RARE EVENTS FOR THE MANNEVILLE-POMEAU MAP

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Abstract. We prove a dichotomy for Manneville-Pomeau maps $f : [0,1] \to [0,1]$: given any point $\zeta \in [0,1]$, either the Rare Events Point Processes (REPP), counting the number of exceedances, which correspond to entrances in balls around ζ, converge in distribution to a Poisson process; or the point ζ is periodic and the REPP converge in distribution to a compound Poisson process. Our method is to use inducing techniques for all points except 0 and its preimages, extending a recent result [HWZ14], and then to deal with the remaining points separately. The preimages of 0 are dealt with applying recent results in [AFV14]. The point $\zeta = 0$ is studied separately because the tangency with the identity map at this point creates too much dependence, which causes severe clustering of exceedances. The Extremal Index, which measures the intensity of clustering, is equal to 0 at $\zeta = 0$, which ultimately leads to a degenerate limit distribution for the partial maxima of stochastic processes arising from the dynamics and for the usual normalising sequences. We prove that using adapted normalising sequences we can still obtain non-degenerate limit distributions at $\zeta = 0$.

1. Introduction

One of the standard ways to investigate the statistical properties of a dynamical system $f : \mathcal{X} \to \mathcal{X}$ with respect to some measure $\mathbb{P}$ is to look at its recurrence to certain points ζ in the system. This can be connected to Extreme Value theory: by taking a suitable observable $\varphi : \mathcal{X} \to \mathbb{R}$ taking its unique maximum u_F at ζ, one can look at the behaviour of the iterates $x, f(x), f^2(x), \ldots$ via the observations

$$X_i = X_i(x) = \varphi \circ f^n(x).$$

If $\mathbb{P}$ is an f-invariant probability measure then $X_0, X_1, \ldots$ is a stationary stochastic process. We furthermore assume that $\mathbb{P}$ is ergodic in order to isolate specific statistical behaviour. So Birkhoff's Ergodic Theorem implies that these random variables will satisfy the law of large numbers. We can now consider the random variable given by the maximum of this process:

$$M_n = M_n(x) := \max\{X_0(x), \ldots, X_{n-1}(x)\}.$$

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Again by the ergodic theorem, if ζ is in the support of $\mathbb{P}$, we expect $M_n \to u_F$, so to obtain a non-trivial limit law, we need to rescale $\{M_n\}_n$. Indeed, we say that we have an *Extreme Value Law* (EVL) for M_n if there is a non-degenerate distribution function $H : \mathbb{R} \to [0,1]$ with $H(0) = 0$ and for every $\tau > 0$ there exists a sequence of levels $u_n = u_n(\tau)$ such that

$$n\mathbb{P}(X_0 > u_n) \to \tau \text{ as } n \to \infty, \tag{1}$$

and for which the following holds:

$$\mathbb{P}(M_n \leqslant u_n) \to \bar{H}(\tau) = 1 - H(\tau) \text{ as } n \to \infty,$$

where the convergence is meant at the continuity points of $H(\tau)$.

In recent years, there has been a great deal of work on EVLs in the context of dynamical systems (see for example [Col01, FF08a, VHF09, FFT10, GHN11, HNT12, LFW12, Kel12, FHN14, AFV14]), the standard form of the observable φ being a function of the distance to ζ, for example $\varphi(x) = -d(x, \zeta)$ for d a metric on $\mathcal{X}$. In many cases it has been shown that for $\mathbb{P}$-a.e. $\zeta \in \mathcal{X}$, this setup gives an EVL with $\bar{H} = e^{-\tau}$. More recently it has been shown that if ζ is a periodic point of period p then $\bar{H} = e^{-\theta\tau}$ where $\theta \in (0,1)$ depends on the Jacobean of the measure for f^p, and is referred to as the *Extremal Index* (EI). The EI is known to measure the intensity of clustering of exceedances of the levels u_n. In fact, in many cases, the EI is equal to the inverse of the average cluster size, so that the EI is equal to 1 when there is no clustering. In the case of a class of uniformly hyperbolic dynamical systems, a stronger property, a dichotomy, has been shown: either ζ is periodic and we have an EVL with some extremal index $\theta \in (0,1)$, or there is an EVL $\bar{H} = e^{-\tau}$. This was shown for f some uniformly expanding interval maps with a finite number of branches in [FP12] (see also [FFT12, Section 6]) and with a countable number of branches in [AFV14]; here, depending on the precise form of the map, the measure can be absolutely continuous with respect to Lebesgue (acip), or an equilibrium state for some Hölder potential. Inducing methods have been used to extend some of these results to non-uniformly hyperbolic dynamical systems (see [FFT13] which built on [BSTV03]), but the results have not thus far extended to such a complete dichotomy.

We can further enrich our process by considering the point process formed by entries into the regions $\{X > u_n\}$, which in good cases gives rise to a Poisson process. An analogous dichotomy can often be shown there also: in the case of a periodic point ζ, we obtain a compound Poisson process. We leave the details of this construction to later.

In this note, we extend the dichotomy to a simple non-uniformly hyperbolic dynamical system, the *Manneville-Pomeau* (MP) map equipped with an absolutely continuous invariant probability measure. The form for such maps given in [LSV99, BSTV03] is, for $\alpha \in (0,1)$, the corresponding MP map is

$$f = f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{for } x \in [0, 1/2) \\ 2x - 1 & \text{for } x \in [1/2, 1] \end{cases}$$

Members of this family of maps are often referred to as Liverani-Saussol-Vaienti maps since their actual equation was first introduced in [LSV99]. As it can be seen for example in [LSV99, You99, Hu04], these maps have polynomial decay of correlations:

$$\left| \int \varphi \cdot (\psi \circ f^t) d\mu_\alpha - \int \varphi d\mu_\alpha \int \psi d\mu_\alpha \right| \leq C \|\varphi\|_{\mathcal{H}_\beta} \|\psi\|_\infty \frac{1}{t^{\frac{1}{\alpha}-1}}, \tag{2}$$

where $\mathcal{H}_\beta$ denotes the space of Hölder continuous functions φ with exponent β equipped with the norm $\|\varphi\|_{\mathcal{H}_\beta} = \|\varphi\|_\infty + |\varphi|_{\mathcal{H}_\beta}$, where

$$|\varphi|_{\mathcal{H}_\beta} = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\beta}.$$

Let $h_\alpha = \frac{d\mu_\alpha}{dx}$. From [Hu04] we know that $\lim_{x\to 0} \frac{h(x)}{x^{-\alpha}} = C_0 > 0$. Hence, for small $s > 0$ we have that

$$\mu_\alpha([0,s)) \sim_c s^{1-\alpha}, \tag{3}$$

where the notation $\sim_c$ is used in the sense that there is $c > 0$ such that $\lim_{s\to\infty} \frac{\mu_\alpha([0,s))}{s^{1-\alpha}} = c$.

In this case there are canonical induced maps which capture all but a countable number of points in the phase space, so with some extra consideration for those points not captured, we can prove the full dichotomy, where for ζ a periodic point of period p, the extremal index is $\theta = 1 - 1/|Df^p(\zeta)|$.

For the special case in which ζ is the indifferent fixed point, we prove that there exists an EI equal to zero, which corresponds to a degenerate limit law, when the usual normalising sequences are used. Moreover, we show that by changing the definition of $(u_n)_n$ given by (1) in a suitable way, we recover a non-degenerate EVL. This latter result relies on information on the transfer operator in [HSV99] as well as a refinement of the techniques for proving EVLs at periodic points developed in [FFT15].

1.1. **Point process of hitting times.** We will use our observations on our dynamical system to generate point processes. Here we adopt the approach and notation of [Zwe07]. Let $\mathcal{M}_p([0,\infty))$ be the space of counting measures on $([0,\infty), \mathcal{B}_{[0,\infty)})$. We equip this space with the vague topology, i.e., $\nu_n \to \nu$ in $\mathcal{M}_p([0,\infty))$ whenever $\nu_n(\psi) \to \nu(\psi)$ for any continuous function $\psi : [0,\infty) \to \mathbb{R}$ with compact support. A *point process* N on $[0,\infty)$ is a random element of $\mathcal{M}_p([0,\infty))$. We will be interested in point processes $N_n : X \to \mathcal{M}_p([0,\infty))$. If we have a fixed measure μ on X, we say that $(N_n)_n$ *converges in distribution* to N if $\mu \circ N_n^{-1}$ converges weakly to $\mu \circ N^{-1}$. We write $N_n \overset{\mu}{\Longrightarrow} N$.

So given $X_0, X_1, X_2, \ldots$ and some $u \in \mathbb{R}$, we begin the construction of our point process $\mathbb{R} \to \mathcal{M}_p([0,\infty))$ as follows. Given $A \subset \mathbb{R}$ we define

$$\mathscr{N}_u(A) := \sum_{i\in A\cap\mathbb{N}_0} \mathbb{1}_{X_i > u}.$$

So $\mathscr{N}_u[0,n)$ counts the number of exceedances of the parameter u among the first n observations of the process $X_0, X_1, \ldots, X_{n-1}$ or, in other words, the number of entrances in $U(u)$ up to time n.

We next re-scale time using the factor $v := 1/\mathbb{P}(X > u)$ given by Kac's Theorem. However, before we give the definition, we need some formalism. Let $\mathcal{S}$ denote the semi-ring of subsets of $\mathbb{R}_0^+$ whose elements are intervals of the type $[a,b)$, for $a, b \in \mathbb{R}_0^+$. Let $\mathcal{R}$ denote the ring generated by $\mathcal{S}$. Recall that for every $J \in \mathcal{R}$ there are $k \in \mathbb{N}$ and k intervals $I_1, \ldots, I_k \in \mathcal{S}$ such that $J = \cup_{i=1}^k I_j$. In order to fix notation, let $a_j, b_j \in \mathbb{R}_0^+$ be such that $I_j = [a_j, b_j) \in \mathcal{S}$.

For $I=[a,b)\in\mathcal{S}$ and $\alpha\in\mathbb{R}$, we denote $\alpha I:=[\alpha a,\alpha b)$ and $I+\alpha:=[a+\alpha,b+\alpha)$. Similarly, for $J\in\mathcal{R}$ define $\alpha J:=\alpha I_1\cup\cdots\cup\alpha I_k$ and $J+\alpha:=(I_1+\alpha)\cup\cdots\cup(I_k+\alpha)$.

We suppose that we are given $\tau>0$ such that $\mathbb{P}(X>u_n)=\frac{\tau}{n}$. We let $U(u_n)=\{X>u_n\}$ and let v_n be the corresponding scaling factor defined above.

Definition. We define the *rare event point process* (REPP) by counting the number of exceedances (or hits to $U(u_n)$) during the (re-scaled) time period $v_nJ\in\mathcal{R}$, where $J\in\mathcal{R}$. To be more precise, for every $J\in\mathcal{R}$, set

$$N_n(J):=\mathscr{N}_{u_n}(v_nJ)=\sum_{j\in v_nJ\cap\mathbb{N}_0}\mathbb{1}_{X_j>u_n}.$$

1.2. Main results.

Theorem 1. *Given $\zeta\in(0,1]$, consider the REPP N_n defined above. Then either*

(1) *ζ is not periodic and N_n converges in distribution to a Poisson process N with intensity 1.*
(2) *ζ is periodic with period p and N_n converges in distribution to a compound Poisson process N with intensity $\theta=1-|D(f^{-p})(\zeta)|$ and multiplicity distribution function π given by $\pi_\kappa=\theta(1-\theta)^{\kappa-1}$, for every $\kappa\in\mathbb{N}_0$.*

Theorem 2. *For $\zeta=0$, consider the maximum function $M_n=M_n(x)$ defined above.*

(1) *Let $(u_n)_n=(u_n(\tau))_n$ be chosen as in (1), then $\mathbb{P}(M_n\leqslant u_n)\to1$ as $n\to\infty$ for any $\tau>0$.*
(2) *If $\alpha\in(0,\sqrt{5}-2)$, then for each $\tau>0$, there exists a sequence of thresholds $(u_n)_n=(u_n(\tau))_n$ so that $\mathbb{P}(M_n\leqslant u_n)\to e^{-\tau}$ as $n\to\infty$.*

Remark 1. *We note that in the case $\zeta=0$, while it is possible to rescale the thresholds to recover an EVL as in Theorem 2 (2), the corresponding REPP remains degenerate. This result will form part of a forthcoming work* [FFR].

1.3. **Comments on history and strategy.** Before discussing our approach we introduce some notation. For a dynamical system $f:\mathcal{X}\to\mathcal{X}$ and a subset $A\subset\mathcal{X}$, for $x\in\mathcal{X}$ define

$$r_A(x):=\inf\{n\in\mathbb{N}:f^n(x)\in A\},$$

the *first hitting time* to A. Note that there is a connection with the behaviour of the variable $r_{U(u_n)}$ and our REPP since we can break that process down into a sequence of first hits to $U(u_n)$. This gives a connection with our REPP and the asymptotics of $r_{U(u_n)}$, the *Hitting Time Statistics* (HTS). One basic difference is that here we are concerned with all hits to $U(u_n)$, not just the first.

Our main result for the case of $\mathbb{P}$-typical points and for periodic points in $(0,1)$ follows quickly from previous works, including works already mentioned above, and indeed in some of these papers mention MP explicitly. We also remark that some of the earliest works on HTS for dynamical systems considered the case of MP maps with $\alpha\geqslant1$, see for example [CG93, CGS92, CI95], with a focus on the behaviour at 0. In these cases, the sets A_n considered were formed from dynamically defined cylinder sets and the analysis was done at $1/2$, the preimage of 0, so that finite measure sets could be used. In this paper we consider the case $\alpha \in (0,1)$, so f has an acip, and we also consider more general points and sets A_n.

We will first consider all points in $(0,1)$, using inducing methods. This will require us to generalise the already very flexible result of [HWZ14] to point processes. Finally we use the approach which goes back to Leadbetter [Lea74] of proving some short range and long range recurrence conditions to prove that we have a degenerate law at 0 (the extremal index is 0).

2. Induced point processes

Here we aim to generalise [HWZ14] to point processes. In that paper, they use [Zwe07, Corollary 5] as one of their key tools. In our, fairly analogous, setting we use [Zwe07, Corollary 6] instead. Note that previous results here include [BSTV03, Theorem 2.1], where they proved that for balls around typical points, the HTS of first return maps are the same as that for the original map - they also remarked, without details, that this can be extended to successive return times. Also in [FFT13], we extended this idea to periodic points. The strengths of the approach in [HWZ14] to HTS are that it covers *all* points, and that the proof is rather short.

We will give our result comparing the point process of the induced system to that coming from the original system in a general setting and then later apply this to our MP example. In this section, we take a dynamical system $f:\mathcal{X}\to\mathcal{X}$ with an ergodic f-invariant probability measure μ, choose a subset $Y\subset\mathcal{X}$ and consider $F_Y : Y\to Y$ to be the first return map f^{r_Y} to Y (note that F may be undefined at a zero Lebesgue measure set of points which do not return to Y, but most of these points are not important, so we will abuse notation here). Let $\mu_Y(\cdot)=\frac{\mu(\cdot\cap Y)}{\mu(Y)}$ be the conditional measure on Y. By Kac's Theorem μ_Y is F_Y-invariant.

Setting $v_n^Y = 1/\mu_Y(X>u_n)$, for the induced process X_i^Y,

$$N_n^Y(J) := \mathscr{N}_{u_n}^Y(v_n^Y J) = \sum_{j\in v_n^Y J\cap\mathbb{N}_0} \mathbb{1}_{X_j^Y>u_n}.$$

In keeping with [HWZ14], we denote our inducing domain by Y. Denote the speeded up return time r_A by $r_{A,Y}$ and the induced measure on Y by μ_Y.

Theorem 3. *For $\eta>0$, setting $J_\eta := \cup_{s\in J} B_\eta(s)$, we assume that $N(J_\eta)$ is continuous in η, for all small η.*

$$N_n^Y \overset{\mu_Y}{\Longrightarrow} N \text{ as } n\to\infty \text{ implies } N_n \overset{\mu}{\Longrightarrow} N \text{ as } n\to\infty.$$

Proof. By [Zwe07, Corollary 6], for a general sequence of point processes $(N_n)_n$ and an ergodic reference measure m, if $P\ll m$ then $N_n \overset{P}{\Longrightarrow} N$ in $\mathcal{M}_p([0,\infty))$ implies $N_n \overset{Q}{\Longrightarrow} N$ in $\mathcal{M}_p([0,\infty))$ for any $Q\ll m$. So replacing both m and Q with μ and replacing P with μ_Y we see that for our sequence of processes, if $N_n \overset{\mu_Y}{\Longrightarrow} N$ in $\mathcal{M}_p([0,\infty))$, then $N_n \overset{\mu}{\Longrightarrow} N$ in $\mathcal{M}_p([0,\infty))$. Thus it suffices to show that for every $J\in\mathcal{R}$ and all $k\in\mathbb{N}$,

$$\mu_Y(N_n^Y(J)\geqslant k)\stackrel{n\to\infty}{\longrightarrow}\mu_Y(N(J)\geqslant k) \text{ implies } \mu_Y(N_n(J)\geqslant k)\stackrel{n\to\infty}{\longrightarrow}\mu_Y(N(J)\geqslant k).$$

For $\delta>0$, let

$$E_M=E_M^\delta:=\left\{\left(\frac{1-\delta}{\mu(Y)}\right)j\leqslant r_Y^j\leqslant\left(\frac{1+\delta}{\mu(Y)}\right)j \text{ for all } j\geqslant M\right\} \text{ and } F_N:=\{r_{U(u_n),Y}\geqslant N\}.$$

As in [HWZ14], $\mu_Y(F_N^c)\leqslant N\mu_Y(U(u_n))\to 0$ as $n\to\infty$. Also the ergodic theorem says that $\mu_Y((E_M^\delta)^c)\to 0$ as $M\to\infty$.

For $x\in E_M^\delta$,

$$r_{U(u_n),Y}^k(x)\left(\frac{1-\delta}{\mu(Y)}\right)\leqslant r_{U(u_n)}^k(x)=\sum_{j=0}^{r_{U(u_n),Y}^k(x)-1}r_Y\circ F_Y^j(x)=r_Y^{r_{U(u_n),Y}^k(x)}(x)\leqslant r_{U(u_n),Y}^k(x)\left(\frac{1+\delta}{\mu(Y)}\right)$$

We can deduce that for $x\in F_N\cap E_M^\delta$,

$$\mu(Y)r_{U(u_n)}^k(x)\in B_{\delta r_{U(u_n),Y}^k(x)}(r_{U(u_n),Y}^k(x)).$$

So if $r_{U(u_n),Y}^k(x)\in v_n^YJ$ then $\mu(Y)r_{U(u_n)}^k(x)\in v_n^YJ_\delta$ and so $r_{U(u_n)}^k(x)\in v_nJ_\delta$. Therefore,

$$\mu_Y\left(\{N_n(J_\delta)\geqslant k\}\cap(E_N\cap F_M)\right)\geqslant\mu_Y\left(\left\{N_n^Y(J)\geqslant k\right\}\cap(E_N\cap F_M)\right).$$

Setting $\delta':=\frac{\delta}{1+\delta}$, we also obtain that

$$\frac{1}{\mu(Y)}r_{U(u_n),Y}^k(x)\in B_{\delta' r_{U(u_n)}^k(x)}(r_{U(u_n)}^k(x))$$

for $x\in F_N\cap E_M$. Analogously to above, this leads us to

$$\mu_Y\left(\left\{N_n^Y(J_{\delta'})\geqslant k\right\}\cap(E_N\cap F_M)\right)\geqslant\mu_Y\left(\{N_n(J)\geqslant k\}\cap(E_N\cap F_M)\right).$$

So since $\varepsilon,\delta>0$ were arbitrary, we are finished. □

3. Application of inducing to Manneville-Pomeau

In this section we prove our main theorem for all points $\zeta\in(0,1)$.

Let $\mathcal{P}$ be the *renewal partition*, that is the partition defined inductively by $Z\in\mathcal{P}$ if $Z=[1/2,1)$ or $f(Z)\in\mathcal{P}$. Now let $Y\in\mathcal{P}$ and let F_Y be the first return map to Y and μ_Y be the conditional measure on Y. It is well-known that (Y,F_Y,μ_Y) is a Bernoulli map and hence, in particular, a Rychlik system (see [Ryc83] or [AFV14, Section 3.2.1] for the essential information about such systems) and so the REPP is understood as in [FFT13, Corollary 3]. Hence by [AFV14] we have the following theorem.

Theorem 4. *Given $\zeta\in Y$, consider the REPP N_n^Y defined above. Then either*

(1) *ζ is not periodic and N_n^Y converges in distribution to a Poisson process N with intensity 1.*

(2) ζ *is periodic with period* p *and* N_n^Y *converges in distribution to a compound Poisson process* N *with intensity* $\theta = 1 - \left|D(F_Y^{-p})(\zeta)\right|$ *and multiplicity d.f.* π *given by* [1] $\pi_\kappa = \theta(1-\theta)^{\kappa-1}$, *for every* $\kappa \in \mathbb{N}_0$.

For points in $Y \setminus \cup_{n\geqslant 1} f^{-n}(0)$, this theorem is Proposition 3.2 of [AFV14]. For the boundary points $\cup_{n\geqslant 1} f^{-n}(0)$, in the language of [AFV14], any such point is called aperiodic non-simple. Hence by Proposition 3.4(1) of that paper, we have a standard extremal index of 1 at all such points. Varying Y means that we have considered all points in $(0,1)$. So combining Theorems 3 and 4 completes the proof of Theorem 1 for $\zeta \neq 0$.

4. Analysis of the indifferent fixed point

The tangency of the graph of the MP map with the identity map, creates an intensive clustering of exceedances of levels $(u_n)_{n\in\mathbb{N}}$, when they are chosen as in (1), that leads to the existence of an EI equal to 0, which leads to a degenerate limit distribution for M_n. However, if we choose the levels $(u_n)_{n\in\mathbb{N}}$ not in the classical way, but rather a sequence of lower thresholds, so that the exceedances that escape the clustering effect have more weight, then we can recover the existence of a non-degenerate distribution for the maxima.

The proof of an EI equal to 0 for the usual normalising sequences follows easily from the existing connections between Return Times Statistics (RTS), Hitting Times Statistics (HTS) and EVL, which we will recall in the next subsection. The proof of the existence of a non degenerate limit, under a different normalising sequence of thresholds, is more complicated and requires some new results from [FFT15], which we will recall below.

4.1. **The usual normalising sequences case.** For any $\zeta \in [0,1]$, let $B_\varepsilon(\zeta) = (\zeta-\varepsilon, \zeta+\varepsilon)\cap[0,1]$. If there exists a non degenerate d.f. G such that for all $t \geqslant 0$,

$$\lim_{\varepsilon\to 0}\mu_\alpha\left(r_{B_\varepsilon(\zeta)} \leq \frac{t}{\mu_\alpha(B_\varepsilon(\zeta))}\right) = G(t),$$

then we say we have *Hitting Time Statistics* (HTS) G for balls, at ζ. Similarly, we can restrict our observations to $U(u)$: if there exists a non degenerate (d.f.) $\tilde{G}$ such that for all $t \geqslant 0$,

$$\lim_{\varepsilon\to 0}\mu_\alpha\left(r_{B_\varepsilon(\zeta)} \leq \frac{t}{\mu_\alpha(B_\varepsilon(\zeta))}\;\middle|\; B_\varepsilon(\zeta)\right) = \tilde{G}(t),$$

then we say we have *Return Time Statistics* (RTS) $\tilde{G}$ for balls, at ζ.

The normalising term in the definition of HTS/RTS is inspired by Kac's Theorem which states that the expected amount of time you have to wait before you return to $B_\varepsilon(\zeta)$ is exactly $\frac{1}{\mu_\alpha(B_\varepsilon(\zeta))}$.

The existence of exponential HTS ($G(t) = 1-\mathrm{e}^{-t}$) is equivalent to the existence of exponential RTS ($\tilde{G}(t) = 1-\mathrm{e}^{-t}$). In fact, according to the Main Theorem in [HLV05], a system has HTS

[1] We note that there is an error in [FFT13, Theorem 1], propagated throughout the main results there: the κ should be replaced by $\kappa - 1$.

G if and only if it has RTS $\tilde{G}$ and

$$G(t) = \int_0^t (1 - \tilde{G}(s))\, ds. \tag{4}$$

This formula was later generalised to obtain a relation between the distributional properties of Hitting Times and Return Times Points Processes in [HLV07].

Moreover, according to the main theorem in [FFT10] the existence of HTS G for balls at ζ is equivalent to the existence of an EVL H for the process generated dynamically by an observable with a maximum at ζ and, in fact, $G = H$.

Let $U = [0, b)$ and $A = [a, b)$, where a is such that $f(a) = b$, *i.e.,* $b = a + 2^\alpha a^{1+\alpha}$. Using (3) we easily get $\mu_\alpha(U) \sim_c a^{1-\alpha} + (1-\alpha)2^\alpha a + o(a)$ and $\mu_\alpha([0, a)) \sim_c a^{1-\alpha}$.

Next we compute the RTS distribution, which we denote by $\tilde{G}(s)$. For $s \leq 0$, we easily have that $\tilde{G}(s) = 0$, since $r_U \geq 1$, by definition of hitting time. Let $s > 0$ then

$$\begin{aligned}\tilde{G}(s) &= \lim_{b\to 0} \mu_U\left(r_U \leq \frac{s}{\mu(U)}\right) = \lim_{b\to 0} \frac{1}{\mu(U)}\mu\left(\left\{r_U \leq \frac{s}{\mu(U)}\right\} \cap U\right)\\ &\geq \lim_{b\to 0} \frac{\mu(U \setminus A)}{\mu(U)} = \lim_{b\to 0} \frac{\mu([0,a))}{\mu([0,b))} = 1\end{aligned}$$

Using the [HLV05] formula, we get $G(t) = \int_0^t 1 - \tilde{F}(s) ds = 0$, which , by [FFT10], corresponds to an EI equal to 0. Recall that $\bar{H}(\tau) = \mathrm{e}^{-\theta\tau} = 1$, which means that, in this case, $H(\tau) = 0$.

4.2. **Adjusted choice of thresholds.** In order to prove the existence of EVLs in a dynamical systems context, there are a couple of conditions on the dependence structure of the stochastic process that if verified allow us to obtain such distributional limits. These conditions are motivated by the conditions $D(u_n)$ and $D'(u_n)$ of Leadbetter but were adapted to the dynamical setting and further developed both in the absence of clustering, such as in [Col01, FF08b, HNT12], and in the presence of clustering in [FFT12]. Very recently, in [FFT15], the authors provided some more general conditions, called $Д(u_n)$ and $Д'_q(u_n)$, which subsumed the previous ones and allowed them to address both the presence ($q \geq 1$) and the absence ($q = 0$) of clustering. To distinguish these conditions the authors used a Cyrillic D to denote them. We recall these conditions here.

Given a sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers satisfying (1) and $q \in \mathbb{N}_0$, set

$$A_n^{(q)} := \{X_0 > u_n, X_1 \leq u_n, \ldots, X_q \leq u_n\}.$$

For $s, \ell \in \mathbb{N}$ and an event B, let

$$\mathscr{W}_{s,\ell}(B) = \bigcap_{i=s}^{s+\ell-1} f^{-i}(B^c). \tag{5}$$

Condition ($Д_q(u_n)$)**.** *We say that* $Д(u_n)$ *holds for the sequence* $X_0, X_1, \ldots$ *if, for every* $\ell, t, n \in \mathbb{N}$

$$\left|\mathbb{P}\left(A_n^{(q)} \cap \mathscr{W}_{t,\ell}\left(A_n^{(q)}\right)\right) - \mathbb{P}\left(A_n^{(q)}\right)\mathbb{P}\left(\mathscr{W}_{0,\ell}\left(A_n^{(q)}\right)\right)\right| \leq \gamma(q, n, t), \tag{6}$$

where $\gamma(q,n,t)$ is decreasing in t and there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(n)$ and $n\gamma(q,n,t_n) \to 0$ when $n \to \infty$.

For some fixed $q \in \mathbb{N}_0$, consider the sequence $(t_n)_{n\in\mathbb{N}}$ given by condition $Д_q(u_n)$ and let $(k_n)_{n\in\mathbb{N}}$ be another sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{7}$$

Condition ($Д'_q(u_n)$)**.** *We say that $Д'_q(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (7) and such that*

$$\lim_{n\to\infty} n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\left(A_n^{(q)} \cap f^{-j}\left(A_n^{(q)}\right)\right) = 0. \tag{8}$$

We note that, when $q = 0$, condition $Д'_q(u_n)$ corresponds to condition $D'(u_n)$ from [Lea74].

Now let

$$\vartheta = \lim_{n\to\infty} \vartheta_n = \lim_{n\to\infty} \frac{\mathbb{P}(A_n^{(q)})}{\mathbb{P}(U_n)}. \tag{9}$$

From [FFT15, Corollary 2.4], it follows that if the stochastic process $X_0, X_1, \ldots$ satisfies both conditions $Д_q(u_n)$ and $Д'_q(u_n)$ and the limit in (9) exists then

$$\lim_{n\to\infty} \mathbb{P}(M_n \le u_n) = \mathrm{e}^{-\vartheta\tau}.$$

Now, we consider the fixed point $\zeta = 0$. For every $n \in \mathbb{N}$, let b_n be such that $U_n = \{X_0 > u_n\} = [0, b_n)$ and $\mu_\alpha(U_n) \sim \tau/n$. Also set $a_n \in U_n$ so that $f_\alpha(a_n) = b_n$, *i.e.*, $b_n = a_n + 2^\alpha a_n^{1+\alpha}$. Using (3) we easily get

$$\mu_\alpha(U_n) \sim_c a_n^{1-\alpha} + (1-\alpha)2^\alpha a_n + o(a_n) \tag{10}$$

$$\mu_\alpha([0, a_n)) \sim_c a_n^{1-\alpha} \tag{11}$$

$$\mu_\alpha([a_n, b_n)) \sim_c (1-\alpha)2^\alpha a_n + o(a_n) \tag{12}$$

Now, since we are assuming that $\mu_\alpha(U_n) \sim \tau/n$, then $a_n \sim_c 1/n^{1/(1-\alpha)}$. Observe that $\mu_\alpha(U_n \cap f_\alpha^{-1}(U_n)) = \mu_\alpha([0, a_n)) \sim_c a_n^{1-\alpha} \sim_c 1/n$. Hence, if we consider $q = 0$, the periodicity of ζ implies that $Д'_q(u_n)$ does not hold since

$$n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\left(U_n \cap f^{-j}\left(U_n\right)\right) \ge n\mu_\alpha(U_n \cap f_\alpha^{-1}(U_n)) > 0,$$

for all $n \in \mathbb{N}$. Hence, here, given that ζ is a periodic point of period 1 the natural candidate for q is $q = 1$. From here on we always assume that $q = 1$.

In this case, $A_n^{(q)} = [a_n, b_n) =: Q_n$. However, if we plug (12) and (10) into (9), we obtain that $\vartheta = 0$, which means that the natural candidate for a limit distribution for $\mu_\alpha(M_n \le u_n)$ is degenerate.

The problem is that the indifferent fixed point creates too much dependence. In [FFT12], under a condition called *SP*, we have seen that when ζ is periodic, the probability of having no entrances in U_n, among the first n observations, is asymptotically equal to the probability of having no entrances in Q_n, among the first n observations, *i.e.*,

$$\lim_{n\to\infty}\mathbb{P}(M_n\le u_n)=\lim_{n\to\infty}\mathbb{P}(\mathscr{W}_{0,n}(U_n))=\lim_{n\to\infty}\mathbb{P}(\mathscr{W}_{0,n}(Q_n)).$$

In [FFT15], it was shown that it is possible to replace U_n by Q_n even without the *SP* condition (see [FFT15, Proposition 2.7]). Making use of this upgraded result, we can now change the normalising sequence of levels $(u_n)_{n\in\mathbb{N}}$ so that we can still obtain a non-degenerate limit for $\mathbb{P}(M_n\le u_n)$. To understand the need to change the normalising sequence in order to obtain a non-degenerate limit, recall that condition (1) guaranteed that M_n was normalised by a sequence of levels that kept the average of exceedances among the first n observations at an (almost) constant value $\tau>0$. When $\vartheta>0$, condition (1) also guarantees that the average number of entrances in Q_n among the first n observations is kept at an (almost) constant value $\theta\tau>0$. Here, since $\vartheta=0$, we need to change u_n so that the average number of entrances in Q_n is controlled, *i.e.*,

$$\lim_{n\to\infty}n\mathbb{P}(A_n^{(q)})=\tau>0. \tag{13}$$

From equations (2.15) and (2.16) from [FFT15] one gets:

$$\left|\mathbb{P}(\mathscr{W}_{0,n}(A_n^{(q)}))-\left(1-\left\lfloor\frac{n}{k_n}\right\rfloor\mathbb{P}(A_n^{(q)})\right)^{k_n}\right|\le 2k_nt_n\mathbb{P}(U_n)+2n\sum_{j=1}^{\lfloor n/k_n\rfloor-1}\mathbb{P}\left(A_n^{(q)}\cap f^{-j}A_n^{(q)}\right)+\gamma(q,n,t_n) \tag{14}$$

Note that since by (13) we have $\lim_{n\to\infty}\left(1-\left\lfloor\frac{n}{k_n}\right\rfloor\mathbb{P}(A_n^{(q)})\right)^{k_n}=\mathrm{e}^{-\tau}$, then if both conditions $Д_q(u_n)$ and $Д'_q(u_n)$ hold, then all the terms on the left of (14) vanish, as $n\to\infty$, and consequently:

$$\lim_{n\to\infty}\mathbb{P}(M_n\le u_n)=\lim_{n\to\infty}\mathbb{P}(\mathscr{W}_{0,n}(A_n^{(q)}))=\mathrm{e}^{-\tau}. \tag{15}$$

Hence, in order to show that we can still obtain a non-degenerate limiting law for the distribution of M_n when $\zeta=0$, we start by taking a sequence $(u_n)_{n\in\mathbb{N}}$ so that (13) holds. Note that this implies that by (12) and (10) we have that $a_n\sim_c 1/n$ and $\mu_\alpha(U_n)\sim_c 1/n^{1-\alpha}$. In particular, this means that $\lim_{n\to\infty}n\mu_\alpha(U_n)=\infty$, which contrasts with the usual case where condition (1) holds.

To prove the existence of the limit in (15) we need to verify conditions $Д_q(u_n)$ and $Д'_q(u_n)$, where $q=1$. We start by the latter, which is more complicated.

4.2.1. *Proof of* $Д'_q(u_n)$. We will next focus on the proof of $Д'_q(u_n)$ in the case of part (2) of Theorem 2. That is, $(a_n)_n$ will be chosen so that $a_n\sim_c 1/n$, as described above. Later we will note that we can change $(a_n)_n$ to recover a degenerate law as in part (1) of that theorem.

We have to estimate the quantity

$$\Delta'_n:=n\sum_{j=1}^{[n/k_n]}\mu_\alpha(Q_n\cap f^{-j}Q_n)$$

where $Q_n = [a_n, b_n)$, for $a_n \sim_c \frac{1}{n}$ and $b_n = f(a_n)$. We follow the proof of [HSV99, Lemma 3.5]. By denoting by P the transfer operator and by $\tau_n \in \mathbb{N}$ the *first return time* of the set Q_n into itself, we have:

$$\Delta'_n \leqslant n\ [n/k_n]\mu_a(Q_n) \sup_{j=\tau_n,\dots,[n/k_n]} \sup_{Q_n} \frac{P^j(\mathbf{1}_{Q_n} h)}{h}$$

where h is the density of μ_α. In order to compute $P^{\tau_n}(\mathbf{1}_{Q_n} h)$ we need to know how many branches of f^{τ_n} have their domain intersecting Q_n. If ξ_0 is the original partition into the sets $[0, 1/2), [1/2, 1]$, we denote with ξ_k the join $\xi_k := \xi_0 \vee f^{-1}\xi_0, \dots, \vee f^{k-1}\xi_0$.

We begin to observe that Q_n contains at most one boundary point of the partition $\xi_{\tau_n - 1}$, otherwise one point of Q_n should be sent into the same set, being f^{τ_n-1} onto on each domain of injectivity. Then when we move to ξ_{τ_n}, the interval Q_n will be crossed by at most 4 cylinders of monotonicity of the partition ξ_{τ_n}. By denoting them with $C_{\tau_n,1}, \dots C_{\tau_n,4}$, we have

$$P^{\tau_n}(\mathbf{1}_{Q_n} h) = \sum_{i=1,4} \frac{h \circ f_i^{-\tau_n} \mathbf{1}_{f_i^{\tau_n} Q_n}}{Df^{\tau_n} \circ f_i^{-\tau_n}}$$

where $f_i^{\tau_n}$ denotes the branch of f^{τ_n} restricted to $C_{\tau_n,i}$. Notice that the density is computed in Q_n whose left boundary point is $1/n$, so h is bounded from above by a constant times n^α. We have now to estimate the derivative $Df^{\tau_n} \circ f_i^{-\tau_n}$ on the sets $Q_n \cap C_{\tau_n,i}$. Let us define r_m as the m-left preimage of 1, $r_m := f_1^{-m}(1)$ and define $m(n)$ as $r_{m(n)} \leqslant a_n \leqslant r_{m(n)-1}$. Then the interval $[a_n, b_n)$ will intersect the two cylinders $(r_{m(n)}, r_{m(n)-1})$ and $(r_{m(n)-1}, r_{m(n)-2})$ and the first return of Q_n will be larger than the first returns of those two cylinders; on the other hand the first return of $(r_{m(n)}, r_{m(n)-1})$ is $m(n)$. The derivative Df^{τ_n} will be computed at a point ι_n which will be in one of those two cylinders; suppose without any restriction that $\iota_n \in (r_{m(n)}, r_{m(n)-1})$. Since we need to bound from below the derivatives, we begin to replace $Df^{\tau_n}(\iota_n)$ with $Df^{m(n)}(\iota_n)$; then we observe that the map $f^{m(n)} : [r_{m(n)}, r_{m(n)-1}] \to [0, 1]$ is onto and we use the distortion bound given, for instance, in [LSY, Lemma 5] which states that there exists a constant C such that for any $m \geqslant 1$ and any $x, y \in [r_m, r_{m-1}]$ we have $\left|\frac{Df^m(x)}{Df^m(y)}\right| \leqslant C$. We finally note that $m(n) \sim_c n^\alpha$. This implies immediately that

$$\frac{1}{Df^{m(n)}(\iota_n)} \leqslant C|r_{m(n)-1} - r_{m(n)}| \sim_c C \frac{1}{m(n)^{\frac{1}{\alpha}+1}} \sim_c C \frac{1}{n^{1+\alpha}}.$$

Consequently (C will continue to denote a constant which could vary from one bound to another)

$$P^{\tau_n}(\mathbf{1}_{Q_n} h) \sim_c \frac{1}{n}$$

We now continue as in [HSV] by getting for the other powers of the transfer operator:

$$\frac{P^j(\mathbf{1}_{Q_n} h)}{h} \leqslant \frac{P^{j-\tau_n}\mathbf{1}}{h} \sup P^{\tau_n}(\mathbf{1}_{Q_n} h) \leqslant \frac{P^{j-\tau_n}\frac{h}{\inf\ h}}{h} \sup P^{\tau_n}(\mathbf{1}_{Q_n} h) \leqslant \frac{C}{\inf\ h} \frac{1}{n}$$

and finally

$$\Delta'_n \leqslant n\ [n/k_n]\mu_a(Q_n) \frac{C}{\inf\ h} \frac{1}{n}$$

We now know that $\mu_a(Q_n) \sim_c \frac{1}{n}$; hence

$$\Delta'_n \leqslant Cn\ [n/k_n]\mu_a(Q_n)^2 \frac{1}{\mu_a(Q_n)}\frac{1}{n} \sim_c [n^2\ \mu_a(Q_n)^2]\ \frac{1}{k_n}.$$

So letting $n \to \infty$, we see that $Д'_q(u_n)$ holds.

4.2.2. *Proof of $Д_q(u_n)$.* This follows since, as in (2), we have decay of correlations of Hölder functions against bounded measurable functions and condition $Д_q(u_n)$ was designed to follow from sufficiently fast decay of correlations, as shown in [Fre13, Proposition 5.2]. In order to compute the required rate of decay of correlations, which will impose a restriction on the domain of the parameter α, we recall here the above-mentioned result so that we can follow the computations closely.

Proposition 1 ([Fre13, Proposition 5.2]). *Assume that $\mathcal{X}$ is a compact subset of $\mathbb{R}^d$ and $f : \mathcal{X} \to \mathcal{X}$ is a system with an acip $\mathbb{P}$, such that $\frac{d\mathbb{P}}{Leb} \in L^{1+\epsilon}$. Assume, moreover, that the system has decay of correlations for all $\varphi \in \mathcal{H}_\beta$ against any $\psi \in L^\infty$ so that there exists some $C > 0$ independent of φ, ψ and t, and a rate function $\varrho : \mathbb{N} \to \mathbb{R}$ such that*

$$\left|\int \varphi \cdot (\psi \circ f^t) d\mathbb{P} - \int \varphi d\mathbb{P} \int \psi d\mathbb{P}\right| \leq C\|\varphi\|_{\mathcal{H}_\beta}\|\psi\|_\infty \varrho(t), \tag{16}$$

and $n^{1+\beta(1+\max\{0,(\epsilon+1)/\epsilon-d\}+\delta)}\varrho(t_n) \to 0$, as $n \to \infty$ for some $\delta > 0$ and $t_n = o(n)$. Then condition $Д_q(u_n)$ holds.

Remark 2. *We note that during the proof, in order to obtain the condition on the rate of decay of correlations, it is assumed that $\mathbb{P}(A_n^{(q)}) \sim_c 1/n$.*

Observe that since we are working in dimension 1, which means $d = 1$, then $\max\{0, (\epsilon + 1)/\epsilon - d\} = 1/\epsilon$. Also, from [Hu04], we may assume that the decay of correlations is written for Lipschitz functions, which allows us to take $\beta = 1$. Hence, for condition $Д_q(u_n)$ hold, we need that the rate of decay of correlations ϱ is sufficiently fast so that there exists some $\delta > 0$ such that

$$\lim_{n\to\infty} n^{2+1/\epsilon+\delta}\varrho(t_n) = 0, \tag{17}$$

where $t_n = o(n)$. From (3), in order that the density $h_\alpha \in L^{1+\epsilon}$, we need that $\epsilon < 1/\alpha - 1$. Since by (2), we have that $\varrho(t) = t^{-(1/\alpha-1)}$, then by (17) it is obvious that we must have $\alpha < 1/2$, which implies that $1/\epsilon < \alpha + 2\alpha^2$. Taking $t_n = n^{1-\alpha}$, we obtain:

$$n^{2+1/\epsilon+\delta}\left(n^{1-\alpha}\right)^{-(1/\alpha-1)} = n^{2+1/\epsilon+\delta}n^{-1/\alpha+2-\alpha} < n^{4+2\alpha^2+\delta-1/\alpha}.$$

Hence, if $\alpha < \sqrt{5} - 2$ we can always find $\delta > 0$ so that (17) holds and consequently condition $Д_q(u_n)$ is verified.

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