EXTREME VALUES FOR MISIURUREWICZ QUADRATIC MAPS

JORGE MILHAZES FREITAS AND ANA CRISTINA MOREIRA FREITAS

Abstract. We consider the quadratic family of maps given by $f_a(x) = 1 - ax^2$ with $x \in [-1, 1]$, where a is a Misiurewicz parameter. On this set of parameters, there is an f_a-invariant probability measure, μ_a, that is absolutely continuous with respect to Lebesgue.

For each of these chaotic dynamical systems we study the extreme value distribution of the stationary stochastic processes $X_0, X_1, \ldots$, given by $X_n = g \circ f_a^n$, for every integer $n \geq 0$, where g is a certain type of continuous random variable on the probability space $([-1, 1], \mathcal{B}, \mu_a)$, with $\mathcal{B}$ denoting the Borel σ-algebra. Using the techniques developed by Benedicks and Carleson, we show that the limiting distribution of $M_n = \max\{X_0, \ldots, X_{n-1}\}$ is the same as that which would apply if the sequence $X_0, X_1, \ldots$ was independent and identically distributed. This result allows us to obtain that the asymptotic distribution of M_n is of Type III (Weibull).

1. Introduction

In broad terms, Dynamical Systems is the study of the long term behavior of typical trajectories (orbits) governed by the laws of the system. Its applications are innumerable, range from the microscopic quantum mechanics to the macroscopic evolution of star systems and touch several different fields of knowledge. The emergence of chaotic dynamics has switched the analysis from a Geometrical and Topological perspective to a Mesure Theoretical and Probabilistic view. The statistical properties of evolving orbits have become a subject of much interest and study. These properties are usually tied in with averages of observable quantities and their asymptotic distributions given by Central Limit Theorems.

However, in certain circumstances, the mean or central statistics do not enclose the information pertinent to the case in question. Take for example the study of river floods. If one wants to evaluate the risk of having a very serious flood, the average level hight of the river, say in a decade, does not tell us as much as do the maximum water level values observed in the days that the river hight has exceeded a given threshold causing a flood. In this type of situation of risk assessment associated with rare events, from the most unlikely as tsunamis or terrorist attacks to the everyday car accidents or failure of a mechanical structure, one needs a different statistical analysis.

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This is where Extreme Value Theory comes in. It is mostly concerned with the study of distributional properties of the higher (lower) order statistics of a sample, like the maximum (minimum) of the sample. Its major classical result states that there are only three types of non-degenerate asymptotic distributions for the maximum of an independent and identically distributed (i.i.d.) sample under linear normalization. This result has been extended to the dependent context. In fact, it has been shown that, under certain conditions on the dependence structure, the same limit laws apply for maxima.

The dynamical systems we consider in this work are the quadratic maps given by $f_a(x) = 1 - ax^2$ on $I = [-1, 1]$, with $a \in \mathcal{M}$, where $\mathcal{M}$ is the Misiurewicz parameter set introduced in [Mis81]. This set $\mathcal{M}$ is contained in a larger set of parameters $\mathcal{BC}$ introduced by Benedicks and Carleson in [BC85]. The set $\mathcal{BC}$ has positive Lebesgue measure and is built in such a way that for every $a \in \mathcal{BC}$ there is exponential growth of the derivative of f_a along the critical orbit, i.e., there is $c > 0$ such that

$$\left|(f_a^n)'\,(f_a(0))\right| \geq \mathrm{e}^{cn},$$

for all $n \in \mathbb{N}$. This property is usually referred to as the Collet-Eckmann condition (see [CE83]) and guarantees not only the non-existence of an attracting periodic orbit but also assures the existence of an ergodic f_a-invariant probability measure μ_a that is absolutely continuous with respect to Lebesgue measure on $[-1, 1]$. The set $\mathcal{M}$ has zero Lebesgue measure but is a dense subset of $\mathcal{BC}$ (see [Thu01]) and has the following extra property: for every $a \in \mathcal{M}$ the critical orbit is non-recurrent, which means that there is $d > 0$ such that

$$f_a^n(0) \notin (-d, d) \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

Most of the times we simply assume that $a \in \mathcal{BC}$ and the main features of these parameters are described together with the Benedicks-Carleson techniques in Section 3. When condition (1.1) is needed, it will always be specifically mentioned. These Benedicks-Carleson systems are chaotic and highly sensitive on initial conditions. In fact, after some iterates the behavior of most orbits becomes erratic and distributed on the set $[-1, 1]$ accordingly to the invariant measure μ_a, or in other words, the frequency of visits payed by the orbit of Lebesgue-almost every point x to a Borel measure set $A \subset [-1, 1]$ tends to $\mu_a(A)$. Hence, it is surely interesting to study the statistical properties of the orbits of these systems and, here, we are particularly concerned with their extreme type behavior.

A natural way to build a stationary stochastic process associated to f_a for some $a \in \mathcal{BC}$ is to consider the random variable (r.v.) Y_0 defined on the probability space $([-1, 1], \mu_a)$, taking values in $[-1, 1]$ with distribution function (d.f.) $F_a(x) = \mu_a\{(-\infty, x] \cap [-1, 1]\}$. Then, we iterate Y_0 using f_a to obtain $Y_1 = f_a(Y_0)$, whose d.f. is also F_a by the f_a-invariance of μ_a. Next, we just repeat the process to obtain the stationary sequence $Y_0, Y_1, Y_2, \ldots$ For future reference we say that this stochastic process is the *natural process* associated to f_a.

For each $a \in \mathcal{BC}$ we may also define a stochastic processe $X_0, X_1, X_2, \ldots$ in the following way:

$$X_n = g \circ f_a^n, \quad \text{for each } n \in \mathbb{N}_0, \tag{1.2}$$

where $g : [-1, 1] \to \mathbb{R}$ is a continuous r.v. on the probability space $(\mathbb{R}, \mathcal{B}, \mu_a)$, with $\mathcal{B}$ denoting the Borel σ-algebra. We will consider two types of r.v. g:

(1) The first type is denoted by g_1. We require that g_1 is C^1 in a an open neighborhood of 1 and
   (a) $g_1(x) < g_1(1)$ for every $x \in [-1, 1)$
   (b) $\lim_{x\to 0^+} \frac{g_1(1)-g_1(1-x)}{x} > 0$.

(2) The second type is denoted by g_2 and is such that
   (a) $g_2(x) < g_2(0)$ for every $x \in [-1, 1] \setminus \{0\}$
   (b) There exists $r > 0$ such that $\bar{g}_2 = g_2|_{(-r,0]}$, $\hat{g}_2 = g_2|_{[0,r)}$ are invertible and for $x \in (-r, r)$ we have $\bar{g}_2(x) = g_2(0) - \mathcal{O}(|x|^{\bar{q}})$ and $\hat{g}_2(x) = g_2(0) - \mathcal{O}\left(x^{\hat{q}}\right)$, for some $\bar{q}, \hat{q} \in \mathbb{N}$, which means that $\frac{g_2(0)-\bar{g}_2(x)}{|x|^{\bar{q}}} \to \text{const} > 0$, $\frac{g_2(0)-\hat{g}_2(x)}{x^{\hat{q}}} \to \text{const} > 0$, as $x \to 0$.
   We set $q = \max\{\bar{q}, \hat{q}\}$.

Observe that the f_a-invariance of the probability measure μ_a implies that every stochastic process above is stationary (see for example [KT66, Section 15.4]) and the common marginal d.f. is given by $G_a(x) = \mu_a \{X_0 \le x\}$. We also remark that taking $g_1(x) = Id(x) = x$ then the process $X_0, X_1, X_2, \dots$ corresponds to the natural stochastic process $Y_0, Y_1, Y_2, \dots$ while if we define $g_2(x) = 1 - ax^2$ we obtain a process corresponding to $Y_1, Y_2, Y_3, \dots$

Our goal is to study the asymptotic distribution of the partial maximum

$$M_n = \max\{X_0, X_1, \dots, X_{n-1}\}, \tag{1.3}$$

when properly normalized. The main results of this work state that the limiting laws of M_n are the same as if $X_0, X_1, \dots$ were independent with the same d.f. G_a. In fact, we verify that under appropriate normalization the asymptotic distribution of M_n is of type III (Weibull). As usual, we denote by G_a^{-1} the generalized inverse of the d.f. G_a, which is to say that $G_a^{-1}(y) := \inf\{x : G_a(x) \ge y\}$.

Theorem A. *For each $a \in \mathcal{M}$ and every stationary stochastic process $(X_i)_{i\in\mathbb{N}_0}$ given by* (1.2) *with $g = g_1$ satisfying conditions* (1a) *and* (1b)*, consider the sequences $b_n = g_1(1)$ and $a_n = \left(1 - G_a^{-1}\left(1 - \frac{1}{n}\right)\right)^{-1}$. Then, we have the following asymptotic behavior:*

$$P\{a_n(M_n - b_n) \le x\} \to H_1(x) = \begin{cases} e^{-(-x)^{1/2}} & , x \le 0 \\ 1 & , x > 0 \end{cases}.$$

Theorem B. *For each $a \in \mathcal{M}$ and every stationary stochastic process $(X_i)_{i\in\mathbb{N}_0}$ given by* (1.2) *with $g = g_2$ satisfying conditions* (2a) *and* (2b)*, consider the sequences $b_n = g_2(0)$ and $a_n = \left(1 - G_a^{-1}\left(1 - \frac{1}{n}\right)\right)^{-1}$. Then, we have the following asymptotic behavior:*

$$P\{a_n(M_n - b_n) \le x\} \to H_2(x) = \begin{cases} e^{-(-x)^{1/q}} & , x \le 0 \\ 1 & , x > 0 \end{cases}.$$

We mention that Haiman [Hai03] has obtained a similar asymptotic result for the natural stochastic process associated with the tent map. We are convinced that the same type of arguments used here for quadratic maps would allow us to obtain a different proof of Haiman's result.

We also refer that a study concerning Extremes for Dynamical Systems, essentially focusing the finite sample behavior of maxima, has already been done by Balakrishnan, Nicolis and Nicolis in [BNN95].

2. Motivation and Strategy

The study of the limit behavior for maxima of a stationary process can be reduced, under adequate conditions on the dependence structure, to the Classical Extreme Value Theory for independent and identically distributed (i.i.d.) sequences of r.v. Hence, to the stationary process $X_0, X_1, \ldots$ we associate an independent sequence of r.v. denoted by $Z_0, Z_1, \ldots$ with common d.f. given by $G_a(x) = P\{X_0 \le x\}$. We also set for each $n \in \mathbb{N}$

$$\hat{M}_n = \max\{Z_0, \ldots, Z_{n-1}\}. \tag{2.1}$$

Let us focus on the conditions that allow us to relate the asymptotic distribution of M_n with that of $\hat{M}_n$. Following [LLR83] we refer to these conditions as $D(u_n)$ and $D'(u_n)$, where u_n is a suitable sequence of thresholds converging to $\max_{x\in[-1,1]} g$, as n goes to ∞, and will be defined below. The first condition, $D(u_n)$, imposes a certain type of distributional mixing property. Essentially it says that the dependence between some special type of events fades away as they become more and more apart in the time line. The second one, $D'(u_n)$, restricts appearance of clusters, that is, it makes the occurrence of consecutive 'exceedances' of the level u_n an unlikely event.

As we have said, $D(u_n)$ is a type of mixing requirement specially adapted to extreme value theory. In this context, the events of interest are those of the form $\{X_i \le u\}$ and their intersections. Observe that $\{M_n \le u\}$ is just $\{X_0 \le u, \ldots, X_{n-1} \le u\}$. A natural mixing condition in this context is the following. Let $G_{i_1,\ldots,i_n}(x_1, \ldots, x_n)$ denote the joint d.f. of $X_{i_1}, \ldots, X_{i_n}$ and set $G_{i_1,\ldots,i_n}(u) = G_{i_1,\ldots,i_n}(u, \ldots, u)$.

Condition ($D(u_n)$)**.** For any integers $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_k$ for which $j_1 - i_p > m$, and any large $n \in \mathbb{N}$,

$$\left|G_{i_1,\ldots,i_p,j_1,\ldots,j_k}(u_n) - G_{i_1,\ldots,i_p}(u_n)G_{j_1,\ldots,j_k}(u_n)\right| \le \gamma(m),$$

where $\gamma(m) \to 0$ as $m \to \infty$.

We remark that the actual definition of $D(u_n)$ appearing in [LLR83, Section 3.2] is a weaker requirement but the one considered here is simpler to formulate and suits our purposes.

Consider a sequence of stationary random variables $\Xi_1, \Xi_2 \ldots$ with common d.f. F. We say that an *exceedance* of the 'level' u_n occurs at time i if $\Xi_i > u_n$. The probability of such an exceedance is $1 - F(u_n)$ and hence the mean value of the number of exceedances occurring up to n is $n(1 - F(u_n))$.

Condition $(D'(u_n))$. We say that $D'(u_n)$ holds for the sequence $X_0, X_1, X_2, \ldots$ if

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} = 0. \tag{2.2}$$

The sequences of levels u_n considered are such that $n(1 - G_a(u_n)) \to \tau$ as $n \to \infty$, for some $\tau \geq 0$, which means that, in a time period of length n, the expected number of exceedances is approximately τ and the average number of exceedances in the time interval $\{0, \ldots, [n/k]\}$ is approximately τ/k, which goes to zero as $k \to \infty$. However, it may happen that the exceedances have a tendency to be concentrated in the time period following the first exceedance at time 0. Condition 2.2 prevents this from happening, i.e., forbids the concentration of exceedances by bounding the probability of more than one exceedance in the time interval $\{0, \ldots, [n/k]\}$. This guarantees that the exceedances should appear scattered through the time period $\{0, \ldots, n-1\}$.

The special relevance of both these conditions is the following: let a_n and b_n be sequences such that $P\{a_n(\hat{M}_n - b_n) \leq x\} \to H(x)$ for some non-degenerate d.f. H; if $D(u_n)$, $D'(u_n)$ are satisfied for the stationary sequence $X_1, X_2, \ldots$, when $u_n = x/a_n + b_n$ for each x, then $P\{a_n(M_n - b_n) \leq x\} \to H(x)$. See [LLR83, Theorem 3.5.2]. This means that if we are able to show that conditions $D(u_n)$ and $D'(u_n)$ hold for the stationary process $X_0, X_1, \ldots$, then M_n and $\hat{M}_n$ share the same asymptotic distribution with the same normalizing sequences. Consequently, our strategy to prove Theorems A and B is the following:

- Compute the limiting distribution of $\hat{M}_n$ and the respective normalizing sequences a_n and b_n.
- Show that conditions $D(u_n)$ and $D'(u_n)$ are satisfied for the stochastic process $X_0, X_1, X_2, \ldots$ defined in (1.2).

The rest of the paper is dedicated to the proof of these assertions and is structured as follows. In Section 3 we describe the properties of the dynamical systems f_a with $a \in \mathcal{BC}$ and the Benedicks-Carleson techniques. Then, in Section 4 we study the asymptotic behavior of the maximum in the i.i.d. case and identify the desired domain of attraction of $\hat{M}_n$ and the respective normalizing sequences a_n and b_n. The validity of condition $D(u_n)$ is a consequence of the very good mixing properties of the systems considered here. Actually, it follows from the fact that these systems possess a weak-Bernoulli generator (see Section 3.8 and Remark 3.1). Hence, we are left with the burden of proving $D'(u_n)$. In Section 5 we use the geometric properties of the systems to show Proposition 5.2 that paves the way for the proof of $D'(u_n)$ that is finally established in Section 6. In Section 7, we present a small simulation study in order to compare the finite sample behavior of the normalized M_n with the asymptotic one.

3. Properties of the Benedicks-Carleson parameters

The Benedicks-Carleson Theorem (see [BC85] or Section 2 of [BC91]) states that there exists a positive Lebesgue measure set of parameters that we denote by $\mathcal{BC}$ verifying

$$\text{there is } c > 0 \ (c \approx \log 2) \text{ such that } |Df_a^n(f_a(0))| \geq \mathrm{e}^{cn} \text{ for all } n \geq 0; \tag{EG}$$

$$\text{there is a small } \alpha > 0 \text{ such that } |f_a^n(0)| \geq \mathrm{e}^{-\alpha n} \text{ for all } n \geq 1. \tag{BA}$$

Before we describe the Benedicks-Carleson strategy we define the *critical region* which is the interval $(-\delta, \delta)$, where $\delta = \mathrm{e}^{-\Delta} > 0$ is chosen small but much larger than $2 - a$. This region is partitioned into the intervals

$$(-\delta, \delta) = \bigcup_{m \geq \Delta} I_m,$$

where $I_m = (\mathrm{e}^{-(m+1)}, \mathrm{e}^{-m}]$ for $m > 0$ and $I_m = -I_{-m}$ for $m < 0$; then each I_m is further subdivided into m^2 intervals $\{I_{m,j}\}$ of equal length inducing the partition $\mathcal{P}_0$ of $[-1, 1]$ into

$$[-1, -\delta) \cup \bigcup_{m,j} I_{m,j} \cup (-\delta, 1]. \tag{3.1}$$

Given $J \in \mathcal{P}$, we let nJ denote the interval n times the length of J centered at J. We also define $U_m := (-\mathrm{e}^{-m}, \mathrm{e}^{-m})$, for every $m \in \mathbb{N}$.

In order to study the growth of $Df_a^n(x)$ for $x \in [-1, 1]$ and $a \in \mathcal{BC}$ we split the orbit in free periods and bound periods. During the former we are certain that the orbit never visits the critical region. The latter begin when the orbit returns to the critical region and initiates a bound to the critical point, accompanying its early iterates. We describe the behavior of the derivative during these periods in Subsections 3.1 and 3.2.

3.1. **Expansion outside the critical region.** There is $c_0 > 0$ and $M_0 \in \mathbb{N}$ such that

(1) If $x, \ldots, f_a^{k-1}(x) \notin (-\delta, \delta)$ and $k \geq M_0$, then $|Df_a^k(x)| \geq \mathrm{e}^{c_0 k}$;
(2) If $x, \ldots, f_a^{k-1}(x) \notin (-\delta, \delta)$ and $f_a^k(x) \in (-\delta, \delta)$, then $|Df_a^k(x)| \geq \mathrm{e}^{c_0 k}$;
(3) If $x, \ldots, f_a^{k-1}(x) \notin (-\delta, \delta)$, then $|Df_a^k(x)| \geq \delta \mathrm{e}^{c_0 k}$.

If we were capable of keeping the orbit of x away from the critical region then it would be in *free period* for ever and the estimates above would apply. However, it is inevitable that almost every $x \in [-1, 1]$ makes a *return* to the critical region. We say that $n \in \mathbb{N}$ is a *return time* of the orbit of x if $f_a^n(x) \in (-\delta, \delta)$. Every free period of x ends with a *free return* to the critical region. We say that the return had a *depth* of $m \in \mathbb{N}$ if $m = [-\log |f_a^n(x)|]$, which is equivalent to saying that $f_a^n(x) \in I_{\pm m}$. Once in the critical region the orbit of x initiates a binding with the critical point.

3.2. **Bound period definition and properties.** Let $\beta = 14\alpha$. For $x \in (-\delta, \delta)$ define $p(x)$ to be the largest integer p such that

$$|f_a^k(x) - f_a^k(0)| < \mathrm{e}^{-\beta k}, \qquad \forall k < p. \tag{3.2}$$

Then

(1) $\frac{1}{2}|m| \leq p(x) \leq 3|m|$, for each $x \in I_m$;

(2) $|Df_a^p(x)| \geq \mathrm{e}^{c'p}$, where $c' = \frac{1-4\beta}{3} > 0$.

The orbit of x is said to be bound to the critical point during the period $0 \leq k < p$. We may assume that p is constant on each $I_{m,j}$. Note that during the bound period the orbit of x may return to the critical region. We call these instants: *bound return times*.

Roughly speaking, the idea behind the proof of Benedicks-Carleson Theorem is that while the orbit of the critical point is outside a critical region we have expansion (see Subsection 3.1); when it returns we have a serious setback in the expansion but then, by continuity, the orbit repeats its early history regaining expansion on account of (EG). To arrange for the exponential growth of the derivative along the critical orbit (EG) one has to guarantee that the losses at the returns are not too drastic hence, by parameter elimination, the basic assumption condition (BA) is imposed. The argument is mounted in a very intricate induction scheme that guarantees both the conditions for the parameters that survive the exclusions. The condition (EG) is usually known as the Collet-Eckmann condition and it was introduced in [CE83].

3.3. **Bookkeeping, essential and inessential returns.** A sequence of partitions $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \ldots$ is built so that points in the same element of the partition $\mathcal{P}_n$ have the same history up to time n. For a detailed description of the construction of this sequence of partitions in the phase space setting we refer to [Fre05, Section 4]. Here, we highlight some of the main aspects of its construction.

For Lebesgue almost every $x \in I$, $\{x\} = \cap_{n\geq 0}\omega_n(x)$, where $\omega_n(x)$ is the element of $\mathcal{P}_n$ containing x. For such x there is a sequence $t_1, t_2, \ldots$ corresponding to the instants when the orbit of x experiences a free *essential return situation*, which means $I_{m,k} \subset f_a^{t_i}(\omega_{t_i}(x))$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$. We have that $\omega_n(x) = \omega_{t_{i-1}}(x)$, for every $t_{i-1} \leq n < t_i$ and $\omega_{t_i}(x) = \omega_0(f^{t_i}(x))$, except for the points at the two ends of $f_a^{t_i}(\omega_{t_{i-1}}(x))$ for which it may occur an adjoining to the neighboring interval. If t_i is an essential return situation for x, then it is either an *essential return time* for x which means that there exists $m \geq \Delta$ and $1 \leq k \leq m^2$ such that $I_{m,k} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset 3I_{m,k}$; or an *escaping time* for x which is to say that $I_{(\Delta-1),1} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset (\delta, 1]$ or $I_{-(\Delta-1),1} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset [-1, -\delta)$, where $I_{\pm(\Delta-1),1}$ is the subinterval of $I_{\pm(\Delta-1)}$ closest to 0.

We remark that every point in $\omega \in \mathcal{P}_n$ has the same history up to n, in the sense that they have the same free periods, return to the critical region simultaneously, with the same depth and their bound periods expire at the same time.

We say that v is a free return time for x of *inessential* type if $f_a^v(\omega_v(x)) \subset 3I_{m,k}$, for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$, but $f_a^v(\omega_v(x))$ is not large enough to contain an interval $I_{m,k}$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$.

3.4. **Distortion of the derivative.** The sequence of partitions described above is designed so that we have bounded distortion in each element of the partition $\mathcal{P}_{n-1}$ up to time n. To be more precise, consider $\omega \in \mathcal{P}_{n-1}$. There exists a constant C independent of ω, n and the parameter such that for every $x, y \in \omega$,

$$\frac{|Df_a^n(x)|}{|Df_a^n(y)|} \leq C. \tag{3.3}$$

See [Fre05, Lemma 4.2] for a proof.

3.5. **Growth of returning and escaping components.** Let t be an essential return time for $\omega \in \mathcal{P}_t$, with $I_{m,k} \subset f_a^t(\omega) \subset 3I_{m,k}$ for some $m \geq \Delta$ and $1 \leq k \leq m^2$. If n is the next free return situation for ω (either essential or inessential) then

$$|f_a^n(\omega)| \geq \mathrm{e}^{c_0 q}\mathrm{e}^{-5\beta|m|}, \tag{3.4}$$

where $q = n - (t+p)$. See [Fre05, Lemma 4.1].

Suppose that $\omega \in \mathcal{P}_t$ is an escape component. Then in the next return situation t_1 for ω we have that

$$\left|f_a^{t_1}(\omega)\right| \geq \mathrm{e}^{-\beta\Delta}. \tag{3.5}$$

See [MS93], [Fre06, Lemma 4.2] or [Mor93, Lemma 5.1] for a proof of a similar statement in the space of parameters.

3.6. **Existence of absolutely continuous invariant measures.** For every $a \in \mathcal{BC}$, the quadratic map $f = f_a$ has an invariant probability measure μ that is absolutely continuous with respect to Lebesgue measure on $[-1, 1]$. The existence of absolutely continuous invariant measures (a.c.i.m) for a positive Lebesgue measure set of parameters was first proved by Jakobson in [Jak81] and others followed. See for example [CE83], [BC85], [Now85], [Ryc88], [BY92], [You92], etc.

The a.c.i.m. $\mu = \rho dx$ has the following properties:

(1) μ is the only a.c.i.m. of f;
(2) (f, μ) is exact;
(3) $\rho = \rho_1 + \rho_2$, where ρ_1 has bounded variation and $0 \leq \rho_2(x) \leq \sum_{j=1}^{\infty} \frac{(1.9)^{-j}}{\sqrt{|x - f^j(0)|}}$;
(4) The support of μ is $[f^2(0), f(0)]$ and $\inf_{x\in[f^2(0),f(0)]} \rho(x) > 0$;

The proof of these statements can be found in [You92, Theorems 1 and 2].

3.7. **Decay of correlations and Central Limit Theorem.** The Benedicks-Carleson quadratic maps have good statistical behavior. In fact, L. S. Young proved that these maps have exponential decay of correlations and satisfy the Central Limit Theorem ([You92, Theorems 3 and 4]). This was also obtained by Keller and Nowicki in [KN92]. To be more precise, for every $a \in B$ we have for $f = f_a$ that there exists $\varsigma \in (0, 1)$ such that for all $\varphi, \psi : [-1, 1] \to \mathbb{R}$ with bounded variation, there is $C = C(\varphi, \psi)$ such that

$$\left|\int \varphi \cdot (\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu\right| \leq C\varsigma^n, \quad \forall n \geq 0. \tag{3.6}$$

Moreover, if $\int \varphi d\mu = 0$ then for every $x \in \mathbb{R}$ we have

$$\mu\left\{\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1} \varphi \circ f^i < x\right\} \xrightarrow[n\to\infty]{} \Phi(x/\sigma), \tag{3.7}$$

where we are assuming that $\sigma := \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left[\int \left(\sum_{i=0}^{n-1} \varphi \circ f^i\right)^2 d\mu\right]^{1/2} > 0$ and $\Phi(\cdot)$ denotes the $N(0, 1)$ distribution function.

3.8. **Exponential weak-Bernoulli mixing.** Keller [Kel94] has obtained a result even sharper than (3.6). Consider the partition of $[-1,1]$ given by $\mathcal{Q}=\{[-1,0),[0,1]\}$. Also, for integers $k<l$, denote by $\mathcal{Q}_k^l$ the join of partitions $\bigvee_{i=k}^l f^{-i}\mathcal{Q}$ and by $\mathcal{F}_k^l$ the σ-algebra generated by $\mathcal{Q}_k^l$. According to [Kel94] the partition $\mathcal{Q}$ is a weak-Bernoulli generator for every $f=f_a$ with $a\in\mathcal{BC}$. This means that the σ-algebra $\mathcal{F}_0^\infty$ coincides, up to sets of Lebesgue measure 0, with the Borel σ-algebra of sets in $[-1,1]$ and that

$$\beta_n(f,\mathcal{Q},\mu)\to 0, \quad \text{as } n\to\infty,$$

where

$$\begin{aligned}\beta_n(f,\mathcal{Q},\mu) &:= 2\sup_{k>0}\int \sup\left\{\left|\mu(A|\mathcal{F}_0^k)-\mu(A)\right| : A\in\mathcal{F}_{k+n}^\infty\right\}d\mu\\ &= \sup_{k\geq 1,\,L\geq 1}\sum_{A\in\mathcal{Q}_0^k,\,B\in\mathcal{Q}_{k+n}^{k+n+L}}\left|\mu(A\cap B)-\mu(A)\mu(B)\right|.\end{aligned}$$

In fact, [Kel94, Theorem 1] states that there are constants $C>0$ and $0<r<1$ such that

$$\beta_n(f,\mathcal{Q},\mu)\leq Cr^n \tag{3.8}$$

for all $n\in\mathbb{N}$.

Remark 3.1. We observe that if we refine the partition $\mathcal{Q}$ by adding one or two points so that $\{X_0>u_n\}\in\mathcal{F}_0$ for each $n\in\mathbb{N}$, where $\mathcal{F}_0$ is the σ-algebra generated by $\mathcal{Q}$, then Keller's argument still holds with the same type of estimate as in (3.8). As a consequence, condition $D(u_n)$ is true for every considered sequence u_n.

4. Domain of attraction of the associated i.i.d. process

We recall that to every stationary stochastic process $X_0,X_1,X_2,\ldots$ defined in (1.2) we associated an i.i.d. sequence of r.v. $Z_0,Z_1,Z_2,\ldots$ with common d.f. given by $G_a(x)=P\{X_0\leq x\}=\mu_a\{g^{-1}((-\infty,x])\}$ (see Section 2). In this Section we will determine the domain of attraction corresponding to the d.f. G_a, i.e., we will compute the limiting distribution of $\hat{M}_n$, defined in (2.1), when properly normalized. For that purpose one must look at the tail behavior of $1-G_a(x)$ as x gets closer to $\sup_{y\in\mathbb{R}}\{G_a(y)<1\}=\max_{y\in[-1,1]}g(y)$. We have to divide the study in two cases corresponding to the two types of r.v. g considered.

Assume g_1 satisfies (1a) and (1b) of Section 1. Since g_1 is C^1 in a neighborhood of 1 and the left-hand derivative at 1 is positive then g_1 is invertible in a neighborhood of 1 and

$$g_1^{-1}\left(g_1(1)-s\right)=1-\mathcal{O}(s), \tag{4.1}$$

meaning that $\lim_{s\to 0^+}\frac{1-g_1^{-1}(g_1(1)-s)}{s}>0$. Attending to Section 3.6 (3) if ω is close to 1 we may write $\rho(\omega)=\mathcal{O}\left(\frac{1}{\sqrt{1-\omega}}\right)$, in the sense that $\frac{\rho(\omega)}{\frac{1}{\sqrt{1-\omega}}}\to c$ for some $c>0$, as $\omega\to 1$. As a consequence we have that $\mu_a\{(\omega,1]\}=\mathcal{O}(\sqrt{1-\omega})$. Hence, for $s>0$ sufficiently close to

0 we have:

$$1 - G_a\left(g_1(1) - s\right) = \mathcal{O}\left(\sqrt{1 - g_1^{-1}\left(g_1(1) - s\right)}\right) = \mathcal{O}(\sqrt{s}), \tag{4.2}$$

which means that $\lim_{s\to 0^+} \frac{1-G_a(g_1(1)-s)}{\sqrt{s}} > 0$. We are now in condition of applying [LLR83, Theorem 1.6.2] to obtain that G_a, in this case, belongs to the domain of attraction of type III (Weibull) with parameter $1/2$, since for every $x > 0$

$$\lim_{h\to 0^+} \frac{1 - G_a\left(g_1(1) - xh\right)}{1 - G_a\left(g_1(1) - h\right)} = \lim_{h\to 0^+} \frac{\sqrt{xh}}{\sqrt{h}} = x^{1/2}.$$

Moreover, according to [LLR83, Corollary 1.6.3] if we consider the sequences defined for each $n \in \mathbb{N}$ by $b_n = g_1(1)$ and $a_n = (g_1(1) - G_a^{-1}(1 - 1/n))$, where $G_a^{-1}(y) = \inf\{x : G_a(x) \geq y\}$, then

$$P\left\{a_n(\hat{M}_n - b_n) \leq x\right\} \to H_1(x) = \begin{cases} \mathrm{e}^{-(-x)^{1/2}} & , x \leq 0 \\ 1 & , x > 0 \end{cases},$$

as $n \to \infty$.

In the second case, when g_2 has the properties (2a) and (2b) of Section 1. It follows that for a sufficiently small $s > 0$ we have

$$\bar{g}_2^{-1}\left(g_2(0) - s\right) = -\mathcal{O}(s^{1/\bar{q}}) \text{ and } \hat{g}_2^{-1}\left(g_2(0) - s\right) = \mathcal{O}(s^{1/\hat{q}}), \tag{4.3}$$

which means that

$$\lim_{s\to 0^+} \frac{\bar{g}_2^{-1}\left(g_2(0) - s\right)}{-s^{1/\bar{q}}} > 0 \text{ and } \lim_{s\to 0^+} \frac{\hat{g}_2^{-1}\left(g_2(0) - s\right)}{s^{1/\hat{q}}} > 0,$$

respectively. Attending to (3) and (4) of Section 3.6 and the fact that for the Misiurewicz parameters (1.1) holds then $C^{-1} \leq \rho(x) \leq C$ for every $x \in (-d, d)$ and some $C > 0$. Hence, for $s > 0$ sufficiently small we have, for $q = \max\{\bar{q}, \hat{q}\}$,

$$1 - G_a\left(g_2(0) - s\right) = \mathcal{O}\left(s^{1/\hat{q}} + s^{1/\bar{q}}\right) = \mathcal{O}\left(s^{1/q}\right), \tag{4.4}$$

which means that $\lim_{s\to 0^+} \frac{1-G_a(g_2(0)-s)}{s^{1/q}} > 0$. We are now in condition of applying [LLR83, Theorem 1.6.2] to obtain that G_a, in this case, belongs to the domain of attraction of type III (Weibull) with parameter $1/q$ since for every $x > 0$

$$\lim_{h\to 0^+} \frac{1 - G_a\left(g_2(0) - xh\right)}{1 - G_a\left(g_2(0) - h\right)} = \lim_{h\to 0^+} \frac{(xh)^{1/q}}{h^{1/q}} = x^{1/q}.$$

Moreover, according to [LLR83, Corollary 1.6.3] if we consider the sequences defined for each $n \in \mathbb{N}$ by $b_n = g_1(1)$ and $a_n = (g_2(0) - G_a^{-1}(1 - 1/n))$, where $G_a^{-1}(y) = \inf\{x : G_a(x) \geq y\}$, then

$$P\left\{a_n(\hat{M}_n - b_n) \leq x\right\} \to H_2(x) = \begin{cases} \mathrm{e}^{-(-x)^{1/q}} & , x \leq 0 \\ 1 & , x > 0 \end{cases},$$

as $n \to \infty$.

5. Probability of an essential return reaching a certain depth

In the study of Extremes one is mostly interested in the probability of occurrence of "exceedances" of the level u_n. In Section 6 we will see how these events are related with the occurrence of deep returns. Thus, in this section we do some preparatory work by estimating the probability of the returns hitting a given depth.

For each $x \in I$, let $v_n(x)$ denote the number of essential return situations of x between 1 and n, $s_n(x)$ be the number of those which are actual essential return times and $\mathfrak{S}_n$ the number of the latter that correspond to *deep essential returns* of the orbit of x, i.e, with return depths above a threshold $\Theta \geq \Delta$. Observe that $v_n(x) - s_n(x)$ is the exact number of escaping situations of the orbit of x, up to n.

Given the integers $0 \leq s \leq \frac{2n}{\Theta}$, $s \leq v \leq n$ and s integers $\gamma_1, \dots, \gamma_s$, each greater than or equal to Θ, we define the event:

$$A^{v,s}_{\gamma_1,\dots,\gamma_s}(n) = \left\{ x \in I \ : \ v_n(x) = v,\, \mathfrak{S}_n(x) = s, \ \begin{array}{r} \text{and the depth of the i-th deep essen-} \\ \text{tial return is } \gamma_i \ \forall i \in \{1, \dots, s\} \end{array} \right\}.$$

Remark 5.1. Observe that the upper bound $\frac{2n}{\Theta}$ for the number of deep essential returns up to time n derives from the fact that each deep essential return originates a bound period of length at least $\frac{1}{2}\Theta$ (see Section 3.2). Since during the bound periods there cannot be any essential return, the number of deep essential returns occurring in a period of length n is at most $\frac{n}{\frac{2}{3}\Theta}$.

Proposition 5.2. *Given the integers $0 \leq s \leq \frac{2n}{\Theta}$ and $s \leq v \leq n$, consider s integers $\gamma_1, \dots, \gamma_s$, each greater than or equal to Θ. If Θ is large enough, then*

$$\lambda\left(A^{v,s}_{\gamma_1,\dots,\gamma_s}(n)\right) \leq \binom{v}{s} Exp\left\{ -(1-6\beta) \sum_{i=1}^{s} \gamma_i \right\}$$

Proof. Fix $n \in \mathbb{N}$ and take $\omega_0 \in \mathcal{P}_0$. Note that the functions v_n, s_n and $\mathfrak{S}_n$ are constant in each $\omega \in \mathcal{P}_n$. Let $\omega \in \omega_0 \cap \mathcal{P}_n$ be such that $v_n(\omega) = v$. Then, there is a sequence $1 \leq t_1 \leq \ldots \leq t_v \leq n$ of essential return situations. Let ω_i denote the element of the partition $\mathcal{P}_{t_i}$ that contains ω. We have $\omega_0 \supset \omega_1 \supset \ldots \supset \omega_v = \omega$. Consider that $\omega_j = \emptyset$ whenever $j > v$. For each $j \in \{0, \dots, n\}$ we define the set:

$$Q_j = \bigcup_{\omega \in \mathcal{P}_n \cap \omega_0} \omega_j,$$

and its partition

$$\mathcal{Q}_j = \{\omega_j : \omega \in \mathcal{P}_n \cap \omega_0\}.$$

Let $\omega \in \mathcal{P}_n$ be such that $\mathfrak{S}_n(\omega) = s$. Then, we may consider $1 \leq r_1 \leq \ldots \leq r_s \leq v$ with r_i indicating that the i-th deep essential return occurs in the r_i-th essential return situation. Now, set $V(0) = Q_0 = \omega_0$. Fix s integers $1 \leq r_1 \leq \ldots \leq r_s \leq v$. Next, for each $j \leq v$ we define recursively the sets $V(j)$. Although the set $V(v)$ will depend on the fixed integers

$1 \leq r_1 \leq \ldots \leq r_s \leq v$, we do not indicate this so that the notation is not overloaded. Suppose that $V(j-1)$ is already defined and $r_{i-1} < j < r_i$. Then, we set

$$V(j) = \bigcup_{\omega \in \mathcal{Q}_j} \omega \bigcap f_a^{-t_j}(I - U_\Theta) \bigcap V(j-1).$$

If $j = r_i$ then we define

$$V(j) = \bigcup_{\omega \in \mathcal{Q}_j} \omega \bigcap f_a^{-t_j}(I_{\gamma_i} \cup I_{-\gamma_i}) \bigcap V(j-1)$$

Observe that for every $j \in \{1, \ldots, v\}$ we have $\frac{|V(j)|}{|V(j-1)|} \leq 1$. Therefore, we concentrate in finding a better estimate for $\frac{|V(r_i)|}{|V(r_i-1)|}$. Consider that $\omega_{r_i} \in \mathcal{Q}_{r_i} \cap V(r_i - 1)$ and let $\omega_{r_i-1} \in \mathcal{Q}_{r_i-1} \cap V(r_i - 1)$ contain ω_{r_i}. We have to consider two situations depending on whether t_{r_i-1} is an escaping situation or an essential return.

Let us suppose first that t_{r_i-1} was an essential return with return depth η. Then,

$$\begin{aligned}
\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} &\leq \frac{|\omega_{r_i}|}{|\widehat{\omega}_{r_i-1}|}, \quad \text{where } \widehat{\omega}_{r_i-1} = \omega_{r_i-1} \cap f_a^{-t_{r_i}}(U_1) \\
&\leq C \frac{\left| f_a^{t_{r_i}}(\omega_{r_i}) \right|}{\left| f_a^{t_{r_i}}(\widehat{\omega}_{r_i-1}) \right|}, \text{ by (3.3)} \\
&\leq C \frac{2\mathrm{e}^{-\gamma_i}}{\mathrm{e}^{-5\beta\eta}}, \text{ by (3.4)}
\end{aligned}$$

Note that when $r_{i-1} = r_i - 1$ then $\eta = \gamma_{i-1} \geq \Theta$. If, on the other hand, $r_{i-1} > r_i - 1$ then t_{r_i-1} is an essential return with depth $\eta < \Theta \leq \gamma_{i-1}$. Then in both situations we have

$$\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} \leq 2C \frac{\mathrm{e}^{-\gamma_i}}{\mathrm{e}^{-5\beta\gamma_{i-1}}}.$$

When t_{r_i-1} is an escape situation instead of using (3.4) we can use (3.5) and obtain

$$\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} \leq 2C \frac{\mathrm{e}^{-\gamma_i}}{\mathrm{e}^{-\beta\Delta}} \leq 2C \frac{\mathrm{e}^{-\gamma_i}}{\mathrm{e}^{-5\beta\gamma_{i-1}}}.$$

Observe also that if $\widehat{\omega}_{r_i-1} \neq \omega_{r_i-1}$ then, because we are assuming that $\omega_{r_i} \neq \emptyset$, we have $\lambda\left(f_a^{t_{r_i}}(\widehat{\omega}_{r_i-1})\right) \geq \mathrm{e}^{-1} - \mathrm{e}^{-\Theta} \geq \mathrm{e}^{-5\beta\gamma_{i-1}}$, for large Θ.

At this point we me have

$$
\begin{aligned}
|V(r_i)| &= \sum_{\omega_{r_i}\in\mathcal{Q}_{r_i}\cap V(r_i-1)} \frac{|\omega_{r_i}|}{|\omega_{r_i-1}|}|\omega_{r_i-1}| \\
&\leq 2C\mathrm{e}^{-\gamma_i}\mathrm{e}^{5\beta\gamma_{i-1}} \sum_{\omega_{r_i}\in\mathcal{Q}_{r_i}\cap V(r_i-1)} |\omega_{r_i-1}| \\
&\leq 2C\mathrm{e}^{-\gamma_i}\mathrm{e}^{5\beta\gamma_{i-1}}|V(r_i-1)|.
\end{aligned}
$$

We are now in conditions to obtain that

$$
|V(v)| \leq (2C)^s \mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\gamma_i\right\}\mathrm{e}^{5\beta\gamma_0}|V(0)|
$$

where γ_0 is given by the interval $\omega_0 \in \mathcal{P}_0$. If $\omega_0 = I_{(\eta_0,k_0)}$ with $|\eta_0| \geq \Delta$ and $1 \leq k_0 \leq \eta_0^2$, then $\gamma_0 = |\eta_0|$. If $\omega_0 = (\delta, 1]$ or $\omega_0 = [-1, -\delta)$, then we can take $\gamma_0 = 0$.

Now, we have to take into account the number of possibilities of having the occurrence of the event $V(v)$ implying the occurrence of the event $A^{v,s}_{\gamma_1,\ldots,\gamma_s}(n)$. The number of possible configurations related with the different values that the integers $r_1, \ldots r_s$ can take is $\binom{v}{s}$. Hence, it follows that

$$
\begin{aligned}
\lambda\left(A^{v,s}_{\gamma_1,\ldots,\gamma_s}(n)\right) &\leq (2C)^s\binom{v}{s}\mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\gamma_i\right\}\sum_{\omega_o\in\mathcal{P}_0}\mathrm{e}^{5\beta|\gamma_0|}|\omega_0| \\
&\leq (2C)^s\binom{v}{s}\mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\gamma_i\right\}\left(2(1-\delta)+\sum_{|\eta_0|\geq\Delta}\mathrm{e}^{5\beta\eta_0}\mathrm{e}^{-|\eta_0|}\right) \\
&\leq 3(2C)^s\binom{v}{s}\mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\gamma_i\right\}, \quad \text{for } \Delta \text{ large enough} \\
&\leq \binom{v}{s}\mathrm{Exp}\left\{-(1-6\beta)\sum_{i=1}^{s}\gamma_i\right\}.
\end{aligned}
$$

The last inequality results from the fact that $s\Theta \leq \sum_{i=1}^{s}\gamma_i$ and the freedom to choose a sufficiently large Θ. □

Given the integers $0 \leq s \leq \frac{2n}{\Theta}$, $s \leq v \leq n$ and the integers γ_0, γ_1, both greater than or equal to Θ, we consider the event:

$$
B^{v,s}_{\gamma_0,\gamma_1}(n) = \left\{ x \in I_{\gamma_0} \;:\; v_n(x) = v,\ \mathfrak{S}_n(x) = s \ \text{ and } n \text{ is a free deep return} \atop \text{with depth } \gamma_1 \right\}.
$$

Corollary 5.3. *Given the integers $0 \leq s \leq \frac{2n}{\Theta}$, $s \leq v \leq n$, and $\gamma_0, \gamma_1 \geq \Theta$. If Θ is large enough, then*

$$\lambda\left(B^{v,s}_{\gamma_0,\gamma_1}(n)\right) \leq \binom{v}{s} \mathit{Exp}\{-(1-6\beta)(\gamma_0+\gamma_1)\}$$

The proof of this statement follows easily from Proposition 5.2 by observing that although n may be an inessential deep return time (instead of an essential deep return) the estimates still prevail and for $\Theta > \Delta$ large enough we have $\sum_{\gamma \geq \Theta} \mathrm{e}^{-(1-6\beta)\gamma} \leq 1$.

6. The condition $D'(u_n)$

Assume that $X_0, X_1, \ldots$ is the stationary stochastic process defined in (1.2) with common d.f. G_a. So far, the two types of conditions imposed on g seem rather arbitrary. The reason for requiring that the maximum of g should be attained in 1 or 0 is that, under these assumptions, there is a nice way of translating exceedances of the level u_n into the occurrence of deep returns.

When g is of type g_2 (see (2) of Section 1) then for $\omega \in [-1,1]$ the event $\{X_j(\omega) > u_n\}$ is simply the set $f_a^{-j}(\{\omega : g_2(\omega) > u_n\})$. If n is sufficiently large then $\{\omega : g_2(\omega) > u_n\} = \left(\bar{g}_2^{-1}(u_n), \hat{g}_2^{-1}(u_n)\right) \subset U_\Delta$. Hence, an exceedance of the level u_n at time j corresponds to a return with depth over the threshold

$$\Theta_2 = \Theta_2(n) = \min\left\{\left[-\log\left(-\bar{g}_2^{-1}(u_n)\right)\right], \left[-\log\left(\hat{g}_2^{-1}(u_n)\right)\right]\right\}, \tag{6.1}$$

where $[y]$ denotes the largest integer not greater than $y \in \mathbb{R}$.

In what regards g_1 one has that for $\omega \in [-1,1]$ the event $\{X_j(\omega) > u_n\}$ is the set $f_a^{-j}(\{\omega : g_1(\omega) > u_n\})$. If n is sufficiently large then $\{\omega : g_1(\omega) > u_n\} = (g_1^{-1}(u_n), 1]$ and $f_a^{-1}(g_1^{-1}(u_n), 1] = (-\sqrt{(1-g_1^{-1}(u_n))/a}, \sqrt{(1-g_1^{-1}(u_n))/a}) \subset U_\Delta$. We may define

$$\Theta_1 = \Theta_1(n) = \left[-\frac{1}{2}\log\frac{1-g_1^{-1}(u_n)}{a}\right]. \tag{6.2}$$

This means that again there is an intimate connection between exceedances and deep returns. In fact, if an exceedance occurs at time j then a deep return with depth over the threshold Θ_1 must have happened at time $j-1$, i.e., if $X_j(\omega) > u_n$ then $f_a^{j-1}(\omega) \in U_{\Theta_1}$.

In what follows statements about Θ apply to both Θ_1 and Θ_2.

Observe that we are dealing with very small perturbations of f_2 for which $f_2^j(0) = -1$ for every $j \geq 2$. Thus, one expects that after a deep return to the critical region (a tight vicinity of 0) it should take a considerable amount of time before another deep return should occur. Since exceedances are related with the occurrence of deep returns then one may have a fair amount of belief that condition (2.2) holds for the sequence $X_0, X_1, \ldots$

Remark 6.1. If the sequence $X_0, X_1, \dots$ was independent then (2.2) would follow easily since

$$n\sum_{j=1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} = n\sum_{j=1}^{[n/k]} P\{X_0 > u_n\}P\{X_j > u_n\} = n\sum_{j=1}^{[n/k]}(1-G_a(u_n))^2$$
$$\le \frac{n^2}{k}(1-G_a(u_n))^2 \xrightarrow[n\to\infty]{} \frac{\tau^2}{k} \xrightarrow[k\to\infty]{} 0$$

Let us give some insight into the argument we use to prove that (2.2) holds for $X_0, X_1, \dots$

(1) We use the exponential decay of correlations (see (3.6)) to compute a turning instant $T = T(n)$ such that the dependence between X_0 and X_j with $j > T$ is negligible. This suggests the splitting of the time interval $\{1, \dots, [n/k]\}$ into $\{1, \dots, T\}$ and $\{T+1, \dots, [n/k]\}$, when n is sufficiently large.
(2) During the time interval $\{T+1, \dots, [n/k]\}$ we use the fact that for $j > T$ the random variable X_j is almost independent from X_0 and argue like in Remark 6.1.
(3) For $j \in \{1, \dots, T\}$ we use Corollary 5.3 to bound $P\{X_0 > u_n \text{ and } X_j > u_n\}$ and then we use the fact that, for n large, $T \ll [n/k]$ to finish the proof.

Step (1)

Taking $\varphi = \psi = \mathbf{1}_{(u_n, \max g]} \circ g$ in (3.6) we get

$$\begin{aligned}|\mu_a\{X_0 > u_n \text{ and } &X_j > u_n\} - [\mu_a\{X_0 > u_n\}]^2| = \\ &= \left|\int \mathbf{1}_{(u_n,\max g]} \circ g \cdot \mathbf{1}_{(u_n,\max g]} \circ g \circ f^j d\mu_a - \left(\int \mathbf{1}_{(u_n,\max g]} \circ g d\mu_a\right)^2\right| \\ &\le C\varsigma^j\end{aligned}$$

where we may assume that C is the same for all $n \in \mathbb{N}$, because $||\mathbf{1}_{(u_n,\max g]} \circ g||_\infty$ and the total variation of $\mathbf{1}_{(u_n,\max g]} \circ g$ are both equal to 1, for every $n \in \mathbb{N}$.

We compute $T = T(n)$ such that for every $j \ge T$ we have

$$C\varsigma^j < \frac{1}{n^3}.$$

Since $C\varsigma^j < \frac{1}{n^3} \Leftrightarrow j > \frac{1}{\log \varsigma^{-1}}(3\log n + \log C)$, we simply take, for n sufficiently large,

$$T = \frac{4}{\log \varsigma^{-1}} \log n. \tag{6.3}$$

For fixed k and n sufficiently large, we have that $T < [n/k]$. Hence, we may write

$$n\sum_{j=1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} =$$

$$= n\sum_{j=1}^{T} P\{X_0 > u_n \text{ and } X_j > u_n\} + n\sum_{j=T+1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\}.$$

In step (2) below we deal with the second term in the sum, leaving the first term for step (3).

Step (2)

Let us show that $\limsup_{n\to\infty} n\sum_{j=T+1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} \to 0$, as $k\to\infty$. By choice of T we have

$$\begin{aligned} n\sum_{j=T+1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} &\leq n\left((1-G_a(u_n))^2 + \frac{1}{n^3}\right)[n/k] \\ &\leq \frac{n^2}{k}(1-G_a(u_n))^2 + \frac{n^2}{kn^3} \end{aligned}$$

Now $\frac{n^2}{k}(1-G_a(u_n))^2 + \frac{n^2}{kn^3} \xrightarrow[n\to\infty]{} \frac{\tau^2}{k} \xrightarrow[k\to\infty]{} 0$ and the result follows.

Step (3)

We are left with the burden of controlling the term $n\sum_{j=1}^{T} P\{X_0 > u_n \text{ and } X_j > u_n\}$. We begin with the following lemma that will enable us to bound the number of exceedances occurring during the time period $\{1,\ldots,T\}$. In what follows we are always assuming that n is large enough so that $\Theta > \Delta$.

Lemma 6.2. *If a deep return occurs at time t (with depth over the threshold Θ), then the next deep return can only occur after $t+\Theta/2$.*

Proof. For every $x\in U_\Theta$, the bound period associated to x is such that $p(x)\geq\Theta/2$, by Section 3.2 (1). For all $j\leq[\Theta/2]$ we have

$$\begin{aligned} \left|f_a^j(x)\right| &\geq \left|f_a^j(0)\right| - \mathrm{e}^{-\beta j} \overset{\text{(BA)}}{\geq} \mathrm{e}^{-\alpha j} - \mathrm{e}^{-\beta j} \geq \mathrm{e}^{-\alpha j}\left(1-\mathrm{e}^{(\alpha-\beta)j}\right) \\ &\geq \mathrm{e}^{-\alpha j}\left(1-\mathrm{e}^{(\alpha-\beta)}\right), \text{ since } \alpha-\beta<0 \\ &\geq \mathrm{e}^{-\alpha\Theta/2}\left(1-\mathrm{e}^{(\alpha-\beta)}\right), \text{ since } j\leq\Theta/2 \\ &\geq \mathrm{e}^{-\alpha\Theta}, \text{ if } n \text{ is large enough so that } 1-\mathrm{e}^{\alpha-\beta}\geq \mathrm{e}^{-\alpha\Theta/2} \\ &\geq \mathrm{e}^{-\Theta}, \text{ since } \alpha<1. \end{aligned}$$

□

As a consequence of Lemma 6.2 we have that the maximum number of exceedances up to time T is at most $2T/\Theta$. At this point we use the fact that we are dealing with Misiurewicz parameters $a \in \mathcal{M} \subset \mathcal{B}C$, for each one of which condition (1.1) holds, which means that there exists $d > 0$ such that $|f_a^j(0)| > d$ for all $j \in \mathbb{N}$. Next lemma shows that, for these kind of parameters, in the bound period following a deep return over the level Θ, deep bound returns do not occur.

Lemma 6.3. *For every $a \in \mathcal{M}$, if Θ is large enough then for every $x \in U_\Theta$ we have that $f^j(x) \notin U_\Theta$ for all $j \leq p(x)$.*

Proof. First observe that if $j > -\frac{1}{\beta}\log(d/2)$ then $\mathrm{e}^{-\beta j} < d/2$.

We choose Θ large enough so that $\mathrm{e}^{-\Theta} < d/2$; so for all $x \in U_\Theta$ and $j \leq -\frac{1}{\beta}\log(d/2)$ we have $|f_a^j(x) - f_a^j(0)| < d/2$.

Let $x \in U_\Theta$ and consider $j \leq p(x)$. If $j > -\frac{1}{\beta}\log(d/2)$ then, by definition of bound period, $|f_a^j(x) - f_a^j(0)| < \mathrm{e}^{-\beta j} < d/2$. If, on the other hand, $j \leq -\frac{1}{\beta}\log(d/2)$ then, by the choice of Θ above, $|f_a^j(x) - f_a^j(0)| < d/2$. Consequently, since f_a is a Misiurewicz quadratic map,

$$|f_a^j(x)| \geq |f_a^j(0)| - |f_a^j(x) - f_a^j(0)| \geq d - d/2 \geq d/2 > e^{-\Theta}.$$

□

As a consequence we have that after a deep return, with depth $m \geq \Theta$, happening at time t, the next deep return can only occur when the bound period initiated at the previous one ceases, i.e. after the instant $t + p(m)$. This way, we may use Corollary 5.3 to estimate $|U_\Theta \cap f_a^{-j}U_\Theta|$, for $j \leq T$.

Observe that in the case of g being of type g_1 and for n large enough, we have

$$\begin{aligned} P\{X_0 > u_n \text{ and } X_j > u_n\} &= \mu_a\left\{(g_1^{-1}(u_n), 1] \cap f_a^{-j}(g_1^{-1}(u_n), 1]\right\} \\ &= \mu_a\left\{f_a^{-1}\left((g_1^{-1}(u_n), 1] \cap f_a^{-(j+1)}(g_1^{-1}(u_n), 1]\right)\right\}, \end{aligned}$$

and $f_a^{-1}(g_1^{-1}(u_n), 1] \cap f_a^{-j-1}(g_1^{-1}(u_n), 1] \subset U_{\Theta_1} \cap f_a^{-j}U_{\Theta_1}$. While in the case of g being of type g_2 and for n large enough, we have

$$P\{X_0 > u_n \text{ and } X_j > u_n\} = \mu_a\left\{\left(\bar{g}_2^{-1}(u_n), \hat{g}_2^{-1}(u_n)\right)\right\}$$

and $\left(\bar{g}_2^{-1}(u_n), (\hat{g}_2)^{-1}(u_n)\right) \subset U_{\Theta_2} \cap f_a^{-j}U_{\Theta_2}$.

Consequently, since $C^{-1} \leq \rho(x) \leq C$ for some $C > 0$ and every $x \in (-d, d)$, once an estimate for $|U_\Theta \cap f_a^{-j}U_\Theta|$ is derived, we may use it to estimate $P\{X_0 > u_n \text{ and } X_j > u_n\}$. In what follows we will use "const" to denote several positive constants independent of n.

Note that

$$U_\Theta \cap f_a^{-j}U_\Theta \subset \bigcup_{s=0}^{2j/\Theta} \bigcup_{v=s}^{j} \bigcup_{\gamma_0 \geq \Theta} \bigcup_{\gamma_1 \geq \Theta} B_{\gamma_0,\gamma_1}^{v,s}(j).$$

Hence, by Corollary 5.3 we have

$$|U_\Theta \cap f_a^{-j} U_\Theta| \le \sum_{s=0}^{2j/\Theta} \sum_{v=s}^{j} \sum_{\gamma_0 \ge \Theta} \sum_{\gamma_1 \ge \Theta} \binom{v}{s} \mathrm{e}^{-(1-6\beta)(\gamma_0+\gamma_1)} \le \text{const} \sum_{s=0}^{2T/\Theta} \sum_{v=s}^{T} \binom{v}{s} \mathrm{e}^{-(1-6\beta)2\Theta}$$
$$\le \text{const} \sum_{s=0}^{2T/\Theta} \sum_{v=s}^{T} \binom{T}{s} \mathrm{e}^{-(1-6\beta)2\Theta} \le \text{const} \sum_{s=0}^{2T/\Theta} T \binom{T}{s} \mathrm{e}^{-(1-6\beta)2\Theta}.$$

Before we continue let us estimate $2T/\Theta$. Remember that the sequence u_n is such that $n(1 - G_a(u_n)) \to \tau$, as $n \to \infty$, which we rewrite as $1 - G_a(u_n) = \mathcal{O}(1/n)$.

In the case g is of type g_1, attending to (4.2), we get $u_n = g_1(1) - \mathcal{O}(1/n^2)$, which by (6.2) and (4.1) leads to $\Theta_1 = \mathcal{O}(\log n)$, meaning that $\frac{\Theta_1}{\log n} \to c$, for some $c > 0$, as $n \to \infty$.

As for the case where g is of type g_2, by (4.4), we have $u_n = g_2(0) - \mathcal{O}(1/n^q)$. Thus, according to (6.1) and (4.3) we get $\Theta_2 = \mathcal{O}(\log n)$, in the sense $\frac{\Theta_2}{\log n} \to c$, for some $c > 0$, as $n \to \infty$.

Recalling (6.3), one easily gets that there exists a constant $C_1 > 0$ such that $2T/\Theta \le C_1$, for n sufficiently large. So, to proceed with the estimation $|U_\Theta \cap f_a^{-j} U_\Theta|$, assume that n is sufficiently large so that $2T/\Theta \le C_1$ and $C_1 \ll T$. Then

$$|U_\Theta \cap f_a^{-j} U_\Theta| \le \text{const} \sum_{s=0}^{2T/\Theta} T \binom{T}{s} \mathrm{e}^{-(1-6\beta)2\Theta} \le \text{const} \sum_{s=0}^{2T/\Theta} T \binom{T}{C_1} \mathrm{e}^{-(1-6\beta)2\Theta}$$
$$\le \text{const} \frac{2T}{\Theta} T \binom{T}{C_1} \mathrm{e}^{-(1-6\beta)2\Theta} \le \text{const} T^{C_1+1} \mathrm{e}^{-(1-6\beta)2\Theta}.$$

Now, since $\mathrm{e}^{-\Theta} \le \text{const}(1 - G_a(u_n))$ we finally conclude:

$$\begin{aligned} n \sum_{j=1}^{T} P\{X_0 > u_n \text{ and } X_j > u_n\} &\le \text{const} \cdot n \sum_{j=1}^{T} T^{C_1+1} (1 - G_a(u_n))^{2(1-6\beta)} \\ &\le \text{const} \cdot n T^{C_1+2} (1 - G_a(u_n))^{2(1-6\beta)} \\ &\le \text{const} \cdot n (\log n)^{C_1+2} (1 - G_a(u_n))^{2-12\beta} \\ &\le \text{const} \cdot [n(1 - G_a(u_n))]^{3/2} (1 - G_a(u_n))^{1/2-12\beta}, \end{aligned}$$

for sufficiently large n. The result follows because

$$\lim_{n\to\infty} [n(1 - G_a(u_n))]^{3/2} (1 - G_a(u_n))^{1/2-12\beta} = \tau^{3/2} \cdot 0 = 0.$$

7. Simulation Study

In this section we present a small simulation study illustrating the finite sample behavior of the normalized M_n, defined in (1.3), for the Misiurewicz quadratic map $f_2(x) = 1 - 2x^2$. We consider the case $g(x) = g_1(x) = x$ for which we have that $P\{X_0 \le x\} = G_2(x) = 1/2 + \arcsin(x)/\pi$. According to Theorem A we have that the normalizing sequences are given by $b_n = 1$ and $a_n = (1 - \cos(\pi/n))^{-1}$, for each $n \in \mathbb{N}$ and the theoretical limiting distribution for $a_n(M_n - b_n)$ is $H_1(x) = \begin{cases} e^{-(-x)^{1/2}} & , x \leq 0 \\ 1 & , x > 0 \end{cases}$.

We performed the following experiment. We picked at random (according to the d.f. G_2) a point ω in the interval $[-1, 1]$, computed its orbit up to time n and calculated $M_n(\omega) = \max\{\omega, f_2(\omega), \ldots, f_2^{n-1}(\omega)\}$. We repeated the process m times to obtain a sample $\{M_n(\omega_1), \ldots, M_n(\omega_m)\}$ and approximated, for certain values of x, $P\{a_n(M_n - b_n) \leq x\}$ by

$$\frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{\{a_n(M_n(\omega_i) - b_n) \leq x\}}, \tag{7.1}$$

where $\mathbf{1}_{\{a_n(M_n(\omega_i)-b_n)\leq x\}} = \begin{cases} 1 & \text{if } a_n(M_n(\omega_i) - b_n) \leq x \\ 0 & \text{if } a_n(M_n(\omega_i) - b_n) > x \end{cases}$, for each $1 \leq i \leq m$.

In Table 1 we present the results obtained by realizing the above experiment, considering different values of x, n and m and compare them with the theoretical limiting ones given by $H_1(x)$.

		$n = 1000$		$n = 10000$		$n = 20000$
x	$H_1(x)$	$m = 1000$	$m = 10000$	$m = 10000$	$m = 20000$	$m = 20000$
-0.001	0.9689	0.976	0.9671	0.9719	0.9677	0.9708
-0.01	0.9048	0.894	0.9079	0.9056	0.9076	0.9052
-0.1	0.7289	0.724	0.7263	0.7323	0.7303	0.7335
-0.3	0.5783	0.569	0.5773	0.5823	0.5840	0.5813
-0.5	0.4931	0.491	0.4906	0.5012	0.4984	0.4941
-0.7	0.4332	0.407	0.4272	0.4403	0.4374	0.4338
-1	0.3679	0.388	0.3631	0.3663	0.3729	0.3678
-3	0.1769	0.164	0.1731	0.1748	0.1833	0.1729
-5	0.1069	0.124	0.1024	0.1092	0.1108	0.1056
-8	0.0591	0.059	0.0510	0.0557	0.0617	0.0580
-10	0.0423	0.049	0.0350	0.0435	0.0438	0.0414
-30	0.0042	0.002	0.0031	0.0033	0.0048	0.0041
-50	0.00085	0.001	0.0007	0.0009	0.0007	0.0009

TABLE 1. Simulation results

As one may verify the results of the experiment are quite close to the asymptotic theoretical ones and there is a general tendency of getting better as n increases which is precisely the behavior we were expecting. It is also noticeable that there is an improvement when m increases, which is also predictable since our approximation (7.1) gets to be more accurate.

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References

[BNN95] V. Balakrishnan, C. Nicolis, and G. Nicolis, *Extreme value distributions in chaotic dynamics*, J. Stat. Phys. **80** (1995), no. 1/2, 307–336.

[BC85] M. Benedicks and L. Carleson, *On iterations of* $1-ax^2$ *on* $(-1,1)$, Ann. Math. **122** (1985), 1–25.

[BC91] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. Math. **133** (1991), 73–169.

[BY92] M. Benedicks and L.S. Young, *Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps*, Ergd. Th. & Dynam. Sys. **12** (1992), 13–27.

[CE83] P. Collet and J. P. Eckmann, *Positive Lyapunov exponents and absolute continuity for maps of the interval*, Ergd. Th. & Dynam. Sys. **3** (1983), 13–46.

[Fre05] J. M. Freitas, *Continuity of SRB measure and entropy for Benedicks-Carleson quadratic maps*, Nonlinearity **18** (2005), 831–854.

[Fre06] J. M. Freitas, *Statistical stability for chaotic dynamical systems*, Ph.D. thesis, Universidade do Porto, 2006, http://www.fc.up.pt/pessoas/jmfreita/homeweb/publications.htm.

[Hai03] G. Haiman, *Extreme values of the tent map process*, Stat. Probab. Lett. **65** (2003), no. 4, 451–456.

[Jak81] M. Jakobson, *Absolutely continuous invariant measures for one parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), 39–88.

[Kel94] G. Keller, *Exponential weak Bernoulli mixing for Collet-Eckmann maps*, Isr. J. Math. **86** (1994), 301–310.

[KN92] G. Keller and T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps*, Comm. Math. Phys. **149** (1992), 31–69.

[KT66] J. F. C. Kingman and S. J. Taylor, *Introduction to Measure and Probability*, Cambridge University Press, 1966.

[LLR83] G. Lindgren, M. R. Leadbetter, and H. Rootzén, *Extremes and related properties of stationary sequences and processes*, Springer Series in Statistics, Springer: New York, 1983.

[MS93] W. de Melo, S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1993.

[Mis81] M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Publ. Math., Inst. Hautes Étud. Sci. **53** (1981), 17–51.

[Mor93] F. J. Moreira, *Chaotic dynamics of quadratic maps*, Informes de Matemática, IMPA, Série A, 092/93, 1993, http://www.fc.up.pt/cmup/fsmoreir/downloads/BC.pdf.

[Now85] T. Nowicki, *Symmetric S-unimodal mappings and positive Liapunov exponents*, Ergd. Th. & Dynam. Sys. **5** (1985), 611–616.

[Ryc88] M. Rychlik, *Another proof of Jakobson's theorem and related results*, Ergd. Th. & Dynam. Sys. **8** (1988), no. 1, 93–109.

[Thu01] H. Thunberg, *Unfolding of chaotic unimodal maps and the parameter dependence of natural measures*, Nonlinearity **14** (2001), no. 2, 323–337.

[You92] L. S. Young, *Decay of correlations for certain quadratic maps*, Comm. Math. Phys. **146** (1992), 123–138.

Jorge Milhazes Freitas, Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: `jmfreita@fc.up.pt`

URL: `http://www.fc.up.pt/pessoas/jmfreita`

Ana Cristina Moreira Freitas, Centro de Matemática & Faculdade de Economia da Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal

E-mail address: `amoreira@fep.up.pt`