MODULI FOR HETEROCLINIC CONNECTIONS INVOLVING SADDLE-FOCI AND PERIODIC SOLUTIONS

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Abstract. Dimension three is the lowest dimension where we can find chaotic behaviour for flows and it may be helpful to distinguish in advance "equivalent" complex dynamics. In this article, we give numerical invariants for the topological equivalence of vector fields on three-dimensional manifolds whose flows exhibit one-dimensional heteroclinic connections involving saddle-foci and/or periodic solutions. Computed as a infinite limit time, these moduli of topological equivalence rely heavily on the local behaviour near the invariant saddles. We also present an alternative proof of the Togawa's Theorem.

1. Introduction

The classification of vector fields according to their topological properties is a major preoccupation in the theory of dynamical systems. The problem of the classification of homo and heteroclinic cycles in low-dimensional spaces has been frequently asked in recent years. For planar vector fields, based on a Bowen's example, Takens [41] described a complete set of moduli of topological equivalence for attracting heteroclinic cycles using asymptotic properties of non-converging time averages.

Dimension three is the lowest dimension where one finds chaotic systems for flows; among them, there is a vast catalogue of exotic phenomena associated with cycles involving either saddle-foci or periodic solutions. One of the most famous examples is the homoclinic cycle associated to a saddle-focus studied by Shilnikov [36, 37, 38]; in its neighbourhood, under an eigenvalue condition, Shilnikov proved the existence of infinitely many periodic solutions of saddle-type, nowadays known as infinitely many suspended horseshoes. Their existence does not require the breaking of the homoclinic connection as in the case of saddles whose linearization has only real eigenvalues. As proved in [3], the exact knowledge of global twisting around cycles involving saddle-foci is not necessary to predict shadowing properties of these cycles, since the spiralling associated to the presence of complex eigenvalues spreads solutions around the cycle – see also [5, 6].

Until the eighties, these spiralling sets have not attracted as much attention as the Lorenz attractor because in general it is difficult to understand the topology associated to non-real eigenvalues. Recently, the dynamics of spiralling attractors have been studied more systematically – see for example [17, 22, 27, 28, 31, 32] and references therein.

In order to distinguish in advance two complex dynamics, topological invariants are gratefully welcomed; they are relevant for comparison of families of vector fields. Roughly speaking, a *topological invariant* or a *modulus of topological equivalence* is a function of the vector field that is invariant under topological equivalence – a more formal definition will be given later in Subsection 2.2 of this paper. An important subject in the study of dynamical systems is the notion of these quantities that remain invariant under C^0–change of coordinates.

The most famous topological invariant associated to a circle $S^1 \equiv \mathbf{R} \pmod{2\pi}$ is the *Poincaré rotation number* [20]: if $f : S^1 \to S^1$ is a homeomorphism, let $F : \mathbf{R} \to \mathbf{R}$ be a lift to $\mathbf{R}$. If $x \in \mathbf{R}$, then:

$$\lim_{n \in \mathbf{N}} \frac{F^n(x) - x}{n}$$

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exists for all x and is equal to $\rho(f) \in \mathbf{R}$, which does not depend neither on the lift nor on $x \in \mathbf{R}$, up to multiples of 2π – this limit is what we call the *Poincaré rotation number*. In particular, if f and $\tilde{f}$ are h–topologically conjugate preserving orientation then:

$$\rho(f) = \rho(h^{-1} \circ f \circ h) = \rho(\tilde{f}).$$

There are some occurrences of moduli in structurally unstable systems with two periodic solutions having a quadratic tangency. The seminal example of moduli appear in 1978: J. Palis [30] proved that heteroclinic connections that have a tangency between the stable and unstable manifolds of two periodic solutions give rise to a modulus that can be expressed as the ratio of the leading Floquet multipliers of the limit cycles. It has also been proved that homoclinic tangencies of the invariant manifolds of periodic solutions give rise to infinitely many topological invariant – an overview of these results may be found in Gonchenko *et al* [18].

Let $C_{\mathbf{p}}, E_{\mathbf{p}}, \alpha_{\mathbf{p}} \in \mathbf{R}^+$. In a three-dimensional flow, in the case of a homoclinic cycle to a saddle-focus $\mathbf{p}$ whose linearization has eigenvalues

$$-C_{\mathbf{p}} \pm i\alpha_{\mathbf{p}} \qquad \text{and} \qquad E_{\mathbf{p}},$$

several authors proved that the ratio $\frac{C_{\mathbf{p}}}{E_{\mathbf{p}}}$ is a topological invariant [11, 12, 42] – this ratio is called the *saddle index* of the equilibrium $\mathbf{p}$. This modulus seems somewhat surprising since the eigenvalues themselves are not topological invariants. The first proof is due to Shilnikov [37]; he did not use this term, but the description of the non-wandering set proved explicitly that its structure depends on this parameter. If the saddle index is greater than 1, then the dynamics near the homoclinic cycle is trivial (the homoclinic cycle is attracting). If the saddle index is lesser than 1, periodic solutions bifurcate from the homoclinic cycle, a phenomenon which may be considered as an internal bifurcation within the Morse-Smale systems. In a classical work, Afraimovich and Ilyashenko [1, Section 5.6] also proved that the saddle index is a modulus of topological equivalence. Other proofs arose: Ceballos and Labarca [11], based on the first return map to a transverse section to the cycle; Dufraine [12] and Togawa [42] used link types of the closed trajectories that appear near the separatrix.

For a heteroclinic cycles associated to a saddle-focus $\mathbf{p}$ as before and a periodic solution $\mathbf{C}$ with real Floquet multipliers $C_{\mathbf{C}}$ and $E_{\mathbf{C}}$ such that $|C_{\mathbf{C}}| < 1$ and $|E_{\mathbf{C}}| > 1$, Beloqui [8] derived that $\frac{C_{\mathbf{p}}}{\alpha_{\mathbf{p}} \ln(E_{\mathbf{C}})}$ is a topological invariant. Heteroclinic connections of codimension two have also been considered by Bonatti and Dufraine [9] and by van Strien [44], who found other topological invariants.

Motivated by these results and using recent results of Aguiar *et al* [3], Rodrigues *et al* [34] and Takens [41], in this article we revive the research of Dufraine [12]. We present local necessary conditions to prove topological equivalence of vector fields defined on a smooth three-dimensional manifold, whose flow exhibits at least one connection involving either a saddle-focus or a periodic solution. The spiralling behaviour associated to the presence of "rotation" spreads solutions near the connections and offers complementary information about the dynamics, as we will observe. Since parts of the spiralling set swirl around the one-dimensional connection, we will make use of the information "encoded" in the neighbourhood of the connections which appears as a phenomenon of codimension one or two. Explicit examples of vector fields where these connections can be found analytically are reported in Aguiar *et al* [2, Section 4] and Rodrigues and Labouriau [33, Section 2].

2. Preliminaries

Let f be a C^2–vector field on a C^∞-Riemannian three-dimensional differential manifold $\mathcal{M}^3$ possibly with boundary, with flow given by the unique solution $x(t) = \varphi(t, x_0) \in \mathcal{M}^3$ of

$$\dot{x} = f(x) \qquad \text{and} \qquad x(0) = x_0. \tag{1}$$

2.1. **Heteroclinic Terminology.** Following Field [15], if A is a compact invariant set for the flow of f, we say that A is an *invariant saddle* if $A \subseteq \overline{W^s(A) \backslash A}$ and $A \subseteq \overline{W^u(A) \backslash A}$, where $\overline{A}$ is the closure of A. In this paper all the saddles will be hyperbolic.

Given two invariant saddles A and B, a *heteroclinic connection* from A to B, denoted $[A \to B]$, is a solution of the Initial Value Problem (1) contained in $W^u(A) \cap W^s(B)$. There may be more than one connection from A to B as shown in [34, Section 8].

Let $\mathcal{S} = \{A_j : j \in \{1, \ldots, k\}\}$ be a finite ordered set of mutually disjoint invariant saddles. We say that there is a *heteroclinic cycle* associated to $\mathcal{S}$ if

$$\forall j \in \{1, \ldots, k\}, W^u(A_j) \cap W^s(A_{j+1}) \neq \emptyset \pmod{k}.$$

If $k = 1$ we say that there is a *homoclinic cycle* associated to A_1. In other words, there is a connection whose trajectories tend to A_1 in both backward and forward time. We also refer to the saddles defining the heteroclinic cycle as *nodes*. The dimension of the local unstable manifold of a saddle A_j will be called the *Morse index* of A_j.

In many articles, all nodes are equilibria, but in this paper we explicitly consider the case in which one of the saddles is a periodic solution with real Floquet multipliers, that we denote by $\mathbf{C}$. The saddles under consideration consist of either saddle-foci or non-trivial closed trajectories – these saddles are what the authors of [3] call *rotating nodes*.

In a three-dimensional manifold, a *Bykov cycle* is a heteroclinic cycle associated to two hyperbolic saddle-foci with different Morse indices, in which the one-dimensional manifolds coincide and the two-dimensional invariant manifolds have a transverse intersection – see Bykov [10]. A Bykov cycle is also called by *T[erminal]–point* because it corresponds to a point on the space of parameters where such cycles appear [14, 27].

2.2. **Modulus of Topological Equivalence.** Following Shilnikov *et al* [39, Chapter 2], two three–dimensional systems:

$$\dot{x} = f_1(x) \qquad \text{and} \qquad \dot{x} = f_2(x)$$

defined in regions $D_1 \subset \mathcal{M}^3$ and $D_2 \subset \mathcal{M}^3$ respectively, are *topologically equivalent* in the subregions $U_1 \subset D_1$ and $U_2 \subset D_2$ if there exists a homeomorphism $h : U_1 \to U_2$, which maps trajectories of the first system into trajectories of the second one.

For $i \in \{1, 2\}$, let $\varphi_i(t, x_0)$ be the unique solution of $\dot{x} = f_i(x)$ with initial condition $x(0) = x_0$. If the homeomorphism h is time preserving, that is, if:

$$\forall y \in \mathcal{M}^3, \qquad \forall t \in \mathbf{R}, \qquad \varphi_1(t, h(y)) = h(\varphi_2(t, y)),$$

we say that the flows are *topologically conjugate* and h is a *conjugacy*. When h is a C^r–diffeomorphism, we say that the flows are C^r–conjugate ($r \in \mathbf{N}_0$).

Remark 1. *The notion of topological conjugacy is restrictive since one requires the solutions $\varphi_1(t, \star)$ and $\varphi_2(t, \star)$ to be topologically conjugate for all t, which is much more than require that trajectories of $\dot{x} = f_1(x)$ are mapped into trajectories of $\dot{x} = f_2(x)$ homeomorphically.*

A vector field f is said to have a *modulus of topological equivalence* (resp. modulus of topological conjugacy) if, in some subspace $\mathcal{U} \supset \{f\}$ of the space of vector fields, a continuous, locally non-constant functional η is defined such that if f and $\tilde{f}$ are topologically equivalent (resp. topologically conjugate), then $\eta(f) = \eta(\tilde{f})$. We require that in the region of the definition of η, there are no open sets in a neighbourhood of f where the functional η takes a constant value – see Gonchenko *et al* [18] and Shilnikov *et al* [40, Chapter 8]. By Remark (1), moduli of topological conjugacy are quite often while moduli of topological equivalence are not.

2.3. **Suspension flows.** Suppose that $\mathcal{M}$ is a smooth compact riemannian manifold and f is a C^∞–diffeomorphism. Consider the space $\Omega = \mathcal{M} \times [0, 1]/ \propto$ where $\propto$ is the identification of $(x, 1)$ with $f(x, 0)$. The standard *suspension* of f is the flow on Ω defined by $\phi_t(y, s) = (y, t + s)$, for $0 \leq t + s < 1$. It is well known that Ω is a smooth compact riemannian manifold and ϕ_t is C^∞. We refer the reader to [24] for more information about suspensions.

3. A tour along the main results

Let $\mathcal{M}^3$ denote a C^∞ Riemannian three-dimensional differential manifold possibly with boundary. Throughout this paper, we assume that h is a homeomorphism mapping the trajectories of the flow induced by the vector field $f : \mathcal{M}^3 \to T\mathcal{M}^3$ into trajectories of the flow induced by $\tilde{f} : \mathcal{M}^3 \to T\mathcal{M}^3$. If $\mathbf{p} \in \mathcal{M}^3$, we set $\tilde{\mathbf{p}} = h(\mathbf{p})$. Note that the homeomorphism h under consideration may reverse the time orientation.

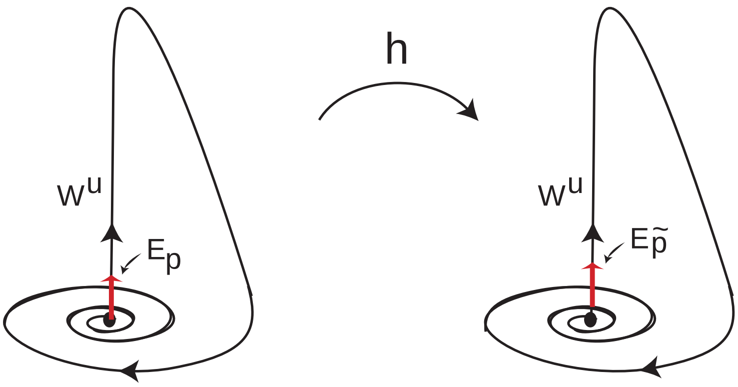

FIGURE 1. Shilnikov cycle $\Gamma_{\mathbf{p}}$: the homeomorphism h preserves time-orientation.

Our first object of study is the set $\mathcal{H}$ of C^2–vector fields $f : \mathcal{M}^3 \to T\mathcal{M}^3$, satisfying the following property – see Figure 1:

(H1) There is a homoclinic cycle $\Gamma_{\mathbf{p}}$ associated to a hyperbolic saddle-focus $\mathbf{p}$, where the imaginary part of the complex conjugate eigenvalues of $Df_{\mathbf{p}}$ is $\pm\alpha_{\mathbf{p}} \neq 0$.

These cycles will be called *Shilnikov cycles*. The equilibrium $\mathbf{p}$ may have Morse index 1 or 2: if its Morse index is 1, then $\mathbf{p}$ is denoted by $\mathbf{v}$ and its eigenvalues are $-C_{\mathbf{v}} \pm \alpha_{\mathbf{v}} i$ and $E_{\mathbf{v}}$ where $C_{\mathbf{v}}, E_{\mathbf{v}} > 0$; otherwise $\mathbf{p}$ is denoted by $\mathbf{w}$ and its eigenvalues are $E_{\mathbf{w}} \pm \alpha_{\mathbf{w}} i$ and $-C_{\mathbf{w}}$ where $C_{\mathbf{w}}, E_{\mathbf{w}} > 0$.

If we do not ask for C^1–smoothness of h, then the eigenvalues of $Df_{\mathbf{p}}$ and $D\tilde{f}_{\tilde{\mathbf{p}}}$ may be different. Hereafter, we use the following convention: if $\lambda_{\mathbf{p}}$ is an eigenvalue of $Df_{\mathbf{p}}$, then $\lambda_{\tilde{\mathbf{p}}}$ is the h–corresponding eigenvalue of $D\tilde{f}_{\tilde{\mathbf{p}}}$. Of course that if h reverses time orientation, then contracting eigenvalues are mapped into expanding eigenvalues; more precisely, according to our notation, if $E_{\mathbf{p}} > 0$ is an eigenvalue of $Df_{\mathbf{p}}$, then $-C_{\tilde{\mathbf{p}}} < 0$ is the h–corresponding eigenvalue of $D\tilde{f}_{\tilde{\mathbf{p}}}$.

The strenght of the swirling motion around the one-dimensional local invariant manifold of $\mathbf{p}$ is measured be the absolute value of the imaginary part of the complex (non-real) eigenvalues of $Df_{\mathbf{p}}$. The first item of Theorem 1 says that this number is a topological invariant in $\mathcal{H}$.

Theorem 1. *Let $f, \tilde{f} \in \mathcal{H}$. If $\tilde{f}$ is h–topologically equivalent to f, then:*

(1) $|\alpha_{\mathbf{p}}| = |\alpha_{\tilde{\mathbf{p}}}|$;
(2) $\frac{C_{\mathbf{p}}}{E_{\mathbf{p}}} = \frac{C_{\tilde{\mathbf{p}}}}{E_{\tilde{\mathbf{p}}}}$, *if h preserves time-orientation;*
(3) $\frac{C_{\mathbf{p}}}{E_{\mathbf{p}}} = \frac{E_{\tilde{\mathbf{p}}}}{C_{\tilde{\mathbf{p}}}}$, *if h reverses time-orientation.*

Item (1) of Theorem 1 has been proved by Dufraine [12] in 2001 using *linking numbers* and *knots* arguments. Item (2) has been shown by Togawa [42] in 1987 and later by Ceballos and Labarca [11] in 1992. Based on the recent work of Rodrigues *et al* [34] and using a similar approach to that of Gaunersdorfer [16] and Takens [41], we address an alternative proof of items (1) and (2) of Theorem 1 in Section 6. Using the topological concept of *spinning in average*, we give a more elegant proof of item (1) of Theorem 1 in Section 7.

There are homeomorphisms that transform a hyperbolic saddle-equilibrium $\mathbf{p}$, where the eigenvalues of $Df_{\mathbf{p}}$ are real, into a saddle-focus. Item (1) of Theorem 1 excludes the possibility that such homeomorphisms transform a homoclinic cycle to a saddle into a homoclinic cycle to a saddle-focus.

Corollary 2. *A homoclinic cycle to a saddle and a homoclinic cycle to a saddle-focus cannot be topologically equivalent.*

Generalizing the above notation, our second goal is the study of the set $\mathcal{B}$ of C^2–vector fields f on $\mathcal{M}^3$ satisfying the following property depicted in Figure 2:

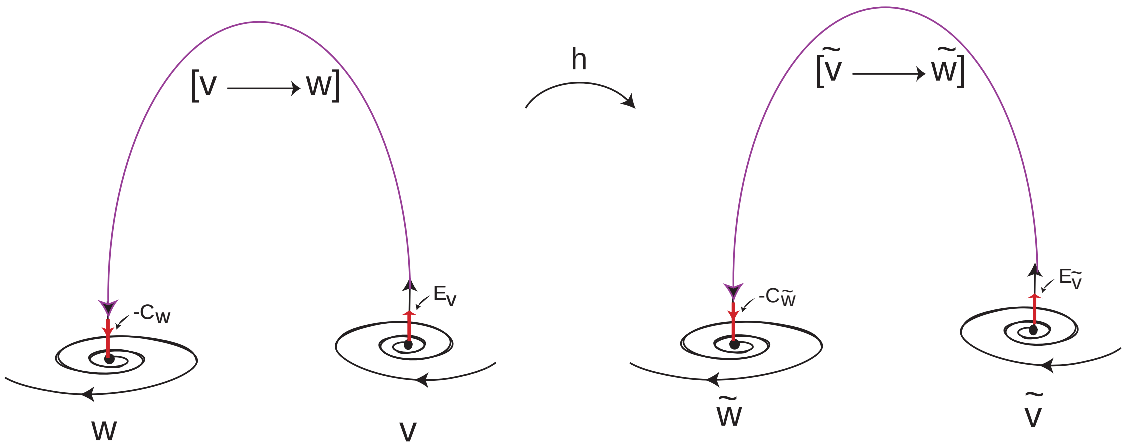

FIGURE 2. Heteroclinic connection associated to two saddle-foci of different Morse indices.

(B1) There is a one-dimensional heteroclinic connection $[\mathbf{v} \to \mathbf{w}]$ associated to two hyperbolic saddle-foci of different Morse index $\mathbf{v}$ and $\mathbf{w}$ (the Morse indices of $\mathbf{v}$ and $\mathbf{w}$ are 1 and 2, respectively). The eigenvalues of $Df_{\mathbf{p}}$ are:
 (a) $-C_{\mathbf{v}} \pm \alpha_{\mathbf{v}} i$ and $E_{\mathbf{v}}$ with $C_{\mathbf{v}}, E_{\mathbf{v}} > 0$, for $\mathbf{p} = \mathbf{v}$;
 (b) $E_{\mathbf{w}} \pm \alpha_{\mathbf{w}} i$ and $-C_{\mathbf{w}}$ with $C_{\mathbf{w}}, E_{\mathbf{w}} > 0$, for $\mathbf{p} = \mathbf{w}$.

Connections satisfying (B1) will be called *Bykov connections*. We now compute a topological invariant for heteroclinic connections associated to two saddle-foci of different Morse indices.

Theorem 3. *Let $f, \tilde{f} \in \mathcal{B}$. If $\tilde{f}$ is h–topologically equivalent to f, then:*
 (1) $\frac{C_{\mathbf{v}}}{E_{\mathbf{w}}} = \frac{C_{\tilde{\mathbf{v}}}}{E_{\tilde{\mathbf{w}}}}$ *if h preserves time-orientation;*
 (2) $\frac{C_{\mathbf{v}}}{E_{\mathbf{w}}} = \frac{E_{\tilde{\mathbf{v}}}}{C_{\tilde{\mathbf{w}}}}$ *if h reverses time-orientation;*
 (3) $(\alpha_{\mathbf{v}} E_{\mathbf{w}} + \alpha_{\mathbf{w}} C_{\mathbf{v}}) \times (C_{\mathbf{v}} + E_{\mathbf{w}})^{-1} = (\alpha_{\tilde{\mathbf{v}}} E_{\tilde{\mathbf{w}}} + \alpha_{\tilde{\mathbf{w}}} C_{\tilde{\mathbf{v}}}) \times (C_{\tilde{\mathbf{v}}} + E_{\tilde{\mathbf{w}}})^{-1}$ *if h preserves time-orientation;*
 (4) $(\alpha_{\mathbf{v}} E_{\mathbf{w}} + \alpha_{\mathbf{w}} C_{\mathbf{v}}) \times (C_{\mathbf{v}} + E_{\mathbf{w}})^{-1} = -(\alpha_{\tilde{\mathbf{v}}} C_{\tilde{\mathbf{w}}} + \alpha_{\tilde{\mathbf{w}}} E_{\tilde{\mathbf{v}}}) \times (E_{\tilde{\mathbf{v}}} + C_{\tilde{\mathbf{w}}})^{-1}$ *if h reverses time-orientation.*

We address the proof of Theorem 3 in Subsection 7.3.

A heteroclinic connection from a saddle-focus to a non-trivial periodic solution is a phenomenon of codimension one in $\mathcal{M}^3$. Our third goal is the study of the set $\mathcal{P}$ of C^2–vector fields f on $\mathcal{M}^3$ such that the flow has a heteroclinic connection between a saddle-focus and a non-trivial periodic solution with real Floquet multipliers – see Figure 10. More precisely, the set $\mathcal{P}$ satisfies the following properties :

(P1) There is a one-dimensional heteroclinic connection $[\mathbf{v} \to \mathbf{C}]$ associated to a saddle-focus $\mathbf{v}$ and a non-trivial periodic solution $\mathbf{C}$ of period $T > 0$ such that:
 (a) the eigenvalues of $Df_{\mathbf{v}}$ are $-C_{\mathbf{v}} \pm \alpha_{\mathbf{v}} i$ and $E_{\mathbf{v}}$ with $C_{\mathbf{v}}, E_{\mathbf{v}} > 0$;
 (b) the Floquet multipliers of $\mathbf{C}$ are real and given by $e^{-C_{\mathbf{C}}} < 1$ and $e^{E_{\mathbf{C}}} > 1$.

(P2) The local manifolds $W^s_{loc}(\mathbf{C})$ and $W^u_{loc}(\mathbf{C})$ are smooth two-dimensional surfaces which are homeomorphic to a cylinder as illustrated in Figure 4.

We do not consider saddle periodic solutions with two-dimensional stable and unstable manifolds which are homeomorphic to a Moebius band. A description of a heteroclinic connection where (P2) does not hold has been given in [39, Chapter 3].

Theorem 4. *Let $f, \tilde{f} \in \mathcal{P}$. If $\tilde{f}$ is h–topologically conjugate to f, then:*
 (1) $\frac{C_{\mathbf{v}}}{E_{\mathbf{C}}} = \frac{C_{\tilde{\mathbf{v}}}}{E_{\tilde{\mathbf{C}}}}$ *if h preserves orientation;*
 (2) $\frac{C_{\mathbf{v}}}{E_{\mathbf{C}}} = \frac{E_{\tilde{\mathbf{v}}}}{C_{\tilde{\mathbf{C}}}}$ *if h reverses orientation;*
 (3) $\left(\alpha_{\mathbf{v}} E_{\mathbf{C}} + \frac{2\pi}{T} C_{\mathbf{v}}\right) \times (E_{\mathbf{C}} + C_{\mathbf{v}})^{-1} = \left(\alpha_{\tilde{\mathbf{v}}} E_{\tilde{\mathbf{C}}} + \frac{2\pi}{\tilde{T}} C_{\tilde{\mathbf{v}}}\right) \times (E_{\tilde{\mathbf{C}}} + C_{\tilde{\mathbf{v}}})^{-1}$ *if h preserves orientation;*

(4) $\left(\alpha_{\mathbf{v}} E_{\mathbf{C}} + \frac{2\pi}{T} C_{\mathbf{v}}\right) \times \left(E_{\mathbf{C}} + C_{\mathbf{v}}\right)^{-1} = -\left(\alpha_{\tilde{\mathbf{v}}} C_{\tilde{\mathbf{C}}} + \frac{2\pi}{\tilde{T}} E_{\tilde{\mathbf{v}}}\right) \times \left(C_{\tilde{\mathbf{C}}} + E_{\tilde{\mathbf{v}}}\right)^{-1}$ *if h reverses orientation.*

The proof of Theorem 4 is similar to that given in Section 7 for Theorem 3; for the sake of completeness, we sketch the main steps of the proof in Subsection 7.4.

The rest of this paper is organized as follows: in Section 4, applying coordinate changes, we C^1–linearize the vector field f around the nodes (saddle-foci and periodic solution); we introduce some notation that will be used in the rest of the article and we obtain a geometrical description of the way the flow transforms a curve of initial conditions lying across the stable manifold of the saddle in Section 5. We show that along one-dimensional heteroclinic connections, the global distortion induced by the transition map is bounded. Based in [13], we present the first proof of Theorem 1 in Section 6, emphasizing that the invariant is topological. Using the concept of *spinning in average*, in Section 7 we present a new proof for item (1) of Theorem 1. Moreover, we prove Togawa's Theorem, we derive a topological invariant for Bykov cycles and conjugacy invariants for cycles involving a non-trivial closed trajectory. We end this article with a short discussion about the results.

Reversing the time-orientation of trajectories does not add significant difficulties. This is why we restrict the proofs to the time-orientation preserving case. When there is no ambiguity, we remove the subscripts which identify the homo/heteroclinic connection under consideration. We have endeavoured to make a self contained exposition bringing together all topics related to the proofs. We have stated short lemmas and we draw illustrative figures to make the paper easily readable.

4. Local Maps

Historically, in order to study homo and heteroclinic bifurcations, two approaches have been taken. In the first one, due to Shilnikov, one rewrites the differential equation in integral form and uses smoothness results for the integral equations to derive approximations of the Poincaré map. A different technique, used by Tresser and coworkers, uses linearization results obtained via a normal form procedure. In this article, we are going to use the second approach; we establish local coordinates near the hyperbolic saddle-foci $\mathbf{p}$ and define some terminology that will be used in the rest of the paper. The letter $\mathbf{p}$ denotes $\mathbf{v}$ or $\mathbf{w}$ when the results hold for both equilibria.

The behaviour of the vector field f in the neighbourhood of each heteroclinic connection $[A \to B]$ is given, up to topological equivalence, by the linear part of f in the neighbourhood of A and B and by the transition map between two discs transversal to the flow in those neighbourhoods. We choose coordinates in the neighbourhoods of A and B in order to put f in canonical form and we assume that the transition map is linear.

The crucial point is the application of Samovol's Theorem [35] to C^1–linearize the flow around the nodes[1]– equilibrium or periodic solution – and to introduce cylindrical coordinates around them – see also Homburg and Sandstede [23, Section 3.1]. We use neighbourhoods with boundary transverse to the linearized flow.

4.1. C^1–linearization comes "for free" for saddlle-foci.

Given two cross sections $\Sigma^{in}_{\mathbf{p}}$ and $\Sigma^{out}_{\mathbf{p}}$ that are transverse to the stable and unstable manifolds of $\mathbf{p}$, respectively, the local transition $\Phi_{\mathbf{p}}$ is the map:

$$\Phi_{\mathbf{p}} : \Sigma^{in}_{\mathbf{p}} \to \Sigma^{out}_{\mathbf{p}},$$

that sends points in the boundary where the flow goes in into points in the boundary where it goes out. The Hartman-Grobman Theorem [19] asserts that there is a continuous change of coordinates h near $\mathbf{p}$ that transforms the solutions of $\dot{x} = f(x)$, in a small neighbourhood of $\mathbf{p}$, into the solutions of the linear system $\dot{y} = Df_{\mathbf{p}}(y - \mathbf{p})$, in a neighbourhood of $h(\mathbf{p})$. With these new coordinates, the return map is a homeomorphism but is not evident the real geometry of the flow. Also, it is not clear how expansions and contractions in $y \in \Sigma^{in}_{\mathbf{p}}$ can be derived. Belitskii [7] and Samovol [35] derived the following eigenvalue condition which ensures the existence of a differentiable change of coordinates that linearizes the original flow:

$$Re(\lambda_i) \neq Re(\lambda_j) + Re(\lambda_k), \tag{2}$$

[1] In Subsection 4.1, we show that in a three-dimensional smooth manifold, if $\mathbf{p}$ is a hyperbolic saddle-focus, then any C^0–linearization is also a C^1–linearization.

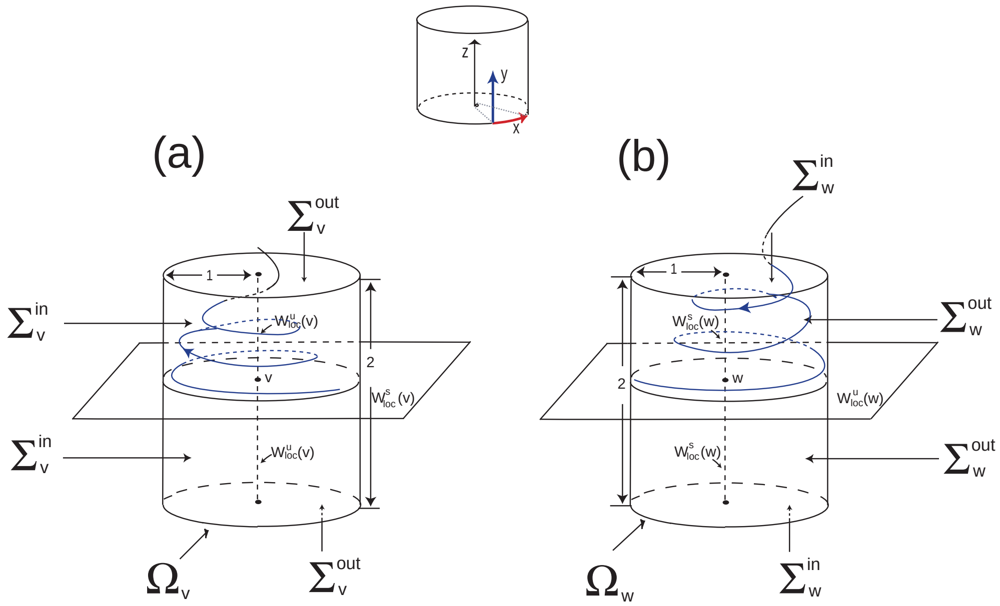

FIGURE 3. Cylindrical neighbourhoods of the saddle-foci $\mathbf{v}$ (a) and $\mathbf{w}$ (b). The flow in the neighbourhood of a saddle-focus swirls around the one-dimensional invariant manifold much as air flow swirls around the core of a *tornado*.

where $\lambda_n's$ are the nonzero eigenvalues of Df at $\mathbf{p}$. When the inequality (2) does not hold, the authors of [39, Section 3.13] say that the eigenvalues exhibit a 2–resonance. If $\mathbf{p}$ is a saddle-focus in $\mathcal{M}^3$, the C^1–linearization holds because condition (2) is always satisfied. After the linearization, we may compute $\Phi_{\mathbf{p}}$ explicitely.

4.2. **Saddle-foci.** Recall that the Morse index of $\mathbf{v}$ and $\mathbf{w}$ is 1 and 2, respectively. Since $\mathbf{v}$ and $\mathbf{w}$ are hyperbolic, by Samovol's Theorem, the vector field f is $C^{(1+\varepsilon)}$–conjugate to its linear part around each saddle-focus $\mathbf{v}$ and $\mathbf{w}$ ($\varepsilon > 0$). As in the classic work of Tresser [43], we choose cylindrical coordinates (ρ, θ, z) near $\mathbf{v}$ and $\mathbf{w}$ so that the linearized vector field can be written as:

$$\dot{\rho} = -C_{\mathbf{v}}\rho \quad \wedge \quad \dot{\theta} = \alpha_{\mathbf{v}} \quad \wedge \quad \dot{z} = E_{\mathbf{v}}z \tag{3}$$

and

$$\dot{\rho} = E_{\mathbf{w}}\rho \quad \wedge \quad \dot{\theta} = \alpha_{\mathbf{w}} \quad \wedge \quad \dot{z} = -C_{\mathbf{w}}z. \tag{4}$$

After a linear rescaling of the local variables, we consider cylindrical neighbourhoods of $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{M}^3$ of radius 1 and height 2 that we denote by V and W, respectively – see Figure 3. Their boundaries consist of three components: the cylinder wall parametrized by $x \in \mathbf{R} \pmod{2\pi}$ and $|y| \leq 1$ with the usual cover $(x, y) \mapsto (1, x, y) = (\rho, \theta, z)$ and two disks (top and bottom). We take polar coverings of these disks $(r, \varphi) \mapsto (r, \varphi, j) = (\rho, \theta, z)$ where $j \in \{-1, +1\}$, $0 \leq r \leq 1$ and $\varphi \in \mathbf{R} \pmod{2\pi}$. By convention, the intersection point of the one-dimensional homo or heteroclinic connection with the wall of the cylinder has zero angular coordinate.

As depicted in Figure 3(a), the cylinder wall of V is denoted by $\Sigma_{\mathbf{v}}^{in}$. The top and the bottom of the cylinder are simply denoted by $\Sigma_{\mathbf{v}}^{out}$. Note that $W_{loc}^s(\mathbf{v})$ corresponds to the circle $y = 0$. The boundary of V can be written as the disjoint union:

$$\partial V = \Sigma_{\mathbf{v}}^{in} \dot{\cup} \Sigma_{\mathbf{v}}^{out} \dot{\cup} \Omega_{\mathbf{v}},$$

where $\Omega_{\mathbf{v}}$ is the part of ∂V where the flow is not transverse. It follows by the above construction that:

Lemma 5. *Solutions starting:*

(1) *at $\Sigma_{\mathbf{v}}^{in}$ go inside the cylinder V in positive time;*
(2) *at $\Sigma_{\mathbf{v}}^{out}$ go outside the cylinder V in positive time;*

(3) *at* $\Sigma^{in}_{\mathbf{v}} \backslash W^s(\mathbf{v})$ *leave the cylindrical neighbourhood* V *at* $\Sigma^{out}_{\mathbf{v}}$.

If $(x,y) \in \Sigma^{in}_{\mathbf{v}} \backslash W^s_{loc}(\mathbf{v})$, let $T_{\mathbf{v}}(x,y)$ be the time of flight through V of the trajectory whose initial condition is (x,y). It only depends on $y \neq 0$ and is given explicitly by

$$T_{\mathbf{v}}(x,y) = \frac{1}{E_{\mathbf{v}}} \ln\left(\frac{1}{|y|}\right). \tag{5}$$

In particular $\lim_{y\to 0} T_{\mathbf{v}}(x,y) = +\infty$. Now, we obtain the expression of the local map that sends points in the boundary where the flow goes in, into points in the boundary where the flows goes out. The local map $\Phi_{\mathbf{v}} : \Sigma^{in}_{\mathbf{v}} \to \Sigma^{out}_{\mathbf{v}}$ near $\mathbf{v}$ is given by

$$\Phi_{\mathbf{v}}(x,y) = \left(y^{\delta_{\mathbf{v}}}, -\tfrac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln|y| + x\right) = (r,\phi) \tag{6}$$

where $\delta_{\mathbf{v}} = \frac{C_{\mathbf{v}}}{E_{\mathbf{v}}} > 0$ is the *saddle index* of $\mathbf{v}$. Observe that for a fixed $x_0 \in \mathbf{R}$, $\lim_{y\to 0} |\Phi_{\mathbf{v}}(x_0,y)| = (0,+\infty)$.

Similarly, after linearizing and rescaling the local variables, we get dual cross sections near $\mathbf{w}$. We omit the details because they are similar to those of $\mathbf{v}$. The set $W^s_{loc}(\mathbf{w})$ is the z–axis, intersecting the top and bottom of the cylinder W at the origin of its coordinates – see Figure 3 (b). The set $W^u_{loc}(\mathbf{w})$ is parametrized by $z = 0$. We denote by $\Sigma^{in}_{\mathbf{w}}$, the two connected components (top and bottom of W). By construction, we may easily conclude that:

Lemma 6. *Solutions starting:*

(1) *at* $\Sigma^{in}_{\mathbf{w}}$ *go inside* W *in positive time;*
(2) *at* $\Sigma^{out}_{\mathbf{w}}$ *go outside the cylinder* W *in positive time;*
(3) *at* $\Sigma^{in}_{\mathbf{w}} \backslash W^s_{loc}(\mathbf{w})$ *leave the cylindrical neighbourhood* W *at* $\Sigma^{out}_{\mathbf{w}}$.

Of course, points in $W^s_{loc}(\mathbf{w}) \cap \Sigma^{in}_{\mathbf{w}}$ do not return to $\Sigma^{out}_{\mathbf{w}}$. If $(r,\varphi) \in \Sigma^{in}_{\mathbf{w}}$, let $T_{\mathbf{w}}(r,\varphi)$ be the time of flight through W of the trajectory whose initial condition is (r,φ). The time of flight only depends on $r \in \mathbf{R}^+$ and is given explicitly by:

$$T_{\mathbf{w}}(r,\varphi) = \frac{1}{E_{\mathbf{w}}} \ln\left(\frac{1}{r}\right). \tag{7}$$

For the connected component of $\Sigma^{in}_{\mathbf{w}} \backslash W^s_{loc}(\mathbf{w})$ with $y > 0$, the local map:

$$\Phi_{\mathbf{w}} : \Sigma^{in}_{\mathbf{w}} \backslash W^s_{loc}(\mathbf{w}) \to \Sigma^{out}_{\mathbf{w}}$$

near $\mathbf{w}$ is given by:

$$\Phi_{\mathbf{w}}(r,\varphi) = \left(-\tfrac{\alpha_{\mathbf{w}}}{E_{\mathbf{w}}} \ln r + \varphi, r^{\delta_{\mathbf{w}}}\right) = (x,y) \tag{8}$$

where $\delta_{\mathbf{w}} = \frac{C_{\mathbf{w}}}{E_{\mathbf{w}}} > 0$ is the *saddle index* of $\mathbf{w}$. The same expression holds for the local map from the other connected component of $\Sigma^{in}_{\mathbf{w}} \backslash W^s_{loc}(\mathbf{w})$ to $\Sigma^{out}_{\mathbf{w}}$ where $y < 0$, with the exception that the second coordinate of $\Phi_{\mathbf{w}}$ changes its sign.

In Homburg [22], the author obtains precise asymptotic expansions for the local map $\Sigma^{in}_{\mathbf{v}} \to \Sigma^{out}_{\mathbf{v}}$, similar to that of Ovsyannikov and Shilnikov [29]. In the present article, we omit high order terms because they are not needed to our purposes.

4.3. **Periodic Solution.** Observing that the Floquet multipliers of $\mathbf{C}$ do not depend on the specific choice of $p \in \mathbf{C}$, let us consider a local cross-section $\Pi^\star$ at a point p in $\mathbf{C}$. Since $\mathbf{C}$ is hyperbolic, by Hartman [19], there is a neighbourhood $V^\star$ of p in $\Pi^\star$ where the first return map π is C^1–conjugated to its linear part. The eigenvalues of $D\pi$ are $e^{E_{\mathbf{C}}}$ and $e^{-C_{\mathbf{C}}}$, where $C_{\mathbf{C}}, E_{\mathbf{C}} > 0$. Suspending the linear map gives rise, in cylindrical coordinates (ρ,θ,z) around $\mathbf{C}$, to the system of differential equations:

$$\begin{cases} \dot\rho = -C_{\mathbf{C}}(\rho - 1) \\ \dot\theta = \frac{2\pi}{T} \\ \dot z = E_{\mathbf{C}} z \end{cases} \tag{9}$$

which is orbitally equivalent to the original flow near $\mathbf{C}$; $T > 0$ is the period of $\mathbf{C}$. In these coordinates, the T–periodic trajectory $\mathbf{C}$ is the circle defined by $\rho = 1$ and $z = 0$, its local stable manifold, $W^s_{loc}(\mathbf{C})$, is the plane $z = 0$ and $W^u_{loc}(\mathbf{C})$ is the surface defined by $\rho = 1$ – see Figure 4.

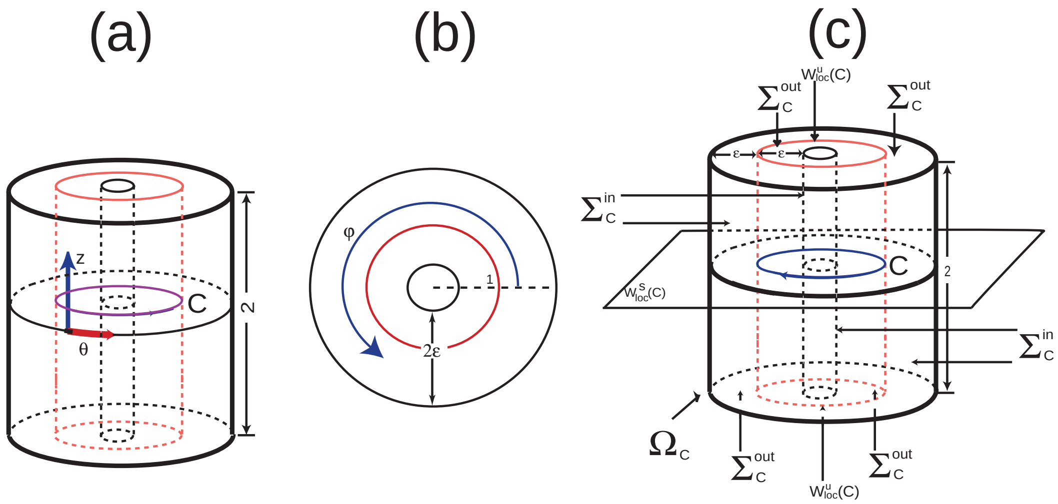

Figure 4. Neighbourhood of the closed trajectory $\mathbf{C}$, invariant manifolds and corresponding cross sections. (a) and (b): general cylindrical coordinates on the neighbourhood of $\mathbf{C}$. (b): the sets $W^s_{loc}(\mathbf{C})$ and $W^u_{loc}(\mathbf{C})$ are smooth two-dimensional surfaces homeomorphic to a cylinder as stated in (P2).

With this kind of suspension in which the ceiling map is constant, we might lose some information about the original dynamics: in particular, it does not preserve the return time of the original trajectories. However, at the limit, the time spent for initial conditions near $W^s_{loc}(\mathbf{C})$ may be approximated by the time of flight through the suspended neighbourhood, which are preserved under conjugacy. This is the only point where the conjugacy is used.

After a rescaling of variables (if necessary), we will work with a hollow three-dimensional cylindrical neighbourhood $V_\mathbf{C}$ of $\mathbf{C}$ contained in the suspension of $V^\star$ (see Figure 4):

$$V_\mathbf{C} = \{(\rho, \theta, z): \quad 1-\varepsilon \leq \rho \leq 1+\varepsilon, \quad -1 \leq z \leq 1 \quad \text{and} \quad \theta \in \mathbf{R} \pmod{2\pi}\} \ .$$

Its boundary is a disjoint union:

$$\partial V_\mathbf{C} = \Sigma^{in}_\mathbf{C} \dot\cup \Sigma^{out}_\mathbf{C} \dot\cup \Omega_\mathbf{C}$$

such that $\Sigma^{in}_\mathbf{C}$ is the union of the walls of the cylinder, $\rho = 1 \pm \varepsilon$, locally separated by $W^u_{loc}(\mathbf{C})$ – see Figure 4. The set $\Sigma^{out}_\mathbf{C}$ is the union of two anuli, the top and the bottom of the cylinder, $z = \pm 1$, locally separated by $W^s_{loc}(\mathbf{C})$. The vector field is transverse to $\Omega_\mathbf{C}$ at all points except at the four circles $\Omega_\mathbf{C} = \overline{\Sigma^{in}_\mathbf{C}} \cap \overline{\Sigma^{out}_\mathbf{C}}$. It follows directly by construction that:

Lemma 7. *Solutions starting:*

(1) *at $\Sigma^{in}_\mathbf{C}$ go inside the hollow cylinder $V_\mathbf{C}$ in positive time;*
(2) *at $\Sigma^{out}_\mathbf{C}$ go outside the hollow cylinder $V_\mathbf{C}$ in positive time.*

The two cylinder walls, $\Sigma^{in}_\mathbf{C}$, are parametrized by the covering maps:

$$(\theta, z) \mapsto (1 \pm \varepsilon, \theta, z) = (\rho, \theta, z),$$

where $\theta \in \mathbf{R} \pmod{2\pi}$, $|z| < 1$. In these coordinates, $\Sigma^{in}_\mathbf{C} \cap W^s_{loc}(\mathbf{C})$ is the union of the two circles $z = 0$. The two anuli $\Sigma^{out}_\mathbf{C}$ are parametrized by the coverings:

$$(\varphi, r) \mapsto (r, \varphi, \pm\varepsilon) = (\rho, \theta, z),$$

for $1-\varepsilon < r < 1+\varepsilon$ and $\varphi \in \mathbf{R} \pmod{2\pi}$ and where $\Sigma^{out}_\mathbf{C} \cap W^u_{loc}(\mathbf{C})$ is the union of the two circles $r = 1$. In these coordinates, the set $\Omega_\mathbf{C}$ is the union of the four circles defined by $\rho = 1 \pm \varepsilon$ and $z = \pm\varepsilon$. Noting that the trajectory whose initial condition is $(\theta, z) \in \Sigma^{in}_\mathbf{C}$ arrives at $\Sigma^{out}_\mathbf{C}$ at time:

$$T_\mathbf{C}(\theta, z) = \frac{1}{E_\mathbf{C}} \ln\left(\frac{1}{|z|}\right), \tag{10}$$

the local maps $\phi_{\mathbf{C}}$ from a connected component of $\Sigma^{in}_{\mathbf{C}}$ into the corresponding connected component of $\Sigma^{out}_{\mathbf{C}}$ are given by:

$$\Phi_{\mathbf{C}}(\theta, z) = \left(\theta - \frac{2\pi}{TE_{\mathbf{C}}} \ln|z|, 1 \pm \varepsilon z^{\delta_{\mathbf{C}}}\right) = (\varphi, r) \qquad \text{where} \quad \delta_{\mathbf{C}} = \frac{C_{\mathbf{C}}}{E_{\mathbf{C}}} > 0\ . \tag{11}$$

The constant $\delta_{\mathbf{C}}$ is called the *saddle index* of $\mathbf{C}$. The signs $\pm$ depend on the component of $\Sigma^{in}_{\mathbf{C}}$ we started at, $+$ for trajectories starting with $\rho > 1$ and $-$ for $\rho < 1$. More details about the local maps may be found in [3, 34].

4.4. **Transition Maps and First Return Maps.** Let A and B denote two nodes not necessarily different. Here, we study the lowest order map for the transition $\Psi_{[A\to B]} : \Sigma^{out}_{A} \to \Sigma^{in}_{B}$ along the connection $[A \to B]$ for the three cases under consideration:

- Shilnikov cycles (Subsection 4.4.1);
- Bykov connections (Subsection 4.4.2);
- heteroclinic connections between a saddle-focus and a periodic solution (Subsection 4.4.2).

By the *Tubular Flow Theorem* [21], solutions starting near $\Sigma^{out}_{A} \cap W^u_{loc}(A)$ follow the one-dimensional homo/heteroclinic connection. The return map is highly nonlinear since the distortion near the hyperbolic saddle-foci is tremendous. Since our topological invariants will be computed as a “limit” involving times of flight, we are interested in the map η_{AB} defined by $\Psi_{[A\to B]} \circ \Phi_A : \Sigma^{in}_{A} \to \Sigma^{in}_{B}$ to compute the ratio of “consecutive” times of flight.

4.4.1. *Homoclinic cycle to* $\mathbf{v}$. The map $\Phi_{\mathbf{v}}$ defined in (6) is given in polar coordinates (r, ϕ) in $\Sigma^{out}_{\mathbf{v}}$. Since the coordinates in $\Sigma^{in}_{\mathbf{v}}$ may be considered as rectangular coordinates, it is required to change them in order to compose the maps. Let $X = r\cos(\phi)$ and $Y = r\sin(\phi)$. In rectangular coordinates, the global transition from $\Sigma^{out}_{\mathbf{v}}$ to $\Sigma^{in}_{\mathbf{v}}$ can be approximated by the linear diffeomorphism given by $\Psi_{[\mathbf{v}\to\mathbf{v}]} : \Sigma^{out}_{\mathbf{v}} \to \Sigma^{in}_{\mathbf{v}}$ where:

$$x = a_{11}X + a_{12}Y \qquad \text{and} \qquad y = a_{21}X + a_{22}Y$$

and $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} \neq 0$. Neglecting high order terms, the first return map:

$$\Pi_{\mathbf{v}} = \Psi_{[\mathbf{v}\to\mathbf{v}]} \circ \Phi_{\mathbf{v}} : \Sigma^{in}_{\mathbf{v}} \to \Sigma^{in}_{\mathbf{v}}$$

is given explicitely by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y^{\delta_{\mathbf{v}}}.\left(a_{11}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y| - x\right) - a_{12}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y| - x\right)\right) \\ y^{\delta_{\mathbf{v}}}.\left(a_{21}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y| - x\right) - a_{22}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y| - x\right)\right)\end{pmatrix} + \ldots, \tag{12}$$

where the dots mean smooth functions which tend to zero as $y \to 0$ to the order of $y^{\delta_{\mathbf{v}}}$. A determination of the precise form of the Poincaré map that also takes higher-order terms into consideration can be found in [40, Chapter 13].

4.4.2. *Heteroclinic connections* $[\mathbf{v} \to \mathbf{w}]$ *and* $[\mathbf{v} \to \mathbf{C}]$. The linear part of the map $\Psi_{[\mathbf{v}\to\mathbf{w}]}$ may be represented, in rectangular coordinates (X, Y), as the composition of a rotation of the coordinate axes and a change of scales. As in Bykov [10] and Homburg and Sandstede [23], after a rotation and a uniform rescaling of the coordinates, we may assume, without loss of generality, that $\Psi_{[\mathbf{v}\to\mathbf{w}]}$ is given in rectangular coordinates by the linear map:

$$\Psi_{[\mathbf{v}\to\mathbf{w}]}\begin{pmatrix} X \\ Y\end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a}\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix} + \ldots \qquad a \in \mathbf{R}^+, \tag{13}$$

where the dots mean smooth functions which tend to zero as $X, Y \to 0$. To compose this map with $\Phi_{\mathbf{w}}$, it is required to change the coordinates. More precisely, let $\eta_{\mathbf{vw}}$ the map defined by:

$$\Psi_{[\mathbf{v}\to\mathbf{w}]} \circ \Phi_{\mathbf{v}} : \Sigma^{in}_{\mathbf{v}} \to \Sigma^{in}_{\mathbf{w}}.$$

The initial condition $(x, y) \in \Sigma^{in}_{\mathbf{v}}$ under the local map near $\mathbf{v}$, $\Phi_{\mathbf{v}}$, is mapped into:

$$\Phi_{\mathbf{v}}(x, y) = \left(y^{\delta_{\mathbf{v}}}, x - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y|\right) = (r, \varphi). \tag{14}$$

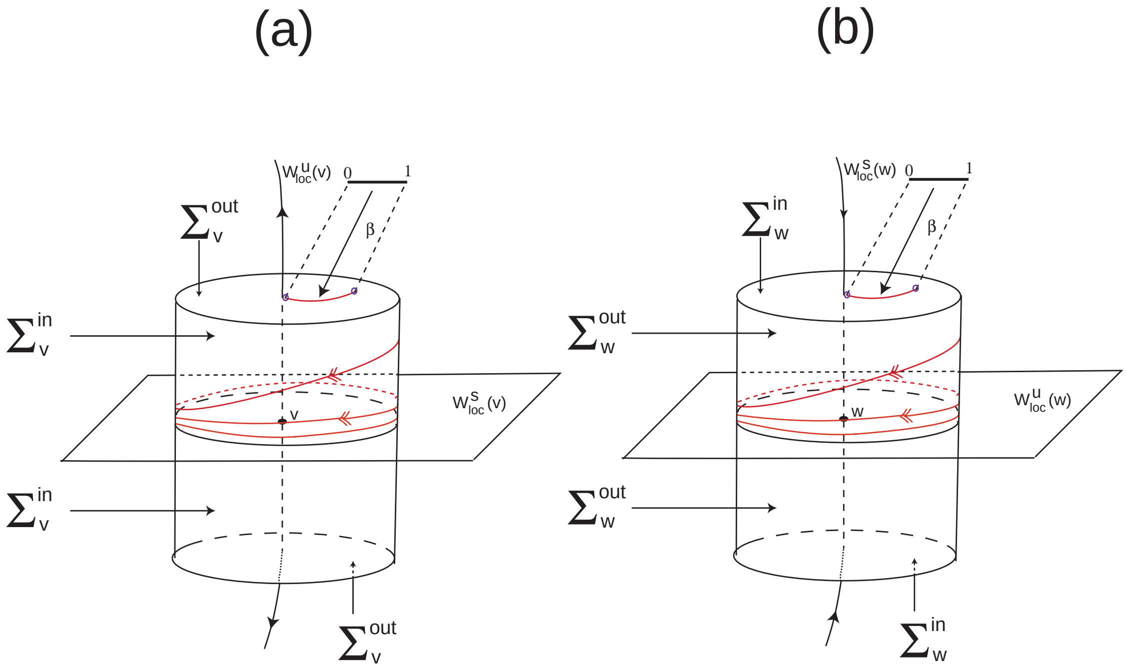

FIGURE 5. The image of a segment β, under $\Phi_{\mathbf{v}}^{-1}$ or $\Phi_{\mathbf{w}}$, is a helix accumulating on the circle defined by $y = 0$ in local coordinates. The double arrows represent the orientation and not the flow: (a) Equilibrium $\mathbf{v}$ with Morse index 1. (b) Equilibrium $\mathbf{w}$ with Morse index 2.

Transforming the polar coordinates of $\Phi_{\mathbf{v}}(x, y)$ into rectangular coordinates, it yields:

$$\left\{ \begin{array}{l} X = r\cos(\varphi) \\ Y = r\sin(\varphi) \end{array} \right. \tag{15}$$

Therefore, we may write the radial component of $\Psi_{[\mathbf{v}\to\mathbf{w}]} \circ \Phi_{\mathbf{v}}(x, y)$ as:

$$R = \sqrt{a^2r^2\cos^2(\varphi) + a^{-2}r^2\sin^2(\varphi)} \tag{16}$$

where $r = y^{\delta_{\mathbf{v}}}$ and $\varphi = x - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y|$. If the argument of $\left(ar\cos(\varphi), \frac{1}{a}r\cos(\varphi)\right)$ lies in the interval $\left(\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right)$, with $k \in \mathbf{N}$, then the argument of $\Psi_{[\mathbf{v}\to\mathbf{w}]}(X, Y)$ lies in the same interval. Formula (16) will be useful to the computation of the times of flight in Subsection 7.3. Similar arguments can be used to conclude that the radial component of the transition map $\Psi_{[\mathbf{v}\to\mathbf{C}]}$ from $\Sigma^{in}_{\mathbf{v}}$ to $\Sigma^{in}_{\mathbf{C}}$ takes a similar form to that of (16).

5. LOCAL GEOMETRY

The coordinates and notations of Section 4 will be used to study the geometry of the local dynamics near each node. This is the main goal of the present section but first we introduce the concept of a *segment* on $\Sigma^{out}_{\mathbf{v}}$ and $\Sigma^{in}_{\mathbf{w}}$.

Definition 1. *A* segment β*:*

(1) on $\Sigma^{out}_{\mathbf{v}}$ *is a smooth regular parametrized curve* $\beta : [0, 1] \to \Sigma^{out}_{\mathbf{v}}$ *that meets* $W^u_{loc}(\mathbf{v})$ *at the point* $\beta(0)$ *and such that, writing* $\beta(s) = (r(s), \phi(s))$*, both* r *and* ϕ *are monotonic and bounded functions of* s *– see Figure 5(a).*

(2) on $\Sigma^{in}_{\mathbf{w}}$ *is a smooth regular parametrized curve* $\beta : [0, 1] \to \Sigma^{in}_{\mathbf{w}}$ *that meets* $W^s_{loc}(\mathbf{w})$ *at the point* $\beta(0)$ *and such that, writing* $\beta(s) = (r(s), \varphi(s))$*, both* r *and* φ *are monotonic and bounded functions of* s *– see Figure 5(b).*

The definition of a *segment* may be relaxed: the components do not need to be monotonic for all $s \in [0,1]$. We use the assumption of monotonicity to simplify the proof of item (1) of Theorem 1. The following definition adapted from [3, 34] will be useful in the sequel.

Definition 2. *Let $a, b \in \mathbf{R}$ such that $a < b$ and let $\Sigma_{\mathbf{w}}^{out}$ or $\Sigma_{\mathbf{v}}^{in}$ be a surface parametrized by a covering $(x, y) \in \mathbf{R} \times [a, b]$ where θ is periodic. A* helix on $\Sigma_{\mathbf{w}}^{out}$ accumulating on the circle $y = y_0$ *is a curve*

$$\gamma : [0,1] \to \Sigma_{\mathbf{v}}^{in} \qquad or \qquad \gamma : [0,1] \to \Sigma_{\mathbf{w}}^{out}$$

such that its coordinates $(x(s), y(s))$ satisfy $\lim_{s\to 0} y(s) = y_0$, $\lim_{s\to 0^+} |x(s)| = +\infty$ and the maps x and y are monotonic.

The next result characterizes the local dynamics near the two types of steady-state.

Lemma 8. *A* segment *β:*

1. *on $\Sigma_{\mathbf{w}}^{in}$ is mapped by $\Phi_{\mathbf{w}}$ into a helix on $\Sigma_{\mathbf{w}}^{out}$ accumulating on the circle defined by $\Sigma_{\mathbf{w}}^{out} \cap W_{loc}^u(\mathbf{w})$;*
2. *on $\Sigma_{\mathbf{v}}^{out}$ is mapped by $\Phi_{\mathbf{v}}^{-1}$ into a helix on $\Sigma_{\mathbf{v}}^{in}$ accumulating on the circle defined by $\Sigma_{\mathbf{v}}^{in} \cap W_{loc}^s(\mathbf{v})$.*

Proof. We prove item (1); the proof of (2) follows straightforwardly reversing the time. Let β be a segment on $\Sigma_{\mathbf{w}}^{in}$ as shown in Figure 5(b). Write $\beta(s) = (r^\star(s), \varphi^\star(s)) \in \Sigma_{\mathbf{w}}^{in}$, where $s \in [0,1]$, $\varphi^\star$ is a decreasing map as function of s and $\lim_{s\to 0^+} r^\star(s) = 0$. We omit the dependence on s to simplify the notation. The function $\Phi_{\mathbf{w}}$ maps the curve $\beta \subset \Sigma_{\mathbf{w}}^{out}$ into the curve defined by:

$$\Phi_{\mathbf{w}}(\beta(s)) = \Phi_{\mathbf{w}}[r^\star(s), \varphi^\star(s)] = \left[-\frac{\alpha_{\mathbf{w}}}{E_{\mathbf{w}}} \ln r^\star(s) + \varphi^\star(s), (r^\star(s))^{\delta_{\mathbf{w}}}\right] = (x(s), y(s)).$$

The map $\Phi_{\mathbf{w}} \circ \beta$ is a helix accumulating on the circle defined by $\Sigma_{\mathbf{w}}^{out} \cap W_{loc}^u(\mathbf{w})$ because $x(s)$ and $y(s)$ are monotonic and

$$\lim_{s\to 0^+} (r^\star(s))^{\delta_{\mathbf{w}}} = 0 \quad \text{and} \quad \lim_{s\to 0^+} \left| -\frac{\alpha_{\mathbf{w}}}{E_{\mathbf{w}}} \ln r^\star(s) + \varphi^\star(s) \right| = +\infty.$$

□

Remark 2. *The coordinates $(x, y) \in \Sigma_{\mathbf{w}}^{out}$ may be chosen so as to make the map $\varphi^\star$ increasing or decreasing, according to our convenience.*

In the context of Theorem 1, recall that by convention, the point $W^u(\mathbf{v}) \cap W^s(\mathbf{v}) \cap \Sigma_{\mathbf{v}}^{in}$ has angular coordinate equal to zero. For $\tau > 0$ very small, we define the two lines:

$$\mathcal{C}_{\mathbf{v}} = \{(x, y) \in \Sigma_{\mathbf{v}}^{in} : \quad y = 0 \quad \text{and} \quad 0 < |x| \le \tau\}.$$

Analogously for $\mathbf{w}$, as depicted in Figure 6, we set:

$$\mathcal{C}_{\mathbf{w}} = \{(x, y) \in \Sigma_{\mathbf{w}}^{out} : \quad y = 0 \quad \text{and} \quad 0 < |x| \le \tau\}.$$

The following geometrical result is the main ingredient for the proof of Theorem 1 in Section 6. Despite the fact that the stable manifold of $\mathbf{v}$ continued along the homoclinic cycle in backward time has a helicoid form, next lemma states that its first intersection with $\Sigma_{\mathbf{v}}^{in}$ has two segments. Similar reasonings work for $\mathbf{w}$. The leading rotation occurs near the invariant saddle and not during the homo/heteroclinic connection; more precisely, the *distortion* induced by $\Psi_{[\mathbf{v}\to\mathbf{v}]}^{-1}$ and $\Psi_{[\mathbf{w}\to\mathbf{w}]}$ is small and controlled. The proof of the following result is an immediate application of the Tubular Flow Box Theorem [21].

Lemma 9. *The set:*

1. *$\mathcal{C}_{\mathbf{v}}$ is mapped by $\Psi_{[\mathbf{v}\to\mathbf{v}]}^{-1}$ into two segments in $\Sigma_{\mathbf{v}}^{out}$;*
2. *$\mathcal{C}_{\mathbf{w}}$ is mapped by $\Psi_{[\mathbf{w}\to\mathbf{w}]}$ into two segments in $\Sigma_{\mathbf{w}}^{in}$.*

Item (2) of Lemma 9 is illustrated in Figure 6. Let $f, \tilde{f} \in \mathcal{H}$ such that f and $\tilde{f}$ are h-topologically equivalent. If $h(\mathbf{v}) =: \tilde{\mathbf{v}}$, then:

1. if h preserves orientation, then each connected component of $\Psi_{[\tilde{\mathbf{v}}\to\tilde{\mathbf{v}}]}^{-1}(\mathcal{C}_{\tilde{\mathbf{v}}}) \backslash W_{loc}^s(\tilde{\mathbf{v}})$ is a segment in $\Sigma_{\tilde{\mathbf{v}}}^{out}$;
2. if h reverses orientation, then each connected component of $\Psi_{[\tilde{\mathbf{v}}\to\tilde{\mathbf{v}}]}(\mathcal{C}_{\tilde{\mathbf{v}}}) \backslash W_{loc}^u(\tilde{\mathbf{v}})$ is a segment in $\Sigma_{\tilde{\mathbf{v}}}^{out}$.

Dual results hold for $\mathbf{w}$.

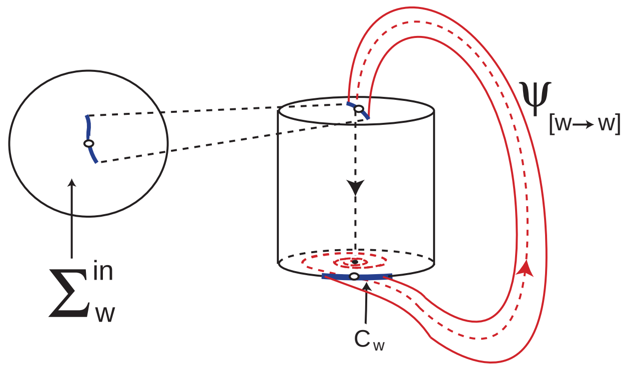

FIGURE 6. The set $\mathcal{C}_{\mathbf{w}}$ is mapped by $\Psi_{[\mathbf{w}\to\mathbf{w}]}$ into two segments in $\Sigma^{in}_{\mathbf{w}}$ meaning that the distortion along the transition map $\Psi_{[\mathbf{w}\to\mathbf{w}]}$ is small.

6. Topological invariants for Shilnikov homoclinic cycles

In this section, we deal with homoclinic cycles of Shilnikov type associated to a saddle-focus $\mathbf{p}$. For convenience, we concentrate our analysis on the case where the Morse index of $\mathbf{p}$ is 2 (equilibrium $\mathbf{w}$) and h preserves time-orientation. The other cases have a similar treatment reversing the time. The following topological proof is based on the works of [3, 13, 16, 41]. Although we are using the time of flight of trajectories through the neighbourhoods V and W, the result does not depend on the conjugacy because the invariants are constructed as a limit.

6.1. Proof of item (1) of Theorem 1: the imaginary part.

Let $f, \tilde{f} \in \mathcal{H}$ and let $\tilde{f}$ be a vector field that is h–topologically equivalent to f (preserving time-orientation) around $\mathbf{w}$. We want to prove that $|\alpha_{\mathbf{w}}| = |\alpha_{\tilde{\mathbf{w}}}|$.

Let us find two neighbourhoods of $\mathbf{w}$ and $\tilde{\mathbf{w}}$ where it is possible to linearize the vector fields around the saddle-foci, as done in Subsection 4.2. If the homeomorphism given by the linearizations are denoted by $\xi_{\mathbf{w}}$ and $\xi_{\tilde{\mathbf{w}}}$, then the map $h^\star := \xi_{\tilde{\mathbf{w}}} \circ h \circ (\xi_{\mathbf{w}})^{-1}$ can be considered as a homeomorphism between two cylinders W and $\tilde{W}$, where formulas of Section 4 can be applied, up to higher order terms. With no loss of generality and in order to simplify the notation, we may consider $h \equiv h^\star$ the homeomorphism between two cylinders as those defined before.

Let α be one of the two segments $\Psi_{[\mathbf{w}\to\mathbf{w}]}(\mathcal{C}_{\mathbf{w}}) \subset \Sigma^{in}_{\mathbf{w}}$ parametrized by $s \in [0,1]$: $\alpha(s) = (r_\alpha(s), \varphi_\alpha(s))$. By definition, it follows that $\varphi_\alpha(s)$ is bounded for $[0,1]$ – in fact, are asking for this restriction for $s \approx 0$. Using (8), we have:

$$\Phi_{\mathbf{w}}(\alpha(s)) = \left(\frac{\alpha_{\mathbf{w}}}{E_{\mathbf{w}}} \ln(r_\alpha(s)) + \varphi_\alpha(s); r^\delta_\alpha(s)\right) = (x, y).$$

Let $\varphi_0 \in [0, 2\pi[$. For $n \in \mathbf{N}$, let $s_n \in \mathbf{R}$ be such that (see Figure 7):

$$-\frac{\alpha_{\mathbf{w}}}{E_{\mathbf{w}}} \ln(r_\alpha(s_n)) + \varphi_\alpha(s_n) = 2n\pi + \varphi_0 \tag{17}$$

and define $A_n = \alpha(s_n)$, $B_n = \Phi_{\mathbf{w}}(\alpha(s_n)) = \Phi_{\mathbf{w}}(A_n) \in \Sigma^{out}_{\mathbf{w}}$ and $\tilde{A}_n = h(A_n) = h(\Phi^{-1}_{\mathbf{w}}(B_n))$. We also set $\tilde{B}_n = \Phi^{-1}_{\tilde{\mathbf{w}}}(\tilde{A}_n)$ as suggested in Figure 7. First we are going to show that for all $n \in \mathbf{N}$, we have:

$$\tilde{\varphi}(\tilde{A}_n) = \tilde{x}(\tilde{B}_n) - \frac{\alpha_{\tilde{\mathbf{w}}}}{\alpha_{\mathbf{w}}}(2n\pi + \varphi_0 - \varphi(A_n)) + o(|r_\alpha(s)|). \tag{18}$$

Indeed, using (8), it follows that:

$$\tilde{\varphi}(\tilde{A}_n) - \tilde{x}(\tilde{B}_n) = \tilde{\varphi}(\tilde{A}_n) + \frac{\alpha_{\tilde{\mathbf{w}}}}{E_{\tilde{\mathbf{w}}}} \ln(\tilde{r}_\alpha(s_n)) - \tilde{\varphi}(\tilde{A}_n). \tag{19}$$

By (7), note that $-\frac{1}{E_{\tilde{\mathbf{w}}}} \ln(\tilde{r}_\alpha(s_n))$ is the time of flight inside $\tilde{W}$ of a trajectory whose initial condition is $(\tilde{r}_\alpha(s_n), \tilde{\varphi}_\alpha(s_n))$, where:

$$\Sigma^{in}_{\tilde{\mathbf{w}}} = h(\Sigma^{in}_{\mathbf{w}}) \qquad \text{and} \qquad \Sigma^{out}_{\tilde{\mathbf{w}}} = h(\Sigma^{out}_{\mathbf{w}}).$$

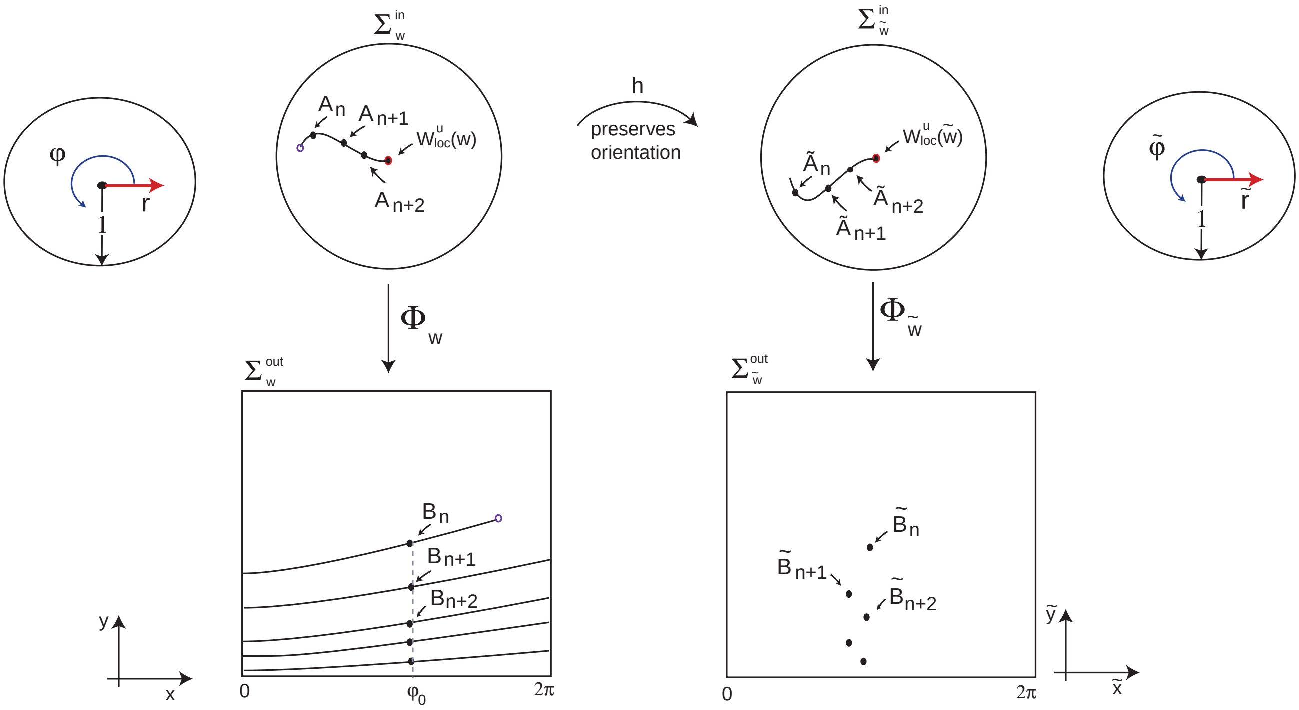

FIGURE 7. Topological equivalence: $A_n = \alpha(t_n)$, $B_n = \Phi_{\mathbf{w}}(\alpha(s_n)) = \Phi_{\mathbf{w}}(A_n) \in \Sigma^{out}_{\mathbf{w}}$, $\tilde{A}_n = h(A_n) = h(\Phi^{-1}_{\mathbf{w}}(B_n)$ and $\tilde{B}_n) = \Phi_{\tilde{\mathbf{w}}}(\tilde{A}_n)$.

Near $W^u_{loc}(\mathbf{w}) \cap \Sigma^{in}_{\mathbf{w}}$ (*ie* when $r_\alpha(s) \approx 0$), up to higher order terms, the (real) time of flight inside a small neighbourhood of $\mathbf{w}$ of a point in $\Sigma^{in}_{\mathbf{w}}$ is essentially governed by $T_{\mathbf{w}}$ given in formula (7). Thus, we may write:

$$\frac{\alpha_{\tilde{\mathbf{w}}}}{E_{\tilde{\mathbf{w}}}} \ln(\tilde{r}_\alpha(s_n)) = \frac{\alpha_{\tilde{\mathbf{w}}}}{E_{\mathbf{w}}} \ln(r_\alpha(s_n)) + o(|r_\alpha(s_n)|). \tag{20}$$

Combining (19) and (20), we have $\tilde{\varphi}(\tilde{A}_n) - \tilde{x}(\tilde{B}_n) = \frac{\alpha_{\tilde{\mathbf{w}}}}{E_{\mathbf{w}}} \ln(r_\alpha(s_n)) + o(|r_\alpha(s_n)|)$. Using now (17), we get:

$$\frac{\alpha_{\tilde{\mathbf{w}}}}{E_{\mathbf{w}}} \ln(r_\alpha(s_n)) + o(|r_\alpha(s_n)|) = (-2n\pi - \varphi_0 + \varphi_\alpha(s_n)) \times \frac{\alpha_{\tilde{\mathbf{w}}}}{\alpha_{\mathbf{w}}}.$$

and so $\tilde{\varphi}(\tilde{A}_n) - \tilde{x}(\tilde{B}_n) = -\frac{\alpha_{\tilde{\mathbf{w}}}}{\alpha_{\mathbf{w}}}(2n\pi + \varphi_0 - \varphi(A_n)) + o(|r_\alpha(s_n)|)$. Formula (18) is shown.

By Lemma 9 and the subsequent remark, $h(\alpha)$ must be a segment on $\Sigma^{in}_{\tilde{\mathbf{w}}}$. Thus $\tilde{\varphi}(h(\alpha(s))$ is bounded; this means that there exists $k \in \mathbf{R}^+$ such that $|\tilde{\varphi}(\tilde{A}_n)| < k$. Equality (18) implies that there exists $k \in \mathbf{R}^+$ such that:

$$\left| \tilde{x}(B_n) - \frac{\alpha_{\tilde{\mathbf{w}}}}{\alpha_{\mathbf{w}}} (2\pi n + \varphi_0 - \varphi(A_n)) + o(|r_\alpha(s_n)|) \right| < k. \tag{21}$$

Taking the limit of (21) for $n \in \mathbf{N}$, it implies that:

$$\lim_{n \in \mathbf{N}} \left| \frac{\tilde{x}(B_n)}{n} - \frac{\alpha_{\tilde{\mathbf{w}}}}{\alpha_{\mathbf{w}}} \left(2\pi + \frac{\varphi_0}{n} - \frac{\varphi(A_n)}{n} \right) + \frac{o(|r_\alpha(s_n)|)}{n} \right| = 0 \tag{22}$$

Since the sequence $\varphi(A_n)$ is bounded, then $\lim_{n \in \mathbf{N}} \frac{\varphi(A_n)}{n} = 0$. Consequently,

$$\lim_{n \in \mathbf{N}} \left| \frac{\tilde{x}(B_n)}{n} \right| = \left| \frac{\alpha_{\tilde{\mathbf{w}}}}{\alpha_{\mathbf{w}}} 2\pi \right|. \tag{23}$$

By construction, one knows that $\lim_{n\in\mathbf{N}} x(B_n) = \varphi_0$, then:

$$\lim_{n\in\mathbf{N}} \tilde{x}(h(B_n)) = \lim_{n\in\mathbf{N}} \tilde{x}(\tilde{B}_n) = \tilde{\varphi}_0,$$

for some $\tilde{\varphi}_0 \in [0, 2\pi[$. Therefore there exists $M \in \mathbf{R}^+$ such that:

$$\tilde{x}(\tilde{B}_n) - 2\pi n < M \quad \text{and} \quad \tilde{x}(\tilde{B}_n) + 2\pi n < M$$

and hence for all $n \in \mathbf{N}$, it follows that $\left|\frac{\tilde{x}(\tilde{B}_n)}{n} \pm 2\pi\right| < \frac{M}{n}$, meaning that $\lim_{n\in\mathbf{N}} \left|\frac{\tilde{x}(\tilde{B}_n)}{n}\right| = 2\pi$. Using (23), it implies that $\alpha_{\mathbf{w}} = \pm\alpha_{\tilde{\mathbf{w}}}$. A novel and more elegant proof of item (1) of Theorem 1 will be given in Section 7.

6.2. **Togawa's Theorem: the saddle index.** Togawa's Theorem states that the saddle index of a saddle-focus is a topological invariant. The idea behind the proof of [42] is the concept of link types of periodic solutions when $\delta_{\mathbf{v}} = \frac{C_{\mathbf{v}}}{E_{\mathbf{v}}} < 1$; more precisely, if $C_{\mathbf{v}} < E_{\mathbf{v}}$, Shilnikov [36, 37, 38] proved that there are infinitely many linked solutions (of saddle-type) arbitrarily close to the homoclinic cycle. The author considered *double round* periodic solutions and knot invariants to count the number of twists around the homoclinic separatrix. The ratio of twists in the first and the second turns determines the ratio of eigenvalues $C_{\mathbf{v}}$ and $E_{\mathbf{v}}$.

The topological invariance of the saddle index reflects the dependence of the configuration of the invariant manifolds on the saddle index. It may have further implications for the bifurcation diagrams of two coexisting saddle-foci as those of [27, 33] but this subject is beyond the scope of this article.

Let $T_{\mathbf{vv}}(x,y)$ denote $T_{\mathbf{v}}(\Pi_{\mathbf{v}}(x,y))$. We will prove that if $(x_n,y_n)_n \in \Sigma^{in}_{\mathbf{v}}$ is a sequence such that:

$$\lim_{n\in\mathbf{N}} (x_n, y_n) \in W^s_{loc}(\mathbf{v}) \qquad (\Leftrightarrow \lim_{n\in\mathbf{N}} y_n = 0), \tag{24}$$

then the limit $\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vv}}(x_n,y_n)}{T_{\mathbf{v}}(x_n,y_n)}$ does not depend on the sequence $(x_n, y_n)_n$ and then it is a topological invariant.

Alternative proof of item (2) of Theorem 1: Let $(x_n, y_n) \in \Sigma^{in}_{\mathbf{v}}$ as in (24). By (5), we have $T_{\mathbf{v}}(x_n, y_n) = -\frac{1}{E_{\mathbf{v}}} \ln|y_n|$. Using the expression (12), we know that:

$$\Pi_{\mathbf{v}}(x_n,y_n) = \begin{pmatrix} |y_n|^{\delta_{\mathbf{v}}}.\left(a_{11}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right) - a_{12}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right)\right) \\ |y_n|^{\delta_{\mathbf{v}}}.\left(a_{21}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right) - a_{22}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right)\right) \end{pmatrix}$$

Using the argument of Takens [41, Section 3.2], it follows that:

$$T_{\mathbf{vv}}(x_n,y_n) = -\frac{1}{E_{\mathbf{v}}}\ln\left(|y_n|^{\delta_{\mathbf{v}}}\left(a_{21}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right) - a_{22}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right)\right)\right) + o(|y_n|^{\delta_{\mathbf{v}}}).$$

Thus, we may write:

$$\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vv}}(x_n,y_n)}{T_{\mathbf{v}}(x_n,y_n)} = \lim_{n\in\mathbf{N}} \frac{\ln\left(|y_n|^{\delta_{\mathbf{v}}}.\left(a_{21}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right) - a_{22}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right)\right)\right) + o(|y_n|^{\delta_{\mathbf{v}}})}{\ln|y_n|}.$$

Using the properties of ln, one concludes that:

$$\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vv}}(x_n,y_n)}{T_{\mathbf{v}}(x_n,y_n)} = \lim_{n\in\mathbf{N}} \frac{\delta_{\mathbf{v}}\ln|y_n| + \ln\left[a_{21}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right) - a_{22}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right)\right]}{\ln|y_n|}. \tag{25}$$

Since $a_{21}\cos\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right) - a_{22}\sin\left(\frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}}\ln|y_n| - x_n\right)$ is bounded and $\lim_{n\in\mathbf{N}} \ln|y_n| = +\infty$ (because $\lim_{n\in\mathbf{N}} y_n = 0$), we focus on the leading terms and we conclude immediately that (25) is equal to $\delta_{\mathbf{v}}$, the saddle index of $\mathbf{v}$. □

If $\delta_{\mathbf{v}} > 1$, the homoclinic cycle $\Gamma_{\mathbf{v}}$ is attracting and $\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vv}}(x_n,y_n)}{T_{\mathbf{v}}(x_n,y_n)} > 1$, which is consistent with previous results in the literature, which say that the time of flight through V increases geometrically

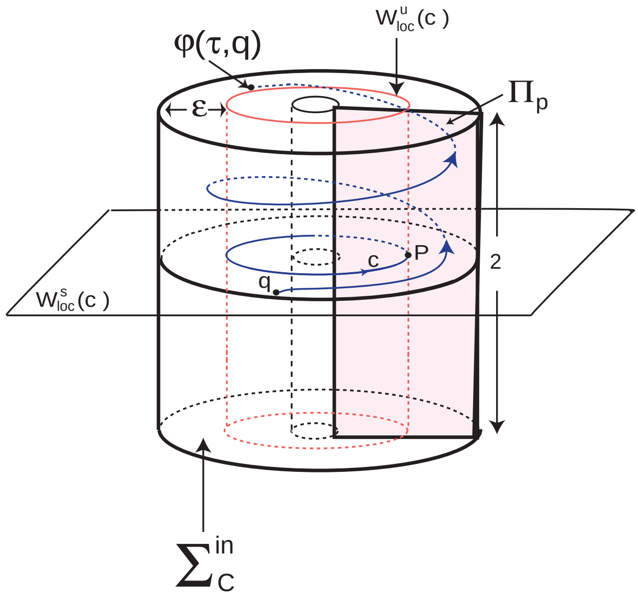

FIGURE 8. Example of a trajectory turning twice around a periodic solution with respect to $\Pi_{\mathbf{p}}$.

in consecutive visits of a solution to V. If $k \in \mathbf{N}$, since $\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{v}}(\Pi_{\mathbf{v}}^{k+1}(x_n,y_n))}{T_{\mathbf{v}}(\Pi_{\mathbf{v}}^{k}(x_n,y_n))} = \delta_{\mathbf{v}}$, then it follows straightforwardly that:

$$\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{v}}(\Pi_{\mathbf{v}}^{k}(x_n,y_n))}{T_{\mathbf{v}}(x_n,y_n)} = \delta_{\mathbf{v}}^{k}.$$

7. Spinning in Average

The following definitions and ideas have been motivated by the concept of *chaotic double cycling* of Rodrigues *et al* [34].

7.1. **Definition and properties.** Throughout this section, let $\mathbf{p}$ denote either a saddle-focus or a periodic solution. Given a homoclinic cycle associated to $\mathbf{p}$, $\Gamma_{\mathbf{p}}$, let $V_{\mathbf{p}}$ be a compact neighbourhood of the node $\mathbf{p}$ such that each boundary $\partial V_{\mathbf{p}}$ is a finite union of smooth manifolds with boundary, that are transverse to the vector field almost everywhere. The set $V_{\mathbf{p}}$ is called an *isolating block for* $\mathbf{p}$.

Consider a codimension one submanifold with boundary $\Pi_{\mathbf{p}} \subset V_{\mathbf{p}}$ of $\mathcal{M}^3$, such that (see Figure 8 where $\mathbf{p}$ is a non-trivial closed trajectory):

- the flow is transverse to $\Pi_{\mathbf{p}}$;
- $\Pi_{\mathbf{p}}$ intersects $\partial V_{\mathbf{p}}$ transversely;
- $W^s_{loc}(\mathbf{p}) \subset \partial \Pi_{\mathbf{p}}$ and $W^u_{loc}(\mathbf{p}) \subset \partial \Pi_{\mathbf{p}}$.

We call $\Pi_{\mathbf{p}}$ a *counting section.* We are interested in trajectories that go inside the neighbourhood $V_{\mathbf{p}}$ in positive time and hit the counting section $\Pi_{\mathbf{p}}$ a finite number of times (which can be zero) until they leave the neighbourhood. Every time the trajectory makes a turn inside $V_{\mathbf{p}}$ it hits the counting section. It is natural to have the following definition where $int(A)$ is the topological interior of $A \subset \mathcal{M}^3$:

Definition 3. *Let* $V_{\mathbf{p}}$ *be an isolating block for* $\mathbf{p}$ *and* $\Pi_{\mathbf{p}}$ *a counting section. Let* $q \in \partial V_{\mathbf{p}}$ *be a point such that the following properties hold:*

- $\exists \tau > 0, \forall t \in (0,\tau), \varphi(t,q) \in int(V_{\mathbf{p}})$
- $\varphi(\tau,q) \in \partial V_{\mathbf{p}}$.

The trajectory of q turns n times around $\mathbf{p}$ in $V_{\mathbf{p}}$, relatively to $\Pi_{\mathbf{p}}$ *if*

$$spin(q, V_{\mathbf{p}}, \Pi_{\mathbf{p}}) := \#\left(\{\varphi(t,q), t \in [0,\tau]\} \cap \Pi_{\mathbf{p}}\right) = n \geq 0 \ . \tag{26}$$

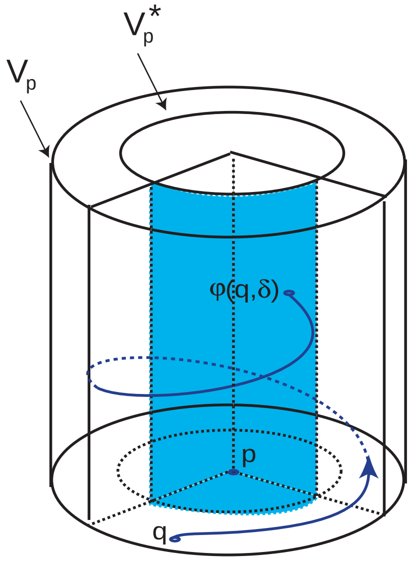

FIGURE 9. The sets $V_{\mathbf{p}}$ and $V_{\mathbf{p}}^{\star}$ are two isolating blocks of $\mathbf{p}$ such that $V_{\mathbf{p}} \supset V_{\mathbf{p}}^{\star}$. The point $\varphi(q, \delta)$ belongs to the wall of the narrower cylinder $V_{\mathbf{p}}^{\star}$.

Hereafter, this integer will be called by *spinning number* of q, which strongly depends on the isolating block and on the counting section. We state the following technical result that we use in the sequel.

Lemma 10. *Let $(x_n, y_n) \in \Sigma_{\mathbf{p}}^{in}$ be a sequence such that $\lim_{n\in\mathbf{N}}(x_n, y_n) \in W_{loc}^s(\mathbf{p})$ and let $T_{\mathbf{p}}(x_n, y_n)$ be the time spent in $V_{\mathbf{p}}$ by the trajectory whose initial condition is (x_n, y_n). The limit*

$$\lim_{n\in\mathbf{N}} \frac{spin((x_n, y_n), V_{\mathbf{p}}, \Pi_{\mathbf{p}})}{T_{\mathbf{p}}(x_n, y_n)} \tag{27}$$

does not depend neither on the isolating block $V_{\mathbf{p}}$ nor on the cross section $\Pi_{\mathbf{p}}$.

Proof. Let $(x_n, y_n) \in \Sigma_{\mathbf{p}}^{in}$ and let $T_{\mathbf{p}}(x_n, y_n)$ be as in the statement. In order to simplify the arguments, let $V_{\mathbf{p}}$ and $V_{\mathbf{p}}^{\star}$ be two isolating blocks of $\mathbf{p}$ such that $V_{\mathbf{p}} \supset V_{\mathbf{p}}^{\star}$ as shown in Figure 9. Let $\Pi_{\mathbf{p}}^{\star} \subset V_{\mathbf{p}}^{\star}$ and $\Pi_{\mathbf{p}} \subset V_{\mathbf{p}}$ be two counting sections at $\mathbf{p}$.

First note that for each $q = (x_n, y_n) \in \Sigma_{\mathbf{p}}^{in}$, there exists a bounded map $k : \Sigma_{\mathbf{p}}^{in} \to \mathbf{Z}$ such that:

$$spin(q, V_{\mathbf{p}}, \Pi_{\mathbf{p}}) = spin(\varphi(q, \delta), V_{\mathbf{p}}^{\star}, \Pi_{\mathbf{p}}^{\star}) + k(q), \tag{28}$$

where $\delta = \min\{t \in \mathbf{R}^+ : \varphi(q, t) \in \partial V_{\mathbf{p}}^{\star}\}$ – it means that $\varphi(q, \delta)$ is the first intersection of $\{\varphi(q, t)\}|_{t>0}$ with the wall of the narrower cylinder $V_{\mathbf{p}}^{\star}$. Using (28), it implies that:

$$\begin{aligned}
\lim_{n\in\mathbf{N}} \frac{spin(\varphi((x_n, y_n), \delta), V_{\mathbf{p}}, \Pi_{\mathbf{p}})}{T_{\mathbf{p}}(x_n, y_n)} &= \lim_{n\in\mathbf{N}} \frac{spin(\varphi((x_n, y_n), \delta), V_{\mathbf{p}}^{\star}, \Pi_{\mathbf{p}}^{\star}) + k(x_n, y_n)}{T_{\mathbf{p}}(x_n, y_n)} \\
&= \lim_{n\in\mathbf{N}} \frac{spin(\varphi((x_n, y_n), \delta), V_{\mathbf{p}}^{\star}, \Pi_{\mathbf{p}}^{\star})}{T_{\mathbf{p}}(x_n, y_n)} + \lim_{n\in\mathbf{N}} \frac{k(x_n, y_n)}{T_{\mathbf{p}}(x_n, y_n)} \\
&= \lim_{n\in\mathbf{N}} \frac{spin(\varphi((x_n, y_n), \delta), V_{\mathbf{p}}^{\star}, \Pi_{\mathbf{p}}^{\star})}{T_{\mathbf{p}}^{\star}(x_n, y_n) + \tau},
\end{aligned}$$

where $\tau > 0$ is the upper limit of time that trajectories spend from $V_{\mathbf{p}}$ to $V_{\mathbf{p}}^{\star}$ and $T_{\mathbf{p}}^{\star}(x_n, y_n)$ is the time of flight of $\varphi((x_n, y_n), \delta)$ through $V_{\mathbf{p}}^{\star}$. The last equality follows because $\lim_{n\in\mathbf{N}} T_{\mathbf{p}}(x_n, y_n) = +\infty$ and k is bounded. $\square$

Limit (27) is what we call *spinning in average*, which does not depend on the choice of coordinates. Observe that a homeomorphism h maps the local invariant manifolds into local invariant manifolds and cylindrical coordinates into cylindrical coordinates. Thus, we may conclude that:

$$\lim_{n\in\mathbf{N}} \frac{spin((x_n,y_n),V_\mathbf{p},\Pi_\mathbf{p})}{T_\mathbf{p}(x_n,y_n)} = \lim_{n\in\mathbf{N}} \frac{spin(h(x_n,y_n),h(V_\mathbf{p}),h(\Pi_\mathbf{p}))}{T_\mathbf{p}(h(x_n,y_n))+o(|y_n|)} = \lim_{n\in\mathbf{N}} \frac{spin(h(x_n,y_n),h(V_\mathbf{p}),h(\Pi_\mathbf{p}))}{T_\mathbf{p}(h(x_n,y_n))}.$$

For an equilibrium point $\mathbf{p}$, from now on we use as counting section the following *rectangle* denoted by $\Pi_\mathbf{p}$:

$$\Pi_\mathbf{p} = \{(\rho,\theta,z): \quad \theta=0, \quad 0\leq\rho\leq 1 \quad \text{and} \quad 0\leq z\leq 1\} \ .$$

The coming step is to compute (27) and show that it does not depend on the sequence $(x_n,y_n)_{n\in\mathbf{N}}$. Denoting by $[a]$ the greatest integer less than or equal to a, we have the following result, whose proof follows from the expression of the angular component of $\Phi_\mathbf{v}$ in (6) and $\Phi_\mathbf{w}$ in (8).

Lemma 11 (Rodrigues *et al* [34], adapted)**.** *The spinning number of $(x_n,y_n)\in\Sigma^{in}_\mathbf{p}$ inside $V_\mathbf{p}$ with respect to $\Pi_\mathbf{p}$ is given by:*

$$spin((x_n,y_n),V_\mathbf{p},\Pi_\mathbf{p}) = \left|\left[\frac{\alpha_\mathbf{p}T_\mathbf{p}(x_n,y_n)+x_n}{2\pi}\right]\right|.$$

which tends to $+\infty$ when y_n tends to zero.

7.2. **Shilnikov cycle.** In the present section, we give an alternative proof of item (1) of Theorem 1, where the absolute value of the imaginary part of the eigenvalues of Df at the saddle-focus may be considered a topological invariant in $\mathcal{H}$.

Alternative Proof of item (1) of Theorem 1. Let $(x_n,y_n)\in\Sigma^{in}_\mathbf{p}$ be a sequence such that $\lim_{n\in\mathbf{N}}(x_n,y_n)\in W^s_{loc}(\mathbf{p})$ and let $T_\mathbf{p}(x_n,y_n)$ be the time spent in $V_\mathbf{p}$ by the trajectory whose initial condition is (x_n,y_n). By Lemma 11, there exists $K\in\mathbf{R}^+$ such that:

$$spin((x_n,y_n),V_\mathbf{p},\Pi_\mathbf{p}) - \left|\frac{\alpha_\mathbf{p}T_\mathbf{p}(x_n,y_n)+x_n}{2\pi}\right| < K.$$

Since $\lim_{n\in\mathbf{N}} T_\mathbf{p}(x_n,y_n)=+\infty$, it implies that:

$$\lim_{n\in\mathbf{N}} \frac{spin((x_n,y_n),V_\mathbf{p},\Pi_\mathbf{p}) - \left|\frac{\alpha_\mathbf{p}T_\mathbf{p}(x_n,y_n)+x_n}{2\pi}\right|}{T_\mathbf{p}(x_n,y_n)} = 0$$

and then:

$$\lim_{n\in\mathbf{N}} \frac{spin((x_n,y_n),V_\mathbf{p},\Pi_\mathbf{p})}{T_\mathbf{p}(x_n,y_n)} = \lim_{n\in\mathbf{N}} \frac{\left|\frac{\alpha_\mathbf{p}T_\mathbf{p}(x_n,y_n)+x_n}{2\pi}\right|}{T_\mathbf{p}(x_n,y_n)} = \lim_{n\in\mathbf{N}} \frac{|\alpha_\mathbf{p}|T_\mathbf{p}(x_n,y_n)}{2\pi T_\mathbf{p}(x_n,y_n)} = \frac{|\alpha_\mathbf{p}|}{2\pi}. \tag{29}$$

Since the spinning in average is preserved under homeomorphisms, it follows that $|\alpha_\mathbf{p}|$ is a modulus of topological equivalence. □

The previous result allows us to conclude that asymptotically we have:

$$\frac{spin((x_n,y_n),V_\mathbf{p},\Pi_\mathbf{p})}{T_\mathbf{p}(x_n,y_n)} \approx \frac{|\alpha_\mathbf{p}|}{2\pi}, \tag{30}$$

which means that up to multiplication by a positive constant, the time of flight of trajectories with initial condition (x_n,y_n) through $V_\mathbf{p}$ (with $\lim_{n\in\mathbf{N}} y_n=0$) can be replaced by the number of revolutions around $\mathbf{p}$ as defined in (26). The one-dimensional invariant manifold of $\mathbf{p}$ behaves as a vortex core curve since solutions spiral around it.

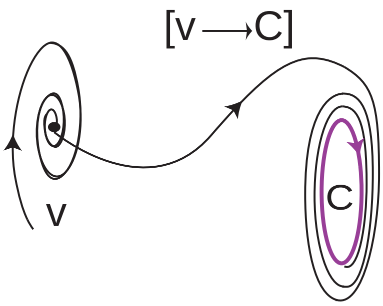

FIGURE 10. Heteroclinic connection involving a saddle-focus $\mathbf{v}$ of Morse index 1 and a periodic solution $\mathbf{C}$.

7.3. **Bykov connection.** Using the concept of *spinning in average*, we prove item (1) of Theorem 3. If $(x, y) \in \Sigma_{\mathbf{v}}^{in}$, let $T_{\mathbf{vw}}(x, y)$ denote $T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x, y))$ – please recall the definiton of $\eta_{\mathbf{vw}}$ in Subsection 4.4.

Proof of item (1) of Theorem 3. Let $(x_n, y_n) \in \Sigma_{\mathbf{v}}^{in}$ such that $\lim_{n\in\mathbf{N}}(x_n, y_n) \in W^s_{loc}(\mathbf{v})$. Using (5), we may write $T_{\mathbf{v}}(x_n, y_n) = -\frac{1}{E_{\mathbf{v}}} \ln |y_n|$. In the polar coordinates of $\Sigma_{\mathbf{w}}^{in}$, the radial component of $\eta_{\mathbf{vw}}(x_n, y_n)$ is given by:

$$R = r\sqrt{a^2 \cos^2(\varphi) + a^{-2} \sin^2(\varphi)},$$

where $r = |y_n|^{\delta_{\mathbf{v}}}$ and $\varphi = x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|$. Writing the expressions explicitely, we have:

$$T_{\mathbf{vw}}(x_n, y_n) = -\frac{1}{E_{\mathbf{w}}} \ln\left(|y_n|^{\delta_{\mathbf{v}}} \times \sqrt{a^2 \cos^2\left(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|\right) + a^{-2} \sin^2\left(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|\right)}\right) + o(|y_n|^{\delta_{\mathbf{v}}})$$

and thus:

$$\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vw}}(x_n, y_n)}{T_{\mathbf{v}}(x_n, y_n)} = \lim_{n\in\mathbf{N}} \frac{\frac{1}{E_{\mathbf{w}}} \ln\left(|y_n|^{\delta_{\mathbf{v}}} \times \sqrt{a^2 \cos^2(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|) + a^{-2} \sin^2(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|)}\right) + o(|y_n|^{\delta_{\mathbf{v}}})}{\frac{1}{E_{\mathbf{v}}} \ln |y_n|}$$

Using the properties of ln, it follows that:

$$\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vw}}(x_n, y_n)}{T_{\mathbf{v}}(x_n, y_n)} = \lim_{n\in\mathbf{N}} \frac{E_{\mathbf{v}}}{E_{\mathbf{w}}} \frac{\delta_{\mathbf{v}} \ln |y_n| + \frac{1}{2} \ln\left(a^2 \cos^2(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|) + a^{-2} \sin^2(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|)\right)}{\ln |y_n|}$$

Noting that $\ln\left(a^2 \cos^2(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|) + a^{-2} \sin^2(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n|)\right)$ is bounded, the previous limit can be written as $\frac{C_{\mathbf{v}}}{E_{\mathbf{w}}}$. □

In order to prove item (3) of Theorem 3, we generalize the notion of *spinning in average* for two nodes connected by a one-dimensional heteroclinic connection of the type $[\mathbf{v} \to \mathbf{w}]$. We are interested in trajectories that go inside a neighbourhood V in positive time and hit the counting section $\Pi_{\mathbf{v}}$ a finite number of times until they leave the neighbourhood and jump for W repeating the process. Every time the trajectory makes a turn inside the neighbourhoods, it hits the corresponding counting sections. This is the motivation for the following definition:

Definition 4. *Let V and W be isolating blocks for $\mathbf{v}$ and $\mathbf{w}$, respectively, and let $\Pi_{\mathbf{v}}$, $\Pi_{\mathbf{w}}$ be the corresponding counting sections. Let $q \in \partial V$ be a point such that the following properties hold:*

- $\exists \tau_1 > 0, \forall t \in (0, \tau_1), \varphi(t, q) \in int(V)$;
- $\varphi(\tau_1, q) \in \partial V$;
- $\exists \tau_2 > 0, \forall t \in (0, \tau_2), \varphi(t, \eta_{\mathbf{vw}}(q)) \in int(W)$;
- $\varphi(\tau_2, \eta_{\mathbf{vw}}(q)) \in \partial W$;
- *between V and W the solution $\varphi(t, \star)$ does not visit the neighbourhood of any other saddle.*

The trajectory of q turns n_1 times around $\mathbf{v}$ in V, relatively to $\Pi_{\mathbf{v}}$ *and n_2 times around $\mathbf{w}$ in W, relatively to $\Pi_{\mathbf{w}}$, if $spin(q, V, \Pi_{\mathbf{v}}) = n_1$ and $spin(\eta_{\mathbf{vw}}(q), W, \Pi_{\mathbf{w}}) = n_2$.*

We may then define the *spinning number* along the heteroclinic connection $[\mathbf{v} \to \mathbf{w}]$ as:

$$spin(q, V, W, \Pi_{\mathbf{v}}, \Pi_{\mathbf{w}}) = spin(q, V, \Pi_{\mathbf{v}}) + spin(\eta_{\mathbf{vw}}(q), W, \Pi_{\mathbf{w}})$$

Roughly speaking, if $q \in \Sigma_{\mathbf{v}}^{in}$, the number $spin(q, V, W, \Pi_{\mathbf{v}}, \Pi_{\mathbf{w}})$ counts the number of hits in $\Pi_{\mathbf{v}}$ and $\Pi_{\mathbf{w}}$ of the trajectory with initial condition q. The following proof has been based on [12, 34, 41].

Proof of item (3) of Theorem 3. Let $(x_n, y_n) \in \Sigma_{\mathbf{v}}^{in}$ be a sequence such that $\lim_{n\in\mathbf{N}}(x_n, y_n) \in W^s_{loc}(\mathbf{v})$. By (30), there exists $k \in \mathbf{R}^+$ such that

$$\left| spin((x_n, y_n), V, W, \Pi_{\mathbf{v}}, \Pi_{\mathbf{w}}) - \frac{|\alpha_{\mathbf{v}}|}{2\pi} T_{\mathbf{v}}(x_n, y_n) - \frac{|\alpha_{\mathbf{w}}|}{2\pi} T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n)) \right| < k.$$

Thus:

$$\lim_{n\in\mathbf{N}} \frac{spin((x_n, y_n), V, W, \Pi_{\mathbf{v}}, \Pi_{\mathbf{w}}) - \left[\frac{|\alpha_{\mathbf{v}}|}{2\pi} T_{\mathbf{v}}(x_n, y_n) + \frac{|\alpha_{\mathbf{w}}|}{2\pi} T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))\right]}{\tau_n(x_n, y_n)} = 0$$

where $\tau_n(x_n, y_n) = T_{\mathbf{v}}(x_n, y_n) + s_n + T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))$ and s_n is the time of flight of the transition between $\Sigma_{\mathbf{v}}^{out}$ and $\Sigma_{\mathbf{w}}^{in}$. Let us compute the following limit:

$$\lim_{n\in\mathbf{N}} \frac{|\alpha_{\mathbf{v}}| T_{\mathbf{v}}(x_n y, y_n) + |\alpha_{\mathbf{w}}| T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))}{T_{\mathbf{v}}(x_n, y_n) + s_n + T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))}$$

Since $\lim_{n\in\mathbf{N}} s_n = s \in \mathbf{R}^+$, then we may deduce the following:

$$\begin{aligned}
& \lim_{n\in\mathbf{N}} \frac{|\alpha_{\mathbf{v}}| T_{\mathbf{v}}(x_n, y_n) + |\alpha_{\mathbf{w}}| T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))}{T_{\mathbf{v}}(x_n, y_n) + s_n + T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))} = \\
= \ & \lim_{n\in\mathbf{N}} \frac{|\alpha_{\mathbf{v}}| T_{\mathbf{v}}(x_n, y_n) + |\alpha_{\mathbf{w}}| T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))}{T_{\mathbf{v}}(x_n, y_n) + T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))} = \\
= \ & \lim_{n\in\mathbf{N}} \frac{|\alpha_{\mathbf{v}}| T_{\mathbf{v}}(x_n, y_n) + |\alpha_{\mathbf{w}}| T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))}{T_{\mathbf{v}}(x_n, y_n)} \times \lim_{n\in\mathbf{N}} \frac{T_{\mathbf{v}}(x_n, y_n)}{T_{\mathbf{v}}(x_n, y_n) + T_{\mathbf{w}}(\eta_{\mathbf{vw}}(x_n, y_n))} = \\
= \ & |\alpha_{\mathbf{v}}| + \lim_{n\in\mathbf{N}} \frac{|\alpha_{\mathbf{w}}| T_{\mathbf{vw}}(x_n, y_n)}{T_{\mathbf{v}}(x_n, y_n)} \times \lim_{n\in\mathbf{N}} \left(1 + \frac{T_{\mathbf{vw}}(x_n, y_n)}{T_{\mathbf{v}}(x_n, y_n)}\right)^{-1} = \\
\overset{(*)}{=} \ & \left(|\alpha_{\mathbf{v}}| + |\alpha_{\mathbf{w}}| \frac{C_{\mathbf{v}}}{E_{\mathbf{w}}}\right) \times \left(1 + \frac{C_{\mathbf{v}}}{E_{\mathbf{w}}}\right)^{-1} = \\
= \ & (|\alpha_{\mathbf{v}}| E_{\mathbf{w}} + |\alpha_{\mathbf{w}}| C_{\mathbf{v}}) \times (C_{\mathbf{v}} + E_{\mathbf{w}})^{-1},
\end{aligned}$$

which does not depend on the sequence $(x_n, y_n)_{n\in\mathbf{N}}$. Equality (*) follows from the proof of item (1) of Theorem 3 where we have shown that $\lim_{n\in\mathbf{N}} \frac{T_{\mathbf{vw}}(x_n, y_n)}{T_{\mathbf{v}}(x_n, y_n)} = \frac{C_{\mathbf{v}}}{E_{\mathbf{w}}}$.

□

On the proof of Theorem 3, the time τ_n can be replaced by the sum of the local time of flight near the two saddles because asymptotically solutions spend almost all the time near the saddles and not during the transition from $\Sigma_{\mathbf{v}}^{out}$ and $\Sigma_{\mathbf{w}}^{in}$.

7.4. **Heteroclinic connection involving a saddle-focus and a periodic solution.** The proof of Theorem 4 is similar to that performed for Theorem 3. We sketch here the main steps of the proof. First of all, observe that Lemmas 10 and 11 hold for the case where $\mathbf{p}$ is a non-trivial closed trajectory. If $(\theta_n, z_n) \in \Sigma_{\mathbf{C}}^{in}$, the *spinning number* of (θ_n, z_n) inside $V_{\mathbf{C}}$, an isolating block of $\mathbf{C}$, with respect to the counting section:

$$\Pi_{\mathbf{C}} = \{(\rho, \theta, z) : \quad \theta = 0, \quad 1 \leq \rho \leq 1 + \varepsilon \quad \text{and} \quad 0 \leq z \leq 1\}$$

is given by:

$$spin((\theta_n, z_n), V_{\mathbf{C}}, \Pi_{\mathbf{C}}) = \left[\frac{\theta_n}{2\pi} - \frac{\ln |z_n|}{T E_{\mathbf{C}}}\right],$$

which tends to $+\infty$ when z_n tends to zero. By (10), one knows that $T_{\mathbf{C}}(\theta_n, z_n) = -\frac{1}{E_{\mathbf{C}}} \ln(z_n)$, hence:

$$\lim_{n\in\mathbf{N}} \frac{spin((\theta_n, z_n), V_{\mathbf{C}}, \Pi_{\mathbf{C}})}{T_{\mathbf{C}}(\theta_n, z_n)} = \lim_{n\in\mathbf{N}} \frac{-\frac{\ln |z_n|}{T E_{\mathbf{C}}}}{T_{\mathbf{C}}(\theta_n, z_n)} = \lim_{n\in\mathbf{N}} \frac{T_{\mathbf{C}}(\theta_n, z_n)}{T \times T_{\mathbf{C}}(\theta_n, z_n)} = \frac{1}{T}$$

The argument used in Subsection 7.3 to prove (1) of Theorem 3 is valid to show that:

$$T_{\mathbf{vC}}(x_n, y_n) = -\frac{1}{E_{\mathbf{C}}} \ln \left(|y_n|^{\delta_{\mathbf{v}}} \times \sqrt{a^2 \cos^2 \left(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n| \right) + a^{-2} \sin^2 \left(x_n - \frac{\alpha_{\mathbf{v}}}{E_{\mathbf{v}}} \ln |y_n| \right)} \right) = \frac{C_{\mathbf{v}}}{E_{\mathbf{C}}}.$$

and then

$$\begin{aligned} \lim_{n \in \mathbf{N}} & \frac{|\alpha_{\mathbf{v}}| T_{\mathbf{v}}(x_n y, y_n) + \frac{2\pi}{T} T_{\mathbf{C}}(\eta_{\mathbf{vC}}(x_n, y_n))}{T_{\mathbf{v}}(x_n, y_n) + s_n + T_{\mathbf{vC}}(x_n, y_n)} = \\ = & \left(|\alpha_{\mathbf{v}}| + \frac{2\pi}{T} \frac{C_{\mathbf{v}}}{E_{\mathbf{C}}} \right) \times \left(1 + \frac{C_{\mathbf{v}}}{E_{\mathbf{C}}} \right)^{-1} = \\ = & \left(|\alpha_{\mathbf{v}}| E_{\mathbf{C}} + \frac{2\pi}{T} C_{\mathbf{v}} \right) \times (E_{\mathbf{C}} + C_{\mathbf{v}})^{-1}. \end{aligned}$$

Theorem 4 is now proved. Before finish this section, we would like to stress that we have used the conjugacy of Theorem 4 just in Subsection 4.3 to construct the suspension around $V_{\mathbf{C}}$. Once we obtain a linear vector field whose flow has a non-trivial closed trajectory of period T, we do not need anymore the conjugacy.

8. Discussion and Final Remarks

In dealing with nonlinear dynamics people are used to ask questions on their asymptotic in time properties. Even most of concepts used in nonlinear dynamics like Lyapunov exponents or decay of correlations involve an infinite time limit. Although general answers to these questions are very difficult to obtain, in this article, in the spirit of the works [16, 34, 41], we have shown that some questions about topological invariants could be partially answered and that the local dynamics near one-dimensional homo/heteroclinic connections could be seen as a signature for the global dynamics.

For three-dimensional flows with connections involving saddle-foci, the spiral patterns contain *vortices*; the strenght of this swirling motion is measured by the absolute value of the imaginary part of the complex eigenvalues. The spiralling dynamics and cycling behaviour described in [3, 33, 34] bring strong properties that should not be neglected in terms of topological dynamics, and constitute the main ingredient for the proofs throughout all article. The rotation around each equilibrium constitutes a different situation from that where all eigenvalues of the linearization at the node are real.

For a system with a homoclinic cycle to a saddle-focus, we gave an alternative proof that the absolute value of the imaginary part of the complex eigenvalues is a topological invariant, in the set of vector fields whose flow has a Shilnikov cycle. Our proof is different to that presented in [12], in which the author used knots-type arguments. Furthermore, we have also constructed numerical invariants for cycles involving rotating nodes, where the saddles are either saddle-foci or periodic solutions. The invariants depend heavily on the eigenvalues of the linearization of the vector fields at the nodes and they allow to distinguish different complex dynamics, up to topologically equivalence. These results generalize those of Gaunersdorfer [16] and Takens [41] for planar vector fields.

A crucial argument of our proofs is the analysis of the dynamics near structurally unstable connections of dimension one. The Flow Box Theorem does not allow "irregular" behaviour. If the heteroclinic connections have codimension zero then transverse connections appear, chaos will occur and we would "lose" information about the "consecutive" times of flight. If the cycle $\Gamma_{\mathbf{vw}}$ were asymptotically stable, then we may generalize Theorem 3: besides $\frac{C_{\mathbf{v}}}{E_{\mathbf{w}}}$, the ratio $\frac{C_{\mathbf{w}}}{E_{\mathbf{C}}}$ would also be a topological invariant and then the constant $\delta = \delta_{\mathbf{v}} \delta_{\mathbf{w}} > 1$ which appear in [16], [25, 26] and [27] would become a modulus. The time averages of continuous functions along trajectories that converge to the heteroclinic cycle typically do not converge – we refer to Takens [41] for a relation between the moduli and these time averages (on the plane). A systematic study on this subject in dimension three is in preparation.

For higher dimensional cycles of rotating nodes, there are works in the literature which describe the local behaviour near the connections – see for instance Rodrigues *et al* [34, Section 6]. The extension relies strongly on the *Center Manifolds for Heteroclinic Cycles* [39]. Structurally unstable heteroclinic connections in higher dimensions may require infinitely many new moduli besides known ones for their description. A lot more needs to be done in order to characterize a complete set of topological invariants for cycles of rotating nodes and their non-wandering dynamics. We hope that this article could be a starting point for further related studies.

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