A radial version of the Kontorovich-Lebedev transform in the unit ball

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Abstract

In this paper we introduce a radial version of the Kontorovich-Lebedev transform in the unit ball. Mapping properties and an inversion formula are proved in L_p.

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1 Introduction

The Kontorovich-Lebedev transform (KL-transform) was introduced by the soviet mathematicians M.J. Kontorovich and N.N. Lebedev in 1938-1939 (see [4]) to solve certain boundary-value problems. The KL-transform arises naturally when one uses the method of separation of variables to solve

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boundary-value problems formulated in terms of cylindrical coordinate systems. It has been tabulated by Erdelyi et al, (see [3]) and Prudnikov et al, (see [11]). Its applications to the Dirichlet problem for a wedge were given by Lebedev in 1965 (see [5]), while Lowndes in 1959 (see [7]) applied a variant of it to a problem of diffraction of transient electromagnetic waves by a wedge. Some other applications can be found, for instance, in Skalskaya and Lebedev in 1974 (see [6]).

This transform was extended by Zemanian in 1975 (see [13]) to the distributional case, by Buggle in 1977 (see [1]) to some larger spaces of generalized functions. A possible extension to the multidimensional case of this index transform was investigated by the first author in his book (see [12]), where it was introduced the essentially multidimensional KL-transform.

The main goal of this work is to introduce a radial version of the KL-transform for the multidimensional case in the unit ball, prove its mapping properties and establish an inversion formula.

Formally, the one dimensional KL-transform is defined as

$$\mathcal{K}_{i\tau}[f] = \int_{\mathbb{R}_+} K_{i\tau}(x)\ f(x)\ dx, \tag{1}$$

where $K_{i\tau}$ denotes the modified Bessel function of pure imaginary index $i\tau$ (also called Macdonald's function). The adjoint operator associated to (1) is

$$f(x) = \frac{2}{\pi^2\ x}\int_{\mathbb{R}_+} \tau\ \sinh(\pi\tau)\ K_{i\tau}(x)\ \mathcal{K}_{i\tau}[f]\ d\tau. \tag{2}$$

As we can see, in expression (2) the integration is realized with respect to the index $i\tau$ of the Macdonald's function. This fact, for instance, carries extra difficulties in the deduction of norm estimates in certain function spaces. For more details about the one-dimensional KL-transform and other index transforms see [12].

The Macdonald's function can be represented by the following Fourier integral (see [2])

$$K_{i\tau}(x) = \int_{\mathbb{R}_+} e^{-x\cosh u}\ \cos(\tau u)\ du, \quad x>0 \tag{3}$$

$$= \frac{1}{2}\int_{\mathbb{R}} e^{-x\cosh u}\ e^{i\tau u}\ du, \quad x>0. \tag{4}$$

Making an extension of the previous integral equation to the strip $\delta \in \left[0, \frac{\pi}{2}\right[$ in the upper half-plane, we have, for $x>0$, the following uniform estimate

$$
\begin{aligned}
|K_{i\tau}(x)| &\leq \frac{e^{-i\tau}}{2} \int_{\mathbb{R}} e^{-x\cos\delta\cosh u}\, du \\
&= e^{-\delta\tau}\; K_0(x\cos\delta), \quad x > 0
\end{aligned} \tag{5}
$$

and in particular

$$
|K_{i\tau}(x)| \leq K_0(x), \quad x > 0,\; \tau \in \mathbb{R}. \tag{6}
$$

The modified Bessel function $K_\nu(x)$ function has the following asymptotic behavior (see [2] for more details) near the origin

$$
K_\nu(x) = O\left(x^{-|\mathrm{Re}(\nu)|}\right), \quad x \to 0,\; \nu \neq 0 \tag{7}
$$

$$
K_0(x) = O\left(\log x\right), \quad x \to 0^+. \tag{8}
$$

Using relation (2.16.52.8) in [11] we have the formulas

$$
\begin{aligned}
&\int_{\mathbb{R}_+} \tau\; \sinh((\pi-\epsilon)\tau)\; K_{i\tau}(x)\; K_{i\tau}(y)\; d\tau \\
&\quad = \frac{\pi xy\sin\epsilon}{2}\; \frac{K_1((x^2+y^2-2xy\cos\epsilon)^{\frac{1}{2}})}{(x^2+y^2-2xy\cos\epsilon)^{\frac{1}{2}}}, \quad x, y > 0,\; 0 < \delta \leq \pi.
\end{aligned} \tag{9}
$$

In the sequel we will appeal to the following definition of homogeneous functions:

Definition 1.1. *(c.f. [8]) Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is said to be homogeneous of degree α in D if and only if $f(\lambda x) = \lambda^\alpha f(x)$, for all $x \in D$, $\lambda > 0$ and $\lambda x \in D$. Here $\alpha \in \mathbb{R}$.*

2 Definition, basic properties and inversion

In this section we introduce the radial KL-transform. Given a function f defined in B^n_+, the radial KL-transform of f is given by

$$
\mathcal{K}_{i\tau}[f] = \int_{B^n_+} K_{i\tau}\left(|x|^2\right)\; f(x)\; dx, \tag{10}
$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$, $dx = dx_1 \wedge \ldots \wedge dx_n$ and

$$
B^n_+ = \left\{x \in \mathbb{R}^n_+ :\; |x| \leq 1\right\}.
$$

We remark that for the case of $n = 1$, the index transform (10) is a similar one used by Naylor in [9]. From (10) and definition of the Macdonald's function (3), we conclude that the KL-transform of a function f is an even function of real variable τ and, without loss of generality, we can consider only nonnegative variable τ. From the asymptotic behavior of the Macdonald's function given by (7), (8) and the Hölder inequality we observe that (10) is absolutely convergent for any function $f \in L_p(B_+^n)$. We have

Lemma 2.1. *Let $f \in L_p(B_+^n)$, with $1 < p < +\infty$. Then the following uniform estimate by $\tau \geq 0$ for the KL-transform* (10) *holds*

$$|\mathcal{K}_{i\tau}[f]| \leq \mathcal{C}_1 \ ||f||_{L_p(B_+^n)}, \tag{11}$$

where C is an absolute positive constant given by

$$\mathcal{C}_1 = \left(\frac{(2\pi)^{2n-3}}{8q}\right)^{\frac{1}{2q}} \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{4q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4q}\right)}, \tag{12}$$

with $q = \dfrac{p}{p-1}$.

Proof. To establish (11) we appeal to (6) and the Hölder inequality in order to obtain

$$\begin{aligned} |\mathcal{K}_{i\tau}[f]| &\leq \int_{B_+^n} K_0(|x|^2) \ |f(x)| \ dx \\ &\leq \left(\int_{B_+^n} K_0^q(|x|^2) \ dx\right)^{\frac{1}{q}} \left(\int_{B_+^n} |f(x)|^p \ dx\right)^{\frac{1}{p}} \\ &= \left(\int_{B_+^n} K_0^q(|x|^2) \ dx\right)^{\frac{1}{q}} \ ||f||_{L_p(B_+^n)}. \end{aligned} \tag{13}$$

Further, using spherical coordinates, generalized Minkowski inequality and relation (2.5.46.6) in Prudnikov et al, [10], we get, in turn,

$$\begin{aligned} \left(\int_{B_+^n} K_0^q(|x|^2) \ dx\right)^{\frac{1}{q}} &\leq \int_{\mathbb{R}_+} \left(\int_{B_+^n} e^{-q|x|^2 \cosh u} \ dx\right)^{\frac{1}{q}} \ du \\ &= \int_{\mathbb{R}_+} \left((2\pi)^{n-2} \int_0^1 e^{-q\rho^2 \cosh u} \rho^{n-1} d\rho\right)^{\frac{1}{q}} \ du \end{aligned}$$

$$\leq \int_{\mathbb{R}_+} \left((2\pi)^{n-2} \int_0^{+\infty} e^{-q\rho^2 \cosh u} d\rho \right)^{\frac{1}{q}} du$$

$$= \left(\frac{(2\pi)^{n-2}}{2} \sqrt{\frac{\pi}{q}} \right)^{\frac{1}{q}} \int_{\mathbb{R}_+} \frac{1}{(\cosh u)^{\frac{1}{2q}}} du$$

$$= \left(\frac{(2\pi)^{2n-3}}{8q} \right)^{\frac{1}{2q}} \left(\frac{\pi}{4} \right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{4q}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{4q}\right)} =: \mathcal{C}_1.$$

□

The previous lemma shows that the KL-transform of a L_p−function is a continuous function on τ in $\mathbb{R}_+$ in view of uniform convergence in (10). Moreover, we can deduce its differential properties. Precisely, performing the differentiation by τ of arbitrary order $k = 0, 1, \ldots$ under the integral representation (4) by Lebesgue's theorem we find

$$\frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x|^2 \cosh u} \, e^{i\tau u} \, (iu)^k \, du, \tag{14}$$

and

$$\left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) \right| \leq \int_{\mathbb{R}_+} e^{-|x|^2 \cosh u} \, u^k \, du. \tag{15}$$

Lemma 2.2. *Under the conditions of Lemma 2.1 the KL-transform* (10) *is an infinitely differentiable function on the nonnegative real axis and for any* $k = 0, 1, \ldots$ *we have the following estimate*

$$\left| \frac{\partial^k}{\partial \tau^k} \mathcal{K}_{i\tau}[f] \right| \leq \mathcal{B}_k \, ||f||_{L_p(B_+^n)}, \tag{16}$$

where

$$\mathcal{B}_k = \left(\frac{(2\pi)^{n-1}}{4\sqrt{\pi q}} \right)^{\frac{1}{q}} \int_{\mathbb{R}_+} \frac{u^k}{(\cosh u)^{\frac{1}{2q}}} du, \quad k = 0, 1, 2, \ldots. \tag{17}$$

Proof. As in Lemma 2.1, making use of the Hölder inequality we derive

$$\left| \frac{\partial^k}{\partial \tau^k} \mathcal{K}_{i\tau}[f] \right| \leq \left(\int_{B_+^n} \left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) \right| dx \right)^{\frac{1}{q}} \, ||f||_{L_p(B_+^n)}.$$

Using estimate (15) it gives

$$\begin{aligned}
\left(\int_{B_+^n}\left|\frac{\partial^k}{\partial\tau^k}K_{i\tau}(|x|^2)\right|\,dx\right)^{\frac{1}{q}} &\le \int_{\mathbb{R}_+} u^k\left(\int_{B_+^n} e^{-q\;|x|^2\;\cosh u}\,dx\right)^{\frac{1}{q}}\,du\\
&\le \int_{\mathbb{R}_+} u^k\left(\frac{(2\pi)^{n-2}}{2}\sqrt{\frac{\pi}{q\cosh u}}\right)^{\frac{1}{q}}\,du\\
&= \left(\frac{(2\pi)^{n-1}}{4\sqrt{\pi q}}\right)^{\frac{1}{q}}\int_{\mathbb{R}_+}\frac{u^k}{(\cosh u)^{\frac{1}{2q}}}\,du\\
&=: \mathcal{B}_k.
\end{aligned}$$

□

From the above properties of the KL-transform (10) one can discuss its belonging to $L_r(\mathbb{R}_+)$ for some $1 < r < +\infty$, investigating only its behavior at infinity.

Lemma 2.3. *The KL-transform* (10) *is a bounded map from any space* $L_p(B_+^n)$, *with* $p \ge 1$, *into the space* $L_r(\mathbb{R}_+)$, *where* $r \ge 1$ *and parameters* p *and* r *have no dependence.*

Proof. Taking into account (5), with $\delta = \frac{\pi}{3}$, we obtain

$$\begin{aligned}
|\mathcal{K}_{i\tau}[f]| &\le e^{-\frac{\pi\tau}{3}}\int_{B_+^n} K_0\left(\frac{|x|^2}{2}\right)\,|f(x)|\,dx\\
&\le e^{-\frac{\pi\tau}{3}}\left(\int_{B_+^n} K_0^q\left(\frac{|x|^2}{2}\right)\,dx\right)^{\frac{1}{q}}\left(\int_{B_+^n}|f(x)|^p\,dx\right)^{\frac{1}{q}}\\
&\le e^{-\frac{\pi\tau}{3}}\int_{\mathbb{R}_+}\left(\int_{B_+^n} e^{-\frac{q|x|^2\cosh u}{2}}\,dx\right)^{\frac{1}{q}}\,du\;||f||_{L_p(B_+^n)}\\
&= e^{-\frac{\pi\tau}{3}}\int_{\mathbb{R}_+}\left((2\pi)^{n-2}\int_0^1 e^{-\frac{q\rho^2\cosh u}{2}}\rho^{n-1}d\rho\right)^{\frac{1}{q}}\,du\;||f||_{L_p(B_+^n)}\\
&\le e^{-\frac{\pi\tau}{3}}\int_{\mathbb{R}_+}\left((2\pi)^{n-2}\int_0^{+\infty} e^{-\frac{q\rho^2\cosh u}{2}}d\rho\right)^{\frac{1}{q}}\,du\;||f||_{L_p(B_+^n)}\\
&= e^{-\frac{\pi\tau}{3}}\left(\frac{(2\pi)^{n-2}}{2}\sqrt{\frac{2\pi}{q}}\right)^{\frac{1}{q}}\int_{\mathbb{R}_+}\frac{1}{(\cosh u)^{\frac{1}{2q}}}\,du\;||f||_{L_p(B_+^n)}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\frac{\pi\tau}{3}} \left(\frac{(2\pi)^{2n-3}}{4q}\right)^{\frac{1}{2q}} \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{4q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4q}\right)} \, ||f||_{L_p(B_+^n)} \\
&= \mathcal{C}_2 \, e^{-\frac{\pi\tau}{3}} \, ||f||_{L_p(B_+^n)}. \qquad (18)
\end{aligned}
$$

□

Corolary 2.4. *The classical L_p−norm for the KL-transform* (10) *in the space $L_r(\mathbb{R}_+)$, with $r \geq 1$ is finite.*

Proof. In fact,

$$
\begin{aligned}
||\mathcal{K}_{i\tau}[f]||_{L_p(\mathbb{R}_+)} &\leq \mathcal{C}_2 \left(\int_0^{+\infty} e^{-p\delta\tau} d\tau\right)^{\frac{1}{p}} ||f||_{L_p(B_+^n)} \\
&= \frac{\mathcal{C}_2}{(p\delta)^{\frac{1}{p}}} \, ||f||_{L_p(B_+^n)},
\end{aligned}
$$

which proves our result. □

Lemmas 2.1, 2.2 and 2.3 show that the KL-transform of an arbitrary L_p−function is a smooth function with L_r−properties and furthermore, its range

$$
\mathcal{K}_{i\tau}(L_p(B_+^n)) = \left\{g: \; g(\tau) = \mathcal{K}_{i\tau}[f]; \; f \in L_p(B_+^n)\right\}, \quad 1 < p < +\infty \qquad (19)
$$

does not coincides with the space $L_r(\mathbb{R}_+)$.

Our next aim is to obtain an inversion formula for the radial KL-transform (10). For this purpose we shall use the regularization operator of type

$$
(I_\epsilon g)(x) = \frac{4|x|^{-n}(\sin\epsilon)^2}{(2\pi)^{n-1}} \int_{\mathbb{R}_+} \tau \, \sinh((\pi-\epsilon)\tau) \, K_{i\tau}(|x|^2) \, g(\tau) \, d\tau, \qquad (20)
$$

where $x \in B_+^n$ and $\epsilon \in]0,\pi[$.

Theorem 2.5. *Let $p > 1$ and $n \in \mathbb{N}$. On functions $g(\tau) = \mathcal{K}_{i\tau}[f]$ which are represented by* (10) *with density function $f \in L_p(B_+^n)$, operator* (20) *has the following representation*

$$
\begin{aligned}
&(I_\epsilon g)(x) \\
&= \frac{|x|^{-n+2}(\sin\epsilon)^3}{(2\pi)^{n-2}} \int_{B_+^n} \frac{K_1((|x|^4+|y|^4-2|x|^2|y|^2\cos\epsilon)^{\frac{1}{2}})}{(|x|^4+|y|^4-2|x|^2|y|^2\cos\epsilon)^{\frac{1}{2}}} \, |y|^2 \, f(y) \, dy, \qquad (21)
\end{aligned}
$$

where $K_1(z)$ is the Macdonald's function of index 1.

Proof. Substituting the value of $g(\tau)$ as the KL-transform (10) into (20), we change the order of integration by Fubini's theorem taking into account the estimate (5)

$$|(I_\epsilon g)(x)| \leq \frac{4K_0(|x|^{2n}\cos\delta_1)(\sin\epsilon)^2}{|x|^n(2\pi)^{n-1}}$$
$$\times \int_{\mathbb{R}_+} \tau\ \sinh((\pi-\epsilon)\tau)\ e^{-(\delta_1+\delta_2)\tau} \int_{B_+^n} K_0(|y|^2\cos\delta_2)\ |f(y)|\ dy\ d\tau, \quad (22)$$

where we choose $\delta_1,\ \delta_2$, such that $\delta_1+\delta_2+\epsilon>\pi$. Hence with (9) we get (21). □

An inversion formula of the KL-transform (10) is established by the following

Theorem 2.6. *Let $p>1$, $g(\tau)=\mathcal{K}_{i\tau}[f]$ and $f\in L_p(B_+^n)$ be a radial function, i.e., $f(x)=h(|x|)$, where h is a homogeneous of degree $2-n$. Then*

$$f(x)=\lim_{\epsilon\to 0}\frac{4|x|^{-n}(\sin\epsilon)^2}{(2\pi)^{n-1}}\int_{\mathbb{R}_+}\tau\ \sinh((\pi-\epsilon)\tau)\ K_{i\tau}(|x|^2)\ g(\tau)\ d\tau \quad (23)$$

where the latter limit is with respect to L_p-norm in $L_p(B_+^n)$.

Proof. Considering the integral (21) and the classical spherical coordinates multiplied by $|x|(\sin\epsilon)^{\frac{1}{2}}$, we find

$$||(I_\epsilon g)-f||_{L_p(B_+^n)}$$

$$=\left|\left|\frac{(\sin\epsilon)^2}{(2\pi)^{n-2}}\underbrace{\int_0^{2\pi}\dots\int_0^{2\pi}}_{n-2\ times}\int_0^{\frac{\pi}{2}}\int_0^{\left[|\cdot|(\sin\epsilon)^{\frac{1}{2}}\right]^{-1}}\frac{R(|\cdot|,\rho,\epsilon)\ \rho^3}{[(\rho^2-\cot\epsilon)^2+1]}h(|\cdot|)\ d\rho\sin\phi\ d\phi\ d\theta_1\dots d\theta_{n-2}\right.\right.$$
$$\left.\left. -h(|\cdot|)\right|\right|_{L_p(B_+^n)}$$

$$=\left|\left|\frac{(\sin\epsilon)^2}{2}\int_0^{\left[|\cdot|^2\sin\epsilon\right]^{-1}}\frac{\rho}{[(\rho-\cot\epsilon)^2+1]}\left[R(|\cdot|,\sqrt{\rho},\epsilon)\ h(|\cdot|)\ -\frac{1}{\mathcal{C}_\epsilon(\cdot)}\ h(|\cdot|)\right]d\rho\right|\right|_{L_p(B_+^n)}$$
$$\leq\frac{(\sin\epsilon)^2}{2}\int_0^{\left[|\cdot|^2\sin\epsilon\right]^{-1}}\frac{\rho}{(\rho-\cot\epsilon)^2+1}\ \left|\left|R(|\cdot|,\sqrt{\rho},\epsilon)h(|\cdot|)-\frac{1}{\mathcal{C}_\epsilon(\cdot)}\ h(|\cdot|)\right|\right|_{L_p(B_+^n)}d\rho,\quad \epsilon>0 \quad (24)$$

where

$$R(|x|,\sqrt{\rho},\epsilon)=|x|^2\sin\epsilon\left[(\rho-\cot\epsilon)^2+1\right]^{\frac{1}{2}}K_1\left(|x|^2\sin\epsilon\left[(\rho-\cot\epsilon)^2+1\right]^{\frac{1}{2}}\right),\quad \epsilon>0,$$

and

$$\begin{aligned}\mathcal{C}_\epsilon(x) &= \sin\epsilon \int_0^{\left[|x|^2 \sin\epsilon\right]^{-1}} \frac{\rho}{(\rho-\cot\epsilon)^2+1}\, d\rho \\ &= \cos\epsilon \left[\arctan\left(\frac{\cos\epsilon}{\sin\epsilon}\right) - \arctan\left(\frac{|x|^2\cos\epsilon - 1}{|x|^2 \sin\epsilon}\right)\right] \\ &\quad + \frac{\sin\epsilon}{2} \ln\left(\frac{(\cos\epsilon - |x|^2)^2 + (\sin\epsilon)^2}{|x|^4}\right), \quad \epsilon > 0.\end{aligned}$$

For sufficiently small $\epsilon > 0$ we have

$$0 < \pi - O(\epsilon) < \mathcal{C}_\epsilon(x) < \pi + O(\epsilon).$$

Taking into account the relations (7) and (8), we have for $R(|x|, \sqrt{\rho}, \epsilon)$ that

$$\lim_{\epsilon\to 0^+} R(|x|, \sqrt{\rho}, \epsilon) = 1,$$

and since $xK_1(x) < 1$, for $x > 0$, we conclude that $R(|x|, \sqrt{\rho}, \epsilon)$ is bounded as a function of three variables. Further, since $R(|x|, \sqrt{\rho}, \epsilon) < 1$ we obtain

$$\begin{aligned}||(I_\epsilon g) - f||_{L_p(B_+^n)} &\le \frac{\sin\epsilon}{2}(\mathcal{C}_\epsilon + 1)||h||_{L_p(B_+^n)} \\ &= O(\epsilon) \to 0, \quad \epsilon \to 0^+, \end{aligned} \tag{25}$$

which leads to the equality (23). □

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