# ON THE PROJECTION OF FUNCTIONS INVARIANT UNDER THE ACTION OF A CRYSTALLOGRAPHIC GROUP

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Abstract. We study functions defined in $(n+1)$-dimensional domains that are invariant under the action of a crystallographic group. We give a complete description of the symmetries that remain after projection into an $n$-dimensional subspace and compare it to similar results for the restriction to a subspace. We use the Fourier expansion of invariant functions and the action of the crystallographic group on the space of Fourier coefficients. Intermediate results relate symmetry groups to the dual of the lattice of periods.

## 1. Introduction

We study real functions with domain $\mathbf{R}^{n+1}$ that are invariant under the action of a crystallographic group $\Gamma$, whose subgroup of translations is a lattice $\mathcal{L}$. We work in $X_\Gamma$, the space of $\Gamma$-invariant functions, that in particular are $\mathcal{L}$-periodic, and that have formal Fourier expansion in terms of the waves $\omega_k(x,y) = \mathrm{e}^{2\pi i <k,(x,y)>}$, with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$, $k \in \mathbf{R}^{n+1}$.

A crystallographic group $\Gamma$ is a subgroup of the Euclidean group $\mathbf{E}(n+1)$, the semi-direct product $\mathbf{E}(n+1) \cong \mathbf{R}^{n+1} \ltimes \mathbf{O}(n+1)$. We denote its elements $\gamma = (v, \delta)$, where $v \in \mathbf{R}^{n+1}$ and $\delta \in \mathbf{O}(n+1)$. We identify all elements of the form $(v+l, \delta) \in \Gamma$, $l \in \mathcal{L}$ and denote them all by $(v_\delta, \delta)$.

Given $\alpha \in \mathbf{O}(n)$, we define the elements of $\mathbf{O}(n+1)$:

$$\sigma = \begin{pmatrix} Id_n & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_- = \sigma\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}.$$

For $y_0 > 0$, consider the region in $\mathbf{R}^{n+1}$ lying between the hyperplanes $y = 0$ and $y = y_0$. The projection operator $\Pi_{y_0}$ integrates $f$ along the width $y_0$, yielding a new function with domain $\mathbf{R}^n$:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy.$$

Our main result, Theorem 1.1, relates the symmetry of the functions $f \in X_\Gamma$ to the symmetry of the projected functions $\Pi_{y_0}(f)$ in the space $\Pi_{y_0}(X_\Gamma)$:

**Theorem 1.1.** *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(I) $((v_\alpha, 0), \alpha_+) \in \Gamma$,
(II) $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
(III) $(0, y_0) \in \mathcal{L}$ *and either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$*, for some* $y_1 \in \mathbf{R}$ .

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Consider the subgroup $\Gamma_0$ of elements of $\Gamma$ with orthogonal part $\alpha_\pm$ for some $\alpha \in \mathbf{O}(n)$ of the form

$$\left((v_\alpha, y_\alpha), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) \qquad \text{with} \qquad \beta = \pm 1$$

and let $\Gamma_\pi$ be the subgroup of $\Gamma_0$ given by:

- If $(0, y_0) \notin \mathcal{L}$ then $\Gamma_\pi$ contains those elements of $\Gamma_0$ where either $y_\alpha = 0$ with $\beta = 1$ or where $y_\alpha = y_0$ with $\beta = -1$.
- If $(0, y_0) \in \mathcal{L}$ then $\Gamma_\pi = \Gamma_0$.

The following corollary to Theorem 1.1 states that the elements in $\Gamma$ that effectively contribute to the symmetry of $\Pi_{y_0}(X_\Gamma)$ are those in $\Gamma_\pi$. It also describes how elements of $\Gamma_\pi$ are transformed by the projection:

**Corollary 1.1.** *Let $\Sigma$ be the group of symmetries of $\Pi_{y_0}(X_\Gamma)$, i.e., the largest subgroup of $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$ that fixes all the elements in $\Pi_{y_0}(X_\Gamma)$. Then $\Sigma$ is the image of $\Gamma_\pi$ by the homomorphism*

$$\begin{array}{ccc} \Gamma_0 & \longrightarrow & \mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n) \\ \left((v_\alpha, y_\alpha), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) & \longmapsto & (v_\alpha, \alpha) \end{array}$$

*whose kernel is given by elements such that $v_\alpha = 0$ and $\alpha = Id_n$.*

The result was motivated by the study of patterns in reaction-diffusion experiments on thin layers, where the observation method carries information from the depth of a layer and thus corresponds to a projection whose role in the formation of a pattern is not always clear (see De Kepper *et al.* [6], Borckmans *et al.* [2] and other articles by the same authors). Gomes [8] proposes that some of these patterns may arise as the projection of a three-dimensional structure. Knowledge of projected patterns is useful when deciding whether this is the case, as in Zhou *et al.* [15]. When the thickness of the layer acts as a bifurcation parameter, as observed by De Kepper *et al.* [6], then the symmetries of the pattern may be subject to change as thickness varies.

Restricting a pattern to a slice allows its visualisation in a lower dimension. In Parker *et al.* [11], for instance, patterns are drawn through restriction to a hyperplane, but projecting a slice would codify more of its symmetries since invariant functions when restricted to a hyperplane have less symmetry than their projections, as we show in Theorem 5.1 in section 5, below.

Symmetries that do not remain after projection may give rise to structures of the projected functions that cannot be described as the invariance under an element of the Euclidean group. An illustrative example of the second case is the quasiperiodic structure obtained by the canonical projection of a periodic one, see Senechal [13, section 2.6].

**Structure of the paper.** After definitions, notation and some preliminary results in section 2, the bulk of the paper contains the proof of Theorem 1.1. Each one of the conditions (I), (II) and (III) of Theorem 1.1 is sufficient by basic properties of the integrals. Thus, we omit the proof of sufficiency for Theorem 1.1, see Pinho [12] for details.

In order to prove that the conditions of Theorem 1.1 are necessary we establish first, in Proposition 3.1, an equivalence between the $(v_\alpha, \alpha)$-invariance of all the functions in $\Pi_{y_0}(X_\Gamma)$ and properties of $\Gamma$ and of the dual lattice $\mathcal{L}^*$. This is done in section 3 using the induced action of $\Gamma$ in the space of Fourier coefficients of $\Gamma$-invariant functions, that appears as relations on the coefficients that may be traced after projection.

Then, in section 4, we show that these properties impose restrictions on $\Gamma$ and on $\mathcal{L}$ by implying the presence of some particular elements in $\Gamma$, establishing Theorem 1.1.

Finally, in section 5 a similar study is carried out for the symmetries of the restriction of invariant function to a hyperplane. This is useful in comparing the restriction and projection of a pattern. The main result of this section, Theorem 5.1, may be proved using simpler versions of the arguments developed for the projection.

The formulation of the results for sufficiently large spaces of $\Gamma$-invariant functions highlights their common characteristic, the symmetry.

## 2. Notation and Preliminary Results

The reader is referred to Armstrong [1] for results on Euclidean and plane crystallographic groups, to Senechal [13] and Miller [10] for results on lattices and crystallographic groups and to Golubitsky *et al.* [7] for results on symmetry. A detailed description appears in Pinho [12].

The action of an element $(v, \delta)$ of the Euclidean group $\mathbf{E}(n+1)$ on $(x, y)$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}$ is given by $(v, \delta) \cdot (x, y) = v + \delta(x, y)$ with the group operation $(v_1, \delta_1) \cdot (v_2, \delta_2) = (v_1 + \delta_1 v_2, \delta_1\delta_2)$, for $(v_1, \delta_1)$, $(v_2, \delta_2) \in \mathbf{E}(n+1)$.

A crystallographic group $\Gamma \leq \mathbf{E}(n+1)$ with lattice $\mathcal{L}$ is a group such that the orbit of the origin by translations $\{v : (v, Id_{n+1}) \in \Gamma\}$ is a $\mathbf{Z}$-module generated by $n+1$ linearly independent vectors $l_1, \ldots, l_{n+1} \in \mathbf{R}^{n+1}$: $\mathcal{L} = \{l_1, \ldots, l_{n+1}\}_{\mathbf{Z}} = \left\{\sum_{i=1}^{n+1} m_i l_i : m_i \in \mathbf{Z}\right\}$. We also use the symbol $\mathcal{L}$ for the subgroup of translations of $\Gamma$ isomorphic to $(\mathcal{L}, +)$.

The projection $(v, \delta) \longmapsto \delta$, of $\Gamma$ into $\mathbf{O}(n+1)$, has kernel $\mathcal{L}$. Its image, $\mathbf{J} = \{\delta : (v, \delta) \in \Gamma \text{ for some } v \in \mathbf{R}^{n+1}\}$, called the point group of $\mathcal{L}$, is isomorphic to the finite quotient $\Gamma/\mathcal{L}$ and is a subgroup of the *holohedry* of $\mathcal{L}$, the largest subgroup of $\mathbf{O}(n+1)$ that leaves $\mathcal{L}$ invariant. Thus, $\mathbf{J}\mathcal{L} = \{\delta l : \delta \in \mathbf{J}, l \in \mathcal{L}\} = \mathcal{L}$.

The set of all the elements in $\Gamma$ with orthogonal component $\delta \in \mathbf{J}$ is the coset $\mathcal{L} \cdot (v, \delta) = \{(l + v, \delta) : l \in \mathcal{L}\}$ for any $v \in \mathbf{R}^{n+1}$ such that $(v, \delta) \in \Gamma$. We use the symbol $(v_\delta, \delta)$ for any element of that coset, *i.e.*, $v_\delta$ is the non-orthogonal component of $(v, \delta) \in \Gamma$ defined up to elements of $\mathcal{L}$. The group $\Gamma$ is thus characterized by the $n+1$ generators of $\mathcal{L}$ plus a finite number of elements $(v_\delta, \delta)$, with $\delta \in \mathbf{J}$.

The action of $\Gamma$ in $\mathbf{R}^{n+1}$ induces the scalar action on functions: $(\gamma \cdot f)(x, y) = f(\gamma^{-1} \cdot (x, y))$ for $\gamma \in \Gamma$ and $(x, y) \in \mathbf{R}^{n+1}$, see Melbourne [9]. A function $f$ is $\Gamma$-*invariant* if $(\gamma \cdot f)(x, y) = f(x, y)$, for all $\gamma \in \Gamma$ and all $(x, y) \in \mathbf{R}^{n+1}$.

The dual lattice of $\mathcal{L}$ is the set of all the elements $k \in \mathbf{R}^{n+1}$ such that $\omega_k$ is $\mathcal{L}$-periodic, given by $\mathcal{L}^* = \{k \in \mathbf{R}^{n+1} :< k, l_i >\in \mathbf{Z},\ i = 1, \ldots, n+1\}$, where $< \cdot, \cdot >$ denotes the usual inner product in $\mathbf{R}^{n+1}$. It may be written as $\mathcal{L}^* = \{l_1^*, \ldots, l_{n+1}^*\}_{\mathbf{Z}}$, where $l_i^* \in \mathbf{R}^{n+1}$ is the dual basis satisfying $< l_i^*, l_j >= \delta_{ij}$ for all $i, j \in \{1, \ldots, n+1\}$. The lattices $\mathcal{L}$ and $\mathcal{L}^*$ have the same holohedry.

The formal Fourier expansion of a function $f \in X_\Gamma$ is

$$f(x, y) = \sum_{k \in \mathcal{L}^*} \omega_k(x, y) C(k)$$

where $C : \mathcal{L}^* \longrightarrow \mathbf{C}$ are the Fourier coefficients. We assume that in $X_\Gamma$ this expansion is unique. For a real function $f$ we have $\overline{C(k)} = C(-k)$. From the action of $\Gamma$ on $X_\Gamma$ we get:

$$\begin{aligned}(v_\delta, \delta) \cdot f(x, y) &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_{\delta k}(x, y) \omega_{\delta k}(-v_\delta) C(k), \text{ by orthogonality of } \delta, \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_k(x, y) \omega_k(-v_\delta) C(\delta^{-1} k), \text{ because } \delta\mathcal{L}^* = \mathcal{L}^*.\end{aligned}$$

By the unicity of the Fourier expansion, this induces an action of $\Gamma$ on the space of Fourier coefficients $(v_\delta,\delta)\cdot C(k)=\omega_k(-v_\delta)C(\delta^{-1}k)$. Analogously, the $(v_\delta,\delta)$-invariance of $f$ implies $C(k)=\omega_k(-v_\delta)C(\delta^{-1}k)$ for all its Fourier coefficients.

The simplest $\Gamma$-invariant functions are the real and imaginary components of $I_k(x,y)=\sum_{\delta\in\mathbf{J}}\omega_{\delta k}(x,y)\omega_{\delta k}(-v_\delta)$, for $k\in\mathcal{L}^*$, and we will assume that they lie in $X_\Gamma$. Each function $I_k$, for $k\in\mathcal{L}^*$, is the sum of the elements in the $\Gamma$-orbit of $\omega_k$.

If $f\in X_\Gamma$ then the projected function satisfies $\Pi_{y_0}(f)(x)=\int_0^{y_0}\sum_{k\in\mathcal{L}^*}\omega_k(x,y)C(k)dy$. When the integral and the summation commute, then

$$\begin{aligned}\Pi_{y_0}(f)(x) &= \textstyle\sum_{k\in\mathcal{L}^*}\int_0^{y_0}\omega_k(x,y)C(k)dy\\ &= \textstyle\sum_{k\in\mathcal{L}^*}\omega_{k_1}(x)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,\end{aligned}$$

where $k=(k_1,k_2)$, with $k_1\in\mathbf{R}^n$ and $k_2\in\mathbf{R}$. Grouping terms with common $n$ first components in $\mathcal{L}^*$, we obtain

$$\begin{aligned}\Pi_{y_0}(f)(x) &= \textstyle\sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(x)\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy\\ &= \textstyle\sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(x)D(k_1),\end{aligned}$$

where $\mathcal{L}_1^*=\{k_1:(k_1,k_2)\in\mathcal{L}^*\}$ and $D(k_1)=\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy$. Note that the coefficients $D(k_1)$ depend on $y_0$.

The functions $\Pi_{y_0}(f)$ may be invariant under the action of some element $(v_\alpha,\alpha)$ of $\mathbf{E}(n)\cong\mathbf{R}^n\ltimes\mathbf{O}(n)$. For $f\in X_\Gamma$ this is equivalent to

$$\sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(x)D(k_1)=\sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(\alpha^{-1}x)\omega_{k_1}(-\alpha^{-1}v_\alpha)D(k_1).$$

This equation imposes restrictions on the coefficients $D(k_1)$, see Lemma 3.3 below.

Summarising, we assume $X_\Gamma$ is a vector space of functions such that:

(1) $\Gamma$ is a $(n+1)$-dimensional crystallographic group with lattice $\mathcal{L}$, dual lattice $\mathcal{L}^*$ and point group $\mathbf{J}$,
(2) if $f\in X_\Gamma$ then:
  (i) $f:\mathbf{R}^{n+1}\longrightarrow\mathbf{R}$ is $\Gamma$-invariant,
  (ii) $f$ has a unique formal Fourier expansion in waves $\omega_k(x,y)$, $k\in\mathcal{L}^*$,
  (iii) the integral and the summation commute in the projection of $f$,
(3) $Re(I_k),Im(I_k)\in X_\Gamma$ for all $k\in\mathcal{L}^*$ with $I_k(x,y)=\sum_{\delta\in\mathbf{J}}\omega_{\delta k}(x,y)\omega_{\delta k}(-v_\delta)$.

## 3. Symmetry of $\Pi_{y_0}(X_\Gamma)$ Related to $\Gamma$ and $\mathcal{L}^*$

For simplicity of notation we write $(v_+,\alpha_+)$ for $(v_{\alpha_+},\alpha_+)$ and $(v_-,\alpha_-)$ for $(v_{\alpha_-},\alpha_-)$. The simultaneous presence of the reflection $(v_\sigma,\sigma)$ and of $(v_+,\alpha_+)$ in a group $\Gamma$ imposes strong restrictions on $\mathcal{L}^*$. One of these restrictions is the subject of the next Lemma.

**Lemma 3.1.** *If both $(v_\sigma,\sigma)\in\Gamma$ and $(v_+,\alpha_+)\in\Gamma$ then $2(\sigma v_+-v_+)\in\mathcal{L}$.*

*Proof.* Since $(v_\sigma,\sigma)\cdot(v_+,\alpha_+)=(v_\sigma+\sigma v_+,\alpha_-)$ and $(v_+,\alpha_+)\cdot(v_\sigma,\sigma)=(v_++\alpha_+v_\sigma,\alpha_-)$, then $v=v_\sigma+\sigma v_+-v_+-\alpha_+v_\sigma\in\mathcal{L}$. As $\sigma\mathcal{L}=\mathcal{L}$ then $v-\sigma v=2(\sigma v_+-v_+)+(Id_{n+1}-\alpha_+-\sigma+\alpha_-)v_\sigma$ also belongs to $\mathcal{L}$. Using $-\alpha_+-\sigma+\alpha_-=-Id_{n+1}$ we get $v-\sigma v=2(\sigma v_+-v_+)$ or, equivalently, $2<k,\sigma v_+-v_+>\in\mathbf{Z}$ for all $k\in\mathcal{L}^*$. □

**Proposition 3.1.** *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha,\alpha)\in\mathbf{R}^n\ltimes\mathbf{O}(n)$ if and only if one of the following conditions holds:*

(A) *$(v_+,\alpha_+)\in\Gamma$ and*
*for each $k\in\mathcal{L}^*$ either $<k,(0,y_0)>\in\mathbf{Z}-\{0\}$ or $<k,v_+-(v_\alpha,0)>\in\mathbf{Z}$,*
(B) *$(v_-,\alpha_-)\in\Gamma$ and*
*for each $k\in\mathcal{L}^*$ either $<k,(0,y_0)>\in\mathbf{Z}-\{0\}$ or $<k,v_--(v_\alpha,y_0)>\in\mathbf{Z}$,*

(C) *both* $(v_\sigma, \sigma) \in \Gamma$ *and* $(v_+, \alpha_+) \in \Gamma$. *Moreover, if* $< k, \sigma v_+ - v_+ > \in \mathbf{Z}$ *then one of the conditions (Ci), (Cii) or (Ciii) below holds and, if* $< k, \sigma v_+ - v_+ > + \frac{1}{2} \in \mathbf{Z}$, *one of the conditions (Ci) or (Civ) holds:*
  (i) $< k, (0, y_0) > \in \mathbf{Z} - \{0\}$,
  (ii) $< k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$,
  (iii) $< k, v_\sigma - (0, y_0) > + \frac{1}{2} \in \mathbf{Z}$,
  (iv) $< k, v_- - (v_\alpha, y_0) > \in \mathbf{Z}$ *and* *either* $< k, v_\sigma - (0, y_0) > + \frac{1}{4} \in \mathbf{Z}$ *or* $< k, v_\sigma - (0, y_0) > - \frac{1}{4} \in \mathbf{Z}$.

A more concise formulation of this result is possible using the subsets of $\mathcal{L}^*$ that we proceed to define. Let $\mathcal{M}^*$, $\mathcal{M}^*_+$ and $\mathcal{M}^*_-$ be the modules

$$\mathcal{M}^* = \{k \in \mathcal{L}^* : < k, \sigma v_+ - v_+ > \in \mathbf{Z}\},$$
$$\mathcal{M}^*_+ = \{k \in \mathcal{L}^* : < k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}\}, \mathcal{M}^*_- = \{k \in \mathcal{L}^* : < k, v_- - (v_\alpha, y_0) > \in \mathbf{Z}\}$$

and let

$$\mathcal{N}^* = \left\{k \in \mathcal{L}^* : < k, \sigma v_+ - v_+ > + \frac{1}{2} \in \mathbf{Z}\right\}$$
$$\mathcal{N}^*_{y_0} = \{k \in \mathcal{L}^* : < k, (0, y_0) > \in \mathbf{Z} - \{0\}\}$$
$$\mathcal{N}^*_\sigma = \left\{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > + \frac{1}{2} \in \mathbf{Z}\right\}$$
$$\mathcal{N}^*_{\tilde\sigma} = \left\{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > \pm \frac{1}{4} \in \mathbf{Z}\right\}.$$

The last four sets are not modules. The smallest modules generated by them are

$$\overline{\mathcal{N}^*} = \mathcal{N}^* \cup \mathcal{M}^* \qquad \overline{\mathcal{N}^*_{y_0}} = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_{y_0} \qquad \overline{\mathcal{N}^*_\sigma} = \mathcal{N}^*_\sigma \cup \mathcal{M}^*_\sigma \qquad \overline{\mathcal{N}^*_{\tilde\sigma}} = \mathcal{N}^*_{\tilde\sigma} \cup \overline{\mathcal{N}^*_\sigma},$$

where $\overline{\mathcal{N}^*} = \mathcal{L}^*$ under the conditions of Lemma 3.1, all the unions are disjoint and $\mathcal{M}^*_{y_0}$ and $\mathcal{M}^*_\sigma$ are the modules

$$\mathcal{M}^*_{y_0} = \{k \in \mathcal{L}^* : < k, (0, y_0) >= 0\} \text{ and } \mathcal{M}^*_\sigma = \{k \in \mathcal{L}^* : < k, v_\sigma - (0, y_0) > \in \mathbf{Z}\}.$$

In the sequel we will use:

**Properties of $\mathcal{N}^*_\sigma$ and $\mathcal{N}^*_{\tilde\sigma}$.** *Let* $m_1, m_2 \in \mathbf{Z}$.

(1) *If* $g_1, g_2 \in \mathcal{N}^*_\sigma$ *then* $m_1 g_1 + m_2 g_2 \in \begin{cases} \mathcal{M}^*_\sigma & \text{if } m_1 + m_2 \text{ even} \\ \mathcal{N}^*_\sigma & \text{if } m_1 + m_2 \text{ odd} \end{cases}$.

(2) *If* $g_1, g_2 \in \mathcal{N}^*_{\tilde\sigma}$ *then* $m_1 g_1 + m_2 g_2 \in \begin{cases} \overline{\mathcal{N}^*_\sigma} & \text{if } m_1 + m_2 \text{ even} \\ \mathcal{N}^*_{\tilde\sigma} & \text{if } m_1 + m_2 \text{ odd} \end{cases}$.

**Lemma 3.2** (Properties of the bases for $\mathcal{L}$ and $\mathcal{L}^*$ and notation)**.** *Let* $\{l_1, \ldots, l_{n+1}\}$ *be a basis for* $\mathcal{L}$ *and* $\{l^*_1, \ldots, l^*_{n+1}\}$ *be its dual basis. The matrices* $M$ *with rows* $l_1, \ldots, l_{n+1}$ *and* $M^*$ *with rows* $l^*_1, \ldots, l^*_{n+1}$, *are related by* $M^* = \left(M^{-1}\right)^T$ *and satisfy:*

(1) *If* $(v_\delta, \delta) \in \Gamma$ *then, given the real numbers* $r_1, \ldots, r_{n+1}$, *we may write* $v_\delta = \sum_{i=1}^{n+1} s_i l_i$ *with* $(s_i - r_i) \in [0, 1[$ *for all* $i \in \{1, \ldots, n+1\}$.

(2) *If* $(0, a) \in \mathcal{L}$ *for some* $a \neq 0$ *then we may choose the basis* $\{l_1, \ldots, l_{n+1}\}$ *for* $\mathcal{L}$ *such that*
  (i) $M = \begin{pmatrix} A & B \\ 0 & b \end{pmatrix}$ *where* $A$ *is an* $n \times n$ *matrix with rows* $a_1, \ldots, a_n \in \mathbf{R}^n$ *and* $B = (b_1, \ldots, b_n)^T$, *with* $b = \frac{a}{m}$ *for some* $m \in \mathbf{Z}$ *and* $b_i \in \mathbf{R}$.
  (ii) $M^* = \begin{pmatrix} A^* & 0 \\ -\frac{1}{b} B^T A^* & \frac{1}{b} \end{pmatrix}$, *where* $A^* = \left(A^{-1}\right)^T$ *has rows* $a^*_1, \ldots, a^*_n$ *with* $< a^*_i, a_j >= \delta_{ij}$ *for* $i, j \in \{1, \ldots, n\}$.
  (iii) *The set* $\{a_1, \ldots, a_n\}$ *is a basis for a lattice in* $\mathbf{R}^n$ *and* $\{a^*_1, \ldots, a^*_n\}$ *is a basis for its dual.*

(iv) $l_i^* = (a_i^*, 0)$ *for* $i \in \{1, \ldots, n\}$ *and* $\mathcal{M}_{y_0}^* = \{l_1^*, \ldots, l_n^*\}_{\mathbf{Z}}$.

(3) *If* $\sigma$ *lies in the holohedry of* $\mathcal{L}$ *then* $\mathcal{L}$ *contains an element of the form* $(0, a)$, $a \neq 0$ *and each entry* $b_i$ *of* $B$ *may be taken to be either zero or* $b/2$.

*Proof.* (1) The set $\{l_1, \ldots, l_{n+1}\}$ is a basis for $\mathbf{R}^{n+1}$ and so $v_\delta = \sum_{i=1}^{n+1} s_i l_i$ with $s_i \in \mathbf{R}$ for all $i \in \{1, \ldots, n+1\}$. As $v_\delta$ is defined up to elements of $\mathcal{L}$ then we may restrict each $s_i$ to an interval $[r_i, r_i + 1[$, where $r_i \in \mathbf{R}$.

(2) Given $(0, a) \in \mathcal{L}$, $a \neq 0$, then $(0, b)$, $b \neq 0$, the smallest element of $\mathcal{L}$ in the direction of $(0, a)$, is a generator and $(0, a) = m(0, b)$ for some $m \in \mathbf{Z}$. Moreover, there are elements $l_1, \ldots, l_n$ in $\mathcal{L}$ such that $\mathcal{L} = \{l_1, \ldots, l_n, (0, b)\}_{\mathbf{Z}}$. For $l_i = (a_i, b_i)$, with $i \in \{1, \ldots, n\}$, and $(0, b) = l_{n+1}$ we obtain the matrix $M$ and $M^*$ has the form given in (2ii). Property (2iv) follows by definition of $\mathcal{M}_{y_0}^*$.

(3) For $(c, d)$ in $\mathcal{L}$ with $d \neq 0$, since $\sigma\mathcal{L} = \mathcal{L}$ then $(c, d) - \sigma(c, d) = (0, 2d) \in \mathcal{L}$ and property (2) is valid. For $l_i = (a_i, b_i)$, with $i \in \{1, \ldots, n\}$, the elements $l_i - \sigma l_i = (0, 2b_i)$ lie in $\mathcal{L}$ and so $(0, 2b_i) = m(0, b)$ for some $m \in \mathbf{Z}$. Therefore $l_i = \left(a_i, \frac{mb}{2}\right)$, and either $l_i = (a_i, 0)$ or $l_i = \left(a_i, \frac{b}{2}\right)$ up to multiples of $(0, b) = l_{n+1}$. □

With the notation above Proposition 3.1 takes the equivalent form:

**Proposition 3.2.** *All functions in* $\Pi_{y_0}(X_\Gamma)$ *are invariant under the action of* $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ *if and only if one of the following conditions holds:*

(A) $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{M}_+^*$,

(B) $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{M}_-^*$,

(C) *both* $(v_\sigma, \sigma)$ *and* $(v_+, \alpha_+)$ *belong to* $\Gamma$ *and, moreover,* $\mathcal{M}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_+^* \cup \mathcal{N}_\sigma^*\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{N}_{y_0}^* \cup \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)\right)$.

There are three main steps in the proof of Proposition 3.1. First, in Lemma 3.3, we write the $(v_\alpha, \alpha)$-invariance of the projection of $f \in X_\Gamma$ as conditions relating the operator $\Pi_{y_0}$ to the projection of the dual lattice $\mathcal{L}^*$ and to the coefficients of the formal Fourier expansion of $f$ in waves. Second, we prove that the conditions (A), (B) and (C) are sufficient, by writing explicitly the restrictions they impose on $\mathcal{L}^*$ and on the Fourier coefficients. Finally we conclude that the conditions of Proposition 3.1 are necessary by the $(v_\alpha, \alpha)$-invariance of the projection of the $\Gamma$-invariant functions $I_k$.

The tools used in this proof are properties of waves and of Fourier coefficients, due to the symmetries in $\Gamma$ and to the symmetry $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$, together with properties of the modules and subsets of $\mathcal{L}^*$ defined above. For $\alpha\mathcal{L}_1^* = \{\alpha k_1 : k_1 \in \mathcal{L}_1^*\}$, we have:

**Lemma 3.3.** *Let* $f \in X_\Gamma$ *and* $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$. *The projection* $\Pi_{y_0}(f)(x)$ *is* $(v_\alpha, \alpha)$*-invariant if and only if for each* $k_1 \in \mathcal{L}_1^*$ *the following conditions hold:*

(1) *if* $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then* $D(k_1) = \omega_{k_1}(-v_\alpha) D(\alpha^{-1} k_1)$,

(2) *if* $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then* $D(k_1) = 0$.

*Proof.* Notice first that the equality

$$\Pi_{y_0}(f)(x) = (v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x) = \Pi_{y_0}(f)(\alpha^{-1}x - \alpha^{-1}v_\alpha)$$

is equivalent to

$$\sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(\alpha^{-1}x) \omega_{k_1}(-\alpha^{-1}v_\alpha) D(k_1), \tag{1}$$

where, by orthogonality, the right hand side equals $\sum_{k_1 \in \mathcal{L}_1^*} \omega_{\alpha k_1}(x) \omega_{\alpha k_1}(-v_\alpha) D(k_1)$ and, for $\tilde{k}_1 = \alpha k_1$, is given by $\sum_{\tilde{k}_1 \in \alpha\mathcal{L}_1^*} \omega_{\tilde{k}_1}(x) \omega_{\tilde{k}_1}(-v_\alpha) D(\alpha^{-1}\tilde{k}_1)$. Thus, by the unicity of the Fourier expansion, expression (1) is valid for all $x \in \mathbf{R}^n$ if and only if, for any $k_1 \in \mathcal{L}_1^*$, the conditions hold. □

*Proof of sufficiency in Proposition 3.1.* We write $D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1)$, in the form

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)G(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,$$

which is zero since, by conditions (A), (B) or (C), either $\int_0^{y_0}\omega_{k_2}(y)dy = 0$, if $< k,(0,y_0) >\in \mathbf{Z}-\{0\}$, or $G(k_1,k_2)$ vanishes. Below we compute $G(k_1,k_2)$ explicitly for each case.

Suppose either condition (A) or condition (B) happens. Since $\alpha_\pm$ (either $\alpha_+$ or $\alpha_-$) is in $\mathbf{J}$ then $\alpha_\pm\mathcal{L}^* = \{\alpha_\pm k : k\in\mathcal{L}^*\} = \mathcal{L}^*$, which implies $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$. Therefore, for any $f\in X_\Gamma$, the projection $\Pi_{y_0}(f)(x) = \sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(x)D(k_1)$ is $(v_\alpha,\alpha)$-invariant if and only if condition (1) of Lemma 3.3 holds for all $k_1\in\mathcal{L}_1^*$.

The $(v_\pm,\alpha_\pm)$-invariance of $f$ implies $C(k) = \omega_k(-v_\pm)C(\alpha_\pm^{-1}k)$ for all its Fourier coefficients. Writing $\mathcal{L}_\pm^* = \{k_2 : (\alpha^{-1}k_1, \pm k_2)\in\mathcal{L}^*\}$ then $D(\alpha^{-1}k_1)$ is

$$\sum_{k_2\in\mathcal{L}_\pm^*} C(\alpha^{-1}k_1,\pm k_2)\int_0^{y_0}\omega_{\pm k_2}(y)dy = \sum_{k_2\in\mathcal{L}_\pm^*}\omega_k(v_\pm)C(k_1,k_2)\int_0^{y_0}\omega_{\pm k_2}(y)dy.$$

As $\{k_2 : (k_1,k_2)\in\mathcal{L}^*\} = \mathcal{L}_\pm^*$ then, using in the minus sign case the property

$$\int_0^{y_0}\omega_{-k_2}(y)dy = \omega_{k_2}(-y_0)\int_0^{y_0}\omega_{k_2}(y)dy, \tag{2}$$

we obtain $G(k_1,k_2) = 1-\omega_k(v_\pm - (v_\alpha,\beta_\pm))$, with $\beta_+ = 0$ and $\beta_- = y_0$, which is zero for $< k, v_\pm - (v_\alpha,\beta_\pm) >\in \mathbf{Z}$.

When (C) happens then $\sigma\in\mathbf{J}$ and so $(k_1,-k_2)\in\mathcal{L}^*$ if $(k_1,k_2)\in\mathcal{L}^*$. Thus $D(k_1)$ is

$$\frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\left(C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy + C(k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy\right)$$

and $D(\alpha^{-1}k_1)$ has a similar expression. By property (2), and by the invariance of $f$ under the action of $(v_+,\alpha_+)$ and $(v_-,\alpha_-)$, as $\alpha_- = \sigma\alpha_+\in\mathbf{J}$, we obtain

$$2G(k_1,k_2) = 1+\omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\left(\omega_k(v_+) + \omega_k(v_-)\omega_{k_2}(-y_0)\right).$$

The hypotheses of Lemma 3.1 are valid and (see the proof of Lemma 3.1)

$$\omega_k(v_-) = \omega_k(v_\sigma)\omega_k(\sigma v_+). \tag{3}$$

If $< k, \sigma v_+ - v_+ >\in\mathbf{Z}$ then $\omega_k(\sigma v_+ - v_+) = 1$ and $2G(k_1,k_2)$ equals, using (3),

$$1+\omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\omega_k(v_+)\left(1+\omega_k(\sigma v_+ - v_+)\omega_k(v_\sigma)\omega_{k_2}(-y_0)\right)$$

$$= \left(1-\omega_k(v_+ - (v_\alpha,0))\right)\left(1+\omega_k(v_\sigma - (0,y_0))\right) = 0$$

because either $1-\omega_k(v_+ - (v_\alpha,0)) = 0$, by condition (Cii), or $1+\omega_k(v_\sigma - (0,y_0)) = 0$, by (Ciii).

If $< k,\sigma v_+ - v_+ > +\frac{1}{2}\in\mathbf{Z}$ then $\omega_k(\sigma v_+)\omega_k(-v_+) = -1$ and

$$\begin{aligned}\omega_{k_1}(-v_\alpha)\omega_k(v_+) &= -\omega_{k_1}(-v_\alpha)\omega_k(\sigma v_+)\\ &= -\omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_k(-v_\sigma),\ \text{by expression (3)}\\ &= -\omega_k(v_- - (v_\alpha,y_0))\omega_k(-v_\sigma + (0,y_0)).\end{aligned}$$

Thus $2G(k_1,k_2)$ is $1+\omega_k(v_\sigma - (0,y_0)) + \omega_k(v_- - (v_\alpha,y_0))\left(\omega_k(-v_\sigma + (0,y_0)) - 1\right) = 0$ because, by condition (Civ), $\omega_k(v_\sigma - (0,y_0)) = \pm i$ and $\omega_k(v_- - (v_\alpha,y_0)) = 1$. Notice that we use the property $\omega_k(-v) = \overline{\omega_k(v)}$ in order to obtain this result. □

*Proof of necessity in Proposition 3.1.* We will show that if the hypothesis of Proposition 3.1 holds for the projection of the real and imaginary parts of $I_k(x,y)$, then one of the three conditions (A), (B) or (C) must hold.

The functions $I_k$ are given by a summation over a $\mathbf{J}$-orbit on $\mathcal{L}^*$. Its projection into $\mathcal{L}_1^*$ is a new orbit that may be used as an index for the summation of $\Pi_{y_0}(I_k)$ writing it in a form suitable for the use of Lemma 3.3.

For $\delta \in \mathbf{O}(n+1)$ and $k \in \mathcal{L}^*$, let $\delta k = (\tilde{k}_1, \tilde{k}_2)$, where $\tilde{k}_1 \in \mathbf{R}^n$ and $\tilde{k}_2 \in \mathbf{R}$. With the notation $\delta k|_1 = \tilde{k}_1$ and $\delta k|_2 = \tilde{k}_2$, we have $\Pi_{y_0}(I_k)(x) = \sum_{\delta\in\mathbf{J}} \omega_{\delta k|_1}(x) D'(\delta,k)$, where $D'(\delta,k) = \omega_{\delta k}(-v_\delta)\int_0^{y_0} \omega_{\delta k|_2}(y)dy$. This corresponds to a summation over the projection of the orbit $\mathbf{J}k$ given by: $\mathbf{J}k|_1 = \{\delta k|_1 :\ \delta \in \mathbf{J}\} \subset \mathcal{L}_1^*$. Grouping terms with the same first $n$ components we obtain

$$\Pi_{y_0}(I_k)(x) = \sum_{\tilde{k}_1\in\mathbf{J}k|_1} \omega_{\tilde{k}_1}(x) \sum_{\tilde{k}_2:(\tilde{k}_1,\tilde{k}_2)\in\mathbf{J}k} D'(\delta,\tilde{k}).$$

For $k=(k_1,k_2)$, the Fourier coefficient of $\Pi_{y_0}(I_k)$ associated to $\omega_{k_1}$ is $\sum_{\delta\in\mathrm{J}^{Id}(k)} D'(\delta,k)$, where $\mathrm{J}^{Id}(k)$ is the subset of $\mathbf{J}$ that preserves $k_1$, given by $\mathrm{J}^{Id}(k) = \{\delta\in\mathbf{J} :\ \delta k|_1 = k_1\}$ and $\mathrm{J}^{\alpha}(k) = \left\{\delta\in\mathbf{J} :\ \delta k|_1 = \alpha^{-1}k_1\right\}$.

Since by hypothesis $\Pi_{y_0}(I_k)$ is $(v_\alpha,\alpha)$-invariant, then by Lemma 3.3 , for all $k=(k_1,k_2)\in\mathcal{L}^*$, the following *invariance conditions* hold:

(a) if $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $\sum_{\delta\in\mathrm{J}^{Id}(k)} D'(\delta,k) = \omega_{k_1}(-v_\alpha)\sum_{\delta\in\mathrm{J}^{\alpha}(k)} D'(\delta,k)$,
(b) if $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $\sum_{\delta\in\mathrm{J}^{Id}(k)} D'(\delta,k) = 0$.

Although these conditions involve the sets $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^{\alpha}(k)$ for all $k\in\mathcal{L}^*$, we will show that for this proof we will only need to consider the sets:

$$\mathbf{J}^{Id} = \{Id_{n+1},\sigma\}\cap\mathbf{J} \quad\text{and}\quad \mathbf{J}^{\alpha} = \left\{\alpha_+^{-1},\alpha_-^{-1}\right\}\cap\mathbf{J}.$$

In Lemma 3.4 we describe all the possibilities for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$ and obtain in each case some consequences for $\mathcal{L}^*$ in terms of the subsets defined before the statement of Proposition 3.2. In Lemma 3.5 we study the set of all $k\in\mathcal{L}^*$ such that either $\mathrm{J}^{Id}(k)\neq\mathbf{J}^{Id}$ or $\mathrm{J}^{\alpha}(k)\neq\mathbf{J}^{\alpha}$. Finally, conditions (A), (B) and (C) are obtained in Lemma 3.6. □

**Properties of $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^{\alpha}(k)$.** *Let $k\in\mathcal{L}^*$.*

(1) $\mathrm{J}^{Id}(k) = \{\delta\in\mathbf{J} :\ \delta k = k \vee \delta k = \sigma k\}$ *and*
$\mathrm{J}^{\alpha}(k) = \left\{\delta\in\mathbf{J} :\ \delta k = \alpha_+^{-1}k \vee \delta k = \alpha_-^{-1}k\right\}$.
(2) $\mathbf{J}^{Id}\subset\mathrm{J}^{Id}(k)$ , $\mathbf{J}^{\alpha}\subset\mathrm{J}^{\alpha}(k)$ *and* $\mathrm{J}^{Id}(0,0)=\mathrm{J}^{\alpha}(0,0)=\mathbf{J}$.

*Proof.* Property (1), for $\mathrm{J}^{Id}(k)$, follows by orthogonality of $\mathbf{J}$, since any element of the orbit $\mathbf{J}(k_1,k_2)$ whose $n$ first components equal $k_1$ is of the form $(k_1,\pm k_2)$. For $\mathrm{J}^{\alpha}(k)$, the elements on $\mathbf{J}(k_1,k_2)$ with $n$ first components $\alpha^{-1}k_1$ are of the form $(\alpha^{-1}k_1,\pm k_2)$, by orthogonality of $\mathbf{J}$ and of $\alpha$.

Property (2) follows directly from the previous one and from the definitions of $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. □

The next lemma describes, under the hypothesis of Proposition 3.1, the set

$$\mathcal{O}^* = \left\{k\in\mathcal{L}^* :\ \mathrm{J}^{Id}(k)=\mathbf{J}^{Id} \ \wedge\ \mathrm{J}^{\alpha}(k)=\mathbf{J}^{\alpha}\right\}$$

according to each of the cases for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. This allows us to restate the invariance conditions (a) and (b) in simpler form in terms of subsets of $\mathcal{L}^*$.

**Lemma 3.4.** *Suppose that the invariance conditions (a) and (b) hold for all $k=(k_1,k_2)\in\mathcal{L}^*$. Then we have one of the following cases:*

(1) $\mathbf{J}^{Id}=\{Id_{n+1}\}$, $\mathbf{J}^{\alpha}=\emptyset$ *and* $\mathcal{O}^*\subset\mathcal{N}_{y_0}^*$,
(2) $\mathbf{J}^{Id}=\{Id_{n+1},\sigma\}$, $\mathbf{J}^{\alpha}=\emptyset$ *and* $\mathcal{O}^*\subset\left(\mathcal{N}_{y_0}^*\cup\mathcal{N}_{\sigma}^*\right)$,

(3) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_+^*\right)$,
(4) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_-^*\right)$,
(5) $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}, \alpha_-^{-1}\}$,
$(\mathcal{O}^* \cap \mathcal{M}^*) \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_+^* \cup \mathcal{N}_{\sigma}^*\right)$ *and* $(\mathcal{O}^* \cap \mathcal{N}^*) \subset \left(\mathcal{N}_{y_0}^* \cup \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)\right)$.

*Proof.* Cases (1) to (5) enumerate all the possibilities for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. This happens because $\mathbf{J}^{Id}$ is a group; if $\alpha_+^{-1}, \alpha_-^{-1} \in \mathbf{J}$ then $\alpha_+\alpha_-^{-1} = \sigma \in \mathbf{J}$ and if $\sigma \in \mathbf{J}$ then either $\mathbf{J}^{\alpha} = \emptyset$ or $\mathbf{J}^{\alpha}$ has two elements.

The equations in (a) and (b) may be written in the form $G(k_1, k_2) \int_0^{y_0} \omega_{k_2}(y)dy = 0$ for $k = (k_1, k_2) \in \mathcal{O}^*$, as in the proof of sufficiency. Thus one of the sufficient conditions appearing in all five cases is $\int_0^{y_0} \omega_{k_2}(y)dy = 0$, implying $k \in \mathcal{N}_{y_0}^*$. For each case we compute $G(k_1, k_2)$ and the constraints that follow when it vanishes, for $k = (k_1, k_2) \in \mathcal{O}^*$.

If $(v_\sigma, \sigma) \in \Gamma$ then $(v_\sigma, \sigma) \cdot (v_\sigma, \sigma) = (v_\sigma + \sigma v_\sigma, I) \in \Gamma$ implying $v_\sigma + \sigma v_\sigma \in \mathcal{L}$ and therefore we have

$$\omega_k(-\sigma v_\sigma) = \omega_k(v_\sigma) \quad \text{if } k \in \mathcal{L}^* \text{ and } (v_\sigma, \sigma) \in \Gamma. \tag{4}$$

If $\mathbf{J}^{\alpha} = \emptyset$ then, for all $k = (k_1, k_2) \in \mathcal{O}^*$, the conditions in the hypothesis of the lemma become $\sum_{\delta \in \mathbf{J}^{Id}} D'(\delta, k) = 0$. Thus, either $G(k_1, k_2) = 1$ or $G(k_1, k_2) = 1 + \omega_k(v_\sigma - (0, y_0))$ according to the absence or presence of $\sigma$ in $\mathbf{J}$ and using (2), orthogonality and property (4) in the second case.

Cases (1) and (2) follow because $1 + \omega_k(v_\sigma - (0, y_0)) = 0$ implies $k \in \mathcal{N}_{\sigma}^*$.

In the remaining cases either $\alpha_+$ or $\alpha_-$ belongs to $\mathbf{J}$. Thus, $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$ and condition (a) must be verified for all $k_1 \in \mathcal{L}_1^*$. For $k \in \mathcal{O}^*$ this means

$$\sum_{\delta \in \mathbf{J}^{Id}} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathbf{J}^{\alpha}} D'(\delta, k). \tag{5}$$

In case (3), $G(k_1, k_2) = 1 - \omega_{k_1}(-v_\alpha)\omega_{\alpha_+^{-1}k}(\alpha_+^{-1}v_+) = 0$ is equivalent, by orthogonality, to $1 - \omega_{k_1}(-v_\alpha)\omega_k(v_+) = 0$, which implies $k \in \mathcal{M}_+^*$.

For case (4), condition (5) leads to $G(k_1, k_2) = 1 - \omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0) = 0$, which implies $k \in \mathcal{M}_-^*$.

For case (5), condition (5) defines, by orthogonality and properties (2) and (4), $2G(k_1, k_2) = 1 + \omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\left(\omega_k(v_+) + \omega_k(v_-)\omega_{k_2}(-y_0)\right)$. In this case we are under the conditions of Lemma 3.1 and so $\mathcal{O}^* \subset (\mathcal{M}^* \cup \mathcal{N}^*)$. If $k = (k_1, k_2) \in \mathcal{M}^*$ then $G(k_1, k_2) = 0$ is equivalent, as shown in the proof of sufficiency, to $\left(1 - \omega_k(v_+ - (v_\alpha, 0))\right)\left(1 + \omega_k(v_\sigma - (0, y_0))\right) = 0$ and the result follows. For $k = (k_1, k_2) \in \mathcal{N}^*$, by the proof of sufficiency the term $2G(k_1, k_2)$ equals $1 + \omega_k(v_\sigma - (0, y_0)) + \omega_k(v_- - (v_\alpha, y_0))\left(\overline{\omega_k(v_\sigma - (0, y_0))} - 1\right)$. For $\omega_k(v_\sigma - (0, y_0)) = z_1$ and $\omega_k(v_- - (v_\alpha, y_0)) = z_2$, quation $G(k_1, k_2) = 0$ is equivalent to $(1 + z_1)/(1 - \overline{z_1}) = z_2$ because $z_1 = 1$ is not a solution of $G(k_1, k_2) = 0$. Therefore, $|(1 + z_1)/(1 - \overline{z_1})| = 1$ which implies $\mathrm{Re}(z_1) = 0 \Leftrightarrow \omega_k(v_\sigma - (0, y_0)) = \pm i$ and $z_2 = \omega_k(v_- - (v_\alpha, y_0)) = 1$, leading to $k \in \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)$. □

Let $\mathcal{P}^*$ be the complement of $\mathcal{O}^*$ in $\mathcal{L}^*$:

$$\mathcal{P}^* = \left\{k \in \mathcal{L}^* : \mathrm{J}^{Id}(k) \neq \mathbf{J}^{Id} \ \vee \ \mathrm{J}^{\alpha}(k) \neq \mathbf{J}^{\alpha}\right\}.$$

In Lemma 3.6 we reformulate the cases of Lemma 3.4 in terms of $\mathcal{L}^*$ instead of $\mathcal{O}^*$. The first two cases of Lemma 3.4 cannot occur since $\mathcal{P}^*$ is too small. In the remaining cases we show that $\mathcal{P}^*$ may be ignored and, therefore, that $\mathcal{L}^*$ can replace $\mathcal{O}^*$ in the expressions given. Thus, the estimate of the size of $\mathcal{P}^*$ in the next lemma is an essential step.

**Lemma 3.5.** *$\mathcal{P}^*$ is the intersection of $\mathcal{L}^*$ with the union of a finite number of vector subspaces of $\mathbf{R}^{n+1}$ with codimension at least one.*

*Proof.* $\mathcal{P}^*$ is the union of the submodules

$$\bigcup_{\delta\in\mathbf{J}-\mathbf{J}^{Id}} \mathcal{M}^*_{\delta,Id} \;\cup\; \bigcup_{\delta\in\mathbf{J}-\mathbf{J}^{\alpha}} \mathcal{M}^*_{\delta,\alpha}$$

where $\mathcal{M}^*_{\delta,Id} = \left\{k \in \mathcal{L}^* : \delta \in \mathrm{J}^{Id}(k)\right\}$ and $\mathcal{M}^*_{\delta,\alpha} = \left\{k \in \mathcal{L}^* : \delta \in \mathrm{J}^{\alpha}(k)\right\}$. This union is finite because $\mathbf{J}$ is finite. Moreover, for all $\xi \neq Id_{n+1} \in \mathbf{O}(n+1)$, $\mathrm{Fix}(\xi) = \left\{(x,y) \in \mathbf{R}^{n+1} : \xi(x,y) = (x,y)\right\}$ is a proper vector subspace of $\mathbf{R}^{n+1}$.

Let $\delta \in \mathbf{J} - \mathbf{J}^{Id}$. If $k \in \mathcal{M}^*_{\delta,Id}$ then either $\delta k = k$ or $\delta k = \sigma k \Leftrightarrow \sigma\delta k = k$, which implies $\mathcal{M}^*_{\delta,Id} = \mathcal{L}^* \cap (\mathrm{Fix}(\delta) \cup \mathrm{Fix}(\sigma\delta))$. Moreover, neither $\delta = Id_{n+1}$ nor $\sigma\delta = Id_{n+1}$, by the hypothesis $\delta \in \mathbf{J}-\mathbf{J}^{Id}$. Thus, the codimensions of the subspaces $\mathrm{Fix}(\delta)$ and $\mathrm{Fix}(\sigma\delta)$ are at least one.

Analogously, if $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$ and $k \in \mathcal{M}^*_{\delta,\alpha}$ then either $\delta k = \alpha_+^{-1} k \Leftrightarrow \alpha_+\delta k = k$ or $\delta k = \alpha_-^{-1} k \Leftrightarrow \alpha_-\delta k = k$. Therefore, $\mathcal{M}^*_{\delta,\alpha} = \mathcal{L}^* \cap (\mathrm{Fix}(\alpha_+\delta) \cup \mathrm{Fix}(\alpha_-\delta))$, where both $\mathrm{Fix}(\alpha_+\delta)$ and $\mathrm{Fix}(\alpha_-\delta)$ have codimensions at least one due to the hypothesis $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$. □

**Lemma 3.6.** *Suppose that the invariance conditions (a) and (b) hold for all $k = (k_1, k_2) \in \mathcal{L}^*$. Then we have one of the following cases:*

(A) $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$,
(B) $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$,
(C) $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}, \alpha_-^{-1}\}$,
$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_{\sigma}\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$.

Notice that the conditions in Lemma 3.6 are the same of Proposition 3.1 as $\delta^{-1} \in \mathbf{J}^{\alpha}$ is equivalent to $(v_\delta, \delta) \in \Gamma$ for some $v_\delta \in \mathbf{R}^{n+1}$, by definition.

*Proof.* At first, we prove the statement:

$$\text{If } (0,a) \in \mathcal{L} \text{ for some } a \neq 0 \text{ then } \mathcal{M}^*_{y_0} \not\subset \mathcal{P}^*. \tag{6}$$

If $(0,a) \in \mathcal{L}$ for some $a \neq 0$ then property (2) of the bases, in Lemma 3.2, ensures that $\mathcal{M}^*_{y_0}$ has $n$ linearly independent generators, $l_i^* = (a_i^*, 0)$ for $i \in \{1,\ldots,n\}$, where $\{a_1^*,\ldots,a_n^*\}_{\mathbf{R}} = \mathbf{R}^n$. If $\mathcal{M}^*_{y_0} \subset \mathcal{P}^*$ then, by Lemma 3.5, the module $\mathcal{M}^*_{y_0}$ is a subset of one of the subspaces forming $\mathcal{P}^*$. Therefore, there is either an element $\delta \in \mathbf{J} - \mathbf{J}^{Id}$ such that $\delta(k_1,0) = (k_1,0)$ for all $(k_1,0) \in \mathcal{M}^*_{y_0}$, or some $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$ such that $\delta(k_1,0) = (\alpha^{-1}k_1, 0)$ for all $(k_1,0) \in \mathcal{M}^*_{y_0}$. By orthogonality of $\delta$ the first case implies either $\delta = I$ or $\delta = \sigma$, which is equivalent to $\delta \in \mathbf{J}^{Id}$. Similarly, the second case implies $\delta \in \mathbf{J}^{\alpha}$, by orthogonality of $\delta$ and $\alpha$, and the statement is proved.

For any element $k \neq (0,0)$ of the dual lattice $\mathcal{L}^*$, let $g \neq (0,0)$ be the smallest element of $\mathcal{L}^*$ in the direction of $k$. Thus, there are elements $g_1,\ldots,g_n \in \mathcal{L}^*$ such that $\mathcal{L}^* = \{g, g_1, \ldots, g_n\}_{\mathbf{Z}}$. Let $\mathcal{M}^*_k$ be the submodule $\mathcal{M}^*_k = \{g_1, g_2, \ldots, g_n\}_{\mathbf{Z}} \subset \mathcal{L}^*$ and, given $h \in \mathcal{M}^*_k$, let $\mathcal{Q}^*_{k,h}$ be the set $\mathcal{Q}^*_{k,h} = \{k + mh : m \in \mathbf{Z}\}$.

We claim that there is some $h \in \mathcal{M}^*_k$ such that $\mathcal{Q}^*_{k,h} \cap \mathcal{P}^*$ is a finite set. Lemma 3.5 asserts that $\mathcal{P}^* \subset \bigcup_{i=1}^m H_i$, where each $H_i$ is a subspace of $\mathbf{R}^{n+1}$ of codimension one. Let $p \in \mathbf{N}$ and consider the subset of $k + \mathcal{M}^*_k$ with $p^n$ elements:

$$W_p = \{k + m_1 g_1 + \cdots + m_n g_n : m_i \in \mathbf{Z}, 1 \leq m_i \leq p\}.$$

Each $H_i$ has at most $p^{n-1}$ elements in $W_p$ and so $W_p \cap \bigcup_{i=1}^m H_i$ has, at most, $mp^{n-1}$ elements. For $p > m$ we have $p^n > mp^{n-1}$ and there is some $h \in \mathcal{M}^*$ such that $k + h \notin \bigcup_{i=1}^m H_i$. For this $h$, let $r$ be a line containing $\mathcal{Q}^*_{k,h}$. Since for each $i$, $r \cap H_i$ is either $r$ or a finite set, and $r$ contains at least the element $k + h \notin H_i$, it follows that $\bigcup_{i=1}^m (r \cap H_i)$ is a finite set. The claim is proved because $\mathcal{Q}^*_{k,h} \cap \mathcal{P}^*$ is a subset of $\bigcup_{i=1}^m (r \cap H_i)$.

Let $k$ be any element of $\mathcal{L}^*-\{(0,0)\}$ and choose some $h \in \mathcal{M}^*_k$ such that $\mathcal{Q}^*_{k,h}\cap\mathcal{P}^*$ is a finite set. For simplicity of notation we write $\mathcal{Q}^*$ instead of $\mathcal{Q}^*_{k,h}$.

Since $\overline{\mathcal{N}^*_{y_0}}$ is a module, the intersection $\mathcal{Q}^* \cap \overline{\mathcal{N}^*_{y_0}}$ is either the empty set or a set with only a point or an infinite set of equally spaced points with a characteristic period, $\tau_{y_0}$. For the set $\mathcal{Q}^* \cap \mathcal{N}^*_\sigma$ there are also the three possible results. Although $\mathcal{N}^*_\sigma$ is not a module, the smallest difference between two elements of $\mathcal{Q}^*\cap\mathcal{N}^*_\sigma$ defines a period $\tau_\sigma \in \mathcal{M}^*_\sigma$, by the properties of $\mathcal{N}^*_\sigma$ stated before Lemma 3.2. An analogous construction may be done for the sets $\mathcal{Q}^* \cap \mathcal{M}^*_+$, $\mathcal{Q}^* \cap \mathcal{M}^*_-$ and $\mathcal{Q}^* \cap \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)$. Thus, if these sets have more than one element we may define characteristic periods $\tau_+$, $\tau_-$ and $\tau_{\tilde{\sigma}}$, respectively.

Under the hypothesis of the Lemma, one of the cases (1) to (5) of Lemma 3.4 must happen.

If case (1) happens then $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{P}^*$, which implies $\mathcal{M}^*_{y_0} \subset \mathcal{P}^*$. Moreover, $\mathcal{Q}^* \cap \mathcal{N}^*_{y_0}$ must be an infinite set because $\mathcal{Q}^* \cap \mathcal{P}^*$ is, by construction, finite. Thus, there exists the period $\tau_{y_0}$ implying that $\mathcal{Q}^* - \overline{\mathcal{N}^*_{y_0}}$ is either the empty set or an infinite set. Since $\left(\mathcal{Q}^* - \overline{\mathcal{N}^*_{y_0}}\right) \subset (\mathcal{Q}^* \cap \mathcal{P}^*)$ is finite, it follows that $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$, which implies that $(0, y_0) \in \mathcal{L}$. However, by the statement (6), under this condition, $\mathcal{M}^*_{y_0}$ cannot be a subset of $\mathcal{P}^*$ and so case (1) cannot occur.

In case (2), $\mathcal{L}^* = \mathcal{N}^*_{y_0}\cup\mathcal{N}^*_\sigma\cup\mathcal{P}^*$ which implies $\mathcal{M}^*_{y_0} \subset (\mathcal{N}^*_\sigma \cup \mathcal{P}^*)$. Moreover, there is an element $(0, a) \in \mathcal{L}$, with $a \neq 0$, due to the existence of $\sigma$ in $\mathbf{J}$, (see properties (2) and (3) of Lemma 3.2), and thus $\mathcal{M}^*_{y_0} \cap \mathcal{N}^*_\sigma \neq \emptyset$. Suppose $\tilde{k} \in \mathcal{M}^*_{y_0} - \mathcal{P}^*$ and $\tilde{k} \neq (0, 0)$. Thus, $\tilde{k} \in \mathcal{N}^*_\sigma$ and $2\tilde{k} \in \mathcal{M}^*_{y_0}$. However, by the properties of $\mathcal{N}^*_\sigma$, $2\tilde{k} \notin \mathcal{N}^*_\sigma$ and, by Lemma 3.5, $2\tilde{k} \notin \mathcal{P}^*$. Therefore, case (2) is also impossible.

For case (3) we follow the arguments of case (1). As $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{P}^*$ then $\mathcal{Q}^* \cap \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ is an infinite set and at least one of the periods $\tau_{y_0}$ or $\tau_+$ must exist. The least common multiple of the existing periods is a period of $\mathcal{Q}^*\cap\left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ which implies that $\mathcal{Q}^*-\left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ is the empty set. Therefore $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ and condition (A) follows by definition of $k$ and because $(0, 0) \in \mathcal{M}^*_+$.

In a similar way, with $\mathcal{M}^*_-$ and $\tau_-$ instead of $\mathcal{M}^*_+$ and $\tau_+$, we prove that case (4) of Lemma 3.4 leads to condition (B).

In case (5) $(\mathcal{Q}^* \cap \mathcal{M}^*) - \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$ must be the empty set by the necessary existence of, at least, one of the periods $\tau_{y_0}$, $\tau_+$ or $\tau_\sigma$ and, analogously, $(\mathcal{Q}^* \cap \mathcal{N}^*) - \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$ is empty due to the least common multiple of the periods $\tau_{y_0}$ and $\tau_{\tilde{\sigma}}$. Besides, either $k \in (\mathcal{Q}^* \cap \mathcal{M}^*)$ or $k \in (\mathcal{Q}^* \cap \mathcal{N}^*)$ and, as $(0, 0) \notin \mathcal{N}^*$, condition (C) follows. □

This completes the proof of Propositions 3.1 and 3.2.

## 4. Proof of Theorem 1.1

In this section we show that conditions (A), (B) and (C) of Proposition 3.1 imply the cases (I), (II) and (III) of Theorem 1.1.

Proposition 3.1 states that elements of $\Gamma$ ensuring symmetry after projection have orthogonal components $\alpha_+$ or $\alpha_-$. Information on the non-orthogonal components ($v_+, v_- \in \mathbf{R}^{n+1}$) appears as constraints on the structure of $\mathcal{L}^*$.

We translate restrictions on $\Gamma$ and $\mathcal{L}^*$ into restrictions on $\Gamma$ and $\mathcal{L}$. The main tool will be to obtain restrictions on a basis of $\mathcal{L}^*$ to find a suitable basis for $\mathcal{L}$.

Each condition of Proposition 3.2 is now treated in a separate lemma.

**Lemma 4.1.** *If $(v_+, \alpha_+) \in \Gamma$ and $\mathcal{L}^* = \mathcal{N}^*_{y_0}\cup\mathcal{M}^*_+$ then one of the conditions holds:*
*I.* $((v_\alpha, 0), \alpha_+) \in \Gamma$,
*III.* $(0, y_0) \in \mathcal{L}$ *and* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$ .

*Proof.* If $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$ then either $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ or $\mathcal{L}^* = \mathcal{M}^*_+$. In the second case $< k, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$ for all $k \in \mathcal{L}^*$, *i.e.* , $v_+ - (v_\alpha, 0) \in \mathcal{L}$, and so

$$(-v_+ + (v_\alpha, 0), I) \cdot (v_+, \alpha_+) = ((v_\alpha, 0), \alpha_+) \in \Gamma.$$

If $\mathcal{L}^* = \overline{\mathcal{N}^*_{y_0}}$ then $(0, y_0) \in \mathcal{L}$ and we may use the basis $\left\{l^*_1, \ldots, l^*_{n+1}\right\}$ for $\mathcal{L}^*$ having the properties (2) in Lemma 3.2. As $\mathcal{M}^*_{y_0} \subset \mathcal{M}^*_+$, it follows that $< l^*_i, v_+ - (v_\alpha, 0) > \in \mathbf{Z}$ for all $i \in \{1, \ldots n\}$. Now we show that $v_+ - (v_\alpha, y_1) \in \mathcal{L}$ for some $y_1 \in \mathbf{R}$. For any element $k \in \mathcal{L}^*$ and any $y_1 \in \mathbf{R}$,

$$\begin{aligned} < k, v_+ - (v_\alpha, y_1) > &= < k, v_+ - (v_\alpha, 0) > - < k, (0, y_1) > \\ &= m_1 + m_2 < l^*_{n+1}, v_+ - (v_\alpha, 0) > - m_2 \tfrac{m}{y_0} y_1, \end{aligned}$$

with $m_1, m_2 \in \mathbf{Z}$. Taking, for instance, $y_1 = < l^*_{n+1}, v_+ - (v_\alpha, 0) > \frac{y_0}{m}$ we obtain $< k, v_+ - (v_\alpha, y_1) > \in \mathbf{Z}$. Thus, $(-v_+ + (v_\alpha, y_1), I) \cdot (v_+, \alpha_+) = ((v_\alpha, y_1), \alpha_+) \in \Gamma$. □

**Lemma 4.2.** *If $(v_-, \alpha_-) \in \Gamma$ and $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$ then one of the conditions holds:*
*II.* $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
*III.* $(0, y_0) \in \mathcal{L}$ *and* $((v_\alpha, y_1), \alpha_-) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$ .

*Proof.* The proof is analogous to that of Lemma 4.1 with $v_- - (v_\alpha, y_0)$ instead of $v_+ - (v_\alpha, 0)$ and $y_1 = < l^*_{n+1}, v_- - (v_\alpha, y_0) > \frac{y_0}{m} + y_0$. □

**Lemma 4.3.** *If both $(v_\sigma, \sigma)$ and $(v_+, \alpha_+)$ belong to $\Gamma$, and if both*

$$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right) \text{ and } \mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right),$$

*then one of the following conditions of Theorem 1.1 holds:*
*I.* $((v_\alpha, 0), \alpha_+) \in \Gamma$,
*II.* $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
*III.* $(0, y_0) \in \mathcal{L}$, *either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$, *for some* $y_1 \in \mathbf{R}$.

*Proof.* Let $v_+ = (v_1, v_2)$ with $v_1 \in \mathbf{R}^n$ and $v_2 \in \mathbf{R}$. Since $\sigma \in \mathbf{J}$, the bases for $\mathcal{L}$ and $\mathcal{L}^*$ satisfy properties (1) to (3) in Lemma 3.2. In particular, $l_1 = (a_1, b_1)$ and $(0, b) \in \mathcal{L}$. We claim:

(a) $v_\sigma + \sigma v_\sigma \in \mathcal{L}$.
(b) $\sigma v_+ - v_+ = -(0, 2v_2)$. Therefore
  (i) $(0, 4v_2) \in \mathcal{L}$ and
  (ii) if $(0, 2v_2) \in \mathcal{L}$ then $\mathcal{N}^* = \emptyset$.
(c) $\mathcal{M}^*_{y_0} \subset \left(\mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$.
(d) Either $v_1 = v_\alpha$ or we may choose $a_1 = 2\,(v_1 - v_\alpha)$.
(e) In both cases of property (d), $l^*_i \in \mathcal{M}^*_+$ for all $i \in \{2, \ldots, n\}$.

We now prove these claims. For (a) see (4). Since Lemma 3.1 holds, (b) follows from the definitions of $\mathcal{M}^*$ and $\mathcal{N}^*$. The hypothesis of Lemma 4.3 implies (c) since $\mathcal{M}^*_{y_0}$ and $\mathcal{N}^*$ are disjoint, by claim (b) and by property (2iv) of the bases (Lemma 3.2). This implies that for all $i \in \{1, \ldots, n\}$, either $< (a^*_i, 0), v_+ - (v_\alpha, 0) > \in \mathbf{Z}$ or $< (a^*_i, 0), v_\sigma - (0, y_0) > + \frac{1}{2} \in \mathbf{Z}$. If $l^*_i \in \mathcal{N}^*_\sigma$ then $2l^*_i \notin \mathcal{N}^*_\sigma$ and so, for all $i \in \{1, \ldots, n\}$,

$$2 < (a^*_i, 0), v_+ - (v_\alpha, 0) > = < a^*_i, 2(v_1 - v_\alpha) > \in \mathbf{Z},$$

therefore, $2(v_1 - v_\alpha) = \sum_{i=1}^n m_i a_i$ with $m_i \in \mathbf{Z}$ for $i \in \{1, \ldots, n\}$. If $v_1 \neq v_\alpha$, we may choose $a_1 = \frac{2(v_1 - v_\alpha)}{m}$ for some $m \in \mathbf{Z}$, by the property (2iii) in Lemma 3.2. If $v_\alpha = \sum_{i=1}^n r_i a_i$, with $r_i \in \mathbf{R}$, by property (1) in Lemma 3.2, $v_+$ may be writen as $\sum_{i=1}^{n+1} s_i l_i$ with $2\,(r_i - s_i) \in [0, 2[$ for $i \in \{1, \ldots, n\}$. Thus, $m = 1$ and (d) follows.

For (e) notice that $v_1 - v_\alpha$ is either zero or $a_1/2$. Therefore, for $i \in \{2, \ldots, n\}$

$$< l^*_i, v_+ - (v_\alpha, 0) > = < a^*_i, v_1 - v_\alpha > = 0.$$

The two cases of Property (d) above are now treated separately.

Suppose $v_1 = v_\alpha$. Then $v_+ - (v_\alpha, 0) = (0, v_2)$ and $l_1^*, \dots, l_n^*$ lie in $\mathcal{M}_+^*$.

If $l_{n+1}^* \in \mathcal{M}_+^*$ then $\mathcal{L}^* = \mathcal{M}_+^*$ and, as in Lemma 4.1, $((v_\alpha, 0), \alpha_+) \in \Gamma$, *i.e.*, condition (I) holds.

If $l_{n+1}^* \in \mathcal{N}_{y_0}^*$ then, by property (2iv) of the bases (Lemma 3.2), $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$ and this implies $(0, y_0) \in \mathcal{L}$. Condition (III) follows since $((v_\alpha, v_2), \alpha_+) \in \Gamma$.

Now suppose that

$$l_{n+1}^* \notin \left(\mathcal{M}_+^* \cup \mathcal{N}_{y_0}^*\right) \quad \Rightarrow \quad l_i^* + l_{n+1}^* \notin \left(\mathcal{M}_+^* \cup \mathcal{N}_{y_0}^*\right) \quad i \in \{1, \dots, n\}. \tag{7}$$

If $(0, 2v_2) \in \mathcal{L}$ we take $2v_2 = b$. Since $\mathcal{N}^* = \emptyset$, then (7) implies that $l_{n+1}^* \in \mathcal{N}_\sigma^*$ and $l_i^* \in \mathcal{M}_\sigma^*$ for $i \in \{1, \dots, n\}$. Moreover, if $v_\sigma = \sum_{i=1}^{n+1} s_i l_i$, with $s_i \in [0, 1[$ for $i \in \{1, \dots, n\}$, then $s_{n+1} - y_0/b + 1/2 \in \mathbf{Z}$ and $s_i = 0$ for $i \in \{1, \dots, n\}$. Therefore, up to multiples of $(0, b)$, we have $v_\sigma = (0, y_0 + b/2) = (0, y_0 + v_2)$ and $((0, y_0 + v_2), \sigma) \cdot ((v_\alpha, v_2), \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma$, *i.e.*, condition (II).

If $(0, 2v_2) \notin \mathcal{L}$ then (7) implies $l_{n+1}^* \in \mathcal{M}_-^*$ and $l_i^* \in \mathcal{M}_-^*$ for $i \in \{1, \dots, n\}$. Thus, $\mathcal{L}^* = \mathcal{M}_-^*$ and, as in Lemma 4.1, $((v_\alpha, y_0), \alpha_-) \in \Gamma$, completing the proof in the case $v_1 = v_\alpha$.

Now suppose $v_1 \neq v_\alpha$ and let $a_1 = 2(v_1 - v_\alpha)$. Since $l_1^* \notin \mathcal{M}_+^*$, from (c) we get $l_1^* \in \mathcal{N}_\sigma^*$ and $l_i^* \in \mathcal{M}_\sigma^*$ for $i \in \{2, \dots, n\}$. Then $v_\sigma = \sum_{i=1}^{n+1} s_i l_i$ may be writen as $s_1 = 1/2$ and $s_i = 0$ for $i \in \{2, \dots, n\}$. Thus, $v_\sigma = l_1/2 + s_{n+1}(0, b)$ and, by (a), $(a_1, 0) \in \mathcal{L}$, *i.e.*, $b_1 = 0$. As $v_\sigma = (a_1/2, 0) + s_{n+1}(0, b) = v_+ - (v_\alpha, 0) + (0, s_{n+1}b - v_2)$, it follows from (a) that $(-\sigma v_+ + (v_\alpha, s_{n+1}b - v_2), \sigma) \in \Gamma$.

If $l_{n+1}^* \in \mathcal{N}_{y_0}^*$ then $(0, y_0) \in \mathcal{L}$. Condition (III) follows from

$$(-\sigma v_+ + (v_\alpha, s_{n+1}b - v_2), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, s_{n+1}b - v_2), \alpha_-) \in \Gamma.$$

Now suppose that $l_{n+1}^* \notin \mathcal{N}_{y_0}^*$ and, consequently, that $l_i^* + l_{n+1}^* \notin \mathcal{N}_{y_0}^*$ for $i \in \{1, \dots, n\}$. If $l_{n+1}^* \in \mathcal{M}_+^*$ then $< l_{n+1}^*, l_1/2 + (0, v_2) >= v_2/b \in \mathbf{Z}$ and $(0, v_2) \in \mathcal{L}$, since $(0, b) \in \mathcal{L}$. Moreover, as $l_1^* \notin \mathcal{M}_+^*$, we must impose $l_1^* + l_{n+1}^* \in \mathcal{N}_\sigma^*$, which implies $s_{n+1} + y_0/b \in \mathbf{Z}$. Therefore, choosing $s_{n+1} = y_0/b$, we get

$$((0, v_2), I) \cdot (-\sigma v_+ + (v_\alpha, y_0 - v_2), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma.$$

For $(0, 2v_2) \in \mathcal{L}$, the only missing case is $l_{n+1}^* \in \mathcal{N}_\sigma^*$, where $s_{n+1} + y_0/b + 1/2 \in \mathbf{Z}$ and, up to multiples of $(0, b)$, $s_{n+1}b - v_2 = y_0$. Condition (II) follows because

$$(-\sigma v_+ + (v_\alpha, y_0), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma.$$

If $(0, 2v_2) \notin \mathcal{L}$ then both $l_{n+1}^*$ and $l_i^* + l_{n+1}^*$ lies in $\mathcal{M}_-^*$ for $i \in \{1, \dots, n\}$ and condition (II) follows. □

## 5. Restriction

In this section we present results for the restriction of functions in $X_\Gamma$ analogous to those obtained for the projection.

For $r \in \mathbf{R}$, let $\Phi_r$ be the operator that maps $f(x, y)$ to its restriction to the affine subspace $\{(x, r) : x \in \mathbf{R}^n\}$ given by $\Phi_r(f)(x) = f(x, r)$. If $f \in X_\Gamma$ then, formally, for $D(k_1) = \sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2)\omega_{k_2}(r)$, the restriction of $f$ is

$$\Phi_r(f)(x) = \sum_{k \in \mathcal{L}^*} \omega_k(x, r) C(k) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x) D(k_1)$$

where $\mathcal{L}_1^* = \{k_1 : (k_1, k_2) \in \mathcal{L}^*\}$.

**Theorem 5.1.** *All functions in $\Phi_r(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(I) $((v_\alpha, 0), \alpha_+) \in \Gamma$,
(II) $((v_\alpha, 2r), \alpha_-) \in \Gamma$.

Given $f \in X_\Gamma$, the formal Fourier series for $\Phi_r(f)$ is similar to that of $\Pi_{y_0}(f)$, with $\omega_{k_2}(r)$ in the restriction corresponding to $\int_0^{y_0} \omega_{k_2}(y)dy$ in the projection. Thus, results concerning $\Phi_r$ are similar those proved in the previous sections for $\Pi_{y_0}$. In particular, the proof of Theorem 5.1 is analogous to that of Theorem 1.1. The condition $\omega_{k_2}(r) = 0$ is never verified and so the sets $\mathcal{N}^*_{y_0}$ and $\mathcal{M}^*_{y_0}$ disappear and we don't have an analogue to the condition $(0, y_0) \in \mathcal{L}$. Moreover, the expression

$$\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_2}(y_0) \int_0^{y_0} \omega_{-k_2}(y)dy = 0$$

has the analogue $\omega_{k_2}(r) - \omega_{k_2}(2r)\omega_{-k_2}(r) = 0$.

The following analogue of Proposition 3.1 is used to prove Theorem 5.1.

**Proposition 5.1.** *All functions in $\Phi_r(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(A) $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}^*_+$,
(B) $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}^*_-$,
(C) *both* $(v_\sigma, \sigma)$ *and* $(v_+, \alpha_+)$ *belong to* $\Gamma$, $\mathcal{M}^* \subset \left(\mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)$.

The analogue of Lemma 3.4 is, for $D'(\delta, k) = \omega_{\delta k}(-v_\delta)\omega_{\delta k|_2}(r)$:

**Lemma 5.1.** *Suppose that*

(a) *if* $k_1 \in \mathcal{L}^*_1 \cap \alpha\mathcal{L}^*_1$ *then* $\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^\alpha(k)} D'(\delta, k)$ *and*
(b) *if* $k_1 \notin \mathcal{L}^*_1 \cap \alpha\mathcal{L}^*_1$ *then* $\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = 0$,

*for all $k = (k_1, k_2) \in \mathcal{L}^*$. Then one of the following cases holds:*

(1) $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^\alpha = \emptyset$ *and* $\mathcal{O}^* \subset \mathcal{N}^*_\sigma$,
(2) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^\alpha = \{\alpha_+^{-1}\}$ *and* $\mathcal{O}^* \subset \mathcal{M}^*_+$,
(3) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^\alpha = \{\alpha_-^{-1}\}$ *and* $\mathcal{O}^* \subset \mathcal{M}^*_-$,
(4) $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^\alpha = \{\alpha_+^{-1}, \alpha_-^{-1}\}$, $(\mathcal{O}^* \cap \mathcal{M}^*) \subset \left(\mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$ *and* $(\mathcal{O}^* \cap \mathcal{N}^*) \subset \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)$.

The proof of Proposition 5.1 also uses the analogue of Lemma 3.6. Under the conditions for the restriction, property (6) in the proof of Lemma 3.6, concerning the set $\mathcal{M}^*_{y_0}$, does not hold. By Lemma 5.1 the case (1) of Lemma 3.6 disappears. For case (2) of Lemma 3.6 the dual lattice is $\mathcal{L}^* = \mathcal{N}^*_\sigma \cup \mathcal{P}^*$ and the arguments concerning $\mathcal{M}^*_{y_0}$ and $\mathcal{N}^*_{y_0}$ must be replaced by: if $\tilde{k} \notin \mathcal{P}^*$ then $\tilde{k} \in \mathcal{N}^*_\sigma$. However both $2\tilde{k} \notin \mathcal{P}^*$ and $2\tilde{k} \notin \mathcal{N}^*_\sigma$, by definition of $\mathcal{P}^*$ and the properties of $\mathcal{N}^*_\sigma$, and so this case is not possible.

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## References

[1] M. A. Armstrong, *Groups and Symmetry*, Springer-Verlag, 1988
[2] P. Borckmans, G. Dewel, A. De Wit, E. Dulos, J. Boissonade, F. Gauffre and P. De Kepper, 'Diffusive instabilities and chemical reactions', *Int. J. Bifurcation and Chaos* **12** (2002) 2307-2332
[3] T. K. Callahan, 'Hopf bifurcation on cubic lattices', in J. Buescu, S. B. S. D. Castro, A. P. S. Dias and I. S. Labouriau (Eds.) *Bifurcations, Symmetry and Patterns* Birkhäuser, Series: Trends in Mathematics (2003) pages 123-127
[4] T. K. Callahan and E. Knobloch, 'Symmetry-breaking bifurcations on cubic lattices', *Nonlinearity* **10** (1997) 1179-1216
[5] T. K. Callahan and E. Knobloch, 'Pattern formation in three-dimensional reaction-diffusion systems', *Physica* D **132** (1999) 339-362

[6] P. De Kepper, E. Dulos, J. Boissonade, A. De Wit, G. Dewel and P. Borckmans, 'Reaction-diffusion patterns in confined chemical systems', *J. Stat. Phys.* **101** (2000) 495-508
[7] M. Golubitsky, I. Stewart and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory* - vol II, Appl. Math. Sci. **69**, Springer-Verlag, 1988
[8] M. G. M. Gomes, 'Black-eye patterns: A representation of three-dimensional symmetries in thin domains', *Phys. Rev.* E **60** (1999) 3741-3747
[9] I. Melbourne, 'Steady-state bifurcation with Euclidean symmetry', *Trans. Amer. Math. Soc.* **351** (1999) 1575-1603
[10] W. Miller, *Symmetry Groups and their Applications*, Academic Press, 1972
[11] M. J. Parker, I. Stewart and M. G. M. Gomes, 'Examples of forced symmetry-breaking to homoclinic cycles in three-dimensional euclidean-invariant systems', *Int. J. Bif. Chaos* **18** (2008) 83–107
[12] E. M. Pinho, *Symmetries of Projected Symmetric Patterns*, Thesis, University of Porto, 2006
[13] M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press, 1995
[14] H. Shoji, K. Yamada, D. Ueyama and T. Ohta, 'Turing patterns in three dimensions', *Phys. Rev.* E **75** (2007) 046212
[15] C. Zhou, H. Guo and Q. Ouyang, 'Experimental study of the dimensionality of black-eye patterns', *Phys. Rev. E* **65** (2002) 036118

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