MULTIFRACTAL ANALYSIS OF THE IRREGULAR SET FOR ALMOST-ADDITIVE SEQUENCES VIA LARGE DEVIATIONS

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Abstract. In this paper we introduce a notion of free energy and large deviations rate function for asymptotically additive sequences of potentials via an approximation method by families of continuous potentials. We provide estimates for the topological pressure of the set of points whose non-additive sequences are far from the limit described through Kingman's sub-additive ergodic theorem and give some applications in the context of Lyapunov exponents for diffeomorphisms and cocycles, and Shannon-McMillan-Breiman theorem for Gibbs measures.

1. Introduction

The study of the thermodynamical formalism for maps with some hyperbolicity has drawn the attention of many researchers in the last decades. A particular topic of interest in ergodic theory is to obtain limit theorems, the characterization of level sets, the velocity of convergence and the set of points that do not converge often called the irregular set. The study of the topological pressure or dimension of the level and the irregular sets can be traced back to Besicovitch and this topic had contributions by many authors in the recent years (see e.g. [6, 14, 3, 12, 16, 18, 23, 24, 27, 26, 25, 29, 19, 1, 20, 13, 10, 8] and the references therein). For additive sequences, level sets carry all ergodic information. In fact, by Birkhoff's ergodic theorem all ergodic measures give full weight to some level set. On the other hand, the irregular set may have full Hausdorff dimension or full topological pressure meaning that it is by no means neglectable from the topological or geometrical point of view (see e.g. [29]). In particular the irregular set associated to Birkhoff sums for maps with some hyperbolicity has a rich multifractal structure (see e.g.[8]).

One of our purposes here is to provide a multifractal analysis of the irregular set in the non-additive setting that we now describe. Fix M a compact metric space and $f : M \to M$ a continuous dynamical system. A sequence $\Phi = \{\varphi_n\} \subset C(M, \mathbb{R})^{\mathbb{N}}$ is a *sub-additive* sequence of potentials if $\varphi_{m+n} \leq \varphi_m + \varphi_n \circ f^m$ for every $m, n \geq 1$. We say that the sequence $\Phi = \{\varphi_n\} \subset C(M, \mathbb{R})^{\mathbb{N}}$ is an *almost additive* sequence of potentials, if there exists a uniform constant $C > 0$ such that $\varphi_m + \varphi_n \circ f^m - C \leq \varphi_{m+n} \leq \varphi_m + \varphi_n \circ f^m + C$ for every $m, n \geq 1$. Finally, we say that $\Phi = \{\varphi_n\} \subset C(M, \mathbb{R})^{\mathbb{N}}$ is an *asymptotically additive* sequence of potentials, if for any $\xi > 0$ there exists a continuous function φ_ξ such that

$$\limsup_{n\to\infty} \frac{1}{n} \left\|\varphi_n - S_n\varphi_\xi\right\|_\infty < \xi \tag{1.1}$$

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where $S_n\varphi_\xi = \sum_{j=0}^{n-1} \varphi_\xi \circ f^j$ denotes the usual Birkhoff sum, and $||\cdot||_\infty$ is the sup norm on the Banach space $C(M, \mathbb{R})$. It follows from the definition that if if $\Phi = \{\varphi_n\}$ is almost additive then there exists $C > 0$ such that the sequence $\Phi_C = \{\varphi_n + C\}$ is sub-additive. Morever, if $\Phi = \{\varphi_n\}$ is almost additive then it is asymptotically additive (see e.g. [33]). By Kingman's subadditive ergodic theorem it follows that for every sub-additive sequence $\Phi = \{\varphi_n\}$ and every f-invariant ergodic probability measure μ so that $\varphi_1 \in L^1(\mu)$ it holds

$$\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \inf_{n\geq 1} \frac{1}{n} \int \varphi_n \, d\mu =: \mathcal{F}_*(\Phi, \mu), \qquad \text{for } \mu\text{-a.e. } x. \tag{1.2}$$

The study of the multifractal spectrum associated to non-additive sequences of potentials arises naturally in the study of Lyapunov exponents for non-conformal dynamical systems. Feng and Huang [16] used the study of subdiferentials of pressure functions to characterize the topological pressure of the level sets

$$\Big\{x \in M : \lim_{n\to\infty} \frac{1}{n}\psi_n(x) = \alpha\Big\}$$

for asymptotically sub-additive and asymptotically additive families $\Psi = \{\psi_n\}_n$. Zhao, Zhang and Cao [33] proved that if f satisfies the specification property then either the irregular set the $X(\{\psi_n\})$ (which consists of the points $x \in M$ such that the limit of $\frac{1}{n}\psi_n(x)$ does not exists) is empty or carries full topological pressure for f with respect to any asymptotically additive sequence continuous potentials Ψ. Taking into account this result we will be most interested in the analysis of the sets

$$\overline{X}_{\mu,\Psi,c} := \Big\{x \in M : \limsup_{n\to\infty} \Big|\frac{1}{n}\psi_n(x) - \mathcal{F}_*(\mu, \Psi)\Big| \geq c\Big\}$$

and

$$\underline{X}_{\mu,\Psi,c} := \Big\{x \in M : \liminf_{n\to\infty} \Big|\frac{1}{n}\psi_n(x) - \mathcal{F}_*(\mu, \Psi)\Big| \geq c\Big\},$$

where $\Psi = \{\psi_n\}$ is an asymptotically additive or sub-additve sequence of observables, $c > 0$ and μ is an equilibrium state. More precisely, what are the properties and regularity of the topological pressure functions $c \mapsto P_{\underline{X}_{\mu,\Psi,c}}(f, \Phi)$ and $c \mapsto P_{\overline{X}_{\mu,\Psi,c}}(f, \Phi)$? Such characterization and interesting applications for sequences $\Psi = \{\psi_n\}$ where $\psi_n = S_n\psi$ are Birkhoff sums were obtained in [8].

One of our purposes here is to characterize the sets $\overline{X}_{\mu,\Psi,c}$ and $\underline{X}_{\mu,\Psi,c}$ thus extending the results from [8] for almost additive sequences of potentials, in which case a thermodynamical formalism is available (see e.g [2, 21, 4, 5]). One motivation is the study of Lyapunov exponents since beyond the one-dimensional and conformal setting the situation is much less understood. The underlying strategy is to use that almost additive sequence Ψ are asymptotically additive and that the sequences $\frac{\psi_n}{n}$ are uniformly approximated by Birkhoff means of sequences of potentials can be chosen to have further regularity (c.f. Proposition 2.2). In the case of uniformly expanding dynamics we choose the approximating potentials to be Hölder continuous. Taking this into account we introduce a free energy function $\mathcal{E}_{f,\Phi,\Psi}(\cdot)$ and a rate function $I_{f,\Phi,\Psi}(\cdot)$ obtained as limit of Legendre transforms that does not depend on the family of approximations chosen and it is strictly convex in a neighborhood of $\mathcal{F}_*(\Psi, \mu_\Phi)$ if and only if Ψ is not cohomologous to a constant. This characterization using the Legendre transform and the variational formulation for the large deviations rate function is enough to obtain a functional analytic expression for the large deviations rate function obtained in [31], opening the way to study its continuous and differentiable dependence. In the case of repellers, when the irregular set $X(\{\psi_n\})$ is nonempty then it carries full topological pressure. We prove that $P_{\underline{X}_{\mu,\Psi,c}}(f,\Phi) \le P_{\overline{X}_{\mu,\Psi,c}}(f,\Phi) < P_{\text{top}}(f,\Phi)$ for any positive $c>0$ meaning that the set $X(\{\psi_n\}) \cap \overline{X}_{\mu,\Psi,c}$ does not have full pressure. This means that irregular points responsable for the topological pressure are those whose values are arbitrarily close to the mean. In fact, in the case that $\Phi=0$ and μ_o denotes the maximal entropy measure we give precise a characterization of the topological entropy of these sets in terms of the large deviations rate function and deduce that $\mathbb{R}_0^+ \ni c \mapsto h_{\underline{X}_{\mu_0,\Psi,c}}(f) = h_{\overline{X}_{\mu_0,\Psi,c}}(f)$ is continuous, strictly decreasing and concave in a neighborhood of zero. (we refer to Section 2 for precise statements).

This paper is organized as follows. In Section 2 we introduce the necessary definitions and notations and state our main results. Section 3.3 is devoted to the definition of these generalized notions of free energy and legendre transforms and to the proof of Theorem A. Section 4 is devoted to the proof of the multifractal analysis of irregular sets. Finally in Section 5 we provide some examples and applications of our results in the study of Lyapunov exponents for linear cocycles, non-conformal repellers and sequences arising from Shannon-McMillan-Breiman theorem for entropy.

2. Statement of the main results

This section is devoted to the statement of the main results. Our first results concern the regularity of the pressure function and the Legendre transform of the free energy function and its consequences to large deviations.

Topological pressure and equilibrium states. Given an asymptotically additive sequence of potentials $\Phi=\{\phi_n\}$ and a arbitrary invariant set $Z\subset M$ it can be defined the topological pressure $P_Z(f,\Phi)$ of Z with respect the f and Φ by means of a Charateodory structure. Let us mention that in the case that $\Phi=\{\phi_n\}$ with $\phi_n = S_n\phi$ for some continuous potential ϕ then $P_Z(f,\Phi)$ is exactly the usual notion of relative topological pressure for f and ϕ on Z introduced by Pesin and Pitskel. We refer the reader to [22] for a complete account on Charateodory structures. Alternativelly, for asymptotically additive sequence of potentials the topological pressure can be defined using the variational principle proved in [16]

$$P_{\text{top}}(f,\Phi) = \sup\{h_\mu(f) + \mathcal{F}_*(\mu,\Phi) : \mu \text{ is a f-invariant probability}, \mathcal{F}_*(\Phi,\mu) \neq -\infty\}$$

(see Subsection 3.1 for more details.) If an invariant probability measure μ_Φ attains the supremum then we say that it is an *equilibrium state* for f with respect to Φ. In this sense, equilibrium states are invariant measures that reflect the topological complexity of the dynamical system. In many cases equilibrium states arise as (weak) Gibbs measures. Given a sequence of functions $\Phi=\{\phi_n\}$, we say that a probability μ is a *weak Gibbs measure* with respect the Φ on $\Lambda\subset M$ if there exists $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$ there exists a positive sequence $(K_n(\varepsilon))_{n\in\mathbb{N}}$ so that $\lim \frac{1}{n}\log K_n(\varepsilon)=0$ such that for every $n\ge 1$ and μ-a.e. $x\in\Lambda$

$$K_n(\varepsilon)^{-1} \le \frac{\mu(B(x,n,\varepsilon))}{e^{-nP+\phi_n(x)}} \le K_n(\varepsilon).$$

If, in addition, $K_n(\varepsilon)=K(\varepsilon)$ does not depend of n we will say that μ is a *Gibbs measure*. Gibbs measures arise naturally in the context of hyperbolic dynamics:

given a basic set Ω for a diffeomorphism f Axiom A (or Ω repeller to f) it is known that every almost additive potential Φ satisfying

$$\text{(bounded variation) } \exists A, \delta > 0 : \sup_{n\in\mathbb{N}} \gamma_n(\Phi, \delta) \leq A, \tag{2.1}$$

where $\gamma_n(\Phi, \delta) := \sup\{|\phi_n(y) - \phi_n(z)| : y, z \in B(x, n, \delta)\}$, admits a unique equilibrium state μ_Φ is a Gibbs measure with respect to Φ on Ω (see [2] and [21] for the proof). This concept in the additive context was introduced by Bowen [9] to prove uniqueness of equilibrium states for expansive maps with the specification property. We will define now a weaker bounded variation condition: we will say that a sequence of continuous functions $\Phi = \{\varphi_n\}$ satisfies the *weak Bowen condition* if there exists $\delta > 0$ so that $\lim_{n\to+\infty} \frac{\gamma_n(\Phi,\delta)}{n} = 0$.

Legendre transforms in the non-additive case. In this section we will assume that M is a Riemannian manifold, $f : M \to M$ is a C^1 map, and $\Lambda \subset M$ is a isolated repeller such that $f\mid_\Lambda$ is topologically mixing. In fact the results for the thermodynamical formalism that we shall use here apply to shifts of finite type and for that reason hold more generally. Nevertheless we will restrict to the context of repellers for simplicity. For any almost additive potential Φ satisfying the bounded variation condition we know by [2] that there is a unique equilibrium state for f with respect the Φ, and we denote it by μ_Φ. Later, Barreira proved also the differentiability of the pressure function.

Proposition 2.1. [5, Theorem 6.3] *Let f be a continuous map on a compact metric space and assume that $\mu \mapsto h_\mu(f)$ is upper semicontinuous. Assume that Φ and Ψ are almost additive sequences satisfying the bounded variation condition and that there exists a unique equilibrium state for the family $\Phi + t\Psi$ for every $t \in \mathbb{R}$. Then the function $\mathbb{R} \ni t \mapsto P_{\text{top}}(f, \Phi + t\Psi)$ is C^1 and $\frac{d}{dt}P(f, \Phi + t\Psi) = \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi})$.*

For any almost additive sequences of potentials Φ and Ψ we define the *free energy function* associated to Φ and Ψ by

$$\mathcal{E}_{f,\Phi,\Psi}(t) := P_{\text{top}}(f, \Phi + t\Psi) - P_{\text{top}}(f, \Phi).$$

for $t \in \mathbb{R}$ such that the right hand side is well defined. It follows from the previous proposition that if Φ and Ψ are almost additive sequences satisfying the bounded variation condition then the free energy function $t \mapsto \mathcal{E}_{f,\Phi,\Psi}$ is C^1.

Proposition 2.2. *Let H be a dense subset of the continuous functions $C(M, \mathbb{R})$ in the usual sup-norm $\|\cdot\|_\infty$. If $\Psi = \{\psi_n\}$ is an asymptotically additive sequence of observables then there exists $(0, 1) \ni \varepsilon \mapsto g_\varepsilon \in H$ so that for any $\varepsilon > 0$*

$$\limsup_{n\to+\infty} \frac{1}{n}||\psi_n - S_n g_\varepsilon||_\infty < \varepsilon.$$

Proof. Since $\Psi = \{\psi_n\}$ is an asymptotically additive sequence of observables there exists a family $(\tilde{g}_\varepsilon)_\varepsilon$ of continuous functions such that for every small $\varepsilon > 0$ we have that $\limsup_{n\to+\infty} \frac{1}{n}||\psi_n - S_n\tilde{g}_\varepsilon||_\infty < \varepsilon/2$. Since $H \subset C(M, \mathbb{R})$ is dense then there exists a family $(g_\varepsilon)_\varepsilon$ of observables in H such that $\|g_\varepsilon - \tilde{g}_\varepsilon\|_\infty < \varepsilon/2$ for all ε. The later implies that the Birkhoff averages are $\varepsilon/2$ close, thus proving the lemma. $\square$

Since the thermodynamical formalism for expanding maps is well adapted the space of Hölder continuous potentials we will take $H = C^\alpha(M, \mathbb{R})$ for some $\alpha \in (0, 1)$. Given $\Psi = \{\psi_n\}_n$ almost additive it follows e.g. from [33, Proposition 2.1] that this sequence is asymptotically additive and thus we can assume the approximations above are by Hölder continuous functions. We will refer to such families of functions as an *admissible family for* Ψ and denote it by $\{g_\varepsilon\}_\varepsilon$. In what follows let $\alpha \in (0,1)$ be fixed.

Definition 2.3. Let Ψ be an almost additive sequence of observables. We will say that Ψ is *cohomologous to a constant* if there exists an admissible family $\{g_\varepsilon\}_\varepsilon$ for Ψ such that g_ε is cohomologous to a constant for every small $\varepsilon \in (0,1)$, that is, there exists a constant c_ε and a *continuous* function u_ε so that $g_\varepsilon = u_\varepsilon \circ f - u_\varepsilon + c_\varepsilon$.

A natural question is to understand which families are cohomologous to a constant. Such characterization is assured by the next lemma.

Lemma 2.4. $\Psi = \{\psi_n\}_n$ *is cohomologous to a constant if only if* $(\frac{\psi_n}{n})_n$ *is uniformly convergent to a constant.*

Proof. On the one hand, if Ψ is cohomologous to a constant then there exists an admissible family $\{g_\varepsilon\}_\varepsilon$ for Ψ such that g_ε is cohomologous to a constant for every small $\varepsilon \in (0,1)$, that is, there are constants $c_\varepsilon \in \mathbb{R}$ and continuous functions u_ε such that $g_\varepsilon = u_\varepsilon \circ f - u_\varepsilon + c_\varepsilon$ and, consequently, $S_n g_\varepsilon = u_\varepsilon \circ f^n - u_\varepsilon + c_\varepsilon n$ for every small ε. Using the convergence given by equation (1.1) it follows that for every small ε

$$\limsup_{n\to\infty} \Big\| \frac{\psi_n}{n} - c_\varepsilon \Big\|_\infty = \limsup_{n\to\infty} \frac{1}{n} \Big\| \psi_n - S_n g_\varepsilon + u_\varepsilon \circ f^n - u_\varepsilon \Big\|_\infty < \varepsilon,$$

which proves that $c = \lim_{\varepsilon\to 0} c_\varepsilon$ does exist and that $(\frac{\psi_n}{n})_n$ is uniformly convergent to the constant c. On the other hand, if $(\frac{\psi_n}{n})_n$ is uniformly convergent to a constant c then take g_ε constant to c and notice that since $S_n g_\varepsilon = cn$ then clearly

$$\limsup_{n\to\infty} \frac{1}{n} \Big\| \psi_n - S_n g_\varepsilon \Big\|_\infty = 0.$$

This finishes the proof of the lemma. □

Remark 2.5. Let us notice that the notion of cohomology for families of observables is slightly different from the corresponding one for a fixed observable. Indeed, for instance by the previous lemma the family $\Psi = \{\psi_n\}_n$ with $\psi_n = \sqrt{n}\, w$ is cohomologous to the constant 0 although the observable $w : M \to \mathbb{R}$ may be chosen to be not cohomologous to a constant.

Observe that it follows from the definition that if Ψ is not cohomologous to a constant then there is a admissible family $\{g_\varepsilon\}_\varepsilon$ for Ψ and a sequence $(\varepsilon_k)_k$ converging to zero such that g_{ε_k} is not cohomologous to a constant for every $k \geq 1$. If this is the case, the family $\varepsilon \mapsto \tilde{g}_\varepsilon$ given by $\tilde{g}_\varepsilon = g_{\varepsilon_k}$ for every $\varepsilon_k \leq \varepsilon < \varepsilon_{k-1}$ is so that $\tilde{g}_\varepsilon$ is not cohomologous to a constant for every small ε (notice that these "step functions" could be chosen in many different ways). We will say that such family $\{\tilde{g}_\varepsilon\}_\varepsilon$ is *not cohomologous* to a constant. Then, for simplicity, given any Ψ is not cohomologous to a constant we shall consider the approximations by admissible sequences $(g_\varepsilon)_\varepsilon$ such that g_ε is not cohomologous to a constant for any small ε.

Assume Φ, Ψ are almost additive sequences of potentials with the bounded variation condition such that Ψ is not cohomologous to a constant and let $(\varphi_\varepsilon)_\varepsilon$ and $\{g_\varepsilon\}_\varepsilon$ be admissible families for Φ and Ψ respectivelly. Then the well defined free energy function $t \mapsto \mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}$ is strictly convex and so it makes sense to compute the

Legendre transform $I_{f,\varphi_\varepsilon,g_\varepsilon}(t)$ for every small $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$. Since each g_ε is not cohomologous to a constant it is a classical result that the following variational property holds

$$I_{f,\varphi_\varepsilon,g_\varepsilon}(\mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t)) = t\mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t) - \mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}(t) \tag{2.2}$$

for every small $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$ (see e.g. [7]). Using this variational property we prove in Section 3.3 that it is possible define the *Legendre transform* of the corresponding free energy functions of Ψ as

$$I_{f,\Phi,\Psi}(s) := \lim_{\varepsilon\to 0} I_{f,\varphi_\varepsilon,g_\varepsilon}(s),$$

for every $s \in \big(\inf_{t\in\mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi})\ ,\ \sup_{t\in\mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi})\big)$, since this limit will not depend of the choices of families $\{\varphi_\varepsilon\}_\varepsilon$ e $\{g_\varepsilon\}_\varepsilon$. We establish some properties of this Legendre transform as follows.

Theorem A. *Let M be a Riemannian manifold, $f : M \to M$ be a C^1-map and $\Lambda \subset M$ be an isolated repeller such that $f\mid_\Lambda$ is topologically mixing. Let Φ and Ψ be almost additive sequences satisfying the bounded variation condition and assume that Ψ is not cohomologous to a constant. The following properties hold:*

i. *the Legendre transform of Ψ satisfies the variational property*

$$I_{f,\Phi,\Psi}(\mathcal{E}'_{f,\Phi,\Psi}(t)) = t\mathcal{E}'_{f,\Phi,\Psi}(t) - \mathcal{E}_{f,\Phi,\Psi}(t),$$

for every $t \in \mathbb{R}$;

ii. *$I_{f,\Phi,\Psi}(\cdot)$ is a non-negative convex function and*

$$\inf_{s\in(a,b)} I_{f,\Phi,\Psi}(s) = \min\{I_{f,\Phi,\Psi}(a), I_{f,\Phi,\Psi}(b)\}$$

for any interval $(a,b) \subset \mathbb{R}$ not containing $\mathcal{F}_*(\Psi, \mu_\Phi)$

iii. *$I_{f,\Phi,\Psi}(s) = \inf_{\eta\in\mathcal{M}_1(f)}\{P_{\text{top}}(f,\Phi) - h_\eta(f) - \mathcal{F}_*(\Phi,\eta) : \mathcal{F}_*(\Psi,\eta) = s\}$*

iv. *$I_{f,\Phi,\Psi}(s) = 0$ if only if $s = \mathcal{F}_*(\Psi,\mu_\Phi)$ and $s \mapsto I_{f,\Phi,\Psi}(s)$ is strictly convex in an open neighborhood of $\mathcal{F}_*(\Psi,\mu_\Phi)$.*

Large deviations results. The variational relation obtained in Theorem A is of particular interest in the study of large deviations. In [31], the first author and Zhao proved several large deviations results for sub-additive and asymptotically additive sequences of potentials. In the case of expanding maps and almost additive sequences of potentials Theorem A leads to the following immediate consequence:

Corollary A. *Let M be a Riemannian manifold, $f : M \to M$ be a C^1-map and $\Lambda \subset M$ be an isolated repeller such that $f\mid_\Lambda$ is topologically mixing. Let $\Phi = \{\varphi_n\}$ be an almost additive sequence of potentials satisfying the bounded variation condition and μ_Φ be the unique equilibrium state for $f\mid_\Lambda$ with respect to Φ. If $\Psi = \{\psi_n\}$ is a family of almost additive potentials satisfying the bounded distortion condition then it satisfies the following large deviations principle: given $F \subset \mathbb{R}$ closed it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\Phi\left(\left\{x \in M : \frac{1}{n}\psi_n(x) \in F\right\}\right) \le -\inf_{s\in F} I_{f,\Phi,\Psi}(s)$$

and also for every open set $E \subset \mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\Phi\left(\left\{x \in M : \frac{1}{n}\psi_n(x) \in E\right\}\right) \ge -\inf_{s\in E} I_{f,\Phi,\Psi}(s).$$

Remark 2.6. Although these quantitative estimates can be expected to hold for more general asymptotically additive sequences, one should mention that an extension of limit theorems from almost-additive to asymptotically additive sequences of potentials is not immediate by no means. In fact, a simple example of an asymptotically additive sequence of potentials can be written as $\psi_n = S_n\psi + a_n$ depending on the sequence of real numbers $(a_n)_n$. If ψ is Hölder continuous and $a_n = o(\sqrt{n})$ then $(\psi_n)_n$ satisfies the central limit theorem. However, the CLT fails in a simple way e.g. if $a_n = n^{\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$.

Multifractal estimates for the irregular set. Given an asymptotically additive sequence of observables $\Psi = \{\psi\}_n$ and $J \subset \mathbb{R}$ we denote

$$\overline{X}_J = \{x \in M : \limsup_{n\to+\infty} \frac{1}{n}\psi_n(x) \in J\}$$

and

$$\underline{X}_J = \{x \in M : \liminf_{n\to+\infty} \frac{1}{n}\psi_n(x) \in J\}.$$

and set $X(J) := \{x \in \Lambda : \lim_{n\to+\infty} \frac{1}{n}\psi_n(x) \in J\}$. For any $\delta > 0$ we denote by J_δ the $\delta-$neighborhood of the set J and for a probability measure μ we define

$$L_{J,\mu} := -\limsup_{n\to+\infty} \frac{1}{n} \log \mu\Big(\{x \in M : \frac{1}{n}\psi_n(x) \in J\}\Big).$$

We are now in a position to state our first main result concerning the multifractal analysis of the irregular set.

Theorem B. *Let M be a compact metric space, $f : M \to M$ be continuous, $\Phi = \{\phi_n\}$ be an almost additive sequence of potentials with $P_{\mathrm{top}}(f,\Phi) > -\infty$. Assume that μ_Φ is the unique equilibrium state of f with respect the Φ, that it is a weak Gibbs measure and that the sequence $\Psi = \{\psi_n\}$ satisfies at least one of the following properties:*

(a) Ψ *is asymptotically additive, or*
(b) Ψ *is a sub-additive sequence so that*
  i. *satisfies the weak Bowen condition;*
  ii. $\inf_{n\geq 1} \frac{\psi_n(x)}{n} > -\infty$ *for every $x \in M$;*
  iii. *the sequence $\{\frac{\psi_n}{n}\}$ is equicontinuous.*

Then, for any closed interval $J \subset \mathbb{R}$ and any small $\delta > 0$,

$$P_{\underline{X}_J}(f,\Phi) \leq P_{\overline{X}_J}(f,\Phi) \leq P_{\mathrm{top}}(f,\Phi) - L_{J_\delta,\mu_\Phi} \leq P(f,\Phi).$$

In Remark 4.2 we indicate few modifications which imply that the estimate $P_{\underline{X}_J}(f,\Phi) \leq P_{\mathrm{top}}(f,\Phi) - L_{J_\delta,\mu_\Phi}$ holds under the assumption that μ_Φ satisfies a *pointwise weak Gibbs property*, namely, whenever there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a positive sequence $(K_n(\varepsilon))_{n\in\mathbb{N}}$ so that $\lim \frac{1}{n}\log K_n(\varepsilon) = 0$ such that for μ_Φ-a.e. $x \in \Lambda$ there exists a subsequence $n_k(x) \to \infty$ (depending on x) satisfying

$$K_{n_k(x)}(\varepsilon)^{-1} \leq \frac{\mu_\Phi(B(x,n_k(x),\varepsilon))}{e^{-n_k(x)P+\phi_{n_k(x)}}} \leq K_{n_k(x)}(\varepsilon). \tag{2.3}$$

From [31, Theorem B] we know that if $\mathcal{F}_*(\Psi,\mu_\Phi) \notin J_\delta$ then $L_{J_\delta,\mu_\Phi} > 0$ and, consequently, the topological pressure of both sets $\underline{X}_J$ and $\overline{X}_J$ is strictly smaller than $P_{\mathrm{top}}(f,\Phi)$. The bound $P_{\overline{X}_{\mu_\Phi,\Psi,c}}(f,\Phi) \leq P_{\mathrm{top}}(f,\Phi) - L_{J,\mu_\Phi}$ holds e.g. if $\delta \mapsto L_{J_\delta,\mu_\Phi}$

is upper semicontinuous. In the additive setting this question is overcomed by means of the functional analytic approach used to define the Legendre transform of the free energy function. Despite that one misses the functional analytic approach our approximation method is still sufficient to obtain finer estimates in the uniformly hyperbolic setting.

Corollary B. *Let M be a Riemannian manifold, $f : M \to M$ be a C^1-map and $\Lambda \subset M$ be an isolated repeller such that $f\mid_\Lambda$ is topologically mixing. Assume $\Phi = 0$ and Ψ is almost additive sequence of potentials satisfying the bounded variation condition, Ψ is not cohomologous to a constant and $\mathcal{F}_*(\Psi, \mu_0) = 0$, where μ_0 is the unique maximal entropy measure for f. Then for any interval $J \subset \mathbb{R}$*

$$h_{\overline{X}_J}(f) \le h_{\mathrm{top}}(f) - I_{f,0,\Psi}(c_*),$$

where c_* belongs to the closure of J is so that $I_{f,0,\Psi}(c_*) = \inf_{s\in J} I_{f,0,\Psi}(s)$. Moreover, if $\overline{X}_J \neq \emptyset$ then c_* is a point in the boundary of J and

$$h_{\overline{X}_J}(f) = h_{\underline{X}_J}(f) = h_{X(c_*)}(f) = h_{X(J)}(f) = h_{\mathrm{top}}(f) - I_{f,0,\Psi}(c_*),$$

In particular $\mathbb{R}_0^+ \ni c \mapsto h_{\overline{X}_{\mu_0,\Psi,c}}(f)$ is continuous, strictly decreasing and concave in a neighborhood of zero.

Let us mention that the previous characterization of the topological entropy of level sets was available in this setting due to Barreira and Doutor [4], while we can expect analogous estimates to hold for the topological pressure provided a generalization of the previous results to the context of the topological pressure.

3. Free energy and Legendre transform

3.1. Non-additive topological pressure for invariant non-compact sets.

In this subsection we describe the notion of topological pressure for asymptotically additive potentials and not necessarily compact invariant sets. Let M be a compact metric space, $f : M \to M$ be a continuous map and $\Phi = \{\phi_n\}_n$ be an asymptotically additive sequence of continuous potentials. The *dynamical ball* of center $x \in M$, radius $\delta > 0$, and length $n \ge 1$ is defined by

$$B(x, n, \delta) := \{y \in M : d(f^j(y), f^j(x)) \le \delta, \text{ for every } 0 \le j \le n\}.$$

Let $\Lambda \subset M$ be, fix $\varepsilon > 0$. Define $\mathcal{I}_n = M \times \{n\}$ and $\mathcal{I} = M \times \mathbb{N}$. For every $\alpha \in \mathbb{R}$ and $N \ge 1$, define

$$m_\alpha(f, \Phi, \Lambda, \varepsilon, N) := \inf_{\mathcal{G}} \Big\{ \sum_{(x,n)\in\mathcal{G}} e^{-\alpha n + \phi_n(x)} \Big\},$$

where the infimum is taken over every finite or enumerable families $\mathcal{G} \subset \cup_{n\ge N} \mathcal{I}_n$ such that the collection of sets $\{B(x, n, \varepsilon) : (x, n) \in \mathcal{G}\}$ cover Λ. Since the sequence is monotone increasing in N there exists the limit

$$m_\alpha(f, \Phi, \Lambda, \varepsilon) := \lim_{N\to+\infty} m_\alpha(f, \Phi, \Lambda, \varepsilon, N)$$

and $P_\Lambda(f, \Phi, \varepsilon) := \inf\{\alpha : m_\alpha(f, \Phi, \Lambda, \varepsilon) = 0\} = \sup\{\alpha : m_\alpha(f, \Phi, \Lambda, \varepsilon) = +\infty\}$. By Cao, Zhang e Zhao [33], the *pressure* of Λ is defined by the limit:

$$P_\Lambda(f, \Phi) = \lim_{\varepsilon\to 0} P_\Lambda(f, \Phi, \varepsilon).$$

If $\Lambda = M$ we have that $P_\Lambda(f,\Phi)$ corresponds to the *topological pressure* of f with respect the Φ and is denoted by $P_{\text{top}}(f,\Phi)$. If we take a continuous potential ϕ we have that $P_\Lambda(f,\{\phi_n\}_n)$, for $\phi_n = \sum_{i=0}^{n-1}\phi\circ f^i$, is equal the usual topological pressure of Λ with respect the f and ϕ. It follows of the definition of relative pressure that if $\Lambda_1 \subset \Lambda_2 \subset M$ we will have that $P_{\Lambda_1}(f,\Phi) \le P_{\Lambda_2}(f,\Phi)$. In the asymptotically additive context also we have the following variational principle:

Proposition 3.1. [16] *Let M be compact metric space, $f : M \to M$ be continuous map and $\Phi = \{\phi_n\}$ a asymptotically additive sequence of potentials. Then*

$$P_{\text{top}}(f,\Phi) = \sup\{h_\mu(f)+\mathcal{F}_*(\Phi,\mu) : \mu \textit{ is a f-invariant probability, } \mathcal{F}_*(\Phi,\mu) \neq -\infty\},$$

where the supremum is taken over all f-invariant probabilities μ and $\mathcal{F}_*(\Phi,\mu) = \lim_{n\to+\infty}\frac{1}{n}\int\phi_n d\mu$.

3.2. **Space of asymptotically additive sequences.** Given a compact metric space M let us define $\mathbb{A} := \{\Psi = \{\psi_n\}_n : \Psi \text{ is asymptotically additive }\}$. The space $\mathbb{A}$ is clearly a vector space with a sum and product by a scalar defined naturally by $\{\psi_{1,n}\}_n + \{\psi_{2,n}\}_n := \{\psi_{1,n} + \psi_{2,n}\}_n$ and $\lambda \cdot \{\psi_{1,n}\}_n := \{\lambda\psi_{1,n}\}_n$ for every $\{\psi_{1,n}\}_n, \{\psi_{2,n}\}_n \in \mathbb{A}$ and $\lambda \in \mathbb{R}$. On this vector space structure we shall consider the seminorm: $||\{\psi_n\}_n||_{\mathbb{A}} := \limsup_{n\to\infty}\frac{1}{n}||\psi_n||_\infty$. If necessary to consider a norm we can consider the space $\mathbb{A}$ endowed with $\|\{\psi_n\}_n\|_{\mathbb{A},0} := \sup_{n\in\mathbb{N}}\frac{1}{n}||\psi_n||_\infty$ which clearly satisfies $\|\{\psi_n\}_n\|_{\mathbb{A}} \le \|\{\psi_n\}_n\|_{\mathbb{A},0}$ for every $\{\psi_n\}_n \in \mathbb{A}$. For that reason we shall consider the continuity results with $\mathbb{A}$ endowed with the weaker topology induced by the semi norm. The balls of the seminorm $||\cdot||_{\mathbb{A}}$ form a basis for a topology on $\mathbb{A}$ that will not be metrizable because it is not Hausdorf. However $\mathbb{A}$ with the aforementioned vector space structure and with this topology is a locally convex topological vector space. We shall consider $\mathbb{A}$ with this topology and the space of almost additive sequences of observables with the natural induced topology.

Proposition 3.2. *Let M be a compact metric space and $f : M \to M$ be a continuous map. Then the following functions are continuous:*

i. $\mathbb{A} \ni \Phi \mapsto P_{\text{top}}(f,\Phi)$;
ii. $\mathcal{M}_1(f) \times \mathbb{A} \ni (\mu,\Psi) \mapsto \mathcal{F}_*(\Psi,\mu)$.

Proof. The first claim (i) is clear from the definition of topological pressure and the one of $||\cdot||_{\mathbb{A}}$. Hence we are left to prove (ii). Given $\Psi_1 = \{\psi_{1,n}\}_n \in \mathbb{A}$ and $\eta_1 \in \mathcal{M}_1(f)$ arbitrary we will prove that $(\mu,\Psi) \mapsto \mathcal{F}_*(\Psi,\mu)$ is continuous in (Ψ_1,η_1). Let $\varepsilon > 0$ be small and fixed.

Since $\Psi_1 \in \mathbb{A}$ there exists a continuous function $g_{\frac{\varepsilon}{6}}$ and $n_0 \in \mathbb{N}$ such that $\frac{1}{n}||\psi_{1,n} - S_n g_{\frac{\varepsilon}{6}}||_\infty < \frac{\varepsilon}{6}$ for all $n \ge n_0$. Moreover, there exists $\delta > 0$ such that if $d(\eta_1,\eta_2) < \delta$ then $|\int g_{\frac{\varepsilon}{6}} d\eta_1 - \int g_{\frac{\varepsilon}{6}} d\eta_2| < \frac{\varepsilon}{6}$. Given $\Psi_2 = \{\psi_{2,n}\}_n \in \mathbb{A}$ and $\eta_2 \in \mathcal{M}_1(f)$ arbitrary in such a way that $||\Psi_1 - \Psi_2||_{\mathbb{A}} < \frac{\varepsilon}{6}$ and $d(\eta_1,\eta_2) < \delta$ then there exists $n_1 = n_1(\Psi_2,\eta_2) \ge n_0$ so that $\frac{1}{n_1}||\psi_{1,n_1} - \psi_{2,n_1}||_\infty < \frac{\varepsilon}{6}$, $|\frac{1}{n_1}\int\psi_{2,n_1}d\eta_2 - \mathcal{F}_*(\Psi_2,\eta_2)| < \frac{\varepsilon}{6}$ and also $|\frac{1}{n_1}\int\psi_{1,n_1}d\eta_1 - \mathcal{F}_*(\Psi_1,\eta_1)| < \frac{\varepsilon}{6}$. Thus, given $\Psi_2 = \{\psi_{2,n}\}_n \in \mathbb{A}$ and $\eta_2 \in \mathcal{M}_1(f)$ such that $||\Psi_1 - \Psi_2||_{\mathbb{A}} < \frac{\varepsilon}{6}$ and $d(\eta_1,\eta_2) < \delta$ we have that

$$\begin{aligned}|\mathcal{F}_*(\Psi_1,\eta_1)-\mathcal{F}_*(\Psi_2,\eta_2)| &\le |\mathcal{F}_*(\Psi_2,\eta_2)-\int g_{\frac{\varepsilon}{6}}\,d\eta_2| + \Big|\int g_{\frac{\varepsilon}{6}}\,d\eta_2-\mathcal{F}_*(\Psi_1,\eta_1)\Big| \\ &\le \Big|\frac{1}{n_1}\int S_{n_1}g_{\frac{\varepsilon}{6}}\,d\eta_2-\frac{1}{n_1}\int \psi_{1,n_1}\,d\eta_2\Big| \\ &+ \Big|\frac{1}{n_1}\int \psi_{1,n_1}\,d\eta_2-\mathcal{F}_*(\Psi_2,\eta_2)\Big| + \Big|\int g_{\frac{\varepsilon}{6}}\,d\eta_2-\int g_{\frac{\varepsilon}{6}}\,d\eta_1\Big| \\ &+ \Big|\int g_{\frac{\varepsilon}{6}}\,d\eta_1-\mathcal{F}_*(\Psi_1,\eta_1)\Big|\end{aligned}$$

and so

$$\begin{aligned}|\mathcal{F}_*(\Psi_1,\eta_1)-\mathcal{F}_*(\Psi_2,\eta_2)| &\le \frac{\varepsilon}{3} + \Big|\frac{1}{n_1}\int \psi_{1,n_1}\,d\eta_2-\frac{1}{n_1}\int \psi_{2,n_1}\,d\eta_2\Big| \\ &+ \Big|\frac{1}{n_1}\int \psi_{2,n_1}\,d\eta_2-\mathcal{F}_*(\Psi_2,\eta_2)\Big| + \Big|\frac{1}{n_1}\int \psi_{1,n_1}\,d\eta_1-\mathcal{F}_*(\Psi_1,\eta_1)\Big| \\ &+ \Big|\int g_{\frac{\varepsilon}{6}}\,d\eta_1-\frac{1}{n_1}\int \psi_{1,n_1}\,d\eta_1\Big|\end{aligned}$$

which is smaller than ε. This proves the continuity of $(\mu,\Psi)\mapsto\mathcal{F}_*(\Psi,\mu)$. □

Now we study some properties of the topological pressure in the case of repellers.

Proposition 3.3. *Let M be a Riemannian manifold, $f: M\to M$ be a C^1-map and $\Lambda\subset M$ be an isolated repeller such that $f\mid_\Lambda$ is topologically mixing. Then:*

i. *If $\Phi\in\mathbb{A}$ then $P_{\text{top}}(f,\Phi)=\lim_{\varepsilon\to 0}P_{\text{top}}(f,g_\varepsilon)$ for any $(g_\varepsilon)_\varepsilon$ admissible family for Φ.*
ii. *If $(\varphi_\varepsilon)_\varepsilon$ is an admissible family for Φ and μ_ε is the unique equilibrium state for f with respect the φ_ε then every accumulation point of μ_ε is a equilibrium state for f with respect the Φ. In particular, if there is a unique equilibrium state μ_Φ for f with respect to Φ then $\mu_\Phi=\lim_{\varepsilon\to 0}\mu_\varepsilon$.*

Proof. Property (i) follows from the corresponding item of Proposition 3.2. Now, since Λ is a repeller we have that $\mu\to h_\mu(f)$ is upper semicontinuous and using the continuity of $(\mu,\Phi)\mapsto\mathcal{F}_*(\mu,\Phi)$ we conclude that every accumulation point of μ_ε is equilibrium state for f with respect to Φ. Using the compactness of the space of invariant probabilities, if there exists a unique equilibrium μ_Φ for f with respect to Φ then the convergence $\mu_\Phi=\lim_{\varepsilon\to 0}\mu_\varepsilon$ should hold. This finishes the proof of the proposition. □

3.3. **Free energy function and Legendre transforms.** We are interested in the regularity of the rate function in the large deviations principles obtained in [31]. Since there exists no direct functional analytic approach using Perron-Frobenius operators, in order to inherit some properties from the classical thermodynamical formalism we will use the approximation by admissible families of Hölder continuous functions.

The next result allows us to define the Legendre transform of Ψ in terms of the Legendre transform associated to any approximating admissible family. For every almost additive sequence of potentials Φ satisfying the bounded variation condition we denote by μ_Φ the unique equilibrium state of f with respect to Φ (for the existence of μ_Φ see [2]). For $\Phi, \Psi \in \mathbb{A}$ consider the free energy function given by $\mathcal{E}_{f,\Phi,\Psi} := P_{\text{top}}(f, \Phi + t\Psi) - P_{\text{top}}(f, \Phi)$ for all $t \in \mathbb{R}$. Observe that

$$\mathcal{E}_{f,\Phi,\Psi} = \lim_{\varepsilon \to 0} \mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon},$$

where $\{\varphi_\varepsilon\}_\varepsilon$ and $\{g_\varepsilon\}_\varepsilon$ are any admissible families for Φ and Ψ respectively. In fact, the pressure function is continuous in the set of all asymptotically additive sequences and so this limit does not depend on the sequence of approximating families and we may take Hölder continuous representatives (admissible families).

Assume Φ, Ψ are almost additive sequences of potentials satisfying bounded variation condition so that Ψ is not cohomologous to a constant and let $\{g_\varepsilon\}_\varepsilon$ be an admissible family for Ψ not cohomologous to a constant and $(\varphi_\varepsilon)_\varepsilon$ be an admissible family for Φ. Then it makes sense to define for every $\varepsilon \in (0,1)$

$$I_{f,\varphi_\varepsilon,g_\varepsilon}(t) = \sup_{s \in \mathbb{R}} \Big(st - \mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}(s) \Big)$$

as the Legendre transform of $\mathcal{E}_{f,\phi_\varepsilon,g_\varepsilon}$. Since each g_ε is not cohomologous to a constant then the previous function is defined over the reals and the variational property yields

$$I_{f,\varphi_\varepsilon,g_\varepsilon}(\mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t)) = t\mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t) - \mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}(t) \tag{3.1}$$

for all $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$, and $\mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}$ is strictly convex (see e.g. [7]). Recalling that

$$\mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon} = P_{\text{top}}(f, \varphi_\varepsilon + tg_\varepsilon) - P_{\text{top}}(f, \varphi_\varepsilon) \quad \text{and} \quad \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t) = \int g_\varepsilon \, d\mu_{\varphi_\varepsilon + tg_\varepsilon}$$

it follows from Propositions 3.2 and 3.3 that for every $t \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t) = \lim_{\varepsilon \to 0} \int g_\varepsilon \, d\mu_{\varphi_\varepsilon + tg_\varepsilon} = \mathcal{F}_*(\Psi, \mu_{f,\Phi+t\Psi}),$$

that $\lim_{\varepsilon \to 0} \mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}(t) = P_{\text{top}}(f, \Phi + t\Psi) - P_{\text{top}}(f, \Phi)$, and that these convergences are uniform in compact sets. Observe that the Legendre transform $I_{f,\varphi_\varepsilon,g_\varepsilon}$ is well defined in the open interval $J_\varepsilon = (\inf_t \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t), \sup_t \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t))$. Thus, we can now define the *Legendre transform*

$$I_{f,\Phi,\Psi}(s) := \lim_{\varepsilon \to 0} I_{f,\varphi_\varepsilon,g_\varepsilon}(s)$$

for any $s \in J_{\Phi,\Psi} := \big(\inf_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) \ , \ \sup_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) \big)$. Note that the previous limiting function and interval does not depend of the chosen families $\{\varphi_\varepsilon\}_\varepsilon$ and $\{g_\varepsilon\}_\varepsilon$. A priori $J_{\Phi,\Psi}$ could be a degenerate interval. However the next lemma assures that the closure of this interval is exactly the spectrum of Ψ and, in particular, $J_{\Phi,\Psi}$ is a degenerate interval if and only if Ψ is cohomologous to a constant.

Lemma 3.4. *Given $\Psi, \Phi \in \mathbb{A}$ we have that*

$$\Big[\inf_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) \ , \ \sup_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) \Big] = \{ \mathcal{F}_*(\Psi, \eta) : \eta \in \mathcal{M}_1(f) \}$$

and the interval is degenerate if only if Ψ is cohomologous to a constant.

Proof. Let $(\varphi_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ be any admissible sequences for Φ and Ψ respectively. We have that $\inf_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) = \lim_{\varepsilon \to 0} \inf_{t \in \mathbb{R}} \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t)$ (analogously for the supremum) and so

$$\bigcap_{\varepsilon_0 \in (0,1)} \bigcup_{\varepsilon \geq \varepsilon_0} \Big(\inf_{t \in \mathbb{R}} \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t) \ , \ \sup_{t \in \mathbb{R}} \mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t) \Big) = \Big(\inf_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) \ , \ \sup_{t \in \mathbb{R}} \mathcal{F}_*(\Psi, \mu_{\Phi+t\Psi}) \Big)$$

and

$$\bigcap_{\varepsilon_0\in(0,1)}\bigcup_{\varepsilon\geq\varepsilon_0}\Big(\inf_{\eta\in\mathcal{M}_1(f)}\int g_\varepsilon d\eta\ ,\ \sup_{\eta\in\mathcal{M}_1(f)}\int g_\varepsilon d\eta\Big)=\Big(\inf_{\eta\in\mathcal{M}_1(f)}\mathcal{F}_*(\Psi,\eta)\ ,\ \sup_{\eta\in\mathcal{M}_1(f)}\mathcal{F}_*(\Psi,\eta)\Big).$$

Let us first prove the result in the additive setting, that is, assuming there are $g,\phi\in C(M,\mathbb{R})$ such that $\varphi_n=S_n\,\phi$ and $\psi_n=S_n\,g$. If this is the case, using the weak* continuity of the equilibrium states with respect to the potential, the image of the function $T_g:\mathbb{R}\to\mathbb{R}$ given by $t\mapsto\int g\,d\mu_{\phi+tg}$ is an interval. In addition, given $\eta\in\mathcal{M}_1(f)$ and $t>0$ we have by the variational principle

$$h_\eta(f)+\int(\phi+tg)\,d\eta\leq h_{\mu_{\phi+tg}}(f)+\int(\phi+tg)\,d\mu_{\phi+tg}$$

and so, dividing by t in both sides and making t tend to infinity in the expression

$$\frac{1}{t}h_\eta(f)+\frac{1}{t}\int\phi\,d\eta+\int g\,d\eta\leq\frac{1}{t}h_{\mu_{\phi+tg}}(f)+\frac{1}{t}\int\phi\,d\mu_{\phi+tg}+\int g\,d\mu_{\phi+tg},$$

we get that $\int g d\eta\leq\limsup_{t\to+\infty}\int g\ d\mu_{\phi+tg}=\int g\ d\mu_*$ for an f-invariant probability μ_* properly chosen as accumulation point of $(\mu_{\phi+tg})_t$. This proves that $\sup_{\eta\in\mathcal{M}_1(f)}\int g\ d\eta=\limsup_{t\to+\infty}\int g\ d\mu_{\phi+tg}$. Proceeding analogously with $-g$ replacing g it follows that $\inf_{\eta\in\mathcal{M}_1(f)}\int g\ d\eta=\liminf_{t\to-\infty}\int g\ d\mu_{\phi+tg}$ and

$$\Big[\inf_{t\in\mathbb{R}}\int g\ d\mu_{\phi+tg}\ ,\ \sup_{t\in\mathbb{R}}\int g\ d\mu_{\phi+tg}\Big]=\Big\{\int g\ d\eta:\eta\in\mathcal{M}_1(f)\Big\}.\tag{3.2}$$

Now, to deal with the general non-additive setting, replacing g by g_ε and also ϕ by φ_ε in equation (3.2), and taking the limit as ε tends to zero it follows that

$$\Big[\inf_{t\in\mathbb{R}}\mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})\ ,\ \sup_{t\in\mathbb{R}}\mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})\Big]=\Big\{\mathcal{F}_*(\Psi,\eta):\eta\in\mathcal{M}_1(f)\Big\}.$$

as claimed. This finishes the first part of the proof of the lemma.

Finally, by [33, Lemma 2.2] we get $\inf_{t\in\mathbb{R}}\mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})=\sup_{t\in\mathbb{R}}\mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})$ if and only if $\frac{\psi_n}{n}$ converges uniformly to a constant, that is, Ψ is cohomologous to a constant. This finishes the proof of the lemma. □

Remark 3.5. It is not hard to check also that there exists a constant $C>0$ (depending only on f) so that $\frac{P(f,\Phi+t\Psi)}{t}=\mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})\pm\frac{C}{t}$ and, consequently, the previous interval is characterized as the interval of limiting slopes for the pressure function $t\mapsto P(f,\Phi+t\Psi)$.

3.4. **Proof of Theorem A.** Let Φ,Ψ be almost additive sequences of Hölder continuous potentials satisfying the bounded variation condition so that Ψ be is not cohomologous to a constant. In particular $t\mapsto\mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})$ is not a constant function. Moreover, the Legendre transform of the free energy function $I_{f,\Phi,\Psi}$ (defined in the previous section) is well defined in an open neighborhood of the mean $\mathcal{F}_*(\Psi,\mu_\Phi)$.

Let $(\varphi_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ be any admissible families for Φ and Ψ, respectivelly. It follows from equation (2.2) that $I_{f,\varphi_\varepsilon,g_\varepsilon}(\mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t))=t\mathcal{E}'_{f,\varphi_\varepsilon,g_\varepsilon}(t)-\mathcal{E}_{f,\varphi_\varepsilon,g_\varepsilon}(t)$ for every $t\in\mathbb{R}$ and so, making ε converging to zero, we obtain that

$$I_{f,\Phi,\Psi}(\mathcal{E}'_{f,\Phi,\Psi}(t))=t\mathcal{E}'_{f,\Phi,\Psi}(t)-\mathcal{E}_{f,\Phi,\Psi}(t),$$

which proves (i). Now, since $I_{f,\varphi_\varepsilon,g_\varepsilon}$ is a non-negative convex function for all $\varepsilon \in (0,1)$ and is pointwise convergent to $I_{f,\Phi,\Psi}$ this is also a non-negative convex function. Clearly, given any interval $(a,b) \subset \mathbb{R}$ not containing $\mathcal{F}_*(\Psi,\mu_\Phi)$ then we know that

$$\inf_{s\in(a,b)} I_{f,\varphi_\varepsilon,g_\varepsilon}(s) = \min\{I_{f,\varphi_\varepsilon,g_\varepsilon}(a), I_{f,\varphi_\varepsilon,g_\varepsilon}(b)\},$$

so the same property will be valid for the limit function $I_{f,\Phi,\Psi}$, which proves (ii).

Concerning property (iv), since $I_{f,\Phi,\Psi}$ is convex, if it was not strictly convex in a neighborhood of $\mathcal{F}_*(\Psi,\mu_\Phi)$ then it would be constant to zero in a open interval containing $\mathcal{F}_*(\mu_\Phi,\Psi)$. We may assume property (iii) for the moment (property (iv) will not be used in the proof of property (iii) later). Since $I_{f,\Phi,\Psi}(s) = 0$ if and only if $s = \mathcal{F}_*(\Psi,\mu_\Phi)$ if $I_{f,\Phi,\Psi}$ was not locally convex this would contradict uniqueness of the equilibrium state μ_Φ.

We are now left to prove (iii), that is, to establish the variational formula

$$I_{f,\Phi,\Psi}(s) = \inf_{\eta\in\mathcal{M}_1(f)} \{P_{\text{top}}(f,\Phi) - h_\eta(f) - \mathcal{F}_*(\Phi,\eta) : \mathcal{F}_*(\Psi,\eta) = s\}$$

for the rate function. The equality is clearly satisfied when $s = \mathcal{F}_*(\Psi,\mu_\Phi)$ by uniqueness of the equilibrium state and the Proposition 3.3. Hence we are reduced to the case where $s \neq \mathcal{F}_*(\Psi,\mu_\Phi)$. From the additive case we already know that for all $s \in \big(\inf_{t\in\mathbb{R}} \mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})\ ,\ \sup_{t\in\mathbb{R}} \mathcal{F}_*(\Psi,\mu_{\Phi+t\Psi})\big)$

$$I_{f,\varphi_\varepsilon,g_\varepsilon}(s) = \inf_{\eta\in\mathcal{M}_1(f)} \big\{P_{\text{top}}(f,\varphi_\varepsilon) - h_\eta(f) - \int \varphi_\varepsilon d\eta : \int g_\varepsilon d\eta = s\big\}$$

and for every small ε. We will use an auxiliary lemma.

Lemma 3.6. *For every s in the interior of $J := \{\mathcal{F}_*(\Psi,\eta) : \eta \in \mathcal{M}_1(f)\}$ it holds:*

$$\lim_{\varepsilon\to0} \sup_{\eta\in\mathcal{M}_1(f)} \{h_\eta(f) + \int \varphi_\varepsilon d\eta : \int g_\varepsilon d\eta = s\} = \sup_{\eta\in\mathcal{M}_1(f)} \{h_\eta(f) + \mathcal{F}_*(\Phi,\eta) : \mathcal{F}_*(\Psi,\eta) = s\}.$$

Proof. We will use the continuity of $\mathcal{F}_*(\Phi,\mu)$ in both coordinates. Let $s \in J$ be fixed and let $\eta \in \mathcal{M}_1(f)$ with $s = \mathcal{F}_*(\Psi,\eta)$. Consider an admissible family $(g_\varepsilon)_\varepsilon$ for Ψ not cohomologous a to constant. We may assume without loss of generality that $\int g_\varepsilon d\eta = s$ for ε small (otherwise just use the admissible family $(\tilde g_\varepsilon)_\varepsilon$ given by $\tilde g_\varepsilon := g_\varepsilon + s - \int g_\varepsilon\, d\eta$ which is also not cohomologous to a constant). In particular,

$$\big\{\eta \in \mathcal{M}_1(f) \colon \int g_\varepsilon \, d\eta = s \text{ for all } \varepsilon \text{ small } \big\}$$

is a closed, non-empty set in $\mathcal{M}_1(f)$, hence compact. In particular, using the compactness and upper semi-continuity of the metric entropy function there exists $\eta_\varepsilon \in \mathcal{M}_1(f)$ such that $\int g_\varepsilon d\eta_\varepsilon = s$ and

$$h_{\eta_\varepsilon}(f) + \int \varphi_\varepsilon d\eta_\varepsilon = \sup_{\eta\in\mathcal{M}_1(f)} \{h_\eta(f) + \int \varphi_\varepsilon d\eta : \int g_\varepsilon d\eta = s\}.$$

Take $\tilde\eta \in \mathcal{M}_1(f)$ be an accumulation point of $(\eta_\varepsilon)_\varepsilon$ and assume for simplicity that $\eta_\varepsilon \to \tilde\eta$ as ε tends to zero. Then Proposition 3.2 yields that $\lim_{\varepsilon\to0} \int \varphi_\varepsilon d\eta_\varepsilon =$

$\mathcal{F}_*(\Phi,\tilde\eta)$ and $\lim_{\varepsilon\to 0}\int g_\varepsilon d\eta_\varepsilon = \mathcal{F}_*(\Psi,\tilde\eta)=s$. Using once more the upper semicontinuity of the metric entropy function

$$\begin{aligned}\lim_{\varepsilon\to0}\sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\int\varphi_\varepsilon d\eta : \int g_\varepsilon d\eta = s\} &= \lim_{\varepsilon\to0}\{h_{\eta_\varepsilon}(f)+\int\varphi_\varepsilon d\eta_\varepsilon\}\\ &\le h_{\tilde\eta}(f)+\mathcal{F}_*(\Phi,\tilde\eta)\\ &\le \sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\mathcal{F}_*(\Phi,\eta):\mathcal{F}_*(\Psi,\eta)=s\}.\end{aligned}$$

To prove the other inequality, let $\tilde\eta\in\mathcal{M}_1(f)$ be that attains the supremum in the right hand side above, that is, so that $s=\mathcal{F}_*(\Psi,\tilde\eta)$ and

$$\sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\mathcal{F}_*(\Phi,\eta):\mathcal{F}_*(\Psi,\eta)=s\} = h_{\tilde\eta}(f)+\mathcal{F}_*(\Phi,\tilde\eta)$$

Let $\delta>0$ be fixed and arbitrary. By Proposition 3.2 there exists $\varepsilon_\delta>0$ such that $\int g_\varepsilon d\tilde\eta\in(s-\delta,s+\delta)$ for all $0<\varepsilon<\varepsilon_\delta$. In particular, using the characterization of rate function $I_{f,\varphi_\varepsilon,g_\varepsilon}(\cdot)$ given by [32]

$$\begin{aligned}h_{\tilde\eta}(f)+\int\varphi_\varepsilon d\tilde\eta &\le \sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\int\varphi_\varepsilon d\eta : \int g_\varepsilon d\eta\in(s-\delta,s+\delta)\}\\ &= -\inf_{t\in(s-\delta,s+\delta)} I_{f,\varphi_\varepsilon,g_\varepsilon}(t)+P_{\mathrm{top}}(f,\varphi_\varepsilon)\end{aligned}$$

for every $0<\varepsilon<\varepsilon_\delta$. Taking the limit as $\varepsilon\to0$ in both sides of the inequality and using the convexity of the Legendre transform

$$\begin{aligned}\sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\mathcal{F}_*(\Phi,\eta):\mathcal{F}_*(\Psi,\eta)=s\} &= \lim_{\varepsilon\to0}(h_{\tilde\eta}(f)+\int\varphi_\varepsilon d\tilde\eta)\\ &\le \lim_{\varepsilon\to0}\,(P_{\mathrm{top}}(f,\varphi_\varepsilon)-\inf_{t\in(s-\delta,s+\delta)} I_{f,\varphi_\varepsilon,g_\varepsilon}(t))\\ &= P_{\mathrm{top}}(f,\Phi)-\min\{I_{f,\Phi,\Psi}(c-\delta),I_{f,\Phi,\Psi}(c+\delta)\}\end{aligned}$$

Since the rate function is continuous, taking δ tend to zero it follows

$$\begin{aligned}\sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\mathcal{F}_*(\Phi,\eta):\mathcal{F}_*(\Psi,\eta)=s\} &\le -I_{f,\Phi,\Psi}(c)+P_{\mathrm{top}}(f,\Phi)\\ &= \lim_{\varepsilon\to0}\{-I_{f,\varphi_\varepsilon,g_\varepsilon}(s)+P_{\mathrm{top}}(f,\varphi_\varepsilon)\}\\ &= \lim_{\varepsilon\to0}\sup_{\eta\in\mathcal{M}_1(f)}\{h_\eta(f)+\int\varphi_\varepsilon d\eta:\int g_\varepsilon d\eta=s\}.\end{aligned}$$

This finishes the proof of the lemma. □

Now, item (iii) is just a consequence of the previous lemma together with the fact that $I_{f,\Phi,\Psi}(s):=\lim_{\varepsilon\to0}I_{f,\varphi_\varepsilon,g_\varepsilon}(s)$. This finishes the proof of Theorem A.

4. Multifractal analysis of irregular sets

This section is devoted to the proof of our multifractal analysis results.

4.1. **Proof of Theorem B.** Let M be a compact metric space, $f : M \to M$ be a continuous map, $\Phi = \{\phi_n\}$ be an almost additive sequence of potentials with $P_{\text{top}}(f, \Phi) > -\infty$. By assumption, the unique equilibrium state μ_Φ of f with respect the Φ is a weak Gibbs measure. Given $J \subset \mathbb{R}$ and $n \geq 1$ set $X_{J,n} = \{x \in M : \frac{1}{n}\psi_n(x) \in J\}$.

Lemma 4.1. *Assume that $\Psi = \{\psi_n\}$ is a sequence of observables that satisfies at least one of the following properties:*

(a) *Ψ is asymptotically additive or;*
(b) *Ψ is a subadditive sequence such that*
  i. *satisfies the weak Bowen condition;*
  ii. *$\inf_{n\geq 1} \frac{\psi_n(x)}{n} > -\infty$ for every $x \in M$; and*
  iii. *the sequence $\{\frac{\psi_n}{n}\}$ is equicontinuous.*

Then Ψ satisfies the tempered variation condition $\lim_{\epsilon\to 0} \lim_{n\to+\infty} \frac{\gamma_n(\Psi,\epsilon)}{n} = 0$. In particular, given $J \subset \mathbb{R}$ be a closed set and $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that if $0 < \varepsilon < \varepsilon_\delta$ then there exists $N = N_{\delta,\epsilon} \in \mathbb{N}$ so that $B(x, n, \varepsilon) \subset X_{J_\delta,n}$ for all $n \geq N$ and every $x \in X_{J,n}$.

Proof. It is immediate that the tempered variation condition is satisfied for sequences of observables satisfying the weak Bowen condition. Moreover, it holds for asymptotically additive sequences as proved in [33, Lemma 2.1].

Let us prove now the second part of the lemma. Let $\delta > 0$ be given. By tempered variation condition there is $\varepsilon_\delta > 0$ such that $\lim_{n\to\infty} \gamma_n(\psi, \varepsilon) < \delta n$ for all $0 < \varepsilon < \varepsilon_\delta$. So, given $0 < \varepsilon < \varepsilon_\delta$ there exists a large $N = N_{\delta,\varepsilon} \in \mathbb{N}$ such that if $n \geq N$ we have $\gamma_n(\psi, \varepsilon) \leq \delta n$. So, if $0 < \varepsilon < \varepsilon_\delta$, $n \geq N$ and $x \in X_{J,n}$, $y \in B(x, n, \varepsilon)$ then

$$\frac{\psi_n(x)}{n} - \frac{\gamma_n(\psi,\varepsilon)}{n} \leq \frac{\psi_n(y)}{n} \leq \frac{\psi_n(x)}{n} + \frac{\gamma_n(\psi,\varepsilon)}{n}$$

and, consequently,

$$\frac{\psi_n(x)}{n} - \delta \leq \frac{\psi_n(y)}{n} \leq \frac{\psi_n(x)}{n} + \delta$$

meaning that $y \in X_{J_\delta,n}$. This finishes the proof of the lemma. $\square$

We can now proceed with the proof of Theorem B. Assuming that $\overline{X}_J \neq \emptyset$, we shall prove that $P_{\overline{X}_J}(f, \Phi) \leq P_{\text{top}}(f, \Phi) - L_{J_\delta,\mu_\Phi}$. If $L_{J_\delta,\mu_\Phi} = 0$ there is nothing to prove so we assume without loss of generality that $L_{J_\delta,\mu_\Phi} > 0$. For our purpose it is enough to prove that for every $\alpha > P_{\text{top}}(f, \Phi) - L_{J_\delta,\mu_\Phi}$, given $\epsilon > 0$ and $N \in \mathbb{N}$ there exists $\mathcal{G}_N \subset \bigcup_{n\geq N} \mathcal{I}_n$ satisfying the covering property $\bigcup_{(x,n)\in\mathcal{G}_N} B(x, n, \epsilon) \supset \overline{X}_I$ and also $\sum_{(x,n)\in\mathcal{G}_N} e^{-\alpha n + \phi_n(x)} \leq a(\epsilon) < \infty$. Let $\alpha > P_{\text{top}}(f, \Phi) - L_{J_\delta,\mu_\Phi}$ and $0 < \varepsilon < \varepsilon_\delta$ fixed, we take $\zeta > 0$ small so that $\alpha > P_{\text{top}}(f, \Phi) - L_{J_\delta,\mu_\Phi} + \zeta$. There exists $N_0 \geq N_{\delta,\epsilon}$ such that $K_n(\varepsilon) \leq e^{\frac{\zeta}{4}n}$, $K_n(\frac{\varepsilon}{2}) \leq e^{\frac{\zeta}{4}n}$ and

$$\mu_\Phi\Big(\{x \in M : \frac{1}{n}\psi_n(x) \in J_\delta\}\Big) \leq e^{-(L_{J_\delta,\mu_\Phi} - \frac{\zeta}{2})n}$$

for all $n \geq N_0$. There is no loss of generality in supposing that $N \geq N_0$. Given $N \geq N_0$ and $x \in \overline{X}_J$ take $m(x) \geq N$ so that $x \in X_{J_{\frac{\delta}{2}},m(x)}$ and consider $\mathcal{G}_N :=$

$\{(x,m(x)) : x \in \overline{X}_J\}$. Now, let $\hat{\mathcal{G}}_N \subset \mathcal{G}_N$ be a maximal (ℓ,ε)-separated set. In particular if (x,ℓ) and (y,ℓ) belong the $\hat{\mathcal{G}}_N$ then $B(x,\ell,\frac{\varepsilon}{2}) \cap B(x,\ell,\frac{\varepsilon}{2}) = \emptyset$. Hence, for $0<\varepsilon<\varepsilon_\delta$ given by Lemma 4.1, using the Gibbs property of μ_Φ we deduce that

$$\begin{aligned}\sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{-\alpha m(x)+\phi_{m(x)}(x)} &= \sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{(P-\alpha)m(x)} e^{-Pm(x)+\phi_{m(x)}(x)} \\ &\le \sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{(P-\alpha)m(x)} K_{m(x)}(\varepsilon)\, \mu_\Phi(B(x,m(x),\varepsilon))\end{aligned}$$

Now, we write $\hat{\mathcal{G}}_N = \cup_{\ell\ge 1}\hat{\mathcal{G}}_{\ell,N}$ with the level sets $\hat{\mathcal{G}}_{\ell,N} := \{(x,\ell) \in \hat{\mathcal{G}}_N\}$. By Lemma 4.1 each dynamical ball $B(x,\ell,\varepsilon)$ is contained in $X_{I_\delta,\ell}$. Thereby, using that $\mu_\Phi(B(x,m(x),\varepsilon)) \le K_{m(x)}(\varepsilon)K_{m(x)}(\varepsilon/2)\mu_\Phi(B(x,m(x),\varepsilon/2)$ then

$$\begin{aligned}\sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{-\alpha m(x)+\phi_{m(x)}(x)} &\le \sum_{(x,m(x))\in\hat{\mathcal{G}}_N} K_{m(x)}(\varepsilon) e^{(P-\alpha)(m(x))}\mu_\Phi(B(x,m(x),\varepsilon)) \\ &= \sum_{\ell\ge N} K_\ell(\varepsilon)e^{(P-\alpha)\ell} \sum_{x\in\hat{\mathcal{G}}_{N,\ell}} \mu_\Phi(B(x,\ell,\varepsilon)) \\ &\le \sum_{\ell\ge N} K_\ell(\varepsilon)K_\ell(\frac{\varepsilon}{2})e^{(P-\alpha)\ell} \sum_{x\in\hat{\mathcal{G}}_{N,\ell}} \mu_\Phi(B(x,\ell,\varepsilon/2)) \\ &\le \sum_{\ell\ge N} K_\ell(\varepsilon)K_\ell(\frac{\varepsilon}{2})e^{(P-\alpha)\ell}\, \mu_\Phi(X_{J_\delta,\ell}) \\ &\le \sum_{\ell\ge N} e^{(P-\alpha-L_{J_\delta,\mu_\Phi}+\zeta)\ell}\end{aligned}$$

that is finite and independent of the choose of N. This proves that for any closed interval $J\subset\mathbb{R}$ and any small $\delta>0$ it holds $P_{\underline{X}_J}(f,\Phi) \le P_{\overline{X}_J}(f,\Phi) \le P_{\text{top}}(f,\Phi) - L_{J_\delta,\mu_\Phi} \le P(f,\Phi)$ and proves the theorem.

Remark 4.2. Let us mention that the argument of Theorem B proving that for any closed interval $J\subset\mathbb{R}$ and any small $\delta>0$,

$$P_{\underline{X}_J}(f,\Phi) \le P_{\text{top}}(f,\Phi) - L_{J_\delta,\mu_\Phi} \le P(f,\Phi) \tag{4.1}$$

carries under the weaker Gibbs condition (2.3). Taking into account the difficulty that the moments where the Gibbs property occurs may depend on the point justifies the fact that the estimate (4.1) holds for set $\underline{X}_J$. Since the proof of this fact is similar to the the one of Theorem B we give only a sketch of proof with main ingredients. In fact, by (2.3) there is $\varepsilon_0>0$ such that: for all $0<\varepsilon<\varepsilon_0$ there exists $K_n(\varepsilon)>0$ such that for μ_Φ-a.e. point x there exists a sequence $n_k(x)\to\infty$ with

$$K_{n_k(x)}(\varepsilon)^{-1} \le \frac{\mu_\Phi(B(x,n_k(x),\varepsilon))}{e^{-n_k(x)P+S_{n_k(x)}\phi(x)}} \le K_{n_k(x)}(\varepsilon).$$

Using $\underline{X}_J \subset \bigcup_{\ell\ge1}\bigcap_{j\ge\ell} X_{J_\delta,j}$ where $X_{J,n}=\{x\in M : \frac{1}{n}S_n\psi(x)\in J\}$ it is not difficulty check that for all $x\in\underline{X}_J$ there is a sequence of positive numbers $(m_j(x))_{j\in\mathbb{N}}$ converging to infinity such that $x\in X_{J_{\frac{\delta}{2}},m_j(x)}$ and $m_j(x)$ is a moment where the Gibbs property holds. Consider $\delta,\zeta>0$ arbitrary small, $\alpha > P_{\text{top}}(f,\Phi) - L_{J_\delta,\mu_\Phi}+\zeta$, $\varepsilon>0$ small and $N\in\mathbb{N}$ large. Take $m(x)\ge N$ so that $x\in X_{J_{\frac{\delta}{2}},m(x)}$, the constants satisfy $K_{m(x)}(\varepsilon) \le e^{\frac{\zeta}{4} m(x)}$, $K_{m(x)}(\frac{\varepsilon}{2}) \le e^{\frac{\zeta}{4} m(x)}$, and

$$\mu_\Phi\Big(\{x \in M : \frac{1}{m(x)} \psi_{m(x)}(x) \in J_\delta\}\Big) \le e^{-(L_{J_\delta,\mu_\Phi} - \frac{\zeta}{2}) m(x)}.$$

Setting $\mathcal{G}_N := \{(x, m(x)) : x \in \underline{X}_J\}$ we prove the result just follow with the same estimates used in the proof of Theorem B and obtain that $P_{\underline{X}_J}(f, \Phi) \le P_{\text{top}}(f, \Phi) - L_{J_\delta,\mu_\Phi}$ as claimed.

4.2. **Proof of Corollary B.** By [2] and [21], since $\Phi = 0$ clearly satisfies the bounded variation condition μ_0 is a Gibbs measure. So Theorem B implies that $h_{\overline{X}_J}(f) \le h_{\text{top}}(f) - L_{J_\delta,\mu_0}$, for all $\delta > 0$ sufficiently small. By the large deviations estimates from [31] and Theorem A we have that

$$h_{\overline{X}_I}(f) \le h_{\text{top}}(f) - \inf_{s \in J_\delta} I_{f,0,\Psi}(s)$$

for all $\delta > 0$ small. The Legendre transform of Ψ is continuous. Hence

$$h_{\overline{X}_I}(f) \le h_{\text{top}}(f) - \inf_{s \in J} I_{f,0,\Psi}(s)$$

For the lower bound we proceed as follows, with an estimate similar to [8, Theorem B]. It follows from Barreira and Doutor [4] that if $X(\alpha) \ne \emptyset$ then $h_{X(\alpha)}(f) = \sup_{\eta \in \mathcal{M}_1(f)}\{h_\eta(f) : \mathcal{F}_*(\Psi, \eta) = \alpha\}$. Thus, if $\overline{X}_J \ne \emptyset$ and $\mathcal{F}_*(\Psi, \mu_0) \notin J$ then Theorem A (item ii.) yields that the infimum of $\inf_{s \in J} I_{f,0,\Psi}(s)$ is realized in a border point c_* of J. Thus:

$$\begin{aligned} h_{\text{top}}(f, \Phi) - I_{f,0,\Psi}(c_*) &= h_{X(c_*)}(f) \le h_{X(J)}(f,) \\ &\le h_{X(\overline{J})}(f) \le h_{\underline{X}_J}(f) \\ &\le h_{\overline{X}_J}(f) \le h_{\text{top}}(f) - I_{f,0,\Psi}(c_*). \end{aligned}$$

In particular, we prove we prove that for $J_c = \mathbb{R} \setminus (\mathcal{F}_*(\Psi, \mu_0) - c, \mathcal{F}_*(\Psi, \mu_0) + c)$ we get $\overline{X_{J_c}} = \overline{X}_{\mu_0,\Psi,c}$ and so

$$h_{\overline{X}_{\mu_0,\Psi,c}}(f) = h_{\text{top}}(f) - \min\{I_{f,0,\Psi}\big(\mathcal{F}_*(\Psi, \mu_0) + c\big)\ ,\ I_{f,0,\Psi}\big(\mathcal{F}_*(\Psi, \mu_0) - c\big)\}$$

whenever the set $\overline{X}_{\mu_0,\Psi,c}$ is not empty. So by Theorem A we deduce that the function $\mathbb{R}_0^+ \ni c \mapsto h_{\overline{X}_{\mu_0,\Psi,c}}(f)$ is strictly decreasing and concave in a neighborhood of zero.

5. Examples and applications

In this section we provide some applications of the theory concerning the study of some classes of non additive sequences of potentials related to either Lyapunov exponents or entropy.

5.1. **Linear cocycles.** Here we consider cocycles over subshifts of finite type as considered by Feng, Lau and Käenmäki [15, 17]. Let $\sigma : \Sigma \to \Sigma$ be the shift map on the space $\Sigma = \{1, \ldots, \ell\}^{\mathbb{N}}$ endowed with the distance $d(x, y) = 2^{-n}$ where $x = (x_j)_j$, $y = (y_j)_j$ and $n = \min\{j \ge 0 : x_j \ne y_j\}$. Consider matrices $M_1, \ldots, M_\ell \in \mathcal{M}_{d\times d}(\mathbb{C})$ such that for every $n \ge 1$ there exists $i_1, \ldots, i_n \in \{1, \ldots, \ell\}$ so that the product matrix $M_{i_1} \ldots M_{i_n} \ne 0$. Then, the topological pressure function is well defined as $P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\iota \in \Sigma_n} \|M_\iota\|^q$ where $\Sigma_n = \{1, \ldots, \ell\}^n$ and for any $\iota = (i_1, \ldots, i_n) \in \Sigma_n$ one considers the matrix $M_\iota = M_{i_n} \ldots M_{i_2} M_{i_1}$. Moreover, for any σ-invariant probability measure μ define also the maximal Lyapunov exponent of μ by

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{\iota\in\Sigma_n} \mu([\iota]) \log \|M_\iota\|$$

and it holds that $P(q) = \sup\{h_\mu(\sigma) + q\, M_*(\mu) : \mu \in \mathcal{M}_\sigma\}$. Notice that this is the variational principle for the potentials $\Phi = \{\varphi_n\}$ where $\varphi_n(x) = q \log \|M_{\iota_n(x)}\|$ and for any $x \in \Sigma$ we set $\iota_n(x) \in \Sigma_n$ as the only symbol such that x belongs to the cylinder $[\iota_n(x)]$. From [17, Proposition 1.2], if the set of matrices $\{M_1, \dots, M_d\}$ is irreducible over $\mathbb{C}^d$, (i.e. there is no non-trivial subspace $V \subset \mathbb{C}^d$ such that $M_i(V) \subset V$ for all $i = 1, \dots, \ell$) there exists a unique equilibrium state μ_q for σ with respect to Φ and it is a Gibbs measure: there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\mu_q([\iota_n])}{e^{-nP(q)} \|M_{\iota_n}\|^q} \le C$$

for all $\iota_n \in \Sigma_n$ and $n \ge 1$. Since the potentials $\varphi_n = \log \|M_{\iota_n(x)}\|$ are constant in n-cylinders the family of potentials Φ clearly satisfies the bounded distortion condition. It follows as a consequence of the large deviations bound in [31] and Theorem B that taking $\Psi = \Phi$ with $q = 1$ and $c > 0$, the set

$$\overline{X}_c = \{x \in \Sigma : \limsup_{n\to\infty} \Big| \frac{1}{n} \log \|M_{\iota_n(x)}\| - M_*(\mu_\Phi) \Big| > c\}$$

of points whose exponential growth of $M_{\iota_n(x)}$ is c-far away from the maximal Lyapunov exponent $M_*(\mu_\Phi)$ for infinitely many values of n has topological pressure strictly smaller than $P_{\text{top}}(f, \Phi)$. Moreover, with respect to the maximal entropy measure μ_0 Corollary B yields that the topological pressure function $c \mapsto h_{\overline{X}_c}(f)$ is strictly decreasing and concave for small positive c.

5.2. **Non-conformal repellers.** The following class of local diffeomorphisms was introduced by Barreira and Gelfert [3] in the study of multifractal analysis for Lyapunov exponents associated to non-conformal repellers. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 local diffeomorphism, and let $J \subset \mathbb{R}^2$ be a compact f-invariant set. Following [3], we say that f satisfies the following *cone condition* on J if there exist a number $b \le 1$ and for each $x \in J$ there is a one-dimensional subspace $E(x) \subset T_x\mathbb{R}^2$ varying continuous with x such that $Df(x)C_b(x) \subset \{0\} \cup \operatorname{int} C_b(fx)$ where $C_b(x) = \{(u,v) \in E(x) \bigoplus E(x)^\perp : \ ||v|| \le b||u||\}$. It follows from [3, Proposition 4] that the later condition implies that both families of potentials given by $\Psi_1 = \{\log \sigma_1(Df^n(x))\}$ and $\Psi_2 = \{\log \sigma_2(Df^n(x))\}$ are almost additive, where $\sigma_1(L) \ge \sigma_2(L)$ stands for the singular values of the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., the eigenvalues of $(L^*L)^{1/2}$ with L^* denoting the transpose of L. Assume that J is a locally maximal topological mixing repeller of f such that:

(i) f satisfies the cone condition on J, and
(ii) f has bounded distortion on J, i.e., there exists some $\delta > 0$ such that

$$\sup_{n\ge1} \frac{1}{n} \log \sup \Big\{ ||Df^n(y)(Df^n(z))^{-1}|| : \ x \in J \text{ and } y, z \in B(x, n, \delta) \Big\} < \infty.$$

Then it follows from [2, Theorem 9] yields that there exists a unique equilibrium state μ_i for (f, Φ_i) which is a weak Gibbs measure with respect to the family of potentials Φ_i, for $i = 1, 2$. Moreover, from [31, Example 4.6], for any $c > 0$ the tail of the convergence to the largest or smallest Lyapunov exponent (corresponding respectively to $j=1$ or $j=2$)

$$\mu_i\left(\left\{x\in M : \left|\frac{1}{n}\log\sigma_j(Df^n(x)) - \lim_{n\to\infty}\frac{1}{n}\int\log\sigma_j(Df^n(x))d\mu_i\right| > c\right\}\right)$$

decays exponentially fast as $n\to\infty$. Moreover, it follows from Corollary A that this exponential decay rate varies continuously with c.

One other consequence is that, although the irregular sets associated to $\Psi_j = \{\log\sigma_j(Df^n(x))\}$ have full topological pressure (using [33] and the fact that f has the specification property) the set of irregular points whose time-n Lyapunov exponents remain c-far away from the corresponding mean have topological pressure strictly smaller than the topological pressure of the system.

5.3. **Entropy and Gibbs measures.** Let $\sigma:\Sigma\to\Sigma$ be the shift map on the space $\Sigma=\{1,\dots,\ell\}^{\mathbb{N}}$ endowed with the distance $d(x,y)=2^{-n}$ where $x=(x_j)_j$, $y=(y_j)_j$ and $n=\min\{j\ge 0 : x_j\neq y_j\}$. Set $\Sigma_n=\{1,\dots,\ell\}^n$ and for any $\iota=(i_1,\dots,i_n)\in\Sigma_n$ consider the n-cylinders $[\iota]=\{x\in\Sigma : x_j=i_j,\ \forall 1\le j\le n\}$.

Let $\Phi=\{\varphi_n\}$ be an almost additive sequence of potentials with the bounded distortion property and μ_Φ be the unique equilibrium state for f with respect to Φ given by [2]. Fix $C>0$ so that for every $x\in\Sigma$

$$\varphi_n(x)+\varphi_m(f^n(x))-C\le\varphi_{m+n}(x)\le\varphi_n(x)+\varphi_m(f^n(x))+C.$$

Since μ_Φ is Gibbs there exists $P\in\mathbb{R}$ and $K>0$ so that

$$\frac{1}{K}\le\frac{\mu_\Phi([\iota_n(x)])}{e^{-Pn+\varphi_n(x)}}\le K$$

for every $n\ge 1$ and every $x\in\Sigma$. In consequence, if $\psi_n(x)=\log\mu_\Phi([\iota_n(x)])$ then

$$\begin{aligned}\exp\psi_{m+n}(x)=\mu([\iota_{m+n}(x)]) &\le K\ e^{-P(m+n)+\varphi_{m+n}(x)}\\ &\le K\ e^C\ e^{-Pn+\varphi_n(x)}\ e^{-Pm+\varphi_m(f^n(x))}\\ &\le K^3\ e^C\ \exp\psi_n(x)\ \exp\psi_m(f^n(x))\end{aligned}$$

for every $n\ge 1$ and $x\in\Sigma$. Thus, $\psi_{m+n}(x)\le\psi_n(x)+\psi_m(f^n(x))+\tilde{C}$ with $\tilde{C}=C+3\log K$. Since the lower bound is completely analogous we deduce that $\Psi=\{\psi_n\}$ is almost additive and satisfies the bounded distortion condition since ψ_n is constant on n-cylinders. In particular these satisfy the hypothesis of Theorem B in [31] to deduce exponential large deviations. In fact it is a simple computation to prove that if μ_Φ is a weak Gibbs measure then the corresponding sequence of functions Ψ as above are asymptotically additive, but we shall not prove or use this fact here. By [33] the irregular set has full topological pressure. Since this set is contained in the set of points for which

$$\limsup_{n\to\infty}\left|-\frac{1}{n}\log\mu_\Phi([\iota_n(x)])-h_{\mu_\Phi}(f)\right|>0$$

this has also full topological pressure. From our Theorem B, for any $c>0$ the set of points so that

$$\limsup_{n\to\infty}\left|-\frac{1}{n}\log\mu_\Phi([\iota_n(x)])-h_{\mu_\Phi}(f)\right|>c$$

has topological pressure strictly smaller than $P_{\text{top}}(f,\Phi)$.

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