial periodic solutions can be slightly adapted for our case and thus we may conclude the existence of a transitive set of initial conditions with the same shape as $\Sigma^\varepsilon_3$, whose trajectories follow the network forwards and backwards and that is at least semi-conjugate to a Markov shift over a finite alphabet. This set lies arbitrarily close to the heteroclinic network $\Sigma^\varepsilon_3$.

The perturbing term $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ in (5.7) is responsible for driving the dynamics far from the equilibria; nevertheless its magnitude is irrelevant for the existence of intermittency. The different parameters $\varepsilon_i$ control the *degree* to which the symmetries are broken: $\varepsilon_1$ breaks the rotational symmetry and $\varepsilon_2$ and $\varepsilon_3$ destroy the reflectional symmetries. Note that the three different symmetry-breaking terms are required for the existence of switching.

[2]The heteroclinic network is embedded in $\mathbf{R}^4$.

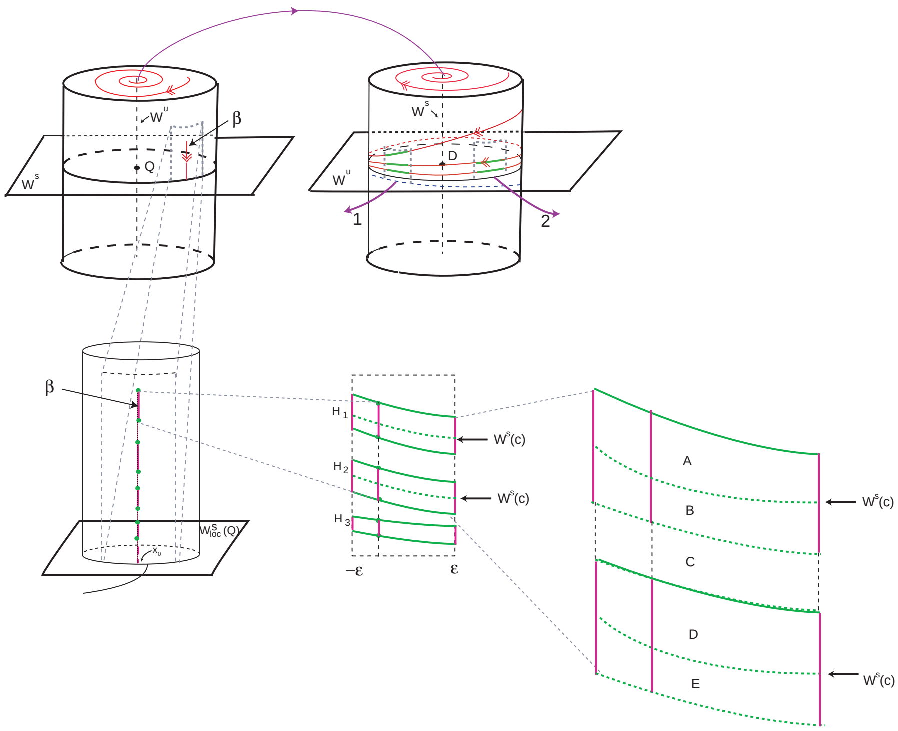


Figure 9: On a segment $\beta \subset In(\mathcal{Q})$, there are infinitely many segments of initial conditions that follow the connections 1 and 2. There are infinitely many points that go away from the network. Varying $x \in [-\varepsilon, \varepsilon]$ ($\varepsilon > 0$ small), each segment gives rise to a horizontal rectangle across $In(\mathcal{Q})$. Caption: A – points that follow the connection 1 and the upper part of $\mathcal{C}$; B – points that follow the connection 1 and the lower part of $\mathcal{C}$; C – points that go away from the network; D – points that follow the connection 2 and the upper part of $\mathcal{C}$; E – points that follow the connection 2 and the lower part of $\mathcal{C}$.

# 6 Discussion

## 6.1 Switching in the context of the Geodynamo Problem

The existence of heteroclinic switching near a general heteroclinic network implies that infinite pseudo-orbits with infinitely many discontinuities may be shadowed. This means that for any admissible sequence of nodes like

$$+\mathcal{Q} \to +\mathcal{D} \to c \to -\mathcal{Q} \to -\mathcal{D} \to c \to \dots,$$

we find at least one solution that visit neighbourhoods of these nodes in the same sequence. In the context of this problem, switching near $\Sigma_3^\varepsilon$ proved in theorem 5 implies that there are aperiodic itineraries visiting randomly the various cycles of the network. Therefore the amplitude coefficients may assume any value in the center manifold $\mathcal{M}^3$ (and more importantly any sign), implying that the magnetic field $B$ may change the orientation.

Due to the chaotic behaviour induced by the presence of suspended horseshoes near the network, we may conclude that there is no satisfactory way to predict the duration of any given polarity. Beyond the existence of geomagnetic reversals, the model in [22] also explains the existence of excursions, *ie*, changes of symmetric modes without changing the orientation of the magnetic field. This behaviour is at least superficially consistent with the real behaviour of the geomagnetic field, in spite of the fact that it should be more chaotic than what is stated in theorem 5.

We present some time series and the projections of a particular solution of (5.7) in figure 10 which are consistent with the analysis of the model presented in [22]. The figure also shows evidence of instant chaos near the perturbed heteroclinic network due to the explosion of suspended horseshoes and homoclinic classes. The initial condition and the parameters are the same in all simulations. The field in the model is predominantly axially dipolar because this is the way the constants have been chosen. We point out that if the constants in the model had been chosen differently, or if the interpretation of $x_2$ and $x_3$ had been swapped, the conclusion could equally well have been that the Earth should have an axial quadrupole magnetic field. Figure 10 has been obtained using the dynamical systems package *Dstool* [13].

In the context of qualitative theory of differential equations, in terms of future work, it would be interesting to investigate the measure of initial conditions whose trajectory stays near this kind of networks for all time – we conjecture that Lebesgue–almost all solutions go away from any given neighbourhood of the network.

## 6.2 Comparison with other models

In the symmetry breaking context, Kirk and Rucklidge [17] analised a codimension three bifurcation and concluded that when all the symmetries are broken, the heteroclinic network is destroyed and orbits may switch, *ie*, they may make traversals near more than one of the original cycles, giving a satisfactory model for the numerical evidences of intermittency. This is what the authors call *switching* – note that in their case, the heteroclinic network is completely broken. While in [17] it is not clear that switching is not a transient behaviour, here we are able to prove that it holds for any infinite sequence of heteroclinic paths.

As in [17], we start with a heteroclinic network with reflectional and rotational symmetry. Nevertheless, in contrast to their work, we present switching in a perturbation that does not break the network. The presence of the non-robust heteroclinic connection has important effects: since the connections $[\mathcal{Q} \to \mathcal{D}]$ are off-centered at the top/bottom of the cylindrical neighbourhood near $\mathcal{D}$, the authors in [17] were only able to conclude the existence of initial conditions following finite heteroclinic paths on the network.

In 1991, in the context of Bénard problem, it has been shown by Armbruster and Chossat [1] that for a class of differential equations with spherical symmetry, intermittency between axisymmetric steady states occurs. In Chossat *et al* [8], it has been demonstrated that in an open set of the parameter space, a robust and Lyapunov-stable heteroclinic network exists, involving $m$-dimensional heteroclinic connections, $m > 1$. Introducing the effect of the Coriolis force in the differential equation, the authors designed a system which is equivariant under a compact Lie group isomorphic to $\mathbf{SO(2)} \times \mathbf{Z}_2$. In its flow, a network (of different nature) persists as long as the norm of the perturbing term is small. Numerical simulations have shown that in certain regions in parameter space, random heteroclinic switching between magnetic dipoles exists. It has been observed that, providing certain inequalities for parameters are satisfied, one of the cycle within the network is *essentially asymptotically stable*: solutions converge to the cycle whenever the initial condition do not belong to a cuspoidal wedge in a neighbourhood of the network.

In contrast to the finding of [8], who claim numerically that the stability is preserved under perturbations, in our case a nested sequence of invariant suspended horseshoes accumulating on the network is created when the symmetry is broken implying that none of the cycles in $\Sigma^\varepsilon$ cannot be Lyapunov-stable. In the present case, since the nodes are either saddle-foci or periodic solutions, all heteroclinic connections are possible and they are equaly possible – there is no a preferred cycle to converge.

Finally, we also point out that recent dynamo experiments have shown pole reversals in a cylindrical shell and where the dynamo is driven by a von Karman swirling flow of liquid sodium. When two impellers are operated in counter-rotation, these flows display various qualities of interest for a potential dynamo. Under generic conditions about the frequency of rotation, stationary magnetic fields as well as random reversals may be generated. Pétrélis and Fauve [23] have proposed a simple and low-dimensional mechanism explaining this behaviour, differing to that of Melbourne *et al* [22] because it does not involve the existence of heteroclinic structures.

## 6.3 Conclusion

Geomagnetic reversals are one of the main interesting points of the geomagnetism, one of the most challenging phenomena in geophysics. Although the details of the reversal process are not completely understood, the occurrence of reversals is well documented by studying the layered of iron-rich lava rocks. Among several phenomenological models that explain georeversals, in this paper, exploiting the symmetries of the model in [22], we proved analytically that a heteroclinic network of rotating nodes exists which is asymptotically stable for an open set of parameters of the three-dimensional normal form representing the evolution of the dominant modes. We proved that slight breaking both symmetries (reflectional and rotational) of this normal form induces intermittent and persistent heteroclinic switching of the dipole for the nearby dynamics, which contributes to the understanding of reversals of the geomagnetic field. In [22], the existence of the network and switching was demonstrated only numerically.

The considered route to the chaos – the addition of symmetry breaking perturbing terms – corresponds to a curious interaction between symmetry breaking, heteroclinic switching and cycling. This dynamical phenomenon is even more interesting because the emergence of heteroclinic switching does not depend on the magnitude of the Lyapunov exponents of the nodes.

Through theorem 5 and corollary 6, we proved that the simple mathematical model [22] reproduces intermittent behaviour of the geomagnetic field and georeversals; the lengths of time intervals of constant polarity and the short duration of each reversal suggested by the model are similar with those of the Earth.

Beside the interest of reviving the Geodynamo question from the mathematical perspective, this work describes the behaviour of a flow associated to a explicit vector field unfolding the symmetry breaking of an equivariant organizing center. In the fully non-equivariant case at first glance the return map is intractable, but here we are able to predict qualitative features of the dynamics by assuming that $X_4^\varepsilon$ is very close to $X_4$. This is an important advantage of studying systems with some symmetry. A lot more needs to be done before the model [22] is well understood; we hope that this article could be a starting point for further related studies.

**Acknowledgments** The author thanks the support and corrections of Isabel Labouriau and Manuela Aguiar, without whom this paper would not have been written. The author is also indebted to Sílvio Gama for helpful discussions about the magnetohydrodynamics equations and to the referees for many improvements they suggested.

Centro de Matemática da Universidade do Porto (CMUP) is supported by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the Fundação para a Ciência e a Tecnologia (FCT) under the project PEst-C/MAT/UI0144/2011. The author was supported by the grant SFRH/BD/28936/2006 of FCT.

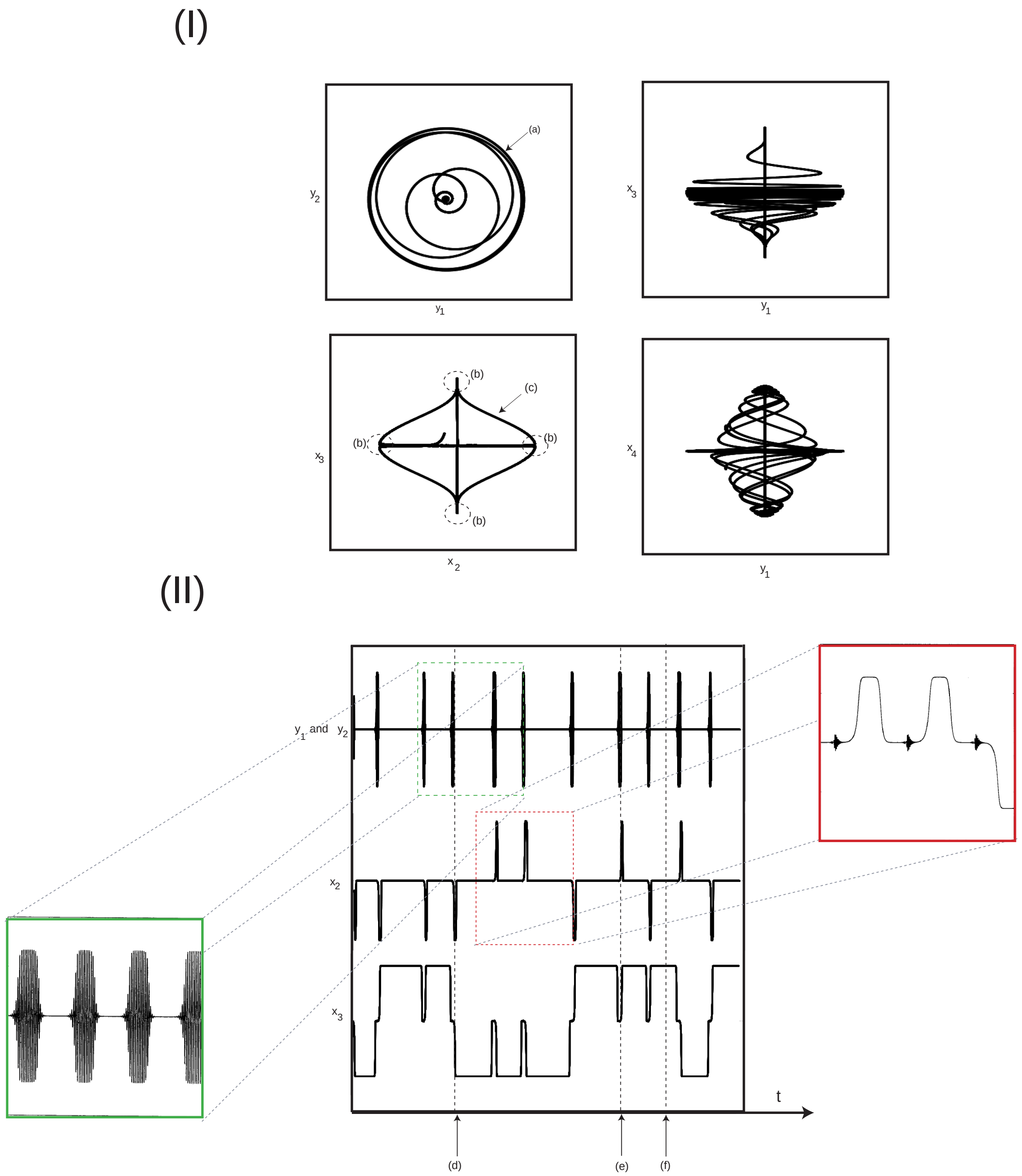


Figure 10: (I) Projection in the $(y_1, y_2)$, $(x_3, x_4)$, $(y_1, x_3)$and $(y_1, x_4)$–planes of the trajectory with initial condition $(-0.5000, 0.0116, -0.1623, -0.2781)$ for the flow corresponding to the vector field (5.7), with $\mu_1 = 0.3$, $\mu_2 = 0.2$, $\mu_3 = 0.4$, $A_{12} = A_{21} = -0.33333$, $A_{13} = A_{31} = -0.5$, $A_{23} = A_{32} = -0.16667$, $\omega_1 = 1$, $\varepsilon_1 = 0.12$, $\varepsilon_2 = 0.1$ and $\varepsilon_3 = 0.001$. The initial condition is $(-0.5000, 0.0116, -0.1623, -0.2781)$. Caption: (a): periodic solution; (b): equilibria; (c): non robust heteroclinic orbit from $\mathcal{Q}$ to $\mathcal{D}$. The interval range for all the variables is $[-3.5, 3.5]$. (II) Time series for the trajectory considered in I. Caption: (d): Reversion (the vector field $B$ change its sign); (e): Excursion (the geometry of the flow associated to $B$ varies without changing the sign); (f): the axial dipole symmetric mode is the predominant situation.

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