GENERIC SINGULARITIES OF THE OPTIMAL AVERAGED PROFIT AMONG STATIONARY STRATEGIES

HELENA MENA-MATOS* AND CÉLIA MOREIRA**

Abstract. We consider the problem of maximizing the (time) averaged profit of a smooth profit density on a smooth compact one dimensional manifold along a trajectory provided by a stationary strategy of a polydynamical system. When the problem depends on a k-dimensional parameter, that optimal averaged profit as a function of the parameter can present singularities (non smoothness points). We present the generic classification of these singularities for $k \leq 3$.

1. Introduction

We consider a *polydynamical system* on a smooth compact one dimensional manifold M (phase space) given by a finite set of smooth vectorfields on M (also called admissible velocities of the system):

$$V(x) = \{v_1(x), ..., v_n(x)\}, \quad x \in M, \ n \geq 2 \tag{1}$$

An *admissible motion* of (1) is an absolutely continuous map $x : t \mapsto x(t)$ from a time interval to the system phase space M for which the velocity of motion $\dot{x}(t)$ (at each moment of differentiability of the map) belongs to $V(x)$.

Remark 1. Because the phase space is compact, any admissible motion of (1) is defined for all $t \in \mathbb{R}$.

Suppose that additionally there is a smooth profit density f on M, then an important control problem is stated as follows:

To maximize the averaged profit on the infinite time horizon

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x(t))dt$$

over all the admissible motions of (1).

Such maximum is called *optimal averaged profit* and a strategy providing it is called *optimal*.

Remark 2. If the last limit does not exist one must take its upper limit.

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In this work we look to this control problem through singularity theory. When the problem depends on parameters, that is when both the polydynamical system and the profit density depend additionally on parameters, then the optimal strategy can vary with the parameters and the optimal averaged profit, as a function of the parameters, can have singularities (points where it is not smooth). For example, this profit can be discontinuous, even when the families of control systems and densities are smooth [1]. We are so led to the problem of classifying such singularities.

This approach was firstly considered in [1] and more recently in [4] and [5], for the time averaged optimization on the circle (phase space = S^1). Those works focuses on two kinds of admissible motions that are crucial for determining the optimal averaged profit on the infinite horizon of a controlled dynamical system, namely

- *a level cycle*: motion using the maximum and minimum velocities when the profit density is less or greater, respectively, than a certain constant, or
- *a stationary strategy*: motion corresponding to an equilibrium point of the controlled dynamical system.

It was proved in [5] that a strategy providing the maximal averaged profit always can be found inside these two kinds of motions. But note that for this statement to be true it is essential the larger concept of equilibrium point of a controlled system considered there, namely, such a point is a point where the convex hull of the admissible velocities of the system contains the zero velocity. Such a point is stationary in the sense that for a control system with a one dimensional phase space there exists an admissible motion circulating close to that point, and converging to it as time goes to infinity. It is clear that the averaged profit on the infinite horizon provided by such motion equals the profit density value at this point, that is, the profit value gained through the permanent staying at the point.

So the classification of the singularities of the optimal averaged profit can be reduced to three cases, namely the singularities for stationary strategies, for level cycles and for transitions between stationary strategies and level cycles.

The generic classification for the one dimensional parameter case is already complete ([1], [4], [5]). For the control problem stated before (when the control system is a polydynamical system) the classification of all generic singularities corresponding to stationary strategies can be found in [11] in the case of one dimensional parameter. The case of $k \leq 3$ dimensional parameter is treated in this paper.

We will use the same definition of *equilibrium point* of a control system as the one given in [5], namely such a point is a point where the convex hull of the admissible velocities contains the zero velocity. A stationary strategy is a choice of an admissible motion converging to an equilibrium point and the *stationary domain* is the union of all such points.

Remark 3. This definition of equilibrium point regards only the polydynamical system. When in the forthcoming text we speak about equilibrium point of a vectorfield or of a parametrized family of vectorfields we mean a point where the vectorfield or the parametrized family of vectorfields vanishes.

It is easy to see that a stationary strategy provides an averaged profit on the infinite horizon that equals the value of the profit density at an equilibrium point (the point to which converges the chosen admissible motion). Moreover, for every point of the stationary domain, it is possible to define a stationary strategy for which the averaged profit on the infinite horizon equals the value of the profit density at the considered point [9].

So for the previous stated problem depending on a k-dimensional parameter p, the optimal averaged profit for stationary strategies is defined as

$$A_s(p) = \max_{x \in S(p)} f(x,p), \tag{2}$$

where $S(p)$ is the set of all phase points x such that (x,p) belongs to the stationary domain $S = \{(x,p) : 0 \in \mathrm{co}V(x,p)\}$. It is defined for all parameter values p such that $S(p)$ is not empty.

So the classification of generic singularities of the maximum averaged profit among stationary strategies can be done in two steps. At first we classify all generic singularities of the stationary domain and then those of the solution of problem (2). It is clear that on the first step we should work with a generic family of polydynamical systems and on the second one we can treat a generic family of profit densities when a generic family of polydynamical systems is fixed.

On the space of our objects (families of vectorfields, families of polydynamical systems, etc.) we introduce the fine smooth Whitney topology. A property is *generic* (or *holds generically*) if it holds for any object belonging to some open everywhere dense subset.

2. Singularities of the Stationary Domain

A family of polydynamical systems on a 1-dimensional compact manifold M is given by a collection of a finite number of smooth families of vectorfields v_i on M parametrized by $p \in P$, where P is a k-dimensional smooth manifold:

$$V(x,p) = \{v_1(x,p), ..., v_n(x,p)\},\ n \geq 2$$

The respective stationary domain $S = \{(x,p) \in M \times P : 0 \in \mathrm{co}V(x,p)\}$ is a closed subset of $M \times P$.

To simplify language we will call the admissible families of vectorfields v_i just *admissible velocities.*

The product space of the phase space M by the parameter space is naturally fibred over the parameter, that is, with fibres $\mathcal{F}_p = M \times \{p\}$, for every parameter value p. Two objects of the same nature defined on a fibred space are $\mathcal{F}-$*equivalent* if one of them can be carried out to the other by a fibered diffeomorphism, i. e., by a diffeomorphism that sends fibres to fibres.

It is clear that the stationary domain around an interior point is locally $\mathcal{F}-$*equivalent* to $\mathbb{R} \times \mathbb{R}^k$. It is also easy to see that at a boundary point of it at least one of the admissible velocities vanishes. So to classify the stationary domain around its boundary points we just have to look to the equilibria of the admissible velocities.

We will call an equilibrium point of a family of vectorfields v on M, an *equilibrium point type* A_i ($i \in \mathbb{N}_0$), if at that point, the germ of the set of equilibria of v is $\mathcal{F}-$*equivalent* to the germ at the origin of

$$x^{i+1} + p_1 x^{i-1} + \cdots + p_i = 0.$$

Generically, every equilibrium point of a k-parameter family of vectorfields on a 1-dimensional manifold is an equilibrium point type A_l with $0 \leq l \leq k$. In fact in a fixed coordinate system we can consider a vectorfield as a function. $\mathcal{F}$-equivalence acts differently on the field and on the respective function but preserves their zero levels. But in a generic case the germ of a k-parametric family of smooth functions on the line at any point of its zero level is $\mathcal{F}$-equivalent to the germ at the origin of $x^{l+1} + p_1 x^{l-1} + \cdots + p_l$, $0 \leq l \leq k$ [2].

Let now Z be the union of the equilibria of all the admissible velocities of the family of polydynamical systems. A point of this set is called a *point type* A_{I_j} with $I_j = (i_1, \cdots, i_j)$, (all $j, i_1, \cdots, i_j$ are nonnegative integers and $0 \leq i_1 \leq \cdots \leq i_j$), if it is an equilibrium point of exactly j admissible velocities w_1,..,w_j which is of type A_{i_1} for w_1,..., A_{i_j} for w_j. Denote $|I_j| = j - 1 + i_1 + \cdots + i_j$.

The proofs of the following results can be found in [10] and are based on Thom Transversality Theorem and on Mather Division Theorem [7].

Theorem 1. *Let Q be a point of the set Z of a k-parameter family of polydynamical systems on a 1-dimensional manifold. Generically*

1. *Q is of one of the types A_{I_j} with $|I_j| \leq k$.*
2. *The germ of the set Z at a point of type A_{I_j} is $\mathcal{F}$-equivalent to the germ at the origin of the set*

$$\left(x^{i_1+1} + \sum_{l=1}^{i_1} p_l x^{i_1-l}\right) \prod_{l=2}^{j} \left(x^{i_l+1} + \sum_{m=|I_l|-i_l}^{|I_l|} p_m x^{|I_l|-m}\right) = 0$$

where x and p_1, $p_2, \ldots$ are local coordinates along the phase space and the parameter space, respectively.

Theorem 2. *The germ of the stationary domain of a generic k-parameter family of polydynamical systems on a 1-dimensional manifold, at any point is, up to $\mathcal{F}$-equivalence, the germ at the origin of one of the sets from the second column of:*

- *Table 1, if* $k = 1$,
- *Tables 1 and 2, if* $k = 2$,
- *Tables 1, 2 and 3, if* $k = 3$.

Moreover, the germs of the stationary domains of a generic family and of any other sufficiently close to it can be reduced one to another by $\mathcal{F}$*-equivalence close to the identity.*

TABLE 1.

N.	Singularities	Type	n	m
0	$\mathbb{R}^{k+1}$	Interior point	≥ 2	0
1	$x \leq 0$	A_0	≥ 2	1
$2_{\pm}$	$\pm(x^2 + p_1) \leq 0$	A_1	≥ 2	2
$3_{\pm}$	$\pm x(x + p_1) \leq 0$	$A_{0,0}$	2	
$4_{\pm}$	$x \leq 0 \vee \pm(x + p_1) \leq 0$		≥ 3	

TABLE 2.

N.	Singularities	Type	n	m
5	$x^3 + p_1x + p_2 \leq 0$	A_2	≥ 2	3
6	$x(x^2 + p_1x + p_2) \leq 0$	$A_{0,1}$	2	
$7_{\pm}$	$x \leq 0 \vee \pm(x^2 + p_1x + p_2) \leq 0$		≥ 3	
8_*	$x(x + p_1) \leq 0 \vee x(x + p_2) \leq 0$	$A_{0,0,0}$	3	
$8_{\pm}$	$\pm x(x + p_1) \leq 0 \vee x(x + p_2) \geq 0$			
$9_{\pm}$	$x \leq 0 \vee x + p_1 \leq 0 \vee \pm(x + p_2) \leq 0$		≥ 4	

TABLE 3.

N.	Singularities	Type	n	m
$10_{\pm}$	$\pm(x^4 + p_1x^2 + p_2x + p_3) \leq 0$	A_3	≥ 2	4
$11_{\pm}$	$\pm x(x^3 + p_1x^2 + p_2x + p_3) \leq 0$	$A_{0,2}$	2	
$12_{\pm}$	$x \leq 0 \vee \pm(x^3 + p_1x^2 + p_2x + p_3) \leq 0$		≥ 3	
$13_{\pm}$	$\pm(x^2 + p_1)(x^2 + p_2x + p_3) \leq 0$	$A_{1,1}$	2	
14_*	$x^2 + p_1 \leq 0 \vee x^2 + p_2x + p_3 \leq 0$		≥ 3	
$14_{\pm}$	$\pm(x^2 + p_1) \leq 0 \vee x^2 + p_2x + p_3 \geq 0$			
$15_{\pm}$	$\pm x(x + p_1) \leq 0 \vee x(x^2 + p_2x + p_3) \leq 0$	$A_{0,0,1}$	3	
$16_{\pm\pm}$	$x \leq 0 \vee \pm(x + p_1) \leq 0 \vee \pm(x^2 + p_2x + p_3) \leq 0$		≥ 4	
$17_{\pm\pm}$	$x(x + p_1) \leq 0 \vee \pm x(x + p_2) \leq 0 \vee \pm x(x + p_3) \leq 0$	$A_{0,0,0,0}$	4	
18_*	$x \leq 0 \vee x + p_1 \leq 0 \vee x + p_2 \leq 0 \vee x + p_3 \leq 0$		≥ 5	
$18_{\pm}$	$x \leq 0 \vee x + p_1 \leq 0 \vee \pm(x + p_2) \leq 0 \vee x + p_3 \geq 0$			

In these tables, the third and the fourth columns show the type of the point and the restriction on the number of admissible velocities, respectively. The last column (m) denotes the codimension in $M \times P$ of the stratum of the respective singularity.

Observe that Tables 2 and 3 correspond to the singularities of the stationary domain at points type A_{I_j} with $|I_j| = 2$ and 3, respectively.

3. Singularities of the Optimal Averaged Profit

We will consider now the optimal averaged profit for stationary strategies, that is given by

$$A_s(p) = \max_{x \in S(p)} f(x,p).$$

To simplify language we will from now on call $A_s(p)$ just optimal profit.

Denote by S^* the subset of the stationary domain whose points provide the optimal profit A_s.

Let p be a parameter value such that $S \cap \mathcal{F}_p \neq \emptyset$. We call p a *value without competition* if the set $\overline{S^*} \cap \mathcal{F}_p$ has a unique element; otherwise, p is called a *value with competition* and the points of such set are said to be *competing* for the profit A_s.

It is clear that the behavior of the function A_s at values without competition requires looking to the family of densities in a neighborhood of the unique point providing the optimal profit; at values with competition it requires looking to the same function in a neighborhood of several points. For this reason, singularities of the optimal averaged profit for stationary strategies at values without competition are called *point singularities* and the other ones, at competition values, are called *competition singularities.* The classification of these singularities will be done separetely.

3.1. **Point Singularities.** Two germs of functions are Γ-*equivalent* if their graphs are $\mathcal{F}$-equivalent, considering the product space of the functions domain by the real axis as a fibred space over the domain. The diffeomorphism carrying one graph into the other can be written in the form $(p,a) \mapsto (\varphi(p), h(p,a))$ where p belongs to the function's domain and $a \in \mathbb{R}$.

R^+-*equivalence* is the particular case of Γ-equivalence when the second component h of the diffeomorphism is of the form $a + c(p)$, where c is a smooth function. It is clear that the germ of a smooth function at a point is R^+-equivalent to the germ of the zero function at the origin.

Theorem 3. *For a generic k-parameter family of pairs of polydynamical systems and profit densities on a one dimensional compact manifold, the germ of the optimal profit at a parameter value p without competition is, up to R^+-equivalence, the germ at the origin of one of the functions from the second column of*

- *Table 4, if $k = 1$;*
- *Tables 4 and 5, if $k = 2$;*
- *Tables 4, 5 and 6, if $k = 3$.*

Table 4.

N	*Singularities*	*Type*	*Conditions*	*c*
1	*0*	*Interior*	$f_x = 0 \neq f_{xx}$	*0*
		A_0	$f_x \neq 0$	
2	$p_1\|p_1\|$	A_0	$f_x = 0 \neq f_{xx}$	*1*
3	$\sqrt{p_1}$	A_1	$f_x \neq 0$	
4	$\|p_1\|$	$A_{0,0}$		

Table 5.

N.	Singularities	Type	Conditions	c
5	$\max\{-x^4+p_1x^2+p_2x : x \in \mathbb{R}\}$	Interior	$f_x = f_{xx} = f_{xxx} = 0 \neq f_{xxxx}$	2
6	$\max\{x^3+p_1x^2+p_2x : x \leq 0\}$	A_0	$f_x = f_{xx} = 0 \neq f_{xxx}$	
7	$\max\{-x^2 : x^2+p_1x+p_2 \leq 0\}$	A_1	$f_x = 0 \neq f_{xx}$	
8	$\sqrt{p_1}\|p_2\|$			
9	$\max\{-x^2 : x^2+p_1x+p_2 \geq 0\}$			
10	$\|p_1p_2\|$	$A_{0,0}$		
11	$\max\{-x^2 : (x+p_1)(x+p_2) \leq 0\}$			
12	$\max\{-x^2 : x \leq \max\{p_1,p_2\}\}$			
13	$\max\{x : x^3+p_1x+p_2 = 0\}$	A_2	$f_x \neq 0$	
14	$\max\{\sqrt{p_1}, p_2\}$	$A_{0,1}$		
15	$\max\{0,p_1,p_2\}$	$A_{0,0,0}$		

On tables 4, 5 and 6, columns 3, 4 and 5 describe the type of singularity of the stationary domain at the point providing the optimal profit, the conditions concerning the family of profit densities at that point and the codimension of the singularities, respectively. The proof of Theorem 3 is done in section 4.

3.2. **Competition Singularities.** Let p be a value with competition, that is, the set $\overline{S^*} \cap \mathcal{F}_p$ has at least two points.

We say that two points of $\overline{S^*} \cap \mathcal{F}_p$ have the *same level* if the family of profit densities has the same value at these points.

For discribing the various situations of competition we will use the following notation: $C(i_1, \cdots, i_N)$ denotes the competition of N points with point singularities $i_1, \cdots, i_N$, respectively, from Tables 4, 5 and 6 ($1 \leq i_j \leq 31$) such that for all $j < N$, the level of the point providing singularity i_j is not higher than the one providing singularity i_{j+1}. Different levels will be marked replacing commas by semicolons. For example, $C(1, 2; 3)$ denotes

TABLE 6.

N	Singularities	Type	Conditions	c
16	$\max\{-x^4+p_1x^3+p_2x^2+p_3x : x \le 0\}$	A_0	$f_x = f_{xx} = f_{xxx} = 0 \neq f_{xxxx}$	3
17	$\max\{x^3+p_1x^2+p_2x : \lvert x\rvert \le \sqrt{p_3}\}$	A_1	$f_x = f_{xx} = 0 \neq f_{xxx}$	
18	$\max\{x^3+p_1x^2+p_2x : \lvert x\rvert \le \lvert p_3\rvert\}$	$A_{0,0}$		
19	$\max\{x^3+p_1x^2+p_2x : x \le \lvert p_3\rvert\}$			
20	$\max\{-x^2 : x^3+p_1x^2+p_2x+p_3 \le 0\}$	A_2	$f_x = 0 \neq f_{xx}$	
21	$\max\{-x^2 : (x+p_1)(x^2+p_2x+p_3) \le 0\}$	$A_{0,1}$		
22	$\max\{-x^2 : x+p_1 \le 0 \vee x^2+p_2x+p_3 \le 0\}$			
23	$\max\{-x^2 : x+p_1 \le 0 \vee x^2+p_2x+p_3 \ge 0\}$			
24	$\max\{p_1^2, p_2^2, p_3^2\}$	$A_{0,0,0}$		
25	$\max\{-x^2 : (x+p_1)(x+p_2) \le 0 \vee (x+p_1)(x+p_3) \le 0\}$			
26	$\max\{-x^2 : x \le \max\{p_1,p_2,p_3\}\}$			
27	$\max\{x : x^4+p_1x^2+p_2x+p_3 = 0\}$	A_3	$f_x \neq 0$	
28	$\max\{x : x(x^3+p_1x^2+p_2x+p_3) = 0\}$	$A_{0,2}$		
29	$\max\{\sqrt{p_1}, \sqrt{p_2}+p_3\}$	$A_{1,1}$		
30	$\max\{0, p_1, \sqrt{p_2}+p_3\}$	$A_{0,0,1}$		
31	$\max\{0, p_1, p_2, p_3\}$	$A_{0,0,0,0}$		

the competition of three points having two distinct levels and providing singularities 1, 2 and 3 of Table 4.

Theorem 4. *For a generic k-parameter family of pairs of polydynamical systems and profit densities on a one dimensional compact manifold, the germ of the optimal profit at a parameter value p with competition is, up to the equivalence pointed out in the third column, the germ at the origin of one of the functions from second column of*

- *Table 7, if $k = 1$;*
- *Tables 7 and 8, if $k = 2$;*
- *Tables 7, 8 and 9, if $k = 3$.*

TABLE 7.

N.	Singularities	Eq.	Situation
1	$\lvert p_1\rvert$	R^+	$C(1,1)$
2	$\max\{0, \sqrt{p_1}+1\}$	Γ	$C(1;3)$

The proof of Theorem 4 is done in the following section.

Table 8.

N.	Singularities	Eq.	Situation
3	$\max\{0, p_1\vert p_1\vert + p_2\}$	R^+	$C(1,2)$
4	$\max\{0, \sqrt{p_1} + p_2\}$	R^+	$C(1,3)$
5	$\max\{0, p_1, p_2\}$	R^+	$C(1,4)$, $C(1,1,1)$
6	$\max\{p_1\vert p_1\vert,\ \sqrt{p_2} + 1\}$	Γ	$C(2;3)$
7	$\max\{\sqrt{p_1},\ \sqrt{p_2} + 1\}$	Γ	$C(3;3)$
8	$\max\{\vert p_1\vert,\ \sqrt{p_2} + 1\}$	Γ	$C(4;3)$, $C(1,1;3)$
9	$\max\{0, -x^2 + 1 :\ x^2 + p_1 x + p_2 \leq 0\}$	Γ	$C(1;7)$
10	$\max\{0,\ \sqrt{p_1}\vert p_2\vert + 1\}$	Γ	$C(1;8)$
11	$\max\{0, \sqrt{p_1} + 1, \sqrt{p_2} + 2\}$	Γ	$C(1;3;3)$

Table 9.

N.	Singularities	Eq.	Situation
12	$\max\{0,\ -x^4 + p_1 x^2 + p_2 x + p_3 : x \in \mathbb{R}\}$	R^+	$C(1,5)$
13	$\max\{0,\ x^3 + p_1 x^2 + p_2 x + p_3 : x \leq 0\}$	R^+	$C(1,6)$
14	$\max\{0, -x^2 + p_3 :\ x^2 + p_1 x + p_2 \leq 0\}$	R^+	$C(1,7)$
15	$\max\{0,\ \sqrt{p_1}\vert p_2\vert + p_3\}$	R^+	$C(1,8)$
16	$\max\{0,\ -x^2 + p_3 : x^2 + p_1 x + p_2 \geq 0\}$	R^+	$C(1,9)$
17	$\max\{0,\ \vert p_1 p_2\vert + p_3\}$	R^+	$C(1,10)$
18	$\max\{0,\ -x^2 + p_3 :\ (x + p_1)(x + p_2) \leq 0\}$	R^+	$C(1,11)$
19	$\max\{0,\ -x^2 + p_3 :\ x \leq \max\{p_1, p_2\}\}$	R^+	$C(1,12)$
20	$\max\{0,\ x + p_3 : x^3 + p_1 x + p_2 = 0\}$	R^+	$C(1,13)$
21	$\max\{\vert p_1\vert,\ \sqrt{p_2} + p_3\}$	R^+	$C(1,14)$, $C(3,4)$, $C(1,1,3)$
22	$\max\{0, p_1, p_2, p_3\}$	R^+	$C(1,15)$, $C(4,4)$, $C(1,1,4)$, $C(1,1,1,1)$
23	$\max\{p_1\vert p_1\vert,\ p_2\vert p_2\vert + p_3\}$	R^+	$C(2,2)$
24	$\max\{p_1\vert p_1\vert,\ \sqrt{p_2} + p_3\}$	R^+	$C(2,3)$
25	$\max\{p_1\vert p_1\vert,\ \vert p_2\vert + p_3\}$	R^+	$C(2,4)$, $C(1,1,2)$
26	$\max\{\sqrt{p_1},\ \sqrt{p_2} + p_3\}$	R^+	$C(3,3)$
27	$\max\{-x^4 + p_1 x^2 + p_2 x : x \in \mathbb{R},\ \sqrt{p_3} + 1\}$	Γ	$C(5;3)$
28	$\max\{x^3 + p_1 x^2 + p_2 x : x \leq 0,\ \sqrt{p_3} + 1\}$	Γ	$C(6;3)$
29	$\max\{-x^2 :\ x^2 + p_1 x + p_2 \leq 0, \sqrt{p_3} + 1\}$	Γ	$C(7;3)$
30	$\max\{\sqrt{p_1}\vert p_2\vert,\ \sqrt{p_3} + 1\}$	Γ	$C(8;3)$
31	$\max\{-x^2 :\ x^2 + p_1 x + p_2 \geq 0, \sqrt{p_3} + 1\}$	Γ	$C(9;3)$
32	$\max\{\vert p_1 p_2\vert, \sqrt{p_3} + 1\}$	Γ	$C(10;3)$

4. Proofs

In this section we will prove Theorems 3 and 4. As seen before, the optimal averaged profit for stationary strategies $A_s(p)$ depends on the behavior of

TABLE 9. (*continued*)

N.	Singularities	Eq.	Situation
33	$\max\{-x^2 : (x+p_1)(x+p_2) \le 0, \sqrt{p_3}+1\}$	Γ	$C(11;3)$
34	$\max\{-x^2 : x \le \max\{p_1, p_2\}, \sqrt{p_3}+1\}$	Γ	$C(12;3)$
35	$\max\{x : x^3+p_1x+p_2=0, \sqrt{p_3}+1\}$	Γ	$C(13;3)$
36	$\max\{0, \sqrt{p_1}+p_2, \sqrt{p_3}+1\}$	Γ	$C(14;3)$, $C(1,3;3)$
37	$\max\{0, p_1, p_2, \sqrt{p_3}+1\}$	Γ	$C(15;3)$, $C(1,4;3)$, $C(1,1,1;3)$
38	$\max\{0, p_1\lvert p_1\rvert+p_2, \sqrt{p_3}+1\}$	Γ	$C(1,2;3)$
39	$\max\{p_1\lvert p_1\rvert, -x^2+1 : x^2+p_2x+p_3 \le 0\}$	Γ	$C(2;7)$
40	$\max\{\sqrt{p_1}, -x^2+1 : x^2+p_2x+p_3 \le 0\}$	Γ	$C(3;7)$
41	$\max\{\lvert p_1\rvert, -x^2+1 : x^2+p_2x+p_3 \le 0\}$	Γ	$C(4;7)$, $C(1,1;7)$
42	$\max\{p_1\lvert p_1\rvert, \sqrt{p_2}\lvert p_3\rvert+1\}$	Γ	$C(2;8)$
43	$\max\{\sqrt{p_1}, \sqrt{p_2}\lvert p_3\rvert+1\}$	Γ	$C(3;8)$
44	$\max\{\lvert p_1\rvert, \sqrt{p_2}\lvert p_3\rvert+1\}$	Γ	$C(4;8)$, $C(1,1;8)$
45	$\max\{0, x^3+p_1x^2+p_2x+1 : \lvert x\rvert \le \sqrt{p_3}\}$	Γ	$C(1;17)$
46	$\max\{0, x+1 : x^4+p_1x^2+p_2x+p_3=0\}$	Γ	$C(1;27)$
47	$\max\{0, \sqrt{p_1}+1, \sqrt{p_2}+p_3+1\}$	Γ	$C(1;29)$, $C(1;3,3)$
48	$\max\{p_1\lvert p_1\rvert, \sqrt{p_2}+1, \sqrt{p_3}+2\}$	Γ	$C(2;3;3)$
49	$\max\{\sqrt{p_1}, \sqrt{p_2}+1, \sqrt{p_3}+2\}$	Γ	$C(3;3;3)$
50	$\max\{\lvert p_1\rvert, \sqrt{p_2}+1, \sqrt{p_3}+2\}$	Γ	$C(4;3;3)$, $C(1,1;3;3)$
51	$\max\{0, -x^2+1 : x^2+p_1x+p_2 \le 0, \sqrt{p_3}+2\}$	Γ	$C(1;7;3)$
52	$\max\{0, \sqrt{p_1}\lvert p_2\rvert+1, \sqrt{p_3}+2\}$	Γ	$C(1;8;3)$
53	$\max\{0, \sqrt{p_1}+1, -x^2+2 : x^2+p_2x+p_3 \le 0\}$	Γ	$C(1;3;7)$
54	$\max\{0, \sqrt{p_1}+1, \sqrt{p_2}\lvert p_3\rvert+2, \}$	Γ	$C(1;3;8)$
55	$\max\{0, \sqrt{p_1}+1, \sqrt{p_2}+2, \sqrt{p_3}+3\}$	Γ	$C(1;3;3;3)$

the family of densities around each point of $\overline{S^*} \cap \mathcal{F}_p$ and on the singularities of the stationary domain around those points. So, before the proof of each of theorems 3 and 4, we will prove some auxiliary results that will permit us to identify which situations must be considered in a generic case.

Lemma 1. *Consider a k-parameter family of pairs (V, f) of polydynamical systems and profit densities on a one-dimensional manifold M, $k \le 3$. Suppose that the stationary domain at a point Q has a codimension m singularity of tables 1-3 and*

$$\frac{\partial f}{\partial x}(Q) = \cdots = \frac{\partial^i f}{\partial x^i}(Q) = 0 \ne \frac{\partial^{i+1} f}{\partial x^{i+1}}(Q) \quad (i \ge 0)$$

where x is a local coordinate on M. Then generically $i + m - 1 \le k$.

Proof. In the jet space $J^5(M\times P,(TM)^n\times \mathbb{R})$, the set of jets of a pair (V,f) at points where the stationary domain has a codimension m singularity of tables 1-3 and where the first i derivatives $\frac{\partial f}{\partial x},\cdots,\frac{\partial^i f}{\partial x^i}$ vanish is a closed submanifold with codimension $m+i$. By Thom Transversality Theorem we conclude that in a generic case $m+i\leq \dim(M\times P)=1+k$. $\square$

Proof of Theorem 3. Let p_0 be a parameter value without competition, and (x_0,p_0) the unique point of $\overline{S^*}\cap\mathcal{F}_{p_0}$ providing the optimal profit $A_s(p_0)$. Because p_0 is a value without competition, to determine $A_s(p)$ for p close to p_0 we just have to look to the family of densities and to the stationary domain in a neighborhood of (x_0,p_0). So we can identify $M\times P$ with $\mathbb{R}\times\mathbb{R}^k$. Firstly we look to the germ of the k-parameter family of densities at (x_0,p_0). By the general singularity theory we conclude that generically it is a versal deformation of $f(\cdot,p_0)$ and so, up to the sum with a C^∞ function depending only on the parameter $p\in\mathbb{R}^k$, it is $\mathcal{F}$-equivalent to the germ at the origin of one of the following functions:

- $F_1=x$, $F_2^{\pm}=\pm x^2$, and $F_3=x^3+p_1x$, if $k=1$ and else
- $F_4^{\pm}=\pm x^4+p_1x^2+p_2x$, if $k=2$ and else
- $F_5=x^5+p_1x^3+p_2x^2+p_3x$, if $k=3$.

So we conclude that the germ of A_s at p_0 is R^+-equivalent to the germ at the origin of

$$\max_{x\in S(p),\,(x,p)\in U} g(x,p)$$

where U is a neighborhood of the origin and g is one of the previous functions F_i^α ($S(p)$ now written in the new coordinates).

Now Theorem 2, Lemma 1, and the fact that $g(\cdot,0)$ restricted to $S(0)$ attains a maximum at the origin, permit us to conclude for each g which types of singularities for the stationary domain have to be considered in a generic case. For example for $k=2$, if $g=x^2$, Theorem 2 and Lemma 1 permit us to conclude that generically the stationary domain at the origin can have only a singularity of codimension ≤ 2. So only singularities of Table 1 for the stationary domain must be considered. It is clear that singularities 0, 2_-, 3_- and 4_- do not take place, because in all those cases, the origin belongs to the interior of $S(0)$ and can not lead to a maximum of x^2. Also it is clear to see that singularities 1 and 4_+ for the stationary domain prevent x^2 to attain a maximum at the origin when restricted to $S(0)$. So for $k=2$ and $g=x^2$, in a generic case we only have to consider singularities 2_+ and 3_+ for the stationary domain.

For each possible case the idea of the rest of the proof is the same. Using Mather Division Theorem and Thom Transversality Theorem, we are able (in a generic case) to put the stationary domain's boundary in normal form without changing g. Comparing then the value of $g(\cdot,p)$ at the boundary of $S(p)$ and at eventual local maxima in its interior we are able to obtain a normal form for the optimal profit $A_s(p)$. To make things clear we will present two cases for $k=2$, namely:

- $g = -x^2$ and the stationary domain with singularity 1 at the origin;
- $g = x^2$ and the stationary domain with singularity 2_+ at the origin.

Consider the first case. If the stationary domain has singularity 1 of Table 1, then its boundary is locally (around the origin) given by $v(x,p) = 0$, where v is an admissible velocity satisfying $v(0,0) = 0 \neq v_x(0,0)$. By Mather Division Theorem the last equation can be written on the form

$$x + r(p) = 0, \tag{3}$$

where r is a smooth function vanishing at the origin. By Thom Transversality Theorem, we conclude that generically the matrix

$$\begin{pmatrix} g_{xx} & g_{xp} \\ v_x & v_p \end{pmatrix}$$

has maximal rank at the origin. Writing $p = (p_1, p_2)$ this means that the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & r_{p_1}(0) & r_{p_2}(0) \end{pmatrix}$$

has maximal rank and so we can always suppose that $r_{p_1}(0) \neq 0$. Hence, an adequate change of coordinates in the parameter space puts equation (3) on the form $x - p_1 = 0$. We can always assume that the stationary domain is given by $x - p_1 \leq 0$. In fact, if this is not the case we consider firstly the change of coordinates $(x, p_1, p_2) \mapsto (-x, -p_1, -p_2)$ that preserves the normal form g. Because g has a local maximum at $x = 0$ we conclude that for $p = (p_1, p_2)$ close to 0, the function A_s is R^+-equivalent to the function

$$\begin{cases} -p_1^2 & p_1 < 0 \\ 0 & p_1 \geq 0 \end{cases}$$

that is clearly R^+-equivalent to $p_1|p_1|$. So we get singularity 2 of Table 4.

In the second case, the stationary domain has at the origin a 2_+ singularity. So its boundary is locally (around the origin) given by $v(x,p) = 0$, where v is an admissible velocity satisfying $v(0,0) = 0 = v_x(0,0) \neq v_{xx}(0,0)$. By Mather Division Theorem the last equation can be written on the form

$$x^2 + \alpha(p)x + \beta(p) = 0, \tag{4}$$

with α and β smooth functions vanishing at the origin. By Thom Transversality Theorem, we conclude that generically the matrix

$$\begin{pmatrix} g_{xx} & g_{xp} \\ v_x & v_p \\ v_{xx} & v_{xp} \end{pmatrix}$$

has maximal rank at the origin. Writing $p = (p_1, p_2)$ this means that the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \beta_{p_1}(0) & \beta_{p_2}(0) \\ 0 & \alpha_{p_1}(0) & \alpha_{p_2}(0) \end{pmatrix}$$

has maximal rank and so an adequate change of coordinates in the parameter space puts equation (4) on the form $x^2+p_1x+p_2=0$. Because we are dealing with singularity 2_+ of Table 1, we conclude that the stationary domain is given by $x^2+p_1x+p_2 \leq 0$. To simplify calculations we consider a new change of coordinates in the parameter space so that the stationary domain is now given by $x^2-2p_1x+p_1^2-p_2 \leq 0$. For $p_2<0$ the set $S(p)$ is empty and for $p_2 \geq 0$ it is given by $S(p)=[p_1-\sqrt{p_2}, p_1+\sqrt{p_2}]$. As g does not have any local maxima we conclude that the maximum is attained at the boundary of $S(p)$. So for $p=(p_1,p_2)$ close to 0 with $p_2>0$ the function $A_s(p)$ is R^+-equivalent to:

$$\max\{(p_1-\sqrt{p_2})^2, (p_1+\sqrt{p_2})^2\} = (|p_1|+\sqrt{p_2})^2,$$

that is clearly R^+-equivalent to $|p_1|\sqrt{p_2}$. So we get singularity 8 of Table 5. □

Lemma 2. *Consider a k-parameter family of pairs (V,f) of polydynamical systems and profit densities on a one-dimensional compact manifold M, $k \leq 3$. Suppose that there are exactly N distinct points Q_i $(i=1,...,N)$ competing for the optimal profit with exactly l distinct levels $(1 \leq l \leq N)$ such that each point Q_i provides a codimension c_i singularity of Tables 4-6. Then, generically*

$$\sum_{i=1}^{N} c_i + N - l \leq k.$$

Proof. On the multijet bundle $J^5_N(M \times P, (TM)^n \times \mathbb{R})$, the set W of multijets of a pair (V,f) at $(Q_1,\cdots,Q_N)$ where $Q_1,\cdots,Q_N$ are distinct points of $M \times P$, satisfying:

- the point Q_i provides a codimension c_i singularity of Tables 4-6,
- $\pi(Q_1)=\pi(Q_2)=\cdots\pi(Q_N)$, where π is the projection on the parameter space,

is a closed submanifold with codimension $\sum_{i=1}^{N}(c_i+1)+(N-1)k$. The subset of W consisting of those multijets at points with exactly l distinct levels is a closed submanifold of $J^5_N(M \times P, (TM)^n \times \mathbb{R})$ with codimension $\mathrm{cod}(Q)+N-l$.

Due to Multijet Transversality Theorem and compactness of the phase space M, we conclude that generically this codimension can not be greater than the dimension $N(1+k)$ of $(M \times P)^N$. So $\sum_{i=1}^{N} c_i + N - l \leq k$. □

Lemma 3. *Consider a k-parameter family of pairs (V,f) of polydynamical systems and profit densities on a one-dimensional compact manifold M,*

$k \leq 3$. *Generically, if there are two points competing for the optimal profit having different levels, then the point having the higher level provides*

- *singularity 3 of Table 4, if* $k = 1$*;*
- *one of singularities 3, 7 and 8 of Tables 4 and 5, if* $k = 2$*;*
- *one of singularities 3, 7, 8, 17, 27 and 29 of Tables 4, 5 and 6, if* $k = 3$.

Proof. All singularities listed in this lemma have in common the fact that any neighborhood of the origin contains values p for which they are not defined. All remaining singularities of Tables 4, 5 and 6 are well defined and are continuous in a neighborhood of the origin. Suppose (x_1, p) and (x_2, p) are two points competing for the optimal profit $A_s(p)$. We shift p to the origin. Let $(x_2, 0)$ be the point having the higher level, that is $f(x_1, 0) < f(x_2, 0)$ and denote

$$G_i(p) = \max_{x \in S(p),\ (x,p) \in U_i} g(x, p)$$

where U_i is a neighborhood of $(x_i, 0)$ not containing other points of $\overline{S^*} \cap \mathcal{F}_0$. By Theorem 3, in a generic case G_i has a singularity from Tables 4-6. So if $(x_2, 0)$ provides a singularity not stated in this lemma, G_2 is continuous in a neighborhood of the origin and because $G_2(0) > G_1(0)$ and G_1 is upper semicontinuous, we conclude that the functions A_s and G_2 coincide in a neighborhood of the origin. But this contradicts the fact that there are two points competing for the optimal profit. $\square$

Proof of Theorem 4.

Let p_0 be a value with competition and let $(x_1, p_0), ..., (x_N, p_0)$ $(N \geq 2)$, be the points competing for the optimal profit $A_s(p_0)$. By Lemma 2 it is easy to see that in a generic case $N \leq k + 1$. We shift p_0 to the origin and define

$$A_s^i(p) = \max_{\substack{x \in S(p) \\ (x,p) \in U_i}} f(x, p)$$

where U_i is a neighborhood of $(x_i, 0)$ not containing other points of $\overline{S^*} \cap \mathcal{F}_0$. It is clear that for p close to 0: $A_s(p) = \max\{A_s^1(p), A_s^2(p), ..., A_s^N(p)\}$. Theorem 3 permits us to obtain the possible generic singularities for each $A_s^i(p)$ at the origin.

Now Lemma 2 and Lemma 3 permit us to conclude which situations $C(i_1, \cdots, i_N)$ must be considered in a generic case. For example for $k = 3$ and $N = 4$, by Lemma 2 we conclude that generically $\sum_{i=1}^{4} c_i + 4 - l \leq 3$. So, if $l = 1$, then $c_1 = \cdots = c_4 = 0$, and we get situation $C(1, 1, 1, 1)$. If $l = 2$, then $\sum_{i=1}^{4} c_i \leq 1$, and by Lemma 3 we conclude that generically only one situation can take place, namely $C(1,1,1;3)$. If $l=3$, then $\sum_{i=1}^{4} c_i \leq 2$ and by Lemma 3, we conclude that only the situation $C(1,1;3;3)$ can occur in a generic case. For $l=4$ we obtain generically also a unique situation, namely $C(1;3;3;3)$. For each possible situation the idea of the rest of the proof is the same. By Multijet Transversality Theorem we conclude that generically it is possible using the same diffeomorphism in the parameter space to put all the functions $A_s^i(p)$ in a preliminary normal form, namely: $A_s^i(p) = g_i(p)+\varphi_i(p)$, where $g_i(p)$ has one of the normal forms presented in Tables 4-6, such that if $i \neq j$ then g_i and g_j depend on different components p_1, p_2,..,p_k of p, and φ_i are smooth functions of the parameter. So, near the origin the optimal profit is R^+-equivalent to: $\max\{g_1(p), g_2(p)+\gamma_2(p), ..., g_N(p)+\gamma_N(p)\}$, where $\gamma_i(p) = \varphi_i(p) - \varphi_1(p)$. Using again Multijet Transversality Theorem we are now able to put A_s in one of the normal forms of Tables 7-9 through R^+-equivalence if all the points in competition have the same level, and through Γ-equivalence if the points in competition have at least two distinct levels.

To make things clear we will present three cases, namely $C(1,4)$, $C(1;3)$ and $C(1;3;3)$ for $k=2$.

Consider firstly situation $C(1,2)$. Let $(x_1,0)$ and $(x_2,0)$ be two points competing for the optimal profit, having the same level and providing singularities 1 and 2 of Table 4, respectively. Around $p=0$, the optimal averaged profit is given by:

$$A_s(p) = \max\{A_s^1(p), A_s^2(p)\}.$$

Suppose that $(x_1,0)$ belongs to the interior of the stationary domain (the other possible case is treated in an analogous manner) and suppose that v is the admissible velocity vanishing at $(x_2,0)$. It is easy to see that there exist local coordinates in the parameter space around $p=0$ such that:

- $f(y,p) = -y^2+\phi(p)$, $v(y,p) = (y-p_1)\cdot V(y,p)$ where y is a local coordinate around x_2 depending smoothly on p in which $x_2=0$, and ϕ and V are smooth functions with $V(0,0) \neq 0$;
- $f_x(x_1,0) = 0 \neq f_{xx}(x_1,0)$.

The stationary domain in a neighbourhood of $(x_2,0)$ is written in the new coordinates as $S=\{y \leq p_1\}$ and so:

$$A_s^2(p) = \max_{y \leq p_1} -y^2+\phi(p).$$

Because $f(\cdot,0)$ has a local maximum at x_1, we have $f_{xx}(x_1,0) < 0$. So $A_s^1(p) = f(\gamma(p),p)$ near the origin, where $x=\gamma(p)$ is the curve of local maxima of the profit density, namely, the solution of equation $f_x=0$ with respect to x near the point $(x_1,0)$. Because $f_{xx}(x_1,0) < 0$, the implicit function theorem guarantees that this solution is unique and smooth.

Then, close to $p=0$, the function A_s is R^+-equivalent to the function

$$\max\{f(\gamma(p),p), \max_{y \leq p_1} -y^2+\phi(p)\},$$

which is clearly R^+-equivalent to the function

$$\max\{0,\, \frac{p_1|p_1| - p_1^2}{2} + \phi(p) - f(\gamma(p), p)\}.$$

Now, because this situation of competition is characterized by two points, (x, p) and (y, p), satisfying the following conditions

$$f_x(x, p) = f_y(y, p) = v(y, p) = f(x, p) - f(y, p) = 0,$$

Multijet Transversality Theorem implies that the matrix

$$\begin{pmatrix} f_{xx}(x,p) & 0 & f_{xp_1}(x,p) & f_{xp_2}(x,p) \\ 0 & f_{yy}(y,p) & f_{yp_1}(y,p) & f_{yp_2}(y,p) \\ 0 & v_y(y,p) & v_{p_1}(y,p) & v_{p_2}(y,p) \\ f_x(x,p) & -f_y(y,p) & f_{p_1}(x,p) - f_{p_1}(y,p) & f_{p_2}(x,p) - f_{p_2}(y,p) \end{pmatrix}$$

has maximal rank. Hence, applying this result at the considered points we conclude that $f_{p_2}(x_1, 0) - \phi_{p_2}(0) \neq 0$, due to the fact that $f_x(x_1, 0) = f_{yp_1}(x_2, 0) = f_{yp_2}(x_2, 0) = v_{p_2}(x_2, 0) = 0$. So, after the coordinate change

$$(p_1, p_2) \mapsto \left(\frac{p_1}{\sqrt{2}}, -\frac{p_1^2}{2} + \phi(p) - f(\gamma(p), p)\right)$$

we obtain singularity 3 of Table 8.

Now let us consider situation $C(1; 3)$. Let $(x_1, 0)$ and $(x_2, 0)$ be two points competing for the optimal profit, having distinct levels and providing singularities 1 and 3 of Table 4, respectively. Suppose that $(x_1, 0)$ belongs to the interior of the stationary domain (the other possible case is treated in an analogous manner) and suppose that v is the admissible velocity vanishing at $(x_2, 0)$. It is easy to see that there exist local coordinates in the parameter space around $p = 0$ such that:

- $f(y, p) = y + \phi(p)$, $v(y, p) = (y^2 - p_1) \cdot V(y, p)$ where y is a local coordinate around x_2 depending smoothly on p in which $x_2 = 0$, and ϕ and V are smooth functions with $V(0, 0) \neq 0$;
- $f_x(x_1, 0) = 0 \neq f_{xx}(x_1, 0)$.

As in the previous case we conclude that, close to $p = 0$, the function A_s is R^+-equivalent to the function

$$\max\{f(\gamma(p), p),\, \max_{y^2 \leq p_1} y + \phi(p)\},$$

with γ smooth satisfying $\gamma(0) = x_1$, which is clearly R^+-equivalent to the function

$$\max\{0,\, \sqrt{p_1} + \phi(p) - f(\gamma(p), p)\}.$$

Observing that $f(x_1, 0) < f(x_2, 0) = \phi(0)$ we conclude that, close to $p = 0$, $\phi(p) - f(\gamma(p), p) > 0$ and so, after the coordinate change

$$(p_1, p_2, a) \mapsto \left(\frac{p_1}{(\phi(p) - f(\gamma(p), p))^2}, p_2, \frac{a}{\phi(p) - f(\gamma(p), p)}\right)$$

we obtain (by Γ-equivalence) singularity 2 of Table 7.

Finally we consider situation $C(1;3;3)$. Let $(x_1,0)$, $(x_2,0)$ and $(x_3,0)$ be the three points competing for the optimal profit, all them having different levels. Suppose that $(x_1,0)$ provides singularity 1 of Table 4, $(x_2,0)$ and $(x_3,0)$ both provide singularity 3 of Table 4 and $f(x_1,0) < f(x_2,0) < f(x_3,0)$. Let v and w be the admissible velocities vanishing at $(x_2,0)$ and $(x_3,0)$ respectively and suppose as before that $(x_1,0)$ belongs to the interior of the stationary domain (the other possible case is treated in an analogous manner). So $f_x(x_1,0) = 0 \neq f_{xx}(x_1,0)$.

As $v(x_2,0) = 0 = v_x(x_2,0)$ and $v_{xx}(x_2,0) \neq 0 \neq f_x(x_2,0)$ it is easy to see, using Mather Division Theorem, that there exists a local coordinate y, around x_2, depending smoothly on p in which $x_2 = 0$, $f(y,p) = y + \varphi(p)$ and $v(y,p) = (y^2 - \alpha(p)) \cdot V(y,p)$ with φ, α and V smooth functions and $V(0,0) \neq 0$. In fact, writing f on the form $f(x,p) = (x - x_2)\psi(x,p) + a(p)$ and considering the new coordinate $\tilde{x} = (x - x_2)\psi(x,p)$, we get $f(\tilde{x},p) = \tilde{x} + a(p)$. By Mather Theorem, the equation $v = 0$ can be written on the form $\tilde{x}^2 + b(p)\tilde{x} + c(p) = 0$ with b and c smooth satisfying $b(0) = c(0) = 0$. Considering now the new coordinate $y = \tilde{x} + b(p)/2$ we get the desired result.

In the same way we conclude the existence of a local coordinate z, around x_3, depending smoothly on p in which $x_3 = 0$, $f(z,p) = z + \phi(p)$ and $w(z,p) = (z^2 - \beta(p)) \cdot W(y,p)$ with ϕ, β and W smooth functions and $W(0,0) \neq 0$.

Now, because this situation of competition is characterized by three points, (x,p), (y,p) and (z,p), satisfying the following conditions

$$f_x(x,p) = v(y,p) = v_y(y,p) = w(z,p) = w_z(z,p) = 0,$$

Multijet Transversality Theorem implies that in a generic case the matrix

$$\begin{pmatrix} f_{xx}(x,p) & 0 & 0 & f_{xp_1}(x,p) & f_{xp_2}(x,p) \\ 0 & v_y(y,p) & 0 & v_{p_1}(y,p) & v_{p_2}(y,p) \\ 0 & v_{yy}(y,p) & 0 & v_{yp_1}(y,p) & v_{yp_2}(y,p) \\ 0 & 0 & w_z(z,p) & w_{p_1}(z,p) & w_{p_2}(z,p) \\ 0 & 0 & w_{zz}(z,p) & w_{zp_1}(z,p) & w_{zp_2}(z,p) \end{pmatrix}$$

has maximal rank. Hence, applying this result at the considered points and due to the fact that $v_y(x_2,0) = w_z(x_3,0) = 0$ and $f_{xx}(x_1,0) \neq 0$, $v_{yy}(x_2,0) \neq 0$ and $w_{zz}(x_3,0) \neq 0$, we conclude that generically the matrix

$$\begin{pmatrix} v_{p_1}(x_2,0) & v_{p_2}(x_2,0) \\ w_{p_1}(x_3,0) & w_{p_2}(x_3,0) \end{pmatrix}$$

has maximal rank and so an adequate change of coordinates in the parameter space puts v and w on the forms:

$$v(y,p) = (y^2 - p_1) \cdot V(y,p) \qquad w(z,p) = (z^2 - p_2) \cdot W(y,p).$$

Notice that after this change of coordinates, the normal forms for f around $(x_2,0)$ and $(x_3,0)$ are the previous ones but with different functions φ and ϕ. So we conclude that close to $p = 0$, the function $A_s(p) =$

$\max\{A_s^1(p), A_s^2(p), A_s^3(p)\}$ is R^+-equivalent to the function

$$\max\{f(\gamma(p),p),\ \max_{y^2\leq p_1} y+\varphi(p),\ \max_{z^2\leq p_2} z+\phi(p)\},$$

with γ smooth satisfying $\gamma(0)=x_1$ obtained as in the previous cases, and $f(x_1,0)<\varphi(0)<\phi(0)$. This function is clearly R^+-equivalent to the function

$$\max\{0,\ \sqrt{p_1}+\varphi(p)-f(\gamma(p),p),\ \sqrt{p_2}+\phi(p)-f(\gamma(p),p)\}.$$

Let $\psi(p,a)$ be a smooth function [8] such that

$$\begin{array}{ll} \psi(p,a)=\varphi(p)-f(\gamma(p),p) & \text{if } a<\varphi(0)-f(x_1,0)+\frac{1}{3}(\phi-\varphi)(0) \\ \psi(p,a)=\frac{\phi(p)-f(\gamma(p),p)}{2} & \text{if } a>\varphi(0)-f(x_1,0)+\frac{2}{3}(\phi-\varphi)(0). \end{array}$$

Observe that $\psi(0,a)>0$, for every real value a. Then, after the coordinate change

$$(p_1,p_2,a)\mapsto\left(\frac{p_1}{(\varphi(p)-f(\gamma(p),p))^2},\ \frac{4p_2}{(\phi(p)-f(\gamma(p),p))^2},\ \frac{a}{\psi(p,a)}\right)$$

we obtain singularity 11 of Table 8. □

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Faculdade de Ciências-Universidade do Porto and Centro de Matemática da Universidade do Porto, Portugal
E-mail address: mmmatos@fc.up.pt

Faculdade de Ciências-Universidade do Porto, Portugal
E-mail address: celi@portugalmail.pt