LARGE DEVIATIONS FOR SEMIFLOWS OVER A NON-UNIFORMLY EXPANDING BASE

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Abstract. We obtain a large deviation bound for continuous observables on suspension semiflows over a non-uniformly expanding base transformation with non-flat singularities or criticalities, where the roof function defining the suspension behaves like the logarithm of the distance to the singular/critical set of the base map. That is, given a continuous function we consider its space average with respect to a physical measure and compare this with the time averages along orbits of the semiflow, showing that the Lebesgue measure of the set of points whose time averages stay away from the space average tends to zero exponentially fast as time goes to infinity.

Suspension semiflows model the dynamics of flows admitting cross-sections, where the dynamics of the base is given by the Poincaré return map and the roof function is the return time to the cross-section. The results are applicable in particular to semiflows modeling the geometric Lorenz attractors and the Lorenz flow, as well as other semiflows with multidimensional non-uniformly expanding base with non-flat singularities and/or criticalities under slow recurrence rate conditions to this singular/critical set. We are also able to obtain exponentially fast escape rates from subsets without full measure.

1. Introduction

The statistical viewpoint on Dynamical Systems provides some of the main tools available for the global study of the asymptotic behavior of transformations or flows. One of the main concepts introduced is the notion of *physical* (or *Sinai-Ruelle-Bowen*) measure for a flow (or a transformation): an invariant probability measure μ for a flow X_t on a compact manifold is a physical probability measure if the points z satisfying

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \psi\Big(X_s(z)\Big)\,ds = \int \psi\,d\mu,$$

this is, points for which the time average of a continuous function (an observable) ψ along the orbit of z coincides with the space average of ψ with respect to μ, form a subset with positive volume (or positive Lebesgue measure) on the ambient space. These

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time averages are in principle physically observable if the flow models a real world phenomenon admitting some measurable features.

For systems admitting such invariant probability measures it is natural to consider the rate of convergence of the time averages to the space average, given by the volume of the subset of points whose time averages stay away from the space average by a prescribed amount up to some evolution time. This rate is closely related to the so-called thermodynamical formalism first developed for (uniformly) hyperbolic diffeomorphisms, borrowed from statistical mechanics by Ruelle, Sinai and Bowen (among others, see e.g. [22, 23, 48, 49, 28, 21]). These authors systematically studied the construction a properties of physical measures for (uniformly) hyperbolic diffeomorphisms and flows. Such measures for non-uniformly hyperbolic maps and flows where obtained more recently [45, 25, 18, 19, 2].

The probabilistic properties of physical measures are an object of intense study, see e.g. [23, 36, 54, 55, 20, 3, 4, 6, 31, 11, 7]. The main insight behind these efforts is that the family $\{\psi \circ X_t\}_{t>0}$ should behave asymptotically in many respects just like a i.i.d. random variable.

The study of suspension (or special) flows is motivated by modeling a flow admitting a cross-section, which is equivalent to a suspension semiflow over the Poincaré return map to the cross-section with roof function given by the return time function for the points in the cross-section. This is one of the main technical tools in the ergodic theory of Axiom A (or uniformly hyperbolic) flows developed by Bowen and Ruelle [23], enabling them to pass from this type of flow to a suspension flow over a shift transformation with finitely many symbols and bounded roof function. Then the properties of the base transformation are used to deduce many results for the suspension flow, which are then passed to the original flow.

Recently this kind of modeling provided results on the rate of decay of correlations for certain flows [13] based on the rate of decay of correlations for suspension semiflows [15] and also general results on the existence and some statistical properties of physical measures for singular-hyperbolic attractors for three-dimensional flows [10] as well as their sensitive dependence on initial conditions. Moreover the classical Lorenz flow [42] was shown to be equivalent to a geometric Lorenz flow by Tucker [51] and so it can be modeled by a suspension semiflow over a non-uniformly hyperbolic transformation with unbounded roof function.

Here we extend part of the results on large deviation rates of Kifer [36] from the uniformly hyperbolic setting to semiflows over non-uniformly expanding base dynamics and unbounded roof function, which model non-hyperbolic flows, like the Lorenz flow, exhibiting equilibria accumulated by regular orbits. We use the properties of non-uniformly expanding transformations, especially the large deviation bound obtained by the author together with Pacifico [7], to obtain a large deviation bound for the suspension semiflow reducing the estimate of the volume of the deviation set to the volume of a certain deviation set for the base transformation. More precisely, if we set $\varepsilon > 0$ as an error margin and consider

$$B_t = \Big\{z : \Big|\frac{1}{t}\int_0^t \psi\Big(X_t(z)\Big) - \int \psi\, d\mu\Big| > \varepsilon\Big\}$$

then we are able to provide conditions under which the Lebesgue measure of B_t decays to zero exponentially fast, i.e. weather there are constants $C, \xi > 0$ such that

$$\mathrm{Leb}\Big(B_t\Big) \le Ce^{-\xi t} \quad \text{for all} \quad t > 0.$$

The values of $C, \xi > 0$ above depend on ε, ψ and on global invariants for the base transformation f, such as the metric entropy and the pressure function of f with respect to the physical measures of f and a certain observable constructed from ψ and X_t, as detailed in the next section. Having this is not difficult to deduce exponential escape rates from subsets of the semiflow.

In order to be able to apply this bound to Lorenz flows we need to admit that the roof function of the suspension flows we consider here is not bounded nor continuous near the singularities of the base dynamical system. This in turn imposes some restrictions on the admissible base dynamics expressed as a slow recurrence rate to the singular set and uniqueness of equilibrium states with respect to the logarithm of the Jacobian of the map. However no cohomology condition on the roof function are needed, while this is essential to obtain fast decay of correlations as in [27, 43, 29].

We present several semiflows with non-uniformly expanding base transformations satisfying all our conditions, including one-dimensional piecewise expanding maps with *Lorenz-like* singularities and quadratic maps but also multidimensional examples. This demanded the detailed study of recurrence rates to the singular set, the study of large deviation bounds for unbounded observables over non-uniformly expanding transformations, and an entropy formula for non-uniformly expanding maps with singularities (which might be of independent interest). Now we give the precise statement of the results.

1.1. **Statement of the results.** Denote by $\|\cdot\|$ a Riemannian norm on the compact boundaryless manifold M, by dist the induced distance and by Leb a Riemannian volume form, which we call *Lebesgue measure* or *volume* and assume Leb to be normalized: $\mathrm{Leb}(M) = 1$.

Given a C^2 local diffeomorphism $f : M \setminus \mathcal{S} \to M$ outside a volume zero non-flat singular set, let $X^t : M_r \to M_r$ be a semiflow with roof function $r : M \setminus \mathcal{S} \to \mathbb{R}$ over the base transformation f, as follows. Set $M_r = \{(x, y) \in M \times [0, +\infty) : 0 \le y < r(x)\}$ and X^0 the identity on M_r. For $x = x_0 \in M$ denote by x_n the nth iterate $f^n(x_0)$ for $n \ge 0$. Denote $S_n\varphi(x_0) = \sum_{j=0}^{n-1} \varphi(x_j)$ for $n \ge 1$ and for any given real function φ in what follows. Then for each pair $(x_0, s_0) \in X_r$ and $t > 0$ there exists a unique $n \ge 1$ such that $S_n r(x_0) \le s_0 + t < S_{n+1} r(x_0)$ and we define

$$X^t(x_0, s_0) = \Big(x_n, s_0 + t - S_n r(x_0)\Big).$$

The non-flatness of the singular set $\mathcal{S}$ is an extension to arbitrary dimensions of the notion of non-flat singular set from one-dimensional dynamics [26] and means that f

behaves like a power of the distance to the singular set. More precisely there are constants $B > 1$ and $0 < \beta < 1$ for which

(S1) $\frac{1}{B}\operatorname{dist}(x,\mathcal{S})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B\operatorname{dist}(x,\mathcal{S})^{-\beta}$;

(S2) $\left|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\right| \le B\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{S})^{\beta}}$;

(S3) $\left|\log|\det Df(x)^{-1}| - \log|\det Df(y)^{-1}|\right| \le B\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{S})^{\beta}}$;

for every $x, y \in M \setminus \mathcal{S}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{S})/2$ and $v \in T_xM \setminus \{0\}$. We also assume an extra condition related to the geometry of $\mathcal{S}$ ensuring that the Lebesgue measure of neighborhoods $\mathcal{S}$ is comparable to a power of the distance to $\mathcal{S}$, that is there exists $C_\kappa, \kappa > 0$ such that for all small $\rho > 0$

(S4) $\operatorname{Leb}\{x \in M : \operatorname{dist}(x,\mathcal{S}) < \rho\} \le C_\kappa \cdot \rho^{\kappa}$.

The singular set $\mathcal{S}$ contains those points x where f is either not defined, is discontinuous, not differentiable or else $Df(x)$ is non-invertible (that is $\mathcal{S}$ contains the set $\mathcal{C}$ of critical points of f). Note that condition (S4) is satisfied in the particular case when $\mathcal{S}$ is a compact submanifold of M, where $\kappa = \dim(M) - \dim(\mathcal{S})$. It is also satisfied for $M = \mathbb{S}^1$ and $\mathcal{S}$ is a denumerable infinite subset with finitely many accumulation points, with $\kappa = 1$. In particular this holds for a piecewise expanding map over the interval or the circle with finitely many domains of monotonicity.

We say that f is *non-uniformly expanding* if there exists $c > 0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}S_n\psi(x) \le -c \quad \text{where} \quad \psi(x) = \log\left\|Df(x)^{-1}\right\|, \tag{1.1}$$

for Lebesgue almost every $x \in M$. This conditions implies in particular that all the lower Lyapunov exponents of the map f are strictly positive Lebesgue almost everywhere.

We say that f has *exponentially slow recurrence to the singular set* $\mathcal{S}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}\log\operatorname{Leb}\left\{x \in M : \frac{1}{n}S_n\Delta_\delta(x) > \varepsilon\right\} < 0, \tag{1.2}$$

where $\Delta_\delta(x) = \left|\log d_\delta(x,\mathcal{S})\right|$ and for any given $\delta > 0$ we define the *smooth δ-truncated distance* from $x \in M$ to $\mathcal{S}$ by

$$d_\delta(x,\mathcal{S}) = \xi_\delta\Big(\operatorname{dist}(x,\mathcal{S})\Big)\cdot\operatorname{dist}(x,\mathcal{S}) + 1 - \xi_\delta\Big(\operatorname{dist}(x,\mathcal{S})\Big)$$

where $\xi_\delta : \mathbb{R} \to [0,1]$ is a standard C^∞ auxiliary function satisfying

$$\xi_\delta(x) = 1 \text{ if } |x| \le \delta \text{ and } \xi_\delta(x) = 0 \text{ if } |x| \ge 2\delta.$$

Observe that Δ_δ is non-negative and continuous away from $\mathcal{S}$ and identically zero 2δ-away from $\mathcal{S}$. Note also that from standard arguments condition (1.2) implies in particular that f has *slow recurrence to* $\mathcal{S}$, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} S_n \Delta_\delta(x) \le \varepsilon \tag{1.3}$$

for Lebesgue almost every $x \in M$.

These notions where presented in [5] and in [5, 1] the following result on existence of finitely many absolutely continuous measures was obtained.

Theorem 1.1. *Let $f : M \to M$ be a C^2 local diffeomorphism outside a non-flat singular set $\mathcal{S}$. Assume that f is non-uniformly expanding with slow recurrence to $\mathcal{S}$. Then there are finitely many ergodic absolutely continuous (in particular* physical *or* Sinai-Ruelle-Bowen*) f-invariant probability measures $\mu_1, \dots, \mu_k$ whose basins cover the manifold Lebesgue almost everywhere, that is $B(\mu_1) \cup \cdots \cup B(\mu_k) = M$,* $\quad \mathrm{Leb} - \mathrm{mod}\ 0$*, and, moreover the support of each measure contains an open disk in M.*

Here the *basin* of an invariant probability measure μ is the subset of points $x \in M$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu$ in the weak* topology.

Large deviation bounds are usually related to measure theoretic entropy and to equilibrium states. We denote by $\mathcal{M}_f$ the family of all invariant probability measures with respect to f. We say that $\mu \in \mathcal{M}_f$ is an *equilibrium state* with respect to the potential $\log J$, where $J = |\det Df|$, if $h_\mu(f) = \mu(\log J)$, that is if μ *satisfies the Entropy Formula*. We denote by $\mathbb{E}$ the subset of $\mathcal{M}_f$ consisting of all equilibrium states for f. It is not difficult to see (Section 5 for more details) that each physical measure provided by Theorem 1.1 belongs to $\mathbb{E}$.

Another standing assumption on f is that *the set $\mathbb{E}$ is formed by a unique f-invariant absolutely continuous probability measure* (see Section 2 for sufficient conditions for this to occur and for examples of application).

We denote by $\nu = \mu \ltimes \mathrm{Leb}^1$ the natural X^t-invariant extension of μ to M_r and by λ the natural extension of Leb to M_r, i.e. $\lambda = \mathrm{Leb} \ltimes \mathrm{Leb}^1$, where Leb^1 is one-dimensional Lebesgue measure on $\mathbb{R}$: for any subset $A \subset M_r$

$$\nu(A) = \frac{1}{\mu(r)} \int d\mu(x) \int_0^{r(x)} ds\, \chi_A(x,s) \quad \text{and} \quad \lambda(A) = \frac{1}{\mathrm{Leb}(r)} \int d\,\mathrm{Leb}(x) \int_0^{r(x)} ds\, \chi_A(x,s).$$

We say that a function $\varphi : M \setminus \mathcal{S} \to \mathbb{R}$ has *logarithmic growth near* $\mathcal{S}$ if there exists $K = K(\varphi) > 0$ such that $|\varphi|\chi_{B(\mathcal{S},\delta)} \le K \cdot \Delta_\delta$ for all small enough $\delta > 0$. We say that f is a *regular map* if for $E \subset M$ such that $\mathrm{Leb}(E) = 0$, then $\mathrm{Leb}\left(f^{-1}(E)\right) = 0$.

Theorem A. *Let X^t be a suspension semiflow over a non-uniformly expanding transformation f on the base M which exhibits exponentially slow recurrence to the non-flat singular set, where the roof function $r : M \setminus \mathcal{S} \to \mathbb{R}$ has logarithmic growth near $\mathcal{S}$. Assume that f is a regular map and that the set $\mathbb{E}$ of equilibrium states is formed by a single measure μ. Let $\psi : M_r \to \mathbb{R}$ be a* *continuous function. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \log \lambda \left\{ z \in M_r : \left| \frac{1}{T} \int_0^T \psi\left(X^t(z)\right) \, dt - \nu(\psi) \right| > \varepsilon \right\} < 0. \tag{1.4}$$

1.2. **Escape rates.** Using the estimate from Theorem A obtained above and that for any compact subset $K \subset M_r$ and a given $\varepsilon > 0$ we can find an open set $W \supset K$ contained in M_r and a continuous function $\varphi : M_r \to \mathbb{R}$ such that $\mathrm{Leb}(W \setminus K) < \varepsilon$ and also $0 \le \varphi \le 1$, $\varphi \mid K \equiv 1$ and $\varphi \mid (M \setminus W) \equiv 0$, then we get for $n \ge 1$

$$\left\{ x \in K : f(x) \in K, \dots, f^{n-1}(x) \in K \right\} \subset \left\{ x \in M : \frac{1}{n} S_n \varphi(x) \ge 1 \right\} \tag{1.5}$$

and so we deduce the following.

Corollary B. *Let X_t be a suspension semiflow over a non-uniformly expanding transformation f on the base M in the same setting as in Theorem A. Let K be a compact subset of M_r such that $\nu(K) < 1$. Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left(\left\{ x \in K : f^j(x) \in K, j = 1, \dots, n-1 \right\} \right) < 0.$$

1.3. **Lorentz and Geometric Lorenz flows.** The Lorenz equations

$$\dot{x} = 10(y - x), \quad \dot{y} = 28x - y - xz, \quad \dot{z} = xy - 8z/3 \tag{1.6}$$

were presented by Lorenz [42] in 1963 as a simplified model of convection of the Earth's atmosphere. It turned out that these equations became one of the main models showing the presence of chaotic dynamics on apparently simple systems. More recently Tucker [51, 52] with a computer assisted proof showed that equations (1.6) and similar equations with nearby parameters define a geometric Lorenz flow, i.e. a three-dimensional flow X_t in $\mathbb{R}^3$ with a hyperbolic singularity at the origin admitting a neighborhood U (a *trapping region*) such that $\overline{X_t(U)} \subset U$ for all $t > 0$ satisfying:

1. (1) the attracting set $\Lambda = \cap_{t>0} X_t(U)$ contains the singularity at 0;
2. (2) Λ contains a regular dense orbit;
3. (3) there exists a square $S = [-1, 1] \times [-1, 1] \times \{1\}$ which is a cross-section for $\Lambda \setminus \{0\}$, that is for every $w \in \Lambda \setminus \{0\}$ there exists $t > 0$ such that $X_t(w) \in S$;
4. (4) the Poincaré first return map to S given by $R : S \setminus \ell \to S$ is C^2 and contracts distances exponentially on the y direction, where $\ell = ([-1, 1] \times \{(0, 1)\})$ is the singular line, so each segment $S \cap \{y = \mathrm{const}\}$ is contained in a stable manifold for X_t. Moreover in general this one-dimensional and co-dimension one foliation of the cross-section S defines a projection P along leaves which is at most $C^{1+\alpha}$ for some $\alpha > 0$;
5. (5) the one-dimensional map $f : [-1, 1] \setminus \{0\} \to [-1, 1]$ obtained from R quotienting out the stable manifolds is a piecewise expanding map with singularities known as *Lorenz-like map*, which is in the setting of the class of examples detailed in Subsection 2.2;

(6) the first return time function $\tau(w)$ for $w \in S$ is Lebesgue integrable over S and has a logarithmic growth near the singular line ℓ.

It is well known that the attractor of the geometric Lorenz flows (and the attractor for the Lorenz equations after the results of Tucker already mentioned) supports a unique ergodic physical measure μ (for more details on this construction see e.g. [53]). Figure 1 gives a visual idea of the geometric Lorenz flow. The reader should consult [32, 33, 47] for proofs of the properties stated above and more details on the construction of such flows. Using τ as a roof function over the base dynamics given by R we see that the

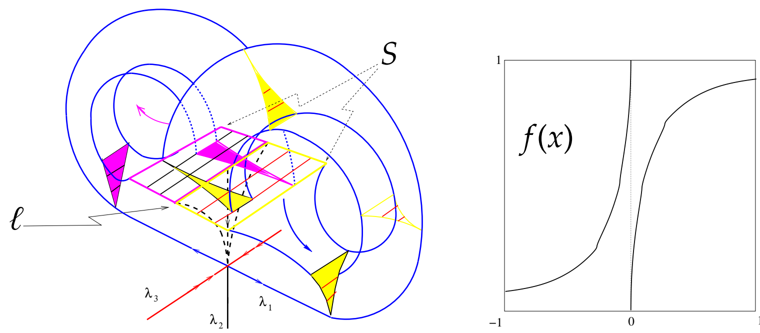

Figure 1. The geometric Lorenz flow and the associated one-dimensional piecewise expanding map

dynamics of geometric Lorenz flow on U is equivalent to a suspension semiflow over R with roof function τ. In addition the uniform contraction along the leaves of the foliation $\{y = \text{const}\}$ together with the uniform expansion of the one-dimensional map f enables us to use Theorem A to deduce

Corollary C. *Let X_t be a flow on $\mathbb{R}^3$ exhibiting a Lorenz or a geometric Lorenz attractor with trapping region U. Denoting by* Leb *the normalized restriction of the Lebesgue volume measure to U, $\psi : U \to \mathbb{R}$ a bounded continuous function and μ the unique physical measure for the attractor, then for any given $\varepsilon > 0$*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathrm{Leb} \left\{ z \in U : \left| \frac{1}{T} \int_0^T \psi\left(X^t(z)\right) dt - \mu(\psi) \right| > \varepsilon \right\} < 0,$$

and consequently for any compact $K \subset U$ such that $\mu(K) < 1$ we also have

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb} \left(\left\{ x \in K : f^j(x) \in K, j = 1, \ldots, n-1 \right\} \right) < 0.$$

1.4. **Comments and organization of the paper.** We note that the smoothness assumption needed for our arguments is only $C^{1+\alpha}$ for some $\alpha \in (0,1)$. Therefore the C^2 condition on f in the statements of results can be relaxed to $C^{1+\alpha}$ throughout.

Kifer [36] together with Newhouse [37] obtain sharp large deviations bounds both from above and from below for uniformly partially hyperbolic attractors for flows and for Axiom A flows, through an estimate of the volume growth of images of balls under the action of the flow near the attractor ("volume lemma", see also [23] and [22]). Moreover to obtain the lower bound an assumption of uniqueness of equilibrium states is necessary and this assumption is also used to prove that the upper bound is strictly negative (see also [54] for uniformly expanding transformations and for partially hyperbolic attractors for diffeomorphisms).

Hence the assumption that $\mathbb{E}$ is formed by a single measure is natural in this setting. The author feels this assumption should not be needed to obtain an expression for the upper bound in terms of entropies, as in [36]. However since the relevant "volume lemmas" are presently not available in the setting of special flows over non-uniformly expanding base and with singularities or criticalities, and also the uniqueness of equilibrium states with respect to a large family of potentials (or observables) is still unknown in general (see [44, 12, 11] for recent progress in this direction), instead of following the approach of [36] we have reduced the problem of estimating the deviations for the suspension flow with respect to a continuous observable to the problem of estimating deviations for the base transformation with respect to an unbounded observable, and then rely on previous work [7] for non-uniformly expanding transformations. To conclude that the upper bound obtained is strictly negative we assume uniqueness of the relevant equilibrium state and to deal with the dynamics near the singularities we impose conditions of very slow recurrence to the singular set $\mathcal{S}$ for the base transformation f and an growth condition of the roof function r near the singularities.

Section 2 shows how the conditions of f and on r are still satisfied by many relevant examples. In particular in Subsection 2.4 it is explained how to obtain a large deviation bound for geometric Lorenz flows using the statement of the Main Theorem applied to suspensions semiflows over piecewise expanding maps with singularities, which are treated in a preliminary fashion in Subsection 2.2 and at length in Section 6. The main result needed for the proof of the Main Theorem is a large deviation bound for observables with logarithmic growth near the singular set for a non-uniformly expanding map, which is proved in Section 3. Then the statement of the Main Theorem about large deviations for a suspension semiflow is reduced to a statement of large deviations for the dynamics of the base transformation in Section 4 concluding the proof of the Main Theorem. Note that in contrast to the results on decay of correlations for Anosov flows or Axiom A flows, here we do not need any coboundary conditions on the roof function for the large deviation bound to hold. This is related to the strategy of reducing the problem from the dynamics of the semiflow to the dynamics of the base.

In Section 5 we present a derivation of the Entropy Formula for non-uniformly expanding maps with slow recurrence to the non-flat singular set, which is used to establish that some examples presented in Section 2 do satisfy our assumptions and which might be interesting in itself.

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2. Examples of application

Here we present some concrete examples where our results can be applied.

2.1. **Suspension semiflows over multidimensional volume expanding and quasi-expanding maps.**

Let $f : M \setminus \mathcal{S} \to M$ be a transitive non-uniformly expanding map with exponentially slow recurrence to $\mathcal{S}$ satisfying $J = |\det Df| > 1$, $\psi \leq 0$ and $\psi = 0$ at finitely many points only (a *quasi-expanding* map). We claim that in this setting $\mathbb{E}$ is a singleton.

Indeed $\mathbb{E}$ is non-empty by Theorem 1.1 since every absolutely continuous invariant probability measure is an equilibrium state (see e.g. Theorem 5.1 in Section 5). Since $f \mid M \setminus \mathcal{S}$ is a local diffeomorphism and the support of such absolutely continuous invariant measures contains open sets, the transitivity together with regularity of the map ensure that there exists only one absolutely continuous invariant measure. For otherwise let μ_i be ergodic absolutely continuous f-invariant probability measures and let $B_i \subset \operatorname{supp}(\mu_i)$ be open sets in the support $i = 1, 2$; by transitivity and continuity there exists a non-empty open subset $B \subset B_1$ and an iterate such that $f^n(B) \subset B_2$ and by smoothness Leb-almost every point in $f^n(B)$ is both a μ_1-generic point and a μ_2-generic point, thus $\mu_1 \equiv \mu_2$. This shows that there exists a unique absolutely continuous invariant probability measure for f.

Note now that every equilibrium state $\nu \in \mathbb{E}$ must be such that $h_\nu(f) = \nu(\log J) > 0$ and since $\psi \leq 0$ and has at most finitely many zeroes, then either $\nu(\psi) < 0$ and by Theorem 5.1 the measure ν must be absolutely continuous, or $\nu(\psi) = 0$ and $\operatorname{supp} \nu \subseteq \psi^{-1}(\{0\})$ is finite thus $h_\nu(f) = 0$, a contradiction.

Therefore by the uniqueness result above ν must coincide with μ. We have shown that $\mathbb{E} = \{\mu\}$, as claimed.

Hence we can apply Theorem A for semiflows over non-uniformly expanding maps with exponentially slow recurrence to the singular set which are also transitive, volume expanding and expanding except at finitely many points, and whose roof function grows with the logarithm of the distance to $\mathcal{S}$.

For examples of multidimensional local diffeomorphisms in this setting see [9]. In this case $\mathcal{S} = \emptyset$ and we can apply Theorem A for semiflows with this type of base transformations plus a continuous (and thus bounded) roof function.

Clearly the same large deviation bound holds for a semiflow over a local diffeomorphisms which is uniformly expanding together with any continuous roof function.

2.2. **Suspension semiflows over piecewise expanding maps with singularities.**

Let M be the circle $\mathbb{S}^1$ or the interval $[0, 1]$ with $\{0, 1\} \subset \mathcal{S}$ and $\mathcal{S} \subset M$ an at most denumerable and non-flat singular set of f such that its closure $\overline{\mathcal{S}}$ has zero Lebesgue measure: $\text{Leb}(\overline{\mathcal{S}}) = 0$.

If we assume that $-\infty < \psi < -c < 0$ on $M \setminus \mathcal{S}$ for some $c > 0$ (so that in particular there are no critical points: $\mathcal{C} = \emptyset$) and that f is transitive with slow recurrence to $\mathcal{S}$, then the set $\mathbb{E}$ of equilibrium states with respect to $\log |f'|$ is formed by a single absolutely continuous invariant probability measure, as shown in Subsection 2.1, since

f is automatically non-uniformly expanding, quasi-expanding and volume expanding as well.

Observe that for C^2 maps in our conditions with finitely many smoothness domains, or with derivative of bounded variation, it is well known that there exists a unique ergodic absolutely continuous invariant probability measure μ with bounded density [34, 50]. Since the function $\log \operatorname{dist}(x, \mathcal{S})$ is Leb-integrable we also have that this function is μ-integrable. Thus for all $\varepsilon > 0$ there is $\delta > 0$ such that $\int \left|\log \operatorname{dist}_\delta(x, \mathcal{S})\right| d\mu(x) < \varepsilon$. By the ergodicity and absolute continuity of μ this means that f has slow recurrence to $\mathcal{S}$ for a positive Lebesgue measure subset of M, which by Theorem 1.1 and [5] ensures that f is in fact non-uniformly expanding with slow recurrence to $\mathcal{S}$. Moreover by [35] the same argument applies to $C^{1+\alpha}$ piecewise expanding maps with finitely many smoothness domains, for some $\alpha \in (0, 1)$.

To be able to apply the Main Theorem we still need exponentially slow recurrence to $\mathcal{S}$. We prove this in Section 6 assuming that $|f'|$ grows as the inverse of some power of the distance to $\mathcal{S}' = \mathcal{S} \cap f(M)$, i.e. besides conditions (S1) through (S4) we impose

(S5) $\left|f'(x)\right| \geq B^{-1} \operatorname{dist}(x, \mathcal{S}')^{-\beta}$ for all $x \in M \setminus \mathcal{S}$,

where $\mathcal{S}'$ is the (sub)set of singularities which matters for the asymptotic dynamics of f.

Hence *a semiflow over a piecewise expanding map with singularities satisfying some technical conditions, and with a roof function having logarithmic growth near the singularities* admits a large deviation bound as in Theorem A.

2.3. **Suspension semiflows over quadratic maps on Benedicks-Carleson parameters.** Set $M = I = [-1, 1]$ and suppose the transformation f is given by $f_a(x) = a - x^2$ for $a \in [a_0, 2]$ in the positive Lebesgue measure subset constructed by Benedicks and Carleson in [16, 17], where $a_0 \approx 2$. The properties of the family f_a have been thoroughly studied by a considerable number of people. We just mention that Freitas in [30] showed that for these parameters f_a is not only a non-uniformly expanding map with $\mathcal{S} = \mathcal{C} = \{0\}$ but also exhibits exponentially slow approximation to the singular set. Actually in [30] only *subexponentially* slow approximation is stated but the same arguments yield an exponential bound as well, as obtained in a much more delicate setting with infinitely many critical points in [8].

Moreover Bruin and Keller [24] show that for this class of maps (specifically for *Collet-Eckman maps*, i.e. such that $\liminf_{n\to\infty} \left|(f_a^n)'(a)\right|^{1/n} > 1$ without extra conditions of recurrence to the criticality) the unique absolutely continuous invariant probability measure is also the unique equilibrium state with respect to $\log |f_a'|$.

Therefore for any given suspension semiflow over such quadratic maps f_a with roof function having logarithmic growth near 0 we can apply Theorem A, and obtain a large deviation bound for these special flows.

2.4. **Lorenz and geometric Lorenz attractors.** The $C^{1+\alpha}$ map $f : [-1,1] \setminus \{0\} \to [-1,1]$ obtained as the quotient map of the Poincaré first return map R presented in Subsection 1.3 through projection along the leaves of the stable foliation satisfies the following conditions, which define a *Lorenz-like map*:

(1) there are constants $c > 0$ and $\sigma > 1$ such that for every $n \geq 1$ and for all $x \in [-1,1] \setminus \{0\}$ we have $(f^n)'(x) \geq c\sigma^n$;
(2) $f(0^+) = -1$, $f(0^-) = 1$, $f(1) \in (0,1)$ and $f(-1) \in (-1,0)$.

Note in particular that there are no critical points and that for some $n \geq 1$ the map $g = f^n$ is in the conditions of Subsection 2.2 so we can obtain a large deviation bound for g which easily gives a large deviation bound for f. To deduce Corollary C, since the reduction to a large deviation bound for the map R is the content of Section 4, all we need to do here is to explain how we deduce a large deviation bound for R from a similar bound for the map f. For this we strongly use the uniform contraction along the leaves of the stable foliation on the global cross-section S, as follows.

According to the construction of geometric Lorenz flows, there are $C > 0$ and $0 < \lambda < 1$ such that given $y \in [-1,1]$ and two distinct values $x_1, x_2 \in [-1,1] \setminus \{0\}$

$$\operatorname{dist}\Big(R^k(x_1,y,1), R^k(x_2,y,1)\Big) \leq C\lambda^k \text{ for all } 1 \leq k \leq n, \tag{2.1}$$

where $n \geq 1$ is the first time the orbit of the points hit the singular line, corresponding to the stable foliation of the singularity of the flow. These *hitting times* depend only on the orbit of y under the map f and correspond to times n for which $f^n(y) = 0$. But $X_0 = \cup_{n\geq 0} f^{-n}(\{0\})$ is denumerable thus the corresponding set of points in S, given by the lines $[-1,1]\times\{(y,1)\}$ for $y \in X_0$, has zero area on S. Therefore for a full Lebesgue measure subset of S we have (2.1) for all $k \geq 1$. The compactness of S enables us to obtain

Lemma 2.1. *Given a continuous $\psi : S \to \mathbb{R}$ and $\varepsilon > 0$ there exists $N > 1$ such that for all $k > N$, $y \in [-1,1] \setminus \{0\}$ and $x_1, x_2 \in [-1,1]$*

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} \Big(\psi(R^j(x_1,y,1)) - \psi(R^j(x_2,y,1) \Big) \right| < \varepsilon.$$

Consider the function $\varphi = \psi \mid \ell$, i.e. $\varphi(y) = \psi(0,y,1)$ for $-1 \leq y \leq 1$ and denote by $P : S \to [-1,1]$ the projection $(x,y,1) \mapsto y$. Note that $P\Big(R^j(0,y,1)\Big) = f^j(y)$ for all $j \geq 0$ and $y = [-1,1] \setminus X_0$. Hence by Lemma 2.1 the asymptotic time average of ψ along the R-orbit of $(x,y,1)$ coincides with the average of φ along the f-orbit of y, for a full Lebesgue measure subset and, crucially, the two averages are uniformly close for all sufficiently big number of iterates of the maps R and f. This is enough to conclude that for a given $\varepsilon > 0$ there exists $N \geq 1$ such that for $k > N$

$$\Big\{ \Big| \frac{1}{k} \sum_{j=0}^{k-1} \psi \circ R^j - \mu(\psi) \Big| > 2\varepsilon \Big\} \subseteq P^{-1} \Big\{ \Big| \frac{1}{k} \sum_{j=0}^{k-1} \varphi \circ f^j - \mu(\psi) \Big| > \varepsilon \Big\}. \tag{2.2}$$

This reduces the problem of estimating the Lebesgue measure of the left hand side set in (2.2) to the estimation of the measure of the right hand side set, transferring the problem to the dynamics of f, which is the subject of Subsection 2.2 and Section 6.

3. Large deviations for observables with logarithmic growth near singularities

The main bound on large deviations for suspension semiflows over a non-uniformly expanding base will be obtained from the following large deviation statement for non-uniformly expanding transformations.

Theorem 3.1. *Let $f : M \to M$ be a regular $C^{1+\alpha}$ local diffeomorphism on $M \setminus \mathcal{S}$ where $\mathcal{S}$ is a non-flat critical set. Assume that f is a non-uniformly expanding map, $0 < \alpha < 1$, with exponentially slow recurrence to the singular set $\mathcal{S}$ and let $\varphi : M \setminus \mathcal{S} \to \mathbb{R}$ be a continuous map which has logarithmic growth near $\mathcal{S}$. Moreover assume that there exists a unique equilibrium state μ with respect to* $\log J$ *which is absolutely continuous. Then for any given $\omega > 0$*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left\{x \in M : \left|\frac{1}{n} S_n\varphi(x) - \mu(\varphi)\right| \geq \omega\right\} < 0.$$

Proof. Fix $\varphi : M \setminus \mathcal{S} \to \mathbb{R}$ as in the statement of Theorem 3.1. Define

$$\varphi_k = \xi_k \circ \varphi \text{ where } \xi_k(x) = \begin{cases} k & \text{if } |x| \geq k \\ x & \text{if } |x| < k \end{cases}, \; k \geq 1.$$

Then $\varphi_k : M \to \mathbb{R}$ is continuous for all $k \geq 1$, $\varphi_k(x) \xrightarrow[k\to\infty]{} \varphi(x)$ for all $x \in M \setminus \mathcal{S}$ and $|\varphi - \varphi_k| = \big|\varphi\big|\chi_{\{|\varphi|>k\}}$. Moreover we clearly have for all $n, k \geq 1$

$$S_n\varphi_k - S_n\big|\varphi - \varphi_k\big| \leq S_n\varphi = S_n\varphi_k + S_n(\varphi - \varphi_k) \leq S_n\varphi_k + S_n\big|\varphi - \varphi_k\big| \tag{3.1}$$

Observe that for any given $c, \varepsilon_0 > 0$ we may choose $\varepsilon_1, \delta_1 > 0$ such that the exponential slow recurrence condition (1.2) is true and $K \cdot \varepsilon_1 \leq \varepsilon_0$. Then choose $k \geq 1$ very big so that $\{|\varphi| > k\} \subseteq B(\mathcal{S}, \delta_1)$. From (3.1) we obtain the following inclusions

$$\begin{aligned}\left\{\frac{1}{n}S_n\varphi > c\right\} &\subseteq \left\{\frac{1}{n}S_n\varphi_k + \frac{1}{n}S_n\big|\varphi - \varphi_k\big| > c\right\} \subseteq \left\{\frac{1}{n}S_n\varphi_k > c - K\varepsilon_1\right\} \cup \left\{\frac{1}{n}S_n\big|\varphi - \varphi_k\big| > K\varepsilon_1\right\} \\ &\subseteq \left\{\frac{1}{n}S_n\varphi_k > c - \varepsilon_0\right\} \cup \left\{\frac{1}{n}S_n\Delta_{\delta_1} \geq \varepsilon_1\right\} \end{aligned} \tag{3.2}$$

where in (3.2) we use the assumption that φ is of logarithmic growth near $\mathcal{S}$ and the choices of ε_1, δ_1. Analogously we get with opposite inequalities

$$\begin{aligned}\left\{\frac{1}{n}S_n\varphi < c\right\} &\subseteq \left\{\frac{1}{n}S_n\varphi_k - \frac{1}{n}S_n\big|\varphi - \varphi_k\big| < c\right\} \subseteq \left\{\frac{1}{n}S_n\varphi_k < c + K\varepsilon_1\right\} \cup \left\{\frac{1}{n}S_n\big|\varphi - \varphi_k\big| > K\varepsilon_1\right\} \\ &\subseteq \left\{\frac{1}{n}S_n\varphi_k < c + \varepsilon_0\right\} \cup \left\{\frac{1}{n}S_n\Delta_{\delta_1} \geq \varepsilon_1\right\} \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we see that *to get the bound for large deviations in the statement of Theorem 3.1 it suffices to obtain a large deviation bound for the continuous function φ_k with* *respect to the same transformation* f and *to have exponentially slow recurrence to the singular set* $\mathcal{S}$.

To prove this we use the following result already obtained for continuous observables over non-uniformly expanding transformations with slow recurrence to a non-flat singular set, see [7].

Theorem 3.2. *Let $f : M \to M$ be a local diffeomorphism outside a non-flat singular set $\mathcal{S}$ which is non-uniformly expanding and has slow recurrence to $\mathcal{S}$. For $\omega_0 > 0$ and a continuous function $\varphi_0 : M \to \mathbb{R}$ there exists $\varepsilon, \delta > 0$ arbitrarily close to 0 such that, writing*

$$A_n = \{x \in M : \frac{1}{n} S_n \Delta_\delta(x) \leq \varepsilon\} \quad \textit{and} \quad B_n = \left\{x \in M : \inf\left\{\left|\frac{1}{n} S_n \varphi_0(x) - \eta(\varphi_0)\right| : \eta \in \mathbb{E}\right\} > \omega_0\right\}$$

we get $\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left(A_n \cap B_n\right) < 0$.

Here $\mathbb{E} = \{\nu \in \mathcal{M}_f : h_\nu(f) = \nu(\log J)\}$ is the set of all equilibrium states of f with respect to the potential $\log J$.

Note that exponentially slow recurrence implies $\limsup_{n\to+\infty} \frac{1}{n} \mathrm{Leb}(M \setminus A_n) < 0$ so that under this assumption Theorem 3.2 ensures that for (ε, δ) close enough to $(0,0)$ we get $\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}(B_n) < 0$. To use this we also need that $\mathbb{E}$ *consists only of the unique absolutely continuous invariant probability measure* μ. Under this uniqueness assumption we have $\mathbb{E} = \{\mu\}$ in Theorem 3.2 and take $\omega, \varepsilon_0 > 0$ small, choose $k \geq 1$ as before, set $\varphi_0 = \varphi_k$ and $\omega_0 = \omega + \varepsilon_0$. In (3.2) set $c = \mu(\varphi_0) - \omega$ and in (3.3) set $c = \mu(\varphi_0) + \omega$. Then we have the inclusion

$$\left\{\left|\frac{1}{n} S_n \varphi - \mu(\varphi)\right| > \omega\right\} \subseteq \left\{\left|\frac{1}{n} S_n \varphi_0 - \mu(\varphi_0)\right| > \omega_0\right\} \cup \left\{\frac{1}{n} S_n \Delta_{\delta_1} \geq \varepsilon_1\right\}, \tag{3.4}$$

and by Theorem 3.2 we may find $\varepsilon, \delta > 0$ small enough so that the exponentially slow recurrence holds also for the pair (ε, δ) and hence

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left\{\left|\frac{1}{n} S_n \varphi_0 - \mu(\varphi_0)\right| > \omega_0\right\} < 0. \tag{3.5}$$

Finally the choice of $\varepsilon_1, \delta_1 >$ according to the exponential slow recurrence to $\mathcal{S}$ ensures that the Lebesgue measure of the right hand subset in (3.4) is also exponentially small when $n \to \infty$ which, together with (3.5), concludes the proof of Theorem 3.1. □

4. Large deviations and the dynamics on the base

Here we show how the large deviation bound for a semiflow over a non-uniformly expanding base, and with a roof function which grows at most logarithmically near the singular set, can be deduced from a large deviation bound for the base dynamics with respect to observables with the same type of logarithmic growth near the singular set.

4.1. **Reduction to the base dynamics.** We can write for a continuous and bounded $\psi : M_r \to \mathbb{R}$, $T > 0$ and $z = (x, s)$ with $x \in M$ and $0 \le s < r(x) < \infty$, and $n = n(x, s, T) \in \mathbb{N}$ such that $S_n r(x) \le s + T < S_{n+1} r(x)$ by definition of X^t

$$\int_0^T \psi\Big(X^t(z)\Big)\,dt = \int_s^{r(x)} \psi\Big(X^t(x,0)\Big)\,dt + \sum_{j=1}^{n-1} \int_0^{r(f^j(x))} \psi\Big(X^t(f^j(x),0)\Big)\,dt + \int_0^{T+s-S_n r(x)} \psi\Big(X^t(f^n(x),0)\Big)\,dt.$$

So if we set $\varphi(x) = \int_0^{r(x)} \psi(x,0)\,dt$ we obtain

$$\frac{1}{T}\int_0^T \psi\Big(X^t(z)\Big)\,dt = \frac{1}{T} S_n\varphi(x) - \frac{1}{T}\int_0^s \psi\Big(X^t(x,0)\Big)\,dt + \frac{1}{T}\int_0^{T+s-S_n r(x)} \psi\Big(X^t(f^n(x),0)\Big)\,dt.$$

Clearly we can bound the sum $I = I(x, s, T)$ of the last two integral terms from above by

$$I = I(x,s,T) \le \left(2\frac{s}{T} + \frac{S_{n+1}r(x) - S_n r(x)}{T}\right)\cdot \|\psi\|, \tag{4.1}$$

where $\|\psi\| = \sup|\psi|$. Observe that for a given $\omega > 0$ and for $0 \le s < r(x)$ and $n = n(x, s, T)$

$$\left\{(x,s) \in M_r : \left|\frac{1}{T}S_n\varphi(x) + I(x,s,T) - \frac{\mu(\varphi)}{\mu(r)}\right| > \omega\right\} \tag{4.2}$$

is contained in (where $z = (x, r)$)

$$\left\{z \in M_r : \left|\frac{1}{T}S_n\varphi(x) - \frac{\mu(\varphi)}{\mu(r)}\right| > \frac{\omega}{2}\right\} \cup \left\{z \in M_r : I(x,s,T) > \frac{\omega}{2}\right\}. \tag{4.3}$$

Observe that if $\psi \equiv 0$ then we need only consider the left hand subset of (4.3) in what follows. Now we bound the λ-measure of each subset above assuming that ψ is not identically zero.

4.2. **Using the roof function as an observable over the base dynamics.** We start with the right hand subset in (4.3). Take $N \ge 1$ big enough so that $N\|\psi\| > 2$ and note that for any given $T, \omega > 0$ using (4.1) and again $n = n(x, s, T)$

$$\begin{aligned}\lambda\Big\{I > \frac{\omega}{2}\Big\} &= \int d\,\mathrm{Leb}(x)\int_0^{r(x)} ds\,\Big(\chi_{(\omega/2,+\infty)} \circ I\Big)(x,s,T)\\ &\le \mathrm{Leb}\left\{r > \frac{\omega T}{2N\|\psi\|}\right\} + \frac{\omega T}{2N\|\psi\|}\,\mathrm{Leb}\left\{\frac{|S_{n+1}r - S_n r|}{T} > \frac{N\|\psi\| - 2}{2N\|\psi\|}\omega\right\}\end{aligned} \tag{4.4}$$

where in the right hand summand we restrict to points $x \in M$ such that $2N\|\psi\|r(x) \le \omega T$ and we have used

$$\frac{2s}{T} < \frac{2r}{T} \le \frac{\omega}{N\|\psi\|} \quad \text{and} \quad \frac{\omega}{2} - \frac{\omega}{N\|\psi\|} = \frac{N\|\psi\| - 2}{2N\|\psi\|}\cdot\omega.$$

On the one hand, since r grows as the logarithm of the distance to $\mathcal{S}$, we have that the left hand summand in (4.4) is bounded by

$$\mathrm{Leb}\left\{x \in \mathcal{S} : \mathrm{dist}\Big(x, \mathcal{S}\Big) \le \exp\Big(- C \cdot \frac{\omega T}{2N\|\psi\|}\Big)\right\} \le e^{-C\cdot\kappa\cdot\omega T\|/(2N\|\psi\|)} \tag{4.5}$$

where $C > 0$ is a constant depending on r only, and we use condition (S4) on the geometry of $\mathcal{S}$.

On the other hand, since r is bounded from below $r \ge r_0 > 0$ we have $T \ge r_0 n$ and so

$$\mathrm{Leb}\left\{\frac{|S_{n+1}r - S_n r|}{n} > \Big(\frac{N\|\psi\| - 2}{2N\|\psi\|} r_0\Big) \cdot \omega\right\} \quad \Big(\text{let } r_0' = \frac{N\|\psi\| - 2}{2N\|\psi\|} r_0\Big)$$
$$\le \mathrm{Leb}\left\{\left|\frac{1}{n} S_n r - \mu(r)\right| > \frac{\omega r_0'}{2}\right\} + \mathrm{Leb}\left\{\left|\frac{1}{n} S_{n+1} r - \mu(r)\right| > \frac{\omega r_0'}{2}\right\}.$$

Altogether we see that $\lambda\{I > \omega/2\}$ is bounded by twice the maximum of the summands in (4.4).

Now because we took r to be μ-integrable, continuous on $M \setminus \mathcal{S}$, with logarithmic growth near $\mathcal{S}$ and since f is a non-uniformly expanding map with exponentially slow recurrence to the non-flat singular set $\mathcal{S}$, then we have a large deviation bound for the observable r with respect to the unique physical measure μ for f (see Section 3). From this we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \lambda\Big\{I > \frac{\omega}{2}\Big\} < 0, \tag{4.6}$$

as long as we take $\omega > 0$ small enough.

4.3. **Using φ as an observable over the base dynamics.** Now for the right hand subset in (4.3), note first that for μ- and Leb-almost every $x \in M$ and every $0 \le s < r(x)$

$$\frac{S_n r(x)}{n} \le \frac{T + s}{n} \le \frac{S_{n+1} r(x)}{n} \quad \text{so} \quad \frac{n(x, s, T)}{T} \xrightarrow[T\to\infty]{} \frac{1}{\mu(r)}. \tag{4.7}$$

We also have, recall $n = n(x, s, T)$

$$\left|\frac{1}{T} S_n\varphi - \frac{\mu(\varphi)}{\mu(r)}\right| \le \left|\frac{n}{T} \cdot \frac{S_n\varphi}{n} - \frac{n}{T}\mu(\varphi)\right| + \left|\frac{n}{T}\mu(\varphi) - \frac{\mu(\varphi)}{\mu(r)}\right| \le \frac{n}{T}\left|\frac{S_n\varphi}{n} - \mu(\varphi)\right| + \mu(\varphi)\left|\frac{n}{T} - \frac{1}{\mu(r)}\right|.$$

Hence the left hand subset in (4.3) is contained in

$$\left\{x \in M : \frac{n}{T}\left|\frac{S_n\varphi}{n} - \mu(\varphi)\right| > \frac{\omega}{4}\right\} \cup \left\{x \in M : \left|\frac{n}{T} - \frac{1}{\mu(r)}\right| > \frac{\omega}{4\mu(\varphi)}\right\}. \tag{4.8}$$

Notice that the Lebesgue measure of the right hand subset of (4.8) satisfies a large deviation bound through the relation (4.7), for small enough $\omega > 0$, as in the previous subsection. Finally the left hand subset of (4.8) is contained in the following union

$$\left\{\left|\frac{T}{n} - \mu(r)\right| > \frac{\mu(r)}{2} \cdot \omega\right\} \cup \left\{\left|\frac{S_n\varphi}{n} - \mu(\varphi)\right| > \frac{\mu(r)}{2} \cdot \frac{\omega}{4}\right\},$$

and again for small $\omega > 0$ the Lebesgue measure of the left hand subset is exponentially small with T, as before, and for the right hand subset we use the large deviation bound for the observable φ with respect to f and Lebesgue measure, since φ has also logarithmic growth near $\mathcal{S}$: in fact $\left|\varphi(x)\right| \le \int_0^{r(x)} |\psi(x,s)|\,dt \le r(x)\cdot\|\psi\|$ for $x \in M\setminus\mathcal{S}$ because $\psi : M_r \to \mathbb{R}$ is bounded. From this we conclude

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda\left\{\left|\frac{1}{T}S_n\varphi - \frac{\mu(\varphi)}{\mu(r)}\right| > \frac{\omega}{2}\right\} < 0. \tag{4.9}$$

Putting (4.6) and (4.9) together, *as long as we have a result on large deviations for continuous observables in $M\setminus\mathcal{S}$ with logarithmic growth near $\mathcal{S}$, with respect to the dynamics of f and the Lebesgue measure*, and *the volume of neighborhoods of $\mathcal{S}$ is comparable to a power of the radius*, we are able to prove the Main Theorem for the suspension flow over f.

Since we have obtained the large deviation bounds needed for the base dynamics in Section 3, the proof of Theorem A is complete.

5. The Entropy Formula for non-uniformly expanding maps

Here we obtain the Entropy Formula when f is a non-uniformly expanding map with slow recurrence to the singular set, which may be formed by a combination of critical points of f and of points where f is either not defined, is not continuous or is not differentiable. Recall from the Introduction that $\psi = \log\|(Df)^{-1}\|$ and that $J = |\det Df|$.

Theorem 5.1. *Let $f : M \to M$ be a non-uniformly expanding map with slow recurrence to the non-flat singular set $\mathcal{S}$. Let $\mu \in \mathcal{M}_f$ be such that μ is f-ergodic, $h_\mu(f) = \mu(\log J)$, $-\infty < \mu(\psi) < 0$ and for every given $\varepsilon > 0$ there exists $\delta > 0$ so that $\mu(\Delta_\delta) < \varepsilon$. Then $\mu \ll \mathrm{Leb}$ and consequently $\mu \in \overline{\mathrm{co}}(\mathbb{F})$.*

Reciprocally, let $\mu \in \mathcal{M}_f$ be such that μ is absolutely continuous with respect to Leb *and assume that Δ_δ is μ-integrable. Then $h_\mu(f) = \mu(\log J)$.*

Clearly this is a particular case of the more general Entropy Formula obtained by Ledrappier and Young [39, 40] applied to maps with singularities and/or criticalities. For C^2 endomorphisms (i.e. smooth maps with criticalities but no singularities) see Bahnmüller and Liu [41, 14] for a general statement. A similar result for piecewise smooth one-dimensional maps with finitely many branches was obtained by Ledrappier [38].

As an easy corollary we deduce that the weak* closure of the convex hull $\overline{\mathrm{co}}(\mathbb{F})$ of the finite set $\mathbb{F}$ of ergodic physical probability measures is isolated among the set $\mathbb{E}$ of all equilibrium states of f with respect to $J = \log\left|\det Df\right|$, which might be of independent interest for the ergodic theory of non-uniformly expanding transformations.

Corollary 5.2. *Let $f : M \to M$ be a non-uniformly expanding map with slow recurrence to the non-flat singular set $\mathcal{S}$. Then there exists a weak* neighborhood $\mathcal{U}$ of $\overline{\mathrm{co}}(\mathbb{F})$ in $\mathcal{M}_f$ such that $\mathcal{U}\cap\mathbb{E} = \overline{\mathrm{co}}(\mathbb{F})$.*

Proof. Take any weak* neighborhood $\mathcal{U}$ of $\overline{\mathrm{co}}(\mathbb{F})$ such that every $\mu \in \mathcal{U}$ satisfies $\mu(\psi) < 0$. Hence every $\mu \in \mathcal{U}\cap\mathbb{E}$ satisfies the conditions of Theorem 5.1, thus $\mu \in \overline{\mathrm{co}}(\mathbb{F})$. □

Note that whenever the Entropy Formula and its reciprocal hold for measures close to $\mathbb{F}$ then the argument proving Corollary 5.2 is applicable and we deduce that $\mathbb{F}$ is isolated in $\mathbb{E}$. The proof of Theorem 5.1 is longer and occupies the rest of this section.

5.1. **Hyperbolic times.** Here we present some essential technical results for the study of non-uniformly expanding maps whose proof can be found in [46, 5, 1].

We say that n is a (σ, δ, b)-*hyperbolic time of* f for a point x if there are $0 < \sigma < 1$ and $b, \delta > 0$ such that $\prod_{j=n-k}^{n-1} \left\| Df\big(f^j(x)\big)^{-1} \right\| \leq \sigma^k$ and $d_\delta\big(f^k(x), \mathcal{S}\big) \geq e^{-bk}$ hold for all $k = 0, \ldots, n-1$.

We now outline the properties of these special times. For detailed proofs see [5, Proposition 2.8] and [3, Proposition 2.6, Corollary 2.7, Proposition 5.2].

Proposition 5.3. *There are constants $C_1, \delta_1 > 0$ depending on (σ, δ, b) and f only such that, if n is (σ, δ, b)-hyperbolic time of f for x, then there are* hyperbolic pre-balls *$V_k(x)$ which are neighborhoods of $f^{n-k}(x)$, $k = 1, \ldots, n$, such that*

(1) *$f^k \mid V_k(x)$ maps $V_k(x)$ diffeomorphically to the ball of radius δ_1 around $f^n(x)$;*
(2) *$d\big(f^{n-k}(y), f^{n-k}(z)\big) \leq \sigma^{k/2} \cdot d\big(f^n(y), f^n(z)\big)$ for every $1 \leq k \leq n$ and $y, z \in V_k(x)$;*
(3) *$C_1^{-1} \leq \big|\det Df^{n-k}(y)\big| / \big|\det Df^{n-k}(z)\big| \leq C_1$ for $y, z \in V_k(x)$.*

The following ensures existence of infinitely many hyperbolic times for μ-almost every point for non-uniformly expanding maps with slow recurrence to the singular set with respect to an ergodic invariant probability measure μ. A complete proof can be found in [5, Section 5].

Theorem 5.4. *Let $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism away from a non-flat singular set $\mathcal{S}$, for some $\alpha \in (0, 1)$, non-uniformly expanding and with slow recurrence to $\mathcal{S}$, with respect to an ergodic invariant probability measure μ, that is there exists $c > 0$ such that*

$$\limsup_{n \to +\infty} \frac{1}{n} S_n \psi \leq -c \quad \mu - \textit{almost everywhere}$$

and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{n} S_n \Delta_\delta(x) \leq \varepsilon \quad \mu - \textit{almost everywhere.}$$

Then there are $\sigma \in (0, 1)$, $\delta, b > 0$ and there exists $\theta = \theta(\sigma, \delta, b) > 0$ such that μ-a.e. $x \in M$ has infinitely many (σ, δ, b)-hyperbolic times. Moreover if we write $0 < n_1 < n_2 < n_2 < \ldots$ for the hyperbolic times of x then their asymptotic frequency satisfies $\liminf_{N \to \infty} \frac{\#\{k \geq 1 : n_k \leq N\}}{N} \geq \theta$ for Leb *-a.e. $x \in M$.*

5.2. **Existence of generating partition.** Let μ be an f-invariant ergodic probability measure in the conditions of the first part of the statement of Theorem 5.1.

Observe first that since $\mu(\psi) < 0$ and μ is ergodic, then f is non-uniformly expanding and moreover by the assumptions on $\mu(\Delta_\delta)$ we see that f has also slow recurrence to $\mathcal{S}$ with respect to μ. Hence by Theorem 5.4 there are $\sigma, \delta, b > 0$ such that μ-almost all $x \in M$ admits infinitely many (σ, δ, b)-hyperbolic times with positive frequency at infinity. Thus there exists a finite partition $\mathcal{P}_0$ of M which is generating with respect to μ.

Indeed let $\mathcal{E} = \{B(x_i, \delta_1/8), i = 1, \dots, l\}$ be a finite open cover of M by $\delta_1/8$-balls whose boundary has zero μ measure. From this we define a finite partition $\mathcal{P}_0$ of M as follows. Start by setting $P_1 = B(x_1, \delta_1/8)$ as the first element of the partition. Then, assuming that $P_1, \dots, P_k$ are already defined, set $P_{k+1} = B(x_{k+1}, \delta_1/8) \setminus (P_1 \cup \cdots \cup P_k)$ for $k = 1, \dots, l-1$. Note that if $P_k \neq \emptyset$ then P_k has non-empty interior, diameter smaller than $\delta_1/4$ and the boundary ∂P_k is a (finite) union of pieces of boundaries of balls in a Riemannian manifold, thus has zero Lebesgue measure and zero μ-measure also. Define $\mathcal{P}_0$ by the elements P_k constructed above which are non-empty. Note that $\mu(\partial\mathcal{P}_0) = \mathrm{Leb}(\partial\mathcal{P}_0) = 0$ and by the existence of infinitely many (σ, δ, b)-hyperbolic times for μ-almost every x it is not difficult to see that $\operatorname{diam}\Big(\bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}_0(x)\Big) \xrightarrow[n\to+\infty]{} 0$.

Therefore since μ satisfies the Entropy Formula we can write

$$\frac{1}{n}\int S_n \log J\,d\mu = \mu(\log J) = h_\mu(f) = h_\mu(f, \mathcal{P}_0) \le \frac{1}{n}H_\mu(\mathcal{P}_n) = \frac{1}{n}\int -\log\mu\big(\mathcal{P}_n(x)\big)\,d\mu$$

where $\mathcal{P}_n = \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}_0$ for $n \ge 1$. Hence by Jensen's Inequality we get, denoting $J_n(x) = \prod_{j=0}^{n-1} J\big(f^j(x)\big)$

$$0 \ge \int \log\Big[J_n(x)\cdot\mu\big(\mathcal{P}_n(x)\big)\Big]\,d\mu(x) \ge \log\int J_n(x)\cdot\mu\big(\mathcal{P}_n(x)\big)\,d\mu(x).$$

Thus defining $Q_\gamma^n = \{x \in M : S_nJ(x)\cdot\mu\big(\mathcal{P}_n(x)\big) > \gamma\}$ we obtain

$$\mu(Q_\gamma^n) \le \gamma^{-1} \quad \text{for all} \quad n \ge 1. \tag{5.1}$$

Now choose $\gamma_n > 0$ such that $\sum_n \gamma_n^{-1} < \infty$. Then for μ-almost every $x \in M$ there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $x \notin Q_{\gamma_n}^n$, i.e. $J_n(x)\cdot\mu\big(\mathcal{P}_n(x)\big) \le \gamma_n$ for all $n \ge n_0 = n_0(x)$. Observe that by the definition and properties of hyperbolic times, we have that there exists $C_1 > 0$ such that

$$C_1^{-1}\cdot\mathrm{Leb}\Big(\mathcal{P}_0(f^n(x))\Big) \le \mathrm{Leb}\Big(\mathcal{P}_n(x)\Big)\cdot J_n(x) \le C_1\cdot\mathrm{Leb}\Big(\mathcal{P}_0(f^n(x))\Big)$$

whenever n is a hyperbolic time for x. This shows that the μ-measure of the atoms of $\mathcal{P}_n$ can be bounded from above by the volume of the same atoms at big enough hyperbolic times

$$\mu\big(\mathcal{P}_n(x)\big) \le C_0\gamma_n\,\mathrm{Leb}(\mathcal{P}_n(x)), \tag{5.2}$$

where $C_0 = C_1\sup_{x\in M}\mathrm{Leb}\Big(\mathcal{P}_0(x)\Big)$. The hyperbolic times satisfying this condition will be called *μ-hyperbolic times*. To use this we need some way to cover any set using atoms of the sequence $(\mathcal{P}_n)_n$ at μ-hyperbolic times.

5.3. **Coverings by hyperbolic times.** Let μ, f and $\Big(\mathcal{P}_n\Big)_{n\ge0}$ be as in the previous subsection. Note that since f is regular and μ is f-invariant the boundary of $g(P)$ still has zero Lebesgue measure and zero μ-measure for every atom $P \in \mathcal{P}_0$ and every inverse branch g of f^n, for any $n \ge 1$.

We can now state the following flexible covering lemma with μ-hyperbolic preballs which will enable us to approximate the μ-measure of a given set through the measure of families of μ-hyperbolic preballs.

Lemma 5.5 (The Hyperbolic Covering Lemma)**.** *Let a measurable set $E \subset M$, $m \geq 1$ and $\zeta > 0$ be given with $\mu(E) > 0$. Let $\theta > 0$ be a lower bound for the density of μ-hyperbolic times for μ-almost every point. Then there are integers $m < n_1 < \cdots < n_k$ for $k = k(\zeta) \geq 1$ and families $\mathcal{E}_i$ of subsets of M, $i = 1, \ldots, k$ such that*

(1) *$\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ is a finite pairwise disjoint family of subsets of M;*
(2) *n_i is a $(\sigma/2, \delta/2)$-μ-hyperbolic time for every point in P, for every element $P \in \mathcal{E}_i$, $i = 1, \ldots, k$;*
(3) *every $P \in \mathcal{E}_i$ is the preimage of some element $Q \in \mathcal{P}$ under an inverse branch of f^{n_i}, $i = 1, \ldots, k$;*
(4) *there is an open set $U_1 \supset E$ containing the elements of $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$ with $\mu(U_1 \setminus E) < \zeta$;*
(5) *$\mu\Big(E \triangle \bigcup_i \mathcal{E}_i\Big) \leq \Big(1 - \frac{\theta}{4}\Big)^k < \zeta$.*

The proof is completely presented in [7, Lemma 3.5] and follows [44, Lemma 8.2] closely.

5.4. **Absolute continuity.** We are now ready to deduce that any measure μ as in the statement of Theorem 5.1 is absolutely continuous. Indeed observe that, by (5.1) and the choice of $(\gamma_n)_{n\geq1}$, for any given $\eta > 0$ we can find $N = N(\eta) \in \mathbb{N}$ such that $\Gamma_\eta = \cap_{n\geq N}\Big(M \setminus Q^n_{\gamma_n}\Big)$ satisfies $\mu(\Gamma_\eta) \geq 1 - \eta$.

Now let $E \subset M$ be given with $\mu(E \cap \Gamma_\eta) > 0$ and let $m = N$ in the statement of the Covering Lemma 5.5 and set $\zeta > 0$ small. Then we get $\mu\Big((E \cap \Gamma_\eta) \triangle \cup_{i=1}^k \mathcal{E}_i\Big) < \zeta$ where all elements of $\mathcal{E}_i$ are μ-preballs and atoms of $\mathcal{P}_{n_i}$ satisfying the bound (5.2). In particular by the choice of m we have $\cup_i \mathcal{E}_i \subset \Gamma_\eta$ and so we may write

$$\mu(E) = \mu\Big(E \cap M \setminus \Gamma_\eta\Big) + \mu\Big(E \cap \Gamma_\eta\Big) \leq \eta + \zeta + \mu\Big(E \cap \cup_i \mathcal{E}_i\Big) \leq \eta + \zeta + C_0 \gamma_{n_k} \operatorname{Leb}\Big(E \cap \cup_i \mathcal{E}_i\Big), \tag{5.3}$$

where n_k is the largest μ-hyperbolic time used in the cover given by the Hyperbolic Covering Lemma.

Hence if we start with a subset Z with $\operatorname{Leb}(Z) = 0$ and assume that $\mu(Z) > 0$, then we set $E = B(Z, \rho)$ with $\rho > 0$ such that $\operatorname{Leb}(E) > 0$ and take $U_1 = B(Z, 2\rho)$ satisfying $\mu(U_1 \setminus E) < \zeta$. Note that now the Covering Lemma provides a cover such that $\mu\Big(E \triangle \cup_i \mathcal{E}_i\Big) < \zeta$ and moreover $\mu(U_1 \setminus \cup_i \mathcal{E}_i) \leq \mu(U_1 \setminus E) + \mu\Big(E \triangle \cup_i \mathcal{E}_i\Big) < 2\zeta$. This shows that if we let $\rho \to 0$ we still get that $\mu(B(Z, \rho) \setminus \cup_i \mathcal{E}_i) \leq \mu(U_1 \setminus \cup_i \mathcal{E}_i) < 2\zeta$.

Observe also that since $\mu(Z) > 0$ there exists η_0 such that $\mu(E \cap \Gamma_\eta) > 0$ for all $0 < \eta \leq \eta_0$. Therefore fixing $\rho, \eta, \zeta > 0$ as above we obtain (5.3). But then we may let $\rho \to 0$ and still get $\mu(Z) \leq \eta + 2\zeta$, for all $0 < \eta \leq \eta_0$, that is $\mu(Z) \leq 2\zeta$. This is a contradiction since we may take $\zeta > 0$ as small as we like.

We have shown that if $\mathrm{Leb}(Z) = 0$ then $\mu(Z) = 0$, i.e. $\mu \ll \mathrm{Leb}$. Then since the basins of the physical measures of f cover M except for a volume zero subset, then it follows easily by the Ergodic Theorem that $\mu = \sum_{i=1}^{k} \mu\Big(B(\mu_i)\Big) \cdot \mu_i$, that is $\mu \in \overline{\mathrm{co}}(\mathbb{F})$.

Reciprocally, let us now assume that μ is an f-invariant absolutely continuous probability measure. Then as above we have $\mu \in \overline{\mathrm{co}}(\mathbb{F})$ and and thus for some constants $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$ we have $h_\mu(f) = \sum_{i=1}^{k} \alpha_i h_{\mu_i}(f) = \sum_{i=1}^{k} \alpha_i \mu_i(\log J) = \mu(\log J)$. This concludes the proof of Theorem 5.1.

6. Exponentially slow approximation to singularities

Let $f : M \to M$ be a $C^{1+\alpha}$ piecewise expanding map with at most countably many smoothness domains for some $\alpha \in (0,1)$ as in Subsection 2.2, that is $|f'| \geq \sigma > 1$ and the non-degenerate singular set $\mathcal{S}$ equals the boundaries of the smoothness domains and satisfies all the conditions (S1) through (S6). Then $\mathcal{S} = \{b_n\}_n$ where we may assume that the sequence is strictly monotonous (in counter-clockwise order if $M = \mathbb{S}^1$).

We consider the middle points $c_n = (b_n + b_{n+1})/2$ for all applicable indexes n to define a Lebesgue modulo zero partition $\mathcal{P}_0$ of M as follows.

6.1. **Initial partition.**

Partition (b_n, c_n) into subintervals

$$M(2n, p) = \Big(b_n + d_{2n}e^{-p}, b_n + d_{2n}e^{-(p-1)}\Big), \tag{6.1}$$

where $d_{2n} = c_n - b_n$ and partition the interval (c_{n-1}, b_n) into the following subintervals

$$M(2n-1, p) = \Big(b_n - d_{2n-1}e^{-(p-1)}, b_n - d_{2n-1}e^{-p}\Big) \tag{6.2}$$

where $d_{2n-1} = b_n - c_{n-1}$, for all $p \geq 1$. The sets defined above form a partition of M Lebesgue modulo zero consisting of small intervals whose length is exponentially small with respect to the distance to $\mathcal{S}$. Let $\mathcal{S}' = \mathcal{S} \cap f(M)$ be the set of singular points of f which matter for the asymptotic dynamics of f.

To define the initial partition consider a threshold $\rho_0 \in \mathbb{N}$ such that

$$e^{-\beta\rho_0} < 1 \quad \text{and} \quad \left(1 + \frac{2}{\rho_0}\right)\left(1 + \frac{\rho_0}{2}\right)^{2/\rho_0} < e^{\beta} \tag{6.3}$$

and let $\mathcal{P}_0$ be formed by the collection of all intervals $M(n,p)$ for all n and every $p \geq \rho_0$ together with the connected components of $M \setminus \Big(\cup_{n;p\geq\rho_0} M(n,p) \cup \{c_n\}_n\Big)$, which will be denoted by $M(n, \rho_0 - 1)$ whenever they are adjacent to $M(n, \rho_0)$.

For each element η of $\mathcal{P}_0$ denote by η^+ the interval obtained by joining η with its two neighboring intervals in $\mathcal{P}_0$. From (6.1) and (6.2) we have the following relations for all k and every $p \geq \rho_0 - 1$

$$\mathrm{Leb}(M(k,p)^+) \leq 9\,\mathrm{Leb}(M(k,p)) = 9d_k \cdot e^{-p}(e-1). \tag{6.4}$$

6.2. **Refining the partition.** The partition $\mathcal{P}_0$ is dynamically refined so that any pair x, y of points in the same atom of the nth refinement $\mathcal{P}_n$ belong to the same element of $\mathcal{P}_0$ during the first consecutive n iterates: $\mathcal{P}_0(f^i(x)) = \mathcal{P}_0(f^i(y))$ for $i = 0, \dots, n-1$ and moreover $f^n \mid \omega$ is a diffeomorphism for every interval $\omega \in \mathcal{P}_n$.

The refinement is defined inductively. Assume that $\mathcal{P}_n$ is already defined and for each $\omega \in \mathcal{P}_n$ there are sets $R_n(\omega)$ of splitting times and $D_n(\omega)$ of corresponding splitting depths, to be defined below.

If $f^{n+1}(\omega)$ intersects three or fewer elements of $\mathcal{P}_0$, then we set $\omega \in \mathcal{P}_{n+1}$, $R_{n+1}(\omega) = R_n(\omega)$ and $D_{n+1}(\omega) = D_n(\omega)$. Otherwise consider the subsets $\eta' = \left(f^{n+1} \mid \omega\right)^{-1}(\eta)$ of the interval ω for all elements η of $\mathcal{P}_0$ which intersect $f^{n+1}(\omega)$.

The family $\{\eta'\}$ obtained above is a partition of ω. Observe that $f^{n+1}(\eta')$ is either equal to some $\eta \in \mathcal{P}_0$ or strictly contained in some $\eta \in \mathcal{P}_0$. In the latter case we redefine the partition joining some of the extreme intervals of $\{\eta'\}$ with its neighbors so that the new partition $\{\zeta\}$ of ω satisfies: for each ζ there exists $\eta = M(k,p) \in \mathcal{P}_0$ such that $\eta \subseteq f^n(\zeta) \subseteq \eta^+$.

Finally we set $\zeta \in \mathcal{P}_{n+1}$, $R_{n+1}(\zeta) = R_n(\zeta) \cup \{n+1\}$ and $D_{n+1}(\zeta) = D_n(\zeta) \cup \{(k,p)\}$, for each element of the partition $\{\zeta\}$ of ω constructed above. For these elements of $\mathcal{P}_{n+1}$ we say that $n+1$ is a *splitting time* and the pairs (k,p) are the corresponding *splitting depths*. Repeat the procedure for each $\omega \in \mathcal{P}_n$. This completes the construction of $\mathcal{P}_{n+1}$ from $\mathcal{P}_n$ for all $n \geq 0$.

6.3. **Bounded distortion.** The uniform expansion of length during n iterates ensures that we have bounded distortion of lengths on atoms of the partition $\mathcal{P}_n$.

Indeed let $\omega \in \mathcal{P}_n$ for some $n \geq 1$ and let $x, y \in \omega$. Then since $f^i \mid \omega$ is a diffeomorphism for $i = 1, \dots, n$, f expands distances at a minimum rate of σ and f' is α-Hölder, there exist constants $C, D > 0$ such that

$$\log\left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| = \sum_{i=0}^{n-1} \Big|\log|f'(f^i(x))| - \log|f'(f^j(y)|\Big| \leq \sum_{i=0}^{n-1} C \cdot \frac{\left|f^i(x) - f^j(y)\right|^\alpha}{\max\{|f'(f^i(x))|, |f'(f^i(y))|\}}$$

$$\leq \frac{C}{\sigma} \sum_{i=0}^{n-1} \sigma^{i-n} \cdot \left|f^n(x) - f^n(y)\right|^\alpha \leq D, \tag{6.5}$$

where D depends only on σ and on the diameter of M.

6.4. **Measure of atoms of $\mathcal{P}_n$ and return depths.** Here we show that we can estimate the measure of an element of $\mathcal{P}_n$ using the information stored in R_n and D_n.

For any given $n \geq 1$ and $\omega \in \mathcal{P}_n$ we have a sequence of times $R_n(\omega) = \{r_1 < \cdots < r_s\}$ with $r_1 \geq 1$ and $r_s \leq n$, and a sequence of intervals $\omega_0 \supsetneq \omega_1 \supsetneq \cdots \supsetneq \omega_s = \omega$ with depths $D_n(\omega) = \{(k_1,p_1), \dots, (k_s,p_s)\}$, where $\omega_0 \in \mathcal{P}_0$ and $\omega_i \in \mathcal{P}_{r_i} \cap \cdots \cap \mathcal{P}_{r_{i+1}-1}$ such that

$$M(k_i,p_i) \subseteq f^{r_i}(\omega_{i-1}) \subseteq M(k_i,p_i)^+ \tag{6.6}$$

for all $i = 0, 1, \dots, s-1$ with $r_0 = 0$ and $s_0 = 0$. These times are the iterates where the images of the previous element of the partition was broken into smaller intervals as in

Subsection 6.2. Using the bounded distortion given by (6.5) we get

$$\frac{\operatorname{Leb}(\omega)}{\operatorname{Leb}(\omega_0)} = \frac{\operatorname{Leb}(w_s)}{\operatorname{Leb}(\omega_{s-1})}\cdots\frac{\operatorname{Leb}(\omega_1)}{\operatorname{Leb}(\omega_0)} \le \prod_{i=1}^{s} D\frac{\operatorname{Leb}\left(f^{r_i}(\omega_i)\right)}{\operatorname{Leb}\left(f^{r_i}(\omega_{i-1})\right)}.$$

Now using (6.6) and (S5) we bound the last expression from above by

$$\prod_{i=1}^{s}\frac{D\operatorname{Leb}\left(M(k_i,p_i)^+\right)}{B^{-1}e^{\beta p_{i-1}}d_{k_{i-1}}^{-\beta}(e-1)^{-\beta}\operatorname{Leb}\left(M(k_{i-1},p_{i-1})\right)}$$

and using (6.4) this can be easily simplified yielding

$$\operatorname{Leb}(\omega) \le \prod_{i=0}^{s-1} d_{k_i}^{\beta}e^{-2\beta p_i} \le \exp\left(-\beta\sum_{i=0}^{s-1}(p_i+q_i)\right) \tag{6.7}$$

where $q_i = [-\log d_{k_i}]$ with $[z] = \max\{k\in\mathbb{Z} : k\le z\}$. We have used $p\ge\rho_0$ and $\log(9BD(e-1)^\beta)/\rho_0 \le \beta$ to compensate the constants on the exponent 2β. Recall also that $\omega_0 = M(k_0,p_0)$. Note also that if $R_n(\omega)=\emptyset$ then since there is no splitting but there is uniform expansion together with distortion control, we get

$$\operatorname{Leb}\left(f^n(\omega)\right) = \int_\omega \left|(f^n)\right| d\operatorname{Leb} \ge D\sigma^n\operatorname{Leb}(\omega) \text{ so } \operatorname{Leb}(\omega)\le D^{-1}\sigma^{-n}. \tag{6.8}$$

6.5. **Distance to $\mathcal{S}$ and splitting depths.** Let again $n\ge 1$ and $\omega\in\mathcal{P}_n$ be given and consider the sets $R_n(\omega)$ and $D_n(\omega)$. Consider the intervals $\omega_0\supsetneq\omega_1\supsetneq\cdots\supsetneq\omega_s=\omega$ as before. Note that for the iterates i between two consecutive times $r<r'$ from R_n, i.e. if $r<i<r'$ then there exists $M(l_i,q_i)\in\mathcal{P}_0$ such that $f^i(\omega_r)\subseteq M(l_i,q_i)^+$ by this choice of i. Moreover by condition (S5) and by (6.1) and (6.2) we deduce

$$\begin{aligned}
9d_{l_1}(e-1)e^{-q_1} &\ge \operatorname{Leb}\left(f^{r+1}(\omega_r)\right) \ge \left(Bd_{k_r}e^{-p_r}\right)^{-\beta}\operatorname{Leb}\left(f^r(\omega_r)\right)\\
&= \left(\frac{B}{e-1}\operatorname{Leb}\left(f^r(\omega_r)\right)\right)^{-\beta}\operatorname{Leb}\left(f^r(\omega_r)\right)\\
&= \left(\frac{e-1}{B}\right)^{\beta}\operatorname{Leb}\left(f^r(\omega_r)\right)^{1-\beta}
\end{aligned}$$

so that $d_{l_1}e^{-q_1-1}\ge\left(9e(e-1)\right)^{-1}\operatorname{Leb}\left(f^{r+1}(\omega_r)\right)$ is the estimate for the minimum distance from $\mathcal{S}$ to $f^{r+1}(\omega_r)$. Let $L_i=\operatorname{Leb}\left(f^{r+i}(\omega_r)\right)$ and $D_i=\operatorname{dist}\left(f^{r+i}(\omega_r),\mathcal{S}\right)$ for $i=0,\dots,r'-r-1$. Then the reasoning above shows that $L_1\ge\left(\frac{e-1}{B}\right)^{\beta}L_0^{1-\beta}$ and $L_{i+1}\ge\left(\frac{9e(e-1)}{B}\right)^{\beta}L_i^{1-\beta}$, and also

$D_i \geq L_i / \big(9e(e-1)\big)$ for $i = 1, \ldots, r' - r - 1$. It is now easy to see that

$$\begin{aligned} -\log L_{i+1} &\leq -(1-\beta)\log L_i + \beta \log \frac{9e(e-1)}{B} \\ &= -\Big(1 - \beta - \beta \log\Big(\frac{9e(e-1)}{B}\Big)\Big/\log L_i\Big)\log L_i = -\gamma \log L_i \end{aligned}$$

where we may assume that $\gamma \in (0,1)$ since it is no restriction to increase the value of B if needed. Hence

$$\begin{aligned} -\sum_{i=1}^{r'-r-1} \log \operatorname{dist}\Big(f^{r+i}(\omega_r), \mathcal{S}\Big) &\leq -\sum_{i=1}^{r'-r-1} \Big(\log L_i - \log\Big(9e(e-1)\Big)\Big) \\ &\leq -\mathrm{Const} \cdot \log L_0 + (r' - r)\log\Big(9e(e-1)\Big) \\ &\leq -\mathrm{Const} \cdot \log L_0, \end{aligned}$$

since by uniform expansion and by definition of r' we have $\sigma^{r'-r} L_0 \leq 1$ and thus $r' - r \leq -\log(L_0)/\log\sigma$. Since $r < r'$ where two arbitrary consecutive elements of $R_n(\omega)$ for $\omega \in \mathcal{P}_n$ we have shown that

$$\sum_{j=0}^{s-1} -\log \operatorname{dist}\Big(f^j(x), \mathcal{S}\Big) \leq -\mathrm{Const} \sum_{(k,p)\in D_s(\omega)} \log\Big(d_k e^{-p}\Big) \tag{6.9}$$

for all $x \in \omega$, where $s < n$ is the last splitting time before n, that is $s = \max R_n(\omega)$. However if $m > n$ is the first integer such that $\omega \notin \mathcal{P}_m$ but $\omega \in \mathcal{P}_l$ for $n < l < m$, then we can write the following disjoint union $\omega = \bigcup_{\omega' \in \mathcal{P}_m} \omega' \cap \omega$. Repeating the argument for $x \in \omega' \cap \omega$ for each $\omega' \in \mathcal{P}_m$ intersecting ω we can obtain a relation like (6.9) with $D_s(\omega)$ replaced by $D_n(\omega)$ as the summation range, where n is between s and m. This shows that *the average of the* log *of the distance to the singular set is bounded by the sum of the depths at splitting times modulo a constant*.

6.6. **Expected value of splitting depths.** Now we estimate the expected value of the splitting depths for deep splitting times up to n iterates of the map. Define for a co-countable set of $x \in M$ the function $\mathcal{D}_n(x) = -\sum_{(k,p)\in D_n(\mathcal{P}_n(x))} \log(d_k e^{-p})$ where $\mathcal{P}_n(x)$ is the unique atom of $\mathcal{P}_n$ which contains $x \in M$. Define also the truncated sum: for any given $\delta > 0$ set for the same points $x \in M$ as above

$$\mathcal{D}_n^\delta(x) = \sum_{\substack{(k,p)\in D_n(\mathcal{P}_n(x)) \\ d_k e^{-p} < \delta}} -\log(d_k e^{-p}). \tag{6.10}$$

By the arguments in Subsection 6.5 and by the definitions (6.1) and (6.2) we obtain

$$\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta\Big(f^j(x), \mathcal{S}\Big) \leq \mathcal{D}_n^\delta(x). \tag{6.11}$$

Define the *number of splittings up to the nth iterate* $t_n(x) = \#R_n\Big(\mathcal{P}_n(\omega)\Big)$ and also the *number of deep splittings among these* $u_n(x) = \#\Big\{(k,p) \in R_n\Big(\mathcal{P}_n(\omega)\Big) : d_k e^{-p} < \delta\Big\}$.

Given x and $n \geq 1$ we let $0 = r_0 < r_1 < \cdots < r_t$ with $t = t_n(x)$ be the splitting times along the orbit of x up to the nth iterate and $0 \leq s_1 < \cdots < s_u$ be indexes corresponding to deep splittings, where $u = u_n(x)$ in what follows. Note that each quantity above is constant on the elements of $\mathcal{P}_n$. Define

$$A^{u,t}_{(\kappa_1,\rho_1),\dots,(\kappa_u,\rho_u)}(n) = \Big\{x \in M : t_n(x) = t,\ u_n(x) = n \text{ and } (k_{s_i}, p_{s_i}) = (\kappa_i, \rho_i),\ i = 1, \dots, u\Big\}$$

the set of points which in n iterates have t splitting times and u deep splittings among these, with the specified depths $(\kappa_1, \rho_1), \dots, (\kappa_u, \rho_u)$.

Lemma 6.1. Leb $\left(A^{u,t}_{(\kappa_1,\rho_1),\dots,(\kappa_u,\rho_u)}(n)\right) \leq \binom{t}{u} \exp\left(-\beta \sum_{i=1}^{u} (\eta_i + \rho_i)\right)$ *where* $\eta_i = [-\log d_{\kappa_i}]$.

Proof. Using the estimate (6.7) we get the following bound for the Lebesgue measure of $A^{u,t}_{(\kappa_1,\rho_1),\dots,(\kappa_u,\rho_u)}(n)$

$$\binom{t}{u} \exp\left(-\beta \sum_{i=1}^{u} (\eta_i + \rho_i)\right) \cdot \exp\Big(-\beta \sum_{\substack{(k_j,p_j) \text{ s.t. } d_k e^{-p_j} \geq \delta \\ j=1,\dots,t-u}} (\nu_j + p_j)\Big)$$

where the binomial coefficient takes into account all the possible orderings of sequences of u deep splitting times among t splitting times and the last exponential bounds the contribution of all the possible $t - u$ non-deep splitting times, with $\nu_j = [-\log d_{k_j}]$. But since $p \geq \rho_0$ was chosen as in (6.3) and $\nu_j \geq 0$ we conclude that the last exponential is smaller than 1. So we obtain the bound in the statement. □

Lemma 6.2. *For any* $z > \beta$ *we have* $\int e^{z\mathcal{D}^\delta_n(x)}\, dx \leq e^{\theta(\delta)n}$ *where* $\theta(\delta)$ *is such that* $\theta(\delta) \searrow 0$ *when* $\delta \searrow 0$.

Proof. By definition

$$\int e^{z\mathcal{D}^\delta_n(x)}\, dx = \sum_{\omega \in \mathcal{P}_n} e^{z\mathcal{D}^\delta_n(\omega)} \cdot \text{Leb}(\omega) \leq \sum_{\substack{\omega_0 \in \mathcal{P}_0 \\ D\sigma^n \text{ Leb}(\omega_0) \leq 1}} \text{Leb}(\omega_0)$$
$$+ \sum_{0 < u \leq t < n} \sum_{\substack{(\kappa_i,\rho_i) \\ i=1,\dots,u}} e^{z\mathcal{D}^\delta_n(\omega)} \text{Leb}\left(A^{u,t}_{(\kappa_1,\rho_1),\dots,(\kappa_u,\rho_u)}(n)\right) \tag{6.12}$$

where we are considering all possible combinations of splitting depths and of deep splittings among all the splitting times for all elements of $\mathcal{P}_n$ in the second sum.

Consider the first term corresponding to the atoms of $\mathcal{P}_0$ which were not split during the first n iterates. Notwithstanding this sum can be separated as follows

$$\sum_{\substack{\omega_0\in\mathcal{P}_0\\ D\sigma^n\,\mathrm{Leb}(\omega_0)\le 1}} \mathrm{Leb}(\omega_0) = \sum_{\substack{D\sigma^n\,\mathrm{Leb}(\omega_0)\le 1\\ d_k\sigma^{n/2}<1}} \mathrm{Leb}(\omega_0) + \sum_{\substack{D\sigma^n\,\mathrm{Leb}(\omega_0)\le 1\\ d_k\sigma^{n/2}\ge 1}} \mathrm{Leb}(\omega_0) \le \mathrm{Leb}\left(B(\mathcal{S},\sigma^{-n/2})\right) + \sum_{p>\log\left(D(e-1)\sigma^{n/2}\right)} e^{-p} \le Ce^{-cn} \tag{6.13}$$

for some constants $C, c > 0$, where we have used expression (6.4) for the length of the atoms of $\mathcal{P}_0$ in terms of (k,p) together with condition (S4) and the obvious $d_k > 0$ and $\sum_k d_k = 1$. Note that if $\mathcal{S}$ is finite then the condition $d_k\sigma^{n/2} < 1$ is always false for big enough n so that in this case we only have the right hand side sum above.

Now we bound the second term (6.12). Considering Lemma 6.1 and taking into account $\mathcal{D}_n^\delta$ we obtain (with $\eta_j = [-\log d_{\kappa_j}]$)

$$\sum_{0<u\le t<n} \sum_{\substack{(\kappa_i,\rho_i)\\ i=1,\dots,u}} \binom{t}{u} e^{-(\beta+z)\sum_i(\eta_i+\rho_i)} \le \sum_{0<u\le t<n} \sum_{h>u\ell(\delta)} \binom{t}{u} uL(h,u)e^{-(\beta+z)h}$$

where $h = \sum_i(\eta_i+\rho_i)$, $\ell(\delta)$ is an integer such that every pair (k,p) satisfying $d_k e^{-p} < \delta$ also satisfies $k + p > \ell(\delta)$, and

$$L(h,n) = \#\left\{\left((\eta_i,\rho_i)\right)_{i=1,\dots,u} \in \mathbb{N}_0^{2u} : \sum_{i=1}^{u}(\eta_i+\rho_i) = h \text{ with } \rho_i \ge \rho_0\right\}.$$

Moreover the factor u bounds the number of distinct d_{k_i} with the same value η_i along the n iterates of the orbit of the points. Observe that

$$L(h,u) \le \#\left\{(h_i) \in \mathbb{N}_0^{2u} : \sum_{i=0}^{2u} h_i = h\right\} = \binom{h+2u-1}{2u-1}$$

and by a standard application of Stirling's Formula

$$L(h,n) \le \left(c^{1/h}\Big(1+\frac{2u-1}{h}\Big)\Big(1+\frac{h}{2u-1}\Big)^{(2u-1)/h}\right)^h \le e^{\beta h} \le e^{zh}$$

where $0 < c < 1$ is a constant independent of the other variables and the last inequalities follow by $h \ge \rho_0 u$, by the choice of ρ_0 in (6.3) and by taking $z > \beta$.

Collecting the bounds we have obtained we conclude that the second sum in (6.12) can be bounded by the following expression

$$\begin{aligned}\sum_{0<u\le t<n} \binom{t}{u} u \sum_{h>u\ell(\delta)} e^{-\beta h} &\le \sum_{u=0}^{n} n\binom{n-1}{u} \cdot ue^{-\beta u\ell(\delta)/2} \cdot \frac{e^{-\beta u\ell(\delta)/2}}{1-e^{-\beta}} \le \sum_{u=0}^{n} \binom{n}{u} C \frac{\left(e^{-\beta\ell(\delta)/2}\right)^u}{1-e^{-\beta}}\\ &\le \left(1+\frac{C}{1-e^{-\beta}}e^{-\beta\ell(\delta)/2}\right)^n\end{aligned}$$

for some constant $C > 0$ bounding $\{ue^{-\beta u\ell(\delta)/2}\}_{u\geq 0}$ which can be taken independently of $\ell(\delta)$. Finally since $\ell(\delta)$ grows without limit when $\delta \searrow 0$, the statement of the lemma follows just by increasing the value of C to take into account the small bound of the first sum (6.13). □

6.7. **Measure of the points with bad recurrence.** We are now ready to deduce exponentially slow approximation to the singular set $\mathcal{S}$. Indeed we just have to use Tchebishev's inequality, as follows: given $\varepsilon, \delta > 0$ we know there exists a constant $C > 0$ as in Subsection 6.5 such that

$$\left\{x \in M : -\frac{1}{n}\sum_{i=0}^{n-1} \log \operatorname{dist}_\delta\left(f^i(x), \mathcal{S}\right) \geq \varepsilon\right\} \subseteq \left\{x : \frac{\mathcal{D}_n^\delta(x)}{n} \geq \frac{\varepsilon}{C}\right\} = \left\{x : e^{z\mathcal{D}_n^\delta(x)} \geq e^{n\varepsilon/C}\right\}$$

hence

$$\operatorname{Leb}\left\{x \in M : -\frac{1}{n}\sum_{i=0}^{n-1} \log \operatorname{dist}_\delta\left(f^i(x), \mathcal{S}\right) \geq \varepsilon\right\} \leq e^{-n\varepsilon/C} \int e^{z\mathcal{D}_n^\delta}\, d\operatorname{Leb} = e^{-n\left(\varepsilon/C - \theta(\delta)\right)}$$

which can be made exponentially small by choosing $\delta > 0$ small enough so that $\varepsilon/C > \theta(\delta)$. This proves that a piecewise expanding map f in our settings has exponentially slow recurrence to the singular set, completing the proof of the statements in Subsection 2.2 and of Corollary C after the reduction procedure of Subsection 2.4.

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