# FINITE DEPTH SUBALGEBRAS IN A HOPF ALGEBRA

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Abstract. Let $k$ be an arbitrary field, $H$ a finite-dimensional Hopf $k$-algebra, $K$ a left coideal subalgebra of $H$ and $V$ their right generalized quotient. We show that finite depth of the subalgebra $K \subseteq H$ is equivalent to the $H$-module coalgebra $V$ representing an algebraic element in the Green ring of $H$. If $K$ is a Hopf subalgebra, we establish a previous claim that the problem of determining if $K$ has finite depth in $H$ is equivalent to determining if $H$ has finite depth in its smash product $V^*\#H$. A necessary condition is obtained for finite depth from stabilization of a descending chain of annihilator ideals of tensor powers of $V$. For a subalgebra pair of finite-dimensional algebras, a necessary condition for finite depth is given in the form of a matrix inequality between products of the matrix of induction and the matrix of restriction. As an application of several of the topics above to a centerless finite group $G$, we determine that the depth of its group $\mathbb{C}$-algebra in the Drinfeld double $D(G)$ is an odd integer coming from the least tensor power of the adjoint representation $V$ that is a faithful $\mathbb{C}G$-module.

## 1. Introduction and Preliminaries

Given a finite Hopf subalgebra pair $R \subseteq H$ over a field $k$, it is interesting to ask when their generalized quotient $V = H/R^+H$ is an algebraic module in the Green ring $A(-)$ of either finite-dimensional Hopf algebra: this is the case if it is a permutation module of a group algebra [14, IX.3.2]. The minimum even depth of $R \subseteq H$ is twice the degree of the minimum polynomial of $V$ in $A(R)$. In this paper we carry the connections established in [23] between subalgebra depth of $R \subseteq H$ and module depth of $V$ further in several directions.

The topics and layout of the paper are as follows. After an introduction of terminology and previously established facts for a Hopf subalgebra $R \subseteq H$, we obtain in Section 2 a necessary condition for finite depth

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involving stabilization of a descending chain of annihilator ideals of tensor powers of $V$ (Proposition 2.3). Given the right generalized quotient $V$ of a left coideal subalgebra $K$ of a finite-dimensional Hopf algebra $H$, we show that finite depth of the subalgebra $K \subseteq H$ is equivalent to the $H$-module coalgebra $V$ being algebraic in Section 6. If $K = R$ is a Hopf subalgebra, we establish previous claims that the problem of determining if $R$ has finite depth in $H$ is equivalent to determining if $H$ has finite depth in its smash product $V^* \# H$ (Theorem 4.2 and and [23, Corollary 5.5]). We note that the minimum depth of a finite group $\mathbb{C}$-algebra in its Drinfeld double is an odd integer determined by the least tensor power of $V$ that is faithful (Section 3 and Corollary 4.3). For a subalgebra pair $B \subseteq A$ of finite-dimensional $k$-algebras, a necessary condition for finite depth is given in Section 5 in the form of a matrix inequality between products of matrices of induction and of restriction, which are related by the Cartan matrices of $A$ and $B$ if $k$ is algebraically closed. In a last Section 7, we establish in direct terms that for the Hopf subalgebra $R \subseteq H$ with quotient $H$-module coalgebra canonical surjection $H \to V$, the subalgebra $V^* \hookrightarrow H^*$ is a left $R^*$-Galois extension with normal basis property.

1.1. **Preliminaries on subalgebra depth.** Let $A$ be a unital associative algebra over a field $k$. In this paper we assume all algebras and modules to be finite-dimensional vector spaces (although several facts below remain true without this assumption [22]). The category of finite-dimensional modules over $A$ will be denoted by $\mathcal{M}_A$. Two modules $M_A$ and $N_A$ are *similar* (or H-equivalent) if $M \oplus * \cong q \cdot N := N \oplus \cdots \oplus N$ ($q$ times) and $N \oplus * \cong r \cdot M$ for some $r, q \in \mathbb{N}$. This is briefly denoted by $M \,|\, q \cdot N$ and $N \,|\, r \cdot M$ for some $q, r \in \mathbb{N} \Leftrightarrow M \sim N$. It is well-known that similar modules have Morita equivalent endomorphism rings.

Let $B$ be a subalgebra of $A$ (always supposing $1_B = 1_A$). Consider the natural bimodules ${}_AA_A$, ${}_BA_A$, ${}_AA_B$ and ${}_BA_B$ where the last is a restriction of the preceding, and so forth. Denote the tensor powers of ${}_BA_B$ by $A^{\otimes_B n} = A \otimes_B \cdots \otimes_B A$ for $n = 1, 2, \ldots$, which is also a natural bimodule over $B$ and $A$ in any one of four ways; set $A^{\otimes_B 0} = B$ which is only a natural $B$-$B$-bimodule.

**Definition 1.1.** *If $A^{\otimes_B (n+1)}$ is similar to $A^{\otimes_B n}$ as $X$-$Y$-bimodules, one says $B \subseteq A$ has*

- *depth $2n+1$ if $X = B = Y$;*
- *left depth $2n$ if $X = B$ and $Y = A$;*
- *right depth $2n$ if $X = A$ and $Y = B$;*

- *h-depth* $2n-1$ *if* $X = A = Y$.

*valid for even depth and h-depth if* $n \geq 1$ *and for odd depth if* $n \geq 0$.

For example, $B \subseteq A$ has depth 1 iff ${}_BA_B$ and ${}_BB_B$ are similar [5, 22]. In this case, it is easy to show that $A$ is algebra isomorphic to $B \otimes_{Z(B)} A^B$ where $Z(B), A^B$ denote the center of $B$ and centralizer of $B$ in $A$. Another example, $B \subset A$ has right depth 2 iff ${}_AA_B$ and ${}_AA \otimes_B A_B$ are similar. If $A = \mathbb{C}\, G$ is a group algebra of a finite group $G$ and $B = \mathbb{C}\, H$ is a group algebra of a subgroup $H$ of $G$, then $B \subseteq A$ has right depth 2 iff $H$ is a normal subgroup of $G$ iff $B \subseteq A$ has left depth 2 [20]; a similar statement is true for a Hopf subalgebra $R \subseteq H$ of finite index and over any field [4].

Note that $A^{\otimes_B n} \mid A^{\otimes_B (n+1)}$ for all $n \geq 2$ and in any of the four natural bimodule structures: one applies 1 and multiplication to obtain a split monic, or split epi oppositely. For three of the bimodule structures, it is true for $n = 1$; as $A$-$A$-bimodules, equivalently $A \mid A \otimes_B A$ as $A^e$-modules, this is the separable extension condition on $B \subseteq A$. But $A \otimes_B A \mid q \cdot A$ as $A$-$A$-bimodules for some $q \in \mathbb{N}$ is the H-separability condition and implies $A$ is a separable extension of $B$ [19]. Somewhat similarly, ${}_BA_B \mid q \cdot {}_BB_B$ implies ${}_BB_B \mid {}_BA_B$ [22]. It follows that subalgebra depth and h-depth may be equivalently defined by replacing the similarity bimodule conditions for depth and h-depth in Definition 1.1 with the corresponding bimodules on

$$A^{\otimes_B (n+1)} \mid q \cdot A^{\otimes_B n} \tag{1}$$

for some positive integer $q$ [3, 21, 22].

Note that if $B \subseteq A$ has h-depth $2n-1$, the subalgebra has (left or right) depth $2n$ by restriction of modules. Similarly, if $B \subseteq A$ has depth $2n$, it has depth $2n+1$. If $B \subseteq A$ has depth $2n+1$, it has depth $2n+2$ by tensoring either $- \otimes_B A$ or $A \otimes_B -$ to $A^{\otimes_B (n+1)} \sim A^{\otimes_B n}$. Similarly, if $B \subseteq A$ has left or right depth $2n$, it has h-depth $2n+1$. Denote the minimum depth of $B \subseteq A$ (if it exists) by $d(B, A)$ [3]. Denote the minimum h-depth of $B \subseteq A$ by $d_h(B, A)$. Note that $d(B, A) < \infty$ if and only if $d_h(B, A) < \infty$; in fact, $|d(B, A) - d_h(B, A)| \leq 2$ if either is finite.

For example, for the permutation groups $\Sigma_n < \Sigma_{n+1}$ and their corresponding group algebras $B \subseteq A$ over any commutative ring $K$, one has depth $d(B, A) = 2n - 1$ [9, 3]. Depths of subgroups in $PGL(2, q)$, twisted group algebras and Young subgroups of $\Sigma_n$ are computed in [16, 13, 17]. If $B$ and $A$ are semisimple complex algebras, the minimum odd depth is computed from powers of an order $r$ symmetric matrix with nonnegative entries $S := MM^t$ where $M$ is the inclusion matrix $K_0(B) \to K_0(A)$ and $r$ is the number of irreducible representations of $B$ in a basic set of $K_0(B)$; the depth is $2n+1$ if $S^n$ and $S^{n+1}$ have an equal number of zero entries [9]. (For example, the matrix $S$ has Frobenius-Perron eigenvector, the dimension vector of $B$-simples with eigenvalue $|A : B|$, the rank of the free $B$-module $A$ if $A$ and $B$ are an algebra extension of finite groups or semisimple Hopf algebras.) Similarly, the minimum h-depth of $B \subseteq A$ is computed from powers of an order $s$ symmetric matrix $T = M^t M$, where $s$ is the rank of $K_0(A)$, and the power $n$ at which the number of zero entries of $T^n$ stabilizes [22]. It follows that the subalgebra pair of semisimple complex algebras $B \subseteq A$ always has finite depth.

1.2. **Depth of Hopf subalgebras and modules.** Let $R \subseteq H$ be a Hopf subalgebra in a finite-dimensional algebra over an arbitrary field $k$. It was shown in [23] that the tensor powers $H^{\otimes_R n}$ reduce to tensor powers of the generalized quotient $V = H/R^+H$ as follows: $H^{\otimes_R n} \xrightarrow{\cong} H \otimes V^{\otimes(n-1)}$ given by the formula in Eq. (21). This is an $H$-$H$-bimodule mapping where the right $H$-module structure on $H \otimes V \otimes \cdots \otimes V$ is given by the diagonal action of $H$: $(y \otimes v_1 \otimes \cdots \otimes v_{n-1}) \cdot h = yh_{(1)} \otimes v_1 h_{(2)} \otimes \cdots \otimes v_{n-1} h_{(n)}$. This shows quite clearly that the following will be of interest to computing $d(R, H)$. Let $W$ be a right $H$-module and $T_n(W) := W \oplus W^{\otimes 2} \oplus \cdots \oplus W^{\otimes n}$.

**Definition 1.2.** *A module $W$ over a Hopf algebra $H$ has depth $n$ if $T_{n+1}(W) \mid q \cdot T_n(W)$ and depth $0$ if $W$ is isomorphic to a direct sum of copies of $k_\varepsilon$, where $\varepsilon$ is the counit. Note that this entails that $W$ also has depth $n+1$, $n+2$, .... Let $d(W, \mathcal{M}_H)$ denote its minimum depth. If $W$ has a finite depth, it is said to be algebraic module.*

Algebraic $H$-modules is a terminology consistent with algebraic module over group algebras for the following reason. Since $T_m(W) \mid T_{m+1}(W)$, the indecomposable summands of $T_m(W)$ occur again (up to isomorphism) in the Krull-Schmidt decomposition of $T_{m+1}(W)$. If $W$ has depth $n$, all $T_m(W)$ and their summands $W^{\otimes m}$ for $m \geq n$ are expressible as sums of the indecomposable summands of $T_n(W)$. This should be compared to [14, Chapter II.5.1] to see that algebraic modules have finite depth and conversely; the proof does not depend on the commutativity of the Green ring of a group algebra. Recall that the Green ring of $H$, denoted by $A(H)$, is the free abelian group with basis consisting of indecomposable $H$-module isoclasses, with addition given by direct sum, and the multiplication in its ring structure given by the tensor product. For example, $K_0(H)$ is a finite rank ideal in $A(H)$, since $P \otimes X$ is projective if $X$ is any module and $P$ is projective (also a well-known fact for finite tensor categories). As shown in [14], a finite depth $H$-module $W$ satisfies a polynomial with integer coefficients in $A(H)$, and conversely.

**Example 1.3.** The paper [12] mentions that the principal block of the simple group $M_{11}$ contains 5-dimensional simple modules that are not algebraic.

The main theorem in [23, 5.1] proves from the basic Eq. (21) that Hopf subalgebra depth and depth of its generalized quotient $V$ are closely related by

$$2d(V, \mathcal{M}_R) + 1 \leq d(R, H) \leq 2d(V, \mathcal{M}_R) + 2. \tag{2}$$

Note that one restricts $V$ to an $R$-module in order to obtain the better result on depth. In Section 6 we need to consider the depth of $V$ as an $H$-module when $R$ is replaced with a left coideal subalgebra $K$ of $H$ (since $K$ is not itself a Hopf algebra). For now we note that h-depth satisfies $d_h(R, H) = 2d(V, \mathcal{M}_H) + 1$ [23, 5.1].

## 2. The descending chain of annihilators of the tensor powers of $V$

In this section $H$ is a finite-dimensional Hopf algebra over a field $k$. Let $R$ be a Hopf subalgebra of $H$. Let $H^+$ denote the kernel of the counit $\varepsilon : H \to k$; then $R^+ = \ker \varepsilon|_R$ is a coideal of $R$. Recall that two right $H$-modules $U$ and $W$ have an $H$-module structure on $U \otimes_k W$ from the diagonal action, $(u \otimes w) \cdot h = uh_{(1)} \otimes wh_{(2)}$. In this section we study the annihilator ideals of the tensor powers of the right $H$-module coalgebra $V := H/R^+H$ and its restriction to right $R$-module coalgebra. The purpose for this is to obtain a necessary condition for finite depth of the subalgebra $R \subseteq H$. Several of the arguments below originate in the pioneering [29, Rieffel] and are illuminated by the related articles by [28, Passman-Quinn], [15, Feldvöss-Klingler] and [11, Chen-Hiss]. A useful fact for finite-dimensional Hopf algebras that we use below is that a bi-ideal $I$ of $H$ is automatically a Hopf ideal; i.e., if $I$ is an (two-sided) ideal and coideal of $H$, then it may be established that $S(I) = I$ for the antipode $S : H \to H$ (e.g., see [28]).

Given the right $R$-module $V = H/R^+H$, its tensor powers $V^{\otimes n} = V \otimes \cdots \otimes V$ ($n$ times $V$) are also $R$-modules, with annihilator ideals denoted by $I_n = \mathrm{Ann}_R V^{\otimes n}$. Thinking of the zeroeth power of $V$ as the trivial $R$-module $k_\varepsilon$, denote $I_0 = R^+$. Now if modules have a monic $U \hookrightarrow W$, one verifies that $\mathrm{Ann}\, W \subseteq \mathrm{Ann}\, U$. Secondly, the $R$-module coalgebra structure of $V$ shows that for each $n \geq 0$, $V^{\otimes n} \mid V^{\otimes(n+1)}$ [23, Prop. 3.8]. It follows that we have a descending chain of ideals,

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n := \mathrm{Ann}_R V^{\otimes n} \supseteq \cdots \tag{3}$$

In a moment we show in the proof of Lemma 2.2 the (also known) fact that $I_n = I_{n+1}$ implies $I_n = I_{n+r}$ for all positive integers $r$; in this case, if $\ell(R)$ denotes the length of $R$ as an $R^e$-module, the chain of ideals of $R$ in (3) must satisfy $I_n = I_{n+1}$ at some $n \leq \ell(R)$. Note that if $t$ is the number of nonisomorphic $R$-simples, then $\ell(R) \geq t$, with equality if and only if $R$ is semisimple [15].

**Example 2.1.** Suppose $I_0 = I_1$. Then $R^+ \subseteq \mathrm{Ann}_R V = \{r \in R^+ : Hr \subseteq R^+ H\}$; i.e., $HR^+ \subseteq R^+ H$, a condition that characterizes left ad-stable Hopf subalgebra as well as right depth two Hopf subalgebra [4]. Thus, $I_0 = I_1$ if and only if $R$ is a normal Hopf subalgebra in $H$ iff $d(R, H) \leq 2$.

Let $I_V := \cap_{n=1}^{\infty} I_n$, an ideal in $R$; indeed $I_V$ is the maximal Hopf ideal contained in $\mathrm{Ann}_R V$, by the next lemma based on venerable arguments given in [29, 28] (and worth giving again in this context).

**Lemma 2.2.** *Each Hopf ideal in* $\mathrm{Ann}_R V$ *is contained in* $I_V$*, which is itself a Hopf ideal. Moreover,* $I_V = I_n$ *for some* $n \leq \ell(R)$.

*Proof.* Suppose $I$ is a Hopf ideal in $\mathrm{Ann}_R V$ and $x \in I$. Then $x$ annihilates $V$, so that $(V \otimes V) \cdot x = (V \otimes V)\Delta(x) = 0$ follows from the coideal property $\Delta(I) \subseteq I \otimes R + R \otimes I$. Similarly $x \in I_n$ for all $n \geq 1$, since the $n-1$'st power of (the coassociative) coproduct satisfies $\Delta^{n-1}(x) \in I^{(n)}$, a subspace in $R^{\otimes n}$ defined generally by

$$I^{(m+1)} := \sum_{i=0}^{m} R^{\otimes i} \otimes I \otimes R^{\otimes (m-i)} \tag{4}$$

(which visibly annihilates $V^{\otimes (m+1)}$).

If $I_n = I_{n+1}$, we show $I_n = I_{n+2}$ and a similar induction argument shows that $I_n = I_{n+r}$ for all $r \geq 0$. If $x \in I_n = I_{n+1}$, then $\Delta(x)$ annihilates $V^{\otimes (n+1)} = V^{\otimes n} \otimes V$, whence $\Delta(x) \in I_n \otimes R + R \otimes I_1$. Then $(\Delta \otimes \mathrm{id}_R)\Delta(x) \in I_n \otimes R \otimes R + R \otimes I_1 \otimes R + R \otimes R \otimes I_1$, which itself annihilates $V^{\otimes n} \otimes V \otimes V = V^{\otimes (n+2)}$. Then $I_n = I_{n+2}$.

From this it follows that $I_V = \cap_{i=1}^{n} I_n = I_n$ and that $I_V$ is a coideal. For suppose $x \in I_n = I_{2n}$. Then $V^{\otimes n} \cdot x = 0 = V^{\otimes 2n} \cdot x$, so writing $V^{\otimes 2n} = V^{\otimes n} \otimes V^{\otimes n}$ shows that $x_{(1)} \otimes x_{(2)} \in I_n \otimes R + R \otimes I_n$, and thus $\Delta(I_V) \subseteq I_V \otimes R + R \otimes I_V$. We conclude that $I_V$ is a bi-ideal in $R$, whence a Hopf ideal, and the maximal Hopf ideal contained in $I_1$. Let $\ell_V$ denote the least $n$ for which $I_V = I_n$, so that $\ell_V \leq \ell(R)$ follows from the general remarks about composition series following (3). $\square$

**Proposition 2.3.** *If a Hopf subalgebra $R$ has depth $2n+2$ in a finite-dimensional Hopf algebra $H$, then* $\mathrm{Ann}_R V^{\otimes n} \subseteq \mathrm{Ann}_R V^{\otimes (n+r)}$ *for all integers $r \geq 0$.*

*Proof.* From the inequality (2), it follows that the depth of $V$ is $n$ or less. Thus $V^{\otimes(n+r)} \sim V^{\otimes n}$ as $R$-modules, and these have equal annihilators. That $I_{n+r} \subseteq I_n$ is always the case. □

Note that

$$\mathrm{Ann}_R V^{\otimes n} = \{r \in R^+ \mid H^{\otimes n}.r \in (R^+H)^{(n)}\} \tag{5}$$

from which it is possible to express the necessary condition for depth $2n+2$ in the proposition in continuation of the condition $HR^+ \subseteq R^+H$ for depth 2. For example, denote $R^{++} := \{r \in R^+ \mid Hr \subseteq R^+H\}$; then a necessary condition that $R \subseteq H$ have depth 4 is

$$(H \otimes H).R^{++} \subseteq (R^+H)^{(2)}, \tag{6}$$

which expresses that $\mathrm{Ann}_R V \subseteq \mathrm{Ann}_R(V \otimes V)$.

**Example 2.4.** Given a finite-dimensional Hopf algebra $H$ over an arbitrary field $k$ with radical ideal $J$, the $H$-module $W = H/J$ may not be a coalgebra if $J$ fails to be a coideal. Of course $\mathrm{Ann}_H W = J$: the annihilator ideals of $W^{\otimes n}$ are shown in [11, Chen-Hiss] to satisfy $\mathrm{Ann}_H W^{\otimes n} = \bigwedge^n J$ (for the wedge product of subspaces of a coalgebra, see for example [26, Chapter 5]), which is also a descending series of ideals. Therefore the lemma applies to $W = H/J$ as well, so the intersection $I_W$ of the annihilators of tensor powers of $W$ is the maximal nilpotent Hopf ideal $J_\omega$ in the radical of $H$, studied in [11]. For example, if $H$ has a projective simple, then $J_\omega = \{0\}$ [11, 2.6(3)] with a partial converse [11, 3.10] involving the condition $\ell_W \leq 2$. On the one hand, if $H$ is a pointed Hopf algebra, then $J_\omega = J$ [26, Chapter 5]; equivalently, $H$ has the Chevalley property [25] (i.e., tensor products of simple modules are semisimple). On the other hand, if $H = kG$ a group algebra over a field $k$ of characteristic $p$, with normal Hopf subalgebra $R = kO_p(G)$, the group algebra of the core $O_p(G)$ of a Sylow $p$-subgroup, then using [28, 11] one notes that $J_\omega(H)$ is the Hopf ideal $R^+H = HR^+$. It is verified in [11, 4.5] that for $k$ algebraically closed of characteristic $p \geq 5$, each of the nonabelian simple groups $G$ has a projective and simple $kG$-module (as suggested by the fact that $O_p(G) = \{1\}$).

Recall that an $R$-module $U$ is *faithful* if $\mathrm{Ann}_R U = \{0\}$.

**Definition 2.5.** *Say that the quotient module $V = H/R^+H$ is conditionally faithful if $I_V = \{0\}$, i.e., the annihilator ideal* $\mathrm{Ann}_R V$ *contains* *no nonzero Hopf ideal in R. By Lemma 2.2 this implies that $V^{\otimes n}$ is faithful as an R-module for all $n \geq \ell_V$.*

It is well-known that an $R$-module $W$ is faithful if and only if $W$ is a generator. For if $W$ is a generator, then for some $n \in \mathbb{N}$, there is $R_R \hookrightarrow W^n$, whence $\mathrm{Ann}_R W \subseteq \mathrm{Ann}_R R = \{0\}$. Conversely, if $W$ is faithful, define a monomorphism $R_R \hookrightarrow W^n$ by $r \mapsto (w_1 r, \ldots, w_n r)$ where $w_1, \ldots, w_n$ is a $k$-basis of $W$. Since $R$ is a (quasi-) Frobenius algebra, $R_R$ is an injective module, and the monomorphism just given is a split monomorphism. The next lemma is classical and follows from the Krull-Schmidt Theorem applied to $R_R \,|\, n \cdot W_R$.

**Lemma 2.6.** *If $W_R$ is faithful, then each projective indecomposable R-module P satisfies $P \,|\, W$.*

**Example 2.7.** Let $R$ be a Hopf algebra where $\dim R \geq 2$. Then the regular representation $R_R$ is faithful and projective, as are the tensor powers $R^{\otimes n}$ for integers $n \geq 1$. From the lemma it follows that $R \sim R^{\otimes n}$ as $R$-modules, so that $\ell_R = 1$ and $d(R, \mathcal{M}_R) = 1$. Similarly, a faithful projective $R$-module $W$ has depth 1; a conditionally faithful projective $R$-module $V$ has depth $\ell_V$.

**Theorem 2.8.** *Suppose $R \subseteq H$ is a Hopf subalgebra with quotient module V a projective, conditionally faithful R-module. Then R is semisimple, $\ell_V \leq t$, where t is the number of irreducible representations of R, and each R-simple $S \,|\, V^{\otimes \ell_V}$. Furthermore, the depth satisfies $d(R, H) \leq 2\ell_V + 2$.*

*Proof.* If $V = H/R^+H$ is a projective right $R$-module, then $R$ is semisimple [23, 3.5]. This may also be seen right away by noting that $k_R \,|\, V_R$, since the counit $\varepsilon_V : V \to k$ is split by the mapping $\mu \mapsto 1\mu + R^+H$. Then $k_R$ is projective, and $R$ is semisimple.

Since $R$ is semisimple, the length $\ell(R)$ of $R_{R^e}$ satisfies $\ell(R) = t$; also, each projective indecomposable is a simple module and conversely. Then $\ell_V \leq t$ follows from Lemma 2.2 , and each $S \,|\, V^{\otimes \ell_V}$ follows from Definition 2.5 and Lemma 2.6.

The last statement of the theorem follows from the inequality for depth Eq. (2). Since the $V^{\otimes(\ell_V + r)}$ are faithful, semisimple $R$-modules for each integer $r \geq 0$, each contains every $R$-simple as noted before (and recalling that $V^{\otimes m} \,|\, V^{\otimes(m+1)}$ for each $m \geq 0$). Consequently, they are similar as $R$-modules: $V^{\otimes \ell_V} \sim V^{\otimes(\ell_V + r)}$ for each $r \geq 0$. It follows that the depth of $V$ satisfies $d(V, \mathcal{M}_R) \leq \ell_V$. □

**Example 2.9.** Suppose $k = \mathbb{C}$ and the Hopf subalgebra $R$ is a group algebra $\mathbb{C}\,G$ where $G$ is a subgroup of grouplike elements in a Hopf algebra $H$. Suppose that $V = H/R^+H$ is conditionally faithful, then its character $\chi_V$ is faithful, i.e., its kernel $\ker \chi_V = \{g \in G | \chi_V(g) = \chi_V(1)\} = N$ is trivial, for if this normal subgroup were nontrivial, then $\mathrm{Ann}_R V$ contains the nontrivial Hopf ideal $I = R\mathbb{C}\, N^+ = \mathbb{C}\, N^+ R$. Note that if $\chi_V(g) = \chi_V(1)$, then $g$ acts like the identity on $V$, whence $1 - g \in \mathrm{Ann}_R V$. Conversely, if the character $\chi_V$ is faithful, the Burnside-Brauer Theorem [18, p. 49] informs us that $V$ is conditionally faithful, for $\chi_i \,|\, \chi_V^m$ for each irreducible character, $\chi_1, \ldots, \chi_t$ of $G$, and $m \leq |\chi_V(G)|$, where $|X|$ denotes the cardinality of a finite set $X$. It follows that $\ell_V \leq |\chi_V(G)|$. (Alternatively for general $k$, if $V_R$ is not conditionally faithful, then $\mathrm{Ann}_R V^{\otimes n}$ stabilizes as $n \to \infty$ on a nonzero Hopf ideal $I$ of the group algebra $R$ necessarily of the form $I = RkN^+ = kN^+R$ [28, 11], where $N$ is a normal subgroup of $G$ in $\ker \chi_V$.)

## 3. Depth of a semisimple group algebra in its Drinfeld double

As an application of Section 2 and the methods sketched in the last subsection of Section 1, we compute the depth of a semisimple group algebra in its Drinfeld double, a smash product of the group algebra and its dual [26]. The computation is very must guided by the ideas in [27, Passman]. A certain portion of this section can be carried further to a general semisimple or cocommutative Hopf algebra in its Drinfeld double; the interested reader should first consult [6].

Suppose $G$ is a finite group, $k$ a field of characteristic not dividing the order of $G$, and consider the group algebra $R = kG$. Denote its Drinfeld double as $H = D(G) = D(R)$ [26] with multiplication given by

$$(p_x \bowtie g)(p_y \bowtie h) = p_x p_{gyg^{-1}} \bowtie gh \tag{7}$$

for all $g, h, x, y \in G$ where $p_x$ denotes the one-point projection in $R^*$. Note that this is the semidirect product of the $R$-module (adjoint representation) algebra $R^*$ with $kG$. Recall that $1_H = \sum_{x \in G} p_x \bowtie 1_G$ and the counit $\varepsilon(p_x \bowtie g) = p_x(1_G) = \delta_{x,1}$. Of course $R$ is identifiable with the subalgebra $1_{R^*} \otimes R$. A short computation with Eq. (7) shows that the centers of $D(G)$ and $G$ satisfy

$$kZ(G) = Z(D(G)) \cap kG. \tag{8}$$

We compute the generalized quotient $V = H/R^+H$ as a right $R$-module. Note that $\dim V = |G|$.

**Lemma 3.1.** *The right $G$-module $V$ is isomorphic to $kG_{\mathrm{ad}}$.*

*Proof.* First compute $R^+H$ from

$$(1_H \bowtie (1-g))(p_y \bowtie h) = p_y \bowtie h - p_{gyg^{-1}} \bowtie gh,$$

for each $1 \neq g, y, h \in G$. Thus in $H/R^+H$ the cosets have a unique representative as follows:

$$\overline{p_y \bowtie h} = \overline{p_{gyg^{-1}} \bowtie gh} = \overline{p_{h^{-1}yh} \bowtie 1_G}$$

Define a $G$-module isomorphism $V \xrightarrow{\cong} R^*$ by $\overline{p_y \bowtie h} \mapsto p_{h^{-1}yh}$. But $kG^*{}_{\mathrm{ad}} \cong kG_{\mathrm{ad}}$ via $p_g \mapsto g$, where the right adjoint is given by $g \cdot x = x^{-1}gx$. □

It is well-known that in characteristic zero, $D(R)$ is a semisimple algebra, if $R$ is semisimple.

**Proposition 3.2** (Burciu [7]). *The module $V = kG_{\mathrm{ad}}$ has depth $n$ if the $kG$-module $V^{\otimes n}$ is faithful for some $n \in \mathbb{N}$. Our converse requires $k$ to be an algebraically closed field of* $\mathrm{char}\, k = 0$ *and that $G$ has trivial center. If $kG \subseteq D(G)$ has depth $2n+1$, then $V^{\otimes n}$ is faithful.*

*Proof.* ($\Leftarrow$) Since $kG$ is a semisimple algebra, the $kG$-modules $V$ and its tensor powers are semisimple modules. Thus if $V^{\otimes n}$ is faithful, it contains each simple $kG$-module by Lemma 2.6. It follows that $V^{\otimes n} \sim V^{\otimes(n+r)}$ for each integer $r \geq 0$. Thus, $V$ has depth $n$.

($\Rightarrow$) Use the Rieffel relation $\overset{R}{\sim}$ between simple $kG$-modules $W, U$ defined by $W \overset{R}{\sim} U$ if $W \otimes_R H$ and $U \otimes_R H$ have an isomorphic nonzero summand in common ([9, p. 139] and [30]). (In terms of the bipartite graph of the semisimple subalgebra pair $R \subseteq H$, the points representing $W$ and $U$ are connected by one irreducible representation of $H$.) Extend $\overset{R}{\sim}$ by transitive closure to an equivalence relation. Note that $\overset{R}{\sim}$ is already a transitive relation iff $R \subseteq H$ has depth 3 [9, Corollary 3.7]. Also, the number of equivalence classes is equal to $\dim Z(H) \cap R$ [9, Corollary 3.3], so by the hypothesis and Eq. (8) there is one equivalence class.

Let $W$ be a left $R$-module (and note that the ${}_R\mathcal{M}$ is isomorphic as tensor categories to $\mathcal{M}_R$ via the inverse). We compute $W \uparrow^{D(R)}\downarrow_R$ from

$$R^* \otimes_k R \otimes_R W \cong R^* \otimes_k W$$

with $G$-action given by $g \cdot p_x \otimes w = p_{gxg^{-1}} \otimes gw$. This implies that the image of $W$ under induction and restriction satisfies

$$W \uparrow^{D(R)}\downarrow_R \cong {}_{\mathrm{ad}}R \otimes W, \tag{9}$$

the right-hand side having the diagonal action by $R$.

Let $\chi_U$ denote the character of a $G$-module $U$, $\chi_{\rm ad}$ be the character of module ${}_{\rm ad}R$, and $\chi_1, \ldots, \chi_t \in \mathrm{Irr}(G)$. If $R \subseteq H$ has depth 3, then $\overset{R}{\sim}$ has one equivalence class, so that the inner product of any irreducible characters, $\chi_U, \chi_W$ of $G$, satisfies $\langle \chi_U \uparrow^{D(G)}, \chi_W \uparrow^{D(G)} \rangle > 0$. By Frobenius reciprocity and Eq. (9) this gives $\langle \chi_V, \chi_{\rm ad}\chi_W \rangle > 0$, so letting $\chi_W = \chi_k$, this shows that ${}_{\rm ad}R$ and $R_{\rm ad}$ are generators, therefore faithful modules.

If $R$ in $H$ has depth 5, then by [9, Proposition 5.4], any two $R$-simples $U, W$ may be reached by a shortest path of length at most two, $U \overset{R}{\sim} X \overset{R}{\sim} W$ for some $R$-simple $X$, and that the entry $\langle \chi_U, \chi_{\rm ad}^2 \chi_W \rangle > 0$ in $S^2$ (where $S$ is the symmetric order $t$ matrix defined in Section 1 by $S_{ij} = \langle \chi_i \uparrow^{D(G)}, \chi_j \uparrow^{D(G)} \rangle$). Thus $V^{\otimes 2}$ is faithful. The rest of the proof is a similar induction argument using [9, Proposition 5.4]. □

Recall from Section 2 that $V$ is conditionally faithful if $\mathrm{Ann}_R V^{\otimes \ell_V} = \{0\}$ for some $\ell_V \geq 1$, while $\mathrm{Ann}_R V^{\otimes m} \neq \{0\}$ for $0 \leq m < \ell_V$.

**Corollary 3.3.** *Suppose $k$ is an algebraically closed field of characteristic zero and $G$ is a finite, centerless group. Then adjoint module $V$ is conditionally faithful and its depth as an $kG$-module is $\ell_V$*

*Proof.* From the hypotheses on $k$, it follows from [9] that $kG \subseteq D(G)$ has a finite depth. Suppose it has depth $2n+1$; then by the proposition, $V^{\otimes n}$ is a faithful $kG$-module. It follows that $n \geq \ell_V$. Since $V^{\otimes \ell_V}$ is a generator, also by Lemma 2.6, $V^{\otimes \ell_V} \sim V^{\otimes(\ell_V + r)}$ for all integers $r \geq 0$. Then $V_R$ has depth $\ell_V$. □

As we will see in Corollary 4.3 the depth is in fact satisfying

$$d(\mathbb{C}\, G, D(G)) = 2\ell_V + 1.$$

**Example 3.4.** Let $k$ be a field of characteristic zero. The paper [27, Passman, Theorem 1.10] shows that for each $n \geq 3$ the symmetric group $S_n$ has a faithful adjoint action on $kS_n$. It follows from Corollary 3.3 that $3 \leq d(kS_n, D(S_n)) \leq 4$ (in fact $d(kS_n, D(S_n)) = 3$ follows from Theorem 4.2 below).

Note that $d(kS_n, D(S_n)) = 3$ for specific $n = 3, 4, \ldots$ also follows from a computation that the symmetric matrix $S > 0$, i.e., has all positive entries. In general the methods above are realized from the $r \times r$ character table $(\chi_i(g_j))$ of a group $G$ with values in $\mathbb{C}$ as follows. The character $\chi_{\rm ad}$ is given by row vector $(|C(g_j)|)_{j=1,\ldots,r}$, where an entry is the number of elements of the conjugacy class of $g_j$. The inner product $\langle \chi_{\rm ad}, \chi_j \rangle$ is the sum $\sum_{i=1}^r \chi_j(g_i)$; e.g. $\langle \chi_{\rm ad}, \chi_1 \rangle = r$, the number of orbits of the permutation module by Burnside's Lemma [18]. That no row of the character table sums to zero is then equivalent to the module $\mathbb{C}\, G_{\text{ad}}$ being faithful. Also the center of $G$ equals the kernel of $\chi_{\text{ad}}$, and is trivial if no $g \neq 1$ satisfies $\chi_{\text{ad}}(g) = \chi_{\text{ad}}(1) = |G|$.

## 4. On depth of a Hopf algebra in a smash product

In this section we show that a Hopf algebra $H$ has finite depth in its smash product algebra $A\#H$ if the left $H$-module algebra $A$ is an algebraic $H$-module.

Suppose $H$ is a Hopf algebra and $A$ is a left $H$-module algebra. Recall that equations such as $h.1_A = \varepsilon(h)1_A$ and $h.(ab) = (h_{(1)}.a)(h_{(2)}.b)$ are satisfied ($a, b \in A$, $h \in H$): briefly, $A$ is an algebra in the tensor category of left $H$-modules. Define the smash product by $A\#H = A \otimes H$ as a linear space with multiplication given by

$$(a\#h)(b\#k) = a(h_{(1)}.b)\#h_{(2)}k \tag{10}$$

Notice how $H$ identifies with the subalgebra $1_A\#H$ in $A\#H$ and if $a = 1_A$, the action of $h$ is the diagonal action.

**Proposition 4.1.** *The $n$-fold tensor powers of $A\#H$ over $H$ are isomorphic as $H$-$H$-bimodules to the following tensor products in the tensor category ${}_H\mathcal{M}$:*

$$(A\#H)^{\otimes_H n} \cong A^{\otimes n} \otimes H \tag{11}$$

*Proof.* The case $n = 1$ follows from the mapping $a\#h \mapsto a \otimes h$, which is clearly right $H$-linear and also left $H$-linear by an application of Eq. (10).

Suppose Eq. (11) holds for an $H$-$H$-bimodule isomorphism for $1 \leq n < m$. Since $H \otimes_H A \cong A$, it follows from induction that

$$(A\#H)^{\otimes_H m} \cong (A\#H)^{\otimes_H (m-1)} \otimes_H A\#H \cong$$
$$A^{\otimes(m-1)} \otimes H \otimes_H A \otimes H \cong A^{\otimes m} \otimes H.$$

Note that the isomorphism becomes $a\#u \otimes_H b\#v \otimes_H \cdots \otimes_H c\#w$

$$\longmapsto a \otimes u_{(1)}.b \otimes \cdots \otimes u_{(n-1)}v_{(n-2)} \cdots .c \otimes u_{(n)}v_{(n-1)} \cdots w \tag{12}$$

for $u, v, w \in H$ and $a, b, c \in A$. □

Define the minimum odd depth of a subalgebra $B \subseteq A$ as $d_{\text{odd}}(B, A) = 2\lceil \frac{d(B,A)-1}{2} \rceil + 1$, which is the least odd integer greater than or equal to the minimum depth $d(B, A)$.

**Theorem 4.2.** *The minimum odd depth of a finite-dimensional Hopf algebra in its smash product satisfies*

$$d_{\text{odd}}(H, A\#H) = 2d(A, {}_H\mathcal{M}) + 1 \tag{13}$$

*Proof.* Since $A$ is a left $H$-module algebra, it follows from applying any of the standard face and degeneracy mappings, which are $H$-module maps, that $A^{\otimes m} \mid A^{\otimes(m+1)}$ for each integer $m \geq 0$. Then the depth $n$ condition for the left $H$-module $A$ given by $T_{n+1}(A) \mid q \cdot T_n(A)$ for some $q \in \mathbb{N}$ is equivalent to $A^{\otimes(n+1)} \mid q \cdot A^{\otimes n}$ for some $q \in \mathbb{N}$. Tensoring this by $- \otimes H$ yields $A^{\otimes(n+1)} \otimes H \mid q \cdot A^{\otimes n} \otimes H$ and thus by Proposition 4.1 $(A\#H)^{\otimes_H (n+1)} \mid q \cdot (A\#H)^{\otimes_H n}$ as $H$-$H$-bimodules. Thus the minimum odd depth $d_{\text{odd}}(H, A\#H) \leq 2d(A, {}_H\mathcal{M}) + 1$ by Definition 1.1.

Conversely, if $(A\#H)^{\otimes_H (n+1)} \mid q \cdot (A\#H)^{\otimes_H n}$ as $H$-$H$-bimodules, we apply Proposition 4.1 and write equivalently $A^{\otimes(n+1)} \otimes H \mid q \cdot A^{\otimes n} \otimes H$. Next apply $- \otimes_H k$ to this, and through the cancellation ${}_H H \otimes_H k \cong {}_H k$ with the unit module in ${}_H\mathcal{M}$, we obtain $A^{\otimes(n+1)} \mid q \cdot A^{\otimes n}$, which is the depth $n$ condition for an $H$-module algebra. Therefore $2d(A, {}_H\mathcal{M}) + 1 \leq d_{\text{odd}}(H, A\#H)$. The conclusion of the theorem follows from the two inequalities established. □

**Corollary 4.3.** *The subalgebra depth and the depth of* $V = kG_{\text{ad}}$ *are related by* $d_{\text{odd}}(kG, D(G)) = 2d(V, \mathcal{M}_{kG}) + 1$. *If* $k$ *is algebraically closed and has characteristic* 0 *and the center of* $G$ *is trivial, then* $V$ *is conditionally faithful and the depth satisfies* $d(kG, D(G)) = 2\ell_V + 1$.

*Proof.* First note from Eq. (7) that $D(G) \cong (kG)^* \# kG$ where the action is the adjoint action, ${}_{\text{ad}}kG^*$, which is isomorphic to $V$. Then Eq. (13) implies that $d_{\text{odd}}(kG, D(G)) = 2d(V, \mathcal{M}_{kG}) + 1$.

For the second statement, note that Corollary 3.3 shows that $d(V, \mathcal{M}_{kG}) = \ell_V$. From the inequality (2) depth of the centerless group algebra in its Drinfeld double satisfies $d(kG, D(G)) = 2\ell_V + 1$ or $2\ell_V + 2$; if $d(kG, D(G)) = 2\ell_V + 2$, then $d_{\text{odd}}(kG, D(G)) = 2\ell_V + 3$. But Theorem 4.2 then implies that $d(V, \mathcal{M}_{kG}) = \ell_V + 1$, a contradiction. □

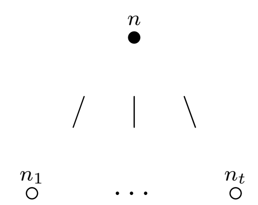
**Example 4.4.** The minimal example suggested in [27, Lemma 1.3] for a centerless group $G$ with adjoint action on $\mathbb{C}\,G$ that is not faithful, is a semidirect product $G$ of a rank 3 elementary 3-group with the Klein 4-group, so that $|G| = 108$ [7]. A long computation by hand of its order 15 character table and $S$-matrix (where $S_{ij} = \langle \chi_i, \chi_{\text{ad}} \chi_j \rangle$) shows that $S$ has zero entries, but $S^2 > 0$, whence there is $q \in \mathbb{N}$ such that $S^2 \leq qS^3$. It follows from Corollary 4.3 that the minimum depth satisfies $d(\mathbb{C}\,G, D(G)) = 5$.

**Example 4.5.** Let $H$ be a Hopf algebra of dimension $n \geq 2$. Let $H^*$ act on $H$ by $f \rightharpoonup h = h_{(1)} f(h_{(2)})$. It is a standard check that $H$ is a left $H^*$-module algebra. Their smash product $H \# H^*$ is the *Heisenberg double* of $H$ [26, Ch. 9]. We compute the depth $d_{\text{odd}}(H^*, H \# H^*)$ next from $d(H, {}_{H^*}\mathcal{M})$ and Theorem 4.2. Since $H^*$ is a Frobenius algebra,

${}_{H^*}H \cong {}_{H^*}H^*$ is isomorphic to the regular representation of $H^*$. It was noted in Example 2.7 that $d(H, {}_{H^*}\mathcal{M}) = 1$. It follows that

$$d_{\text{odd}}(H^*, H\#H^*) = 3. \tag{14}$$

This result on depth makes good sense, since $H\#H^* \cong M_n(k)$ via the (Galois) algebra isomorphism $\lambda : H\#H^* \xrightarrow{\cong} \text{End}_k H$ given by $\lambda(h\#f)(x) = h(f \rightharpoonup x)$. Thus $H\#H^*$ is an Azumaya $k$-algebra; then $H^* \hookrightarrow H\#H^*$ is an H-separable extension if the extension is split and projective (cf. [21]). In this case $d_h(H^*, H\#H^*) = 1$ and $d(H^*, H\#H^*) = 2$. If $H^*$ is a semisimple complex algebra, that $2 = d(H^*, H\#H^*)$ may also be seen from the bipartite graph of the inclusion [9] pictured below (where $n_1, \ldots, n_t$ denote the dimensions of the simples of $H^*$).




## 5. Depth of subalgebras projective in a finite-dimensional algebra

Let $A$ be a finite-dimensional algebra over a field $k$. Denote the principal right $A$-modules, or projective indecomposables of $A$, by $P_1, \ldots, P_s$. (We sometimes confuse objects and their isoclasses for the sake of brevity.) Let $J$ denote the radical ideal of $A$. Then each $P_i$ is the projective cover of $P_i/P_iJ := S_i$, the simple $A$-modules where $i = 1, \ldots, s$. Recall that the Cartan matrix $C$ of $A$ is an $s \times s$-matrix of nonnegative entries whose rows give the multiplicity of each simple $S_j$ in the composition factors of $P_i$; one may view $C$ as the matrix of a linear mapping $K_0(A) \to G_0(A)$ corresponding to sending a projective into a sum of its simple composition factors with multiplicity. Recall that $K_0(A) \cong Z^s$ is a free abelian group on the basis $P_1, \ldots, P_s$, such that a projective $X$ in $K_0(A)$ is a nonnegative sum of the $P_i$ corresponding to its Krull-Schmidt decomposition; also recall that $G_0(A) \cong Z^s$ is the free abelian group on the basis $S_1, \ldots, S_s$ (the Grothendieck group of $A$) such that a module $Y$ in $G_0(A)$ is a nonnegative sum of the $S_i$ corresponding to the multiplicity of its composition factors. If $k$ is an algebraically closed field, $\dim_k \text{Hom}_A(P_i, X)$ equals the multiplicity of (the isomorphism class of) $S_i$ as a composition factor in a finite-dimensional module $X$ [2, p. 45]: in this case, the Cartan matrix entry $c_{ij} = \dim \mathrm{Hom}_A(P_i, P_j)$ for each $i, j = 1, \ldots, s$.

Suppose $B \subseteq A$ is a subalgebra of $A$ such that the natural module $A_B$ is projective. Denote the projective indecomposables of $B$ by $Q_1, \ldots Q_r$, the Cartan matrix of $B$ by $D$, which has entries $d_{ij} = \dim \mathrm{Hom}_B(Q_i, Q_j)$ in case $k$ is algebraically closed.

Of interest to us are two $r \times s$-matrices with nonnegative entries. (For both matrices, we use the Krull-Schmidt Theorem for finite length modules of Artin algebras.) First define the *matrix of restriction* $M$ with entries given by $m_{ij}$ defined by

$$P_j \downarrow_B \cong \oplus_{i=1}^r m_{ij} \cdot Q_i \tag{15}$$

since each projective $A$-module restricts to a projective $B$-module by the hypothesis that $A_B$ is projective. Secondly, define the *matrix of induction* for the subalgebra $B \subseteq A$ as the $r \times s$-matrix $N$ with row entries $n_{ij} \in \mathbb{N}$ given by inducing each of the projective indecomposable $B$-modules,

$$Q_i \otimes_B A \cong \oplus_{j=1}^s n_{ij} \cdot P_j \tag{16}$$

**Lemma 5.1.** *Suppose $k$ is algebraically closed. Then the matrices of restriction $M$ and induction $N$ are related by*

$$DM = NC \tag{17}$$

*where $C$ and $D$ denote the Cartan matrices of $A$ and $B$, respectively.*

*Proof.* From the Hom-Tensor adjoint relation it follows that

$$\mathrm{Hom}_A(Q_i \otimes_B A, P_j) \cong \mathrm{Hom}_B(Q_i, P_j \downarrow_B)$$

[19]. Substitution of Eqs. (16) and (15) reduces to

$$\oplus_{k=1}^s n_{ik} \cdot \mathrm{Hom}_A(P_k, P_j) \cong \oplus_{q=1}^r m_{qj} \cdot \mathrm{Hom}_B(Q_i, Q_q).$$

Taking the dimension of both sides yields $\sum_{k=1}^s n_{ik} c_{kj} = \sum_{q=1}^r m_{qj} d_{iq}$. for each $i = 1, \ldots, r$, $j = 1, \ldots, s$, from which the lemma follows. □

**Example 5.2.** Suppose $A$ and $B$ are semisimple algebras with $B$ a subalgebra of $A$. Then $P_i = S_i$ so that the Cartan matrix of $A$ is the identity matrix, $C = I_s$; similarly, the Cartan matrix of $B$ satisfies $D = I_r$. It follows from the lemma that if the ground field $k$ is algebraically closed, $M = N$, which is then the induction-restriction matrix studied in [9] for $k$ additionally of characteristic zero, or the induction-restriction table studied in [1] for subgroup pairs of finite complex group algebras. That $M = N$ also follows from the proof of

Lemma 5.1 by applying Schur's Lemma for algebraically closed fields to $\mathrm{Hom}_A(S_i, S_j) \cong k\delta_{ij}$ and similarly $\dim \mathrm{Hom}_B(Q_i, Q_j) = \delta_{ij}$.

**Example 5.3.** Let $A = T_n(k)$ be the upper triangular $n \times n$-matrices over an algebraically closed field $k$. Let $B = \mathrm{Diag}_n(k)$ the diagonal matrices of order $n$, a semisimple subalgebra of $A$. The Cartan matrix $D = I_n$ is immediate. Let $J$ denote the radical ideal of $A$, so that the obvious algebra epimorphism $A \to B$ is equal to the canonical epi $A \to A/J$. Denote the simples of $A$ by $S_1, \ldots, S_n$ which are then also the simples of $B$ by restriction. Thus $Q_i = S_i \downarrow_B$ for each $i = 1, \ldots, n$. The projective indecomposable right $A$-modules are given in terms of matrix units by $P_1 = e_{11}A, \ldots, P_n = e_{nn}A$, which are the projective covers of $S_1, \ldots, S_n$, respectively. Then the matrix of induction from $B$ to $A$ is $N = I_n$, since $S_i \otimes_B A \cong P_i$ is immediate from writing $S_i = Be_{ii}$.

The composition series of $P_i$ is given by $P_i \supset P_i J \supset P_i J^2 \supset \cdots \supset P_i J^{n-i+1} = \{0\}$ with simple factors $P_i/P_iJ \cong S_i, P_iJ/P_iJ^2 \cong S_{i+1}$, and so forth, obtaining the Cartan matrix $C = \sum_{i \leq j} e_{ij}$ for $A$. Restriction of the principal modules, $P_1 \downarrow_B \cong Q_1 \oplus \cdots \oplus Q_n$ is clear from writing $P_1 = \sum_{j=1}^n e_{1j}k$ and the matrix unit equations $e_{ij}e_{qk} = \delta_{jq}e_{ik}$. Similarly, $P_i \downarrow_B \cong Q_i \oplus \cdots \oplus Q_n$, whence the restriction matrix of $B \subset A$ is $M = \sum_{i \leq j} e_{ij}$. Indeed $M = C$ as implied by Lemma 5.1.

The theorem below does not require that $k$ is algebraically closed. Set the zeroeth power of a square matrix equal to the identity matrix.

**Theorem 5.4.** *Suppose $B \subseteq A$ is a subalgebra pair of finite-dimensional $k$-algebra with $A_B$ assumed projective. If the subalgebra $B \subseteq A$ has left depth $2n$ (respectively, depth $2n+1$), then*

$$(MN^t)^n M \leq t(MN^t)^{n-1}M \quad (\textit{resp. } (MN^t)^{n+1} \leq t(MN^t)^n) \tag{18}$$

*for some $t \in \mathbb{N}$.*

*Proof.* Suppose $B \subseteq A$ has depth 1. Then for some $B$-$B$-bimodule $W$, we have

$$ {}_BA_B \oplus {}_BW_B \cong t \cdot {}_BB_B \tag{19}$$

for some positive $t \in \mathbb{N}$. Tensoring Eq. (19) to the right $B$-projective indecomposable $Q_i$, one obtains after a standard cancellation,

$$Q_i \otimes_B A \downarrow_B \oplus\, Q_i \otimes_B W_B \cong t \cdot Q_i. \tag{20}$$

By the Krull-Schmidt Theorem, there is $w_i \in \mathbb{N}$ such that $Q_i \otimes_B W_B \cong w_i \cdot Q_i$ for each $i = 1, \ldots, r$; and using Eqs. (16) and (15), $Q_i \otimes_B A_B \cong (\sum_{j=1}^s n_{ij}m_{ij}) \cdot Q_i$. It follows from $w_i \geq 0$ and Eq. (20) that $MN^t \leq tI_r$. The rest of the proof is a similar application of the matrices of restriction and induction to the characterization of depth $2n, 2n+1$ subalgebra in Eq. (1). □

In [9, 2.1, 3.5] the matrix inequality (18) with $M = N$ characterizes a depth $n$ semisimple complex algebra-subalgebra pair $B \subseteq A$.

**Example 5.5.** Example 5.3 provides a counterexample to the converse for Theorem 5.4. Recall that $A$ is the upper triangular matrix algebra and $B$ is the subalgebra of diagonal matrices. Then the minimum depth $d(B, A)$ is computed in [24] as the semisimple subalgebra of quiver vertices within the path algebra for the quiver

$$1 \to 2 \to \cdots \to n-1 \to n.$$

The depth satisfies $d(B, A) = 3$ as a corollary of [24, Section 6, first paragraph]. However, we computed the $n \times n$ restriction matrix $M = \sum_{i \leq j} e_{ij}$ in terms of matrix units, and the induction matrix $N = I_n$. It follows that $MN^t = M$, all of whose powers satisfy $M^s \leq tM^{s-1}$ for integers $s \geq 2$ and some positive $t \in \mathbb{N}$ (depending on $s$), since the set of upper triangular matrices with only positive entries is closed under matrix multiplication. In particular, the subalgebra $B$ does not have depth two in $A$, although it satisfies the depth two matrix inequality $M^2 \leq nM$ (taking $t = n$) in Theorem 5.4.

## 6. Depth of a left coideal subalgebra in a finite-dimensional Hopf algebra

Let $K$ be a left coideal subalgebra of a finite-dimensional Hopf algebra $H$. In this case we only know that $\Delta(K) \subset H \otimes K$, and $K$ might not be a Hopf algebra. However, we generalize the results in [23, 3, 3.6] and generalize the h-depth result in [23, 5.1]. Below we use $\otimes$ to denote the tensor in the finite tensor category $\mathcal{M}_H$.

Let $K^+$ denote the kernel of the counit $\varepsilon$ restricted to the subalgebra $K$. Although not completely obvious, it is well-known that the right $H$-module $V := H/K^+H$ is in fact a right $H$-module coalgebra: see for example [31]. Denoting the canonical epi $H \to V$ by $\pi(h) = \overline{h}$, note that for any $x \in K, h \in H$ we have the useful identity in $V$, $\overline{xh} = \varepsilon(x)\overline{h}$.

**Lemma 6.1.** *Let $A$ be an arbitrary $k$-algebra. For any $A$-$H$ bimodule $M$, there is an isomorphism of $A$-$H$-bimodules, $M \otimes_K H \cong M \otimes V$.*

*Proof.* The mapping $M \otimes_K H \to M \otimes V$ given by $m \otimes_K h \mapsto mh_{(1)} \otimes \overline{h_{(2)}}$ is well-defined, since for $x \in K$, we compute

$$m \otimes_K xh \mapsto mx_{(1)}h_{(1)} \otimes \overline{x_{(2)}h_{(2)}} = mxh_{(1)} \otimes \overline{h_{(2)}}$$

noting that $\Delta(x) \in H \otimes K$. This map has an obvious inverse mapping $M \otimes V \to M \otimes_K H$ given by $m \otimes \overline{h} \mapsto mS(h_{(1)}) \otimes h_{(2)}$ where $S : H \to H$ denotes the antipode of $H$. The inverse mapping is well-defined since for $x \in K^+$

$$mS(x_{(1)}h_{(1)}) \otimes_K x_{(2)}h_{(2)} = mS(h_{(1)})S(x_{(1)})x_{(2)} \otimes_K h_{(2)} = 0.$$

The rest of the proof is similarly straightforward. □

This lemma is noted for a Hopf subalgebra $R \subseteq H$ by [32, Ulbrich], who also shows that the category $\mathcal{M}_R$ is equivalent to a category $\mathcal{M}_H^V$ of module-comodules over the $H$-module coalgebra $V$.

**Proposition 6.2.** *The n-fold tensor powers of $H$ over a left coideal subalgebra $K$ satisfies the $H$-$H$-bimodule isomorphism,*

$$\begin{aligned} H^{\otimes_K n} &\xrightarrow{\cong} H \otimes V^{\otimes (n-1)}, \\ x \otimes y \otimes \cdots \otimes z &\longmapsto xy_{(1)} \cdots z_{(1)} \otimes \overline{y_{(2)} \cdots z_{(2)}} \otimes \cdots \otimes \overline{z_{(n)}} \end{aligned} \tag{21}$$

*for integers $n \geq 2$ and $x, y, \ldots, z \in H$.*

*Proof.* The case $n = 2$ is done in Lemma 6.1 for $M = H$. Assume Eq. (21) holds for $2 \leq n < m$. Then

$$H^{\otimes_K m} \cong H^{\otimes_K (m-1)} \otimes_K H \cong (H \otimes V^{\otimes (m-2)}) \otimes_K H \cong H \otimes V^{\otimes (m-1)}$$

where we apply the induction hypothesis and then Lemma 6.1. □

**Theorem 6.3.** *The h-depth of the left coideal subalgebra $K \subseteq H$ satisfies $d_h(K, H) = 2d(V, \mathcal{M}_H) + 1$.*

*Proof.* Suppose depth $d(V, \mathcal{M}_H) = n$. Then $V^{\otimes (n+1)} \mid q \cdot V^{\otimes n}$. Applying the additive functor ${}_H H \otimes -$ to this yields $H \otimes V^{\otimes (n+1)} \mid q \cdot H \otimes V^{\otimes n}$ as $H$-$H$-bimodules, whence by Proposition 6.2 we obtain the h-depth $2n + 1$ condition on $H$-$H$-bimodules, $H^{\otimes_K (n+2)} \mid q \cdot H^{\otimes_K (n+1)}$. Then $d_h(K, H) \leq 2d(V, \mathcal{M}_H) + 1$.

Suppose h-depth $d_h(K, H) = 2n + 1$. Then as $H$-$H$-bimodules, $H^{\otimes_K (n+2)} \mid q \cdot H^{\otimes_K (n+1)}$; equivalently, $H \otimes V^{\otimes (n+1)} \mid q \cdot H \otimes V^{\otimes n}$ by Proposition 6.2. Tensoring this by $k \otimes_H -$, and applying the cancellations $k \otimes_H H \cong k_\varepsilon$ and $k \otimes V \cong V$, we obtain the depth $n$ condition $V^{\otimes (n+1)} \mid q \cdot V^{\otimes n}$. This shows that $d_h(K, H) \geq 2d(V, \mathcal{M}_H) + 1$, which finishes the proof. □

Let $A(H)$ denote the Green ring of $H$, where multiplication and addition are given by tensor and direct sum, and $[W]$ denotes the isomorphism class of a module $W \in \mathcal{M}_H$ in $A(H)$. An $H$-module $W$ is said to be an *algebraic module* if $[W]$ satisfies an integer coefficient polynomial in $A(H)$. This is equivalent to $W$ having finite depth [14], where the minimum depth $d(W, \mathcal{M}_H)$, defined in Section 1, is equal to one less the degree of a minimum polynomial of $[W]$ in $A(H)$. We let $A_{\mathbb{C}}(H) = A(H) \otimes_{\mathbb{Z}} \mathbb{C}$ denote the Green algebra, which has basis consisting of all isoclasses of indecomposable finitely-generated modules. The projective indecomposables span the finite-dimensional ideal $K_0(H) \otimes_{\mathbb{Z}} \mathbb{C}$. Note that a module $W$ is algebraic if $[W]$ is contained in a finite-dimensional ideal in $A_{\mathbb{C}}(H)$.

**Corollary 6.4.** *The subalgebra pair $K \subseteq H$ defined above has finite depth if and only if the generalized quotient $V$ is an algebraic $H$-module.*

The proof follows directly from the equality in Theorem 6.3 and the implications h-depth $2n+1 \Rightarrow$ depth $2n+2$, and depth $2n \Rightarrow$ h-depth $2n+1$ discussed in Section 1.

## 7. Galois theory for $\pi^* : V^* \hookrightarrow H^*$ of a Hopf subalgebra

Let $R \subseteq H$ be a Hopf subalgebra of a finite-dimensional Hopf algebra. Let $V = H/R^+H$ be the generalized quotient and right $H$-module coalgebra [10]. The canonical coalgebra epi $\pi : H \to V$, where $\pi(h) = \overline{h} = h + R^+H$ has an interesting dual algebra monomorphism $\pi^* : V^* \hookrightarrow H^*$; in [23] it is noted that $H^*$ is a Frobenius extension of $V^*$. The next lemma follows directly from freeness and that the Hopf algebra $H^*$ is a Frobenius algebra.

**Proposition 7.1.** *The algebra $V^*$ defined above is a Frobenius algebra.*

*Proof.* The natural right $V^*$-module $H^*$ (via $\pi^*$) is free since Schneider's result is that $H \cong R \otimes V$ as left $R$-modules, and right $V$-comodules (equivalently, left $V^*$-modules) [26, Ch. 8]. Thus, as right $V^* \otimes R$-modules,

$$H^* \cong V^*_{V^*} \otimes R^*_R \tag{22}$$

by standard duality for finite-dimensional algebras and modules. Now a Frobenius extension free on one side is necessarily free on the other side: it follows that ${}_{V^*}H^*$ is also free.

The proof now follows from Pareigis's argument using Krull-Schmidt (cf. [19, p. 68]). □

Note that $H^*$ is left $R^*$-comodule algebra under the left $R^*$-comodule structure stemming from restriction of its dual coproduct: the left coaction $\rho : H^* \to R^* \otimes H^*$ is defined by $\rho(h^*) = {h^*}_{(1)}|_R \otimes {h^*}_{(2)}$.

**Theorem 7.2.** *The algebra extension given by $\pi^* : V^* \hookrightarrow H^*$ is a left $R^*$-Galois extension with normal basis property.*

*Proof.* Note that $\pi : H \to V$ may be viewed as coextension of left $R$-module coalgebras [31]. At first we note that the associated canonical mapping

$$\beta : R \otimes H \to H \Box_V H, \quad r \otimes h \mapsto rh_{(1)} \otimes h_{(2)} \tag{23}$$

(well-defined as one easily checks) is injective, for $H \Box_V H \subseteq H \otimes H$ and there is a left inverse $H \otimes H \to H \otimes H$ defined by $h \otimes h' \mapsto hS(h'_{(1)}) \otimes h'_{(2)}$.

We compute the dual of $\beta$,

$$\beta^* : H^* \otimes_{V^*} H^* \to R, \quad f \otimes g \longmapsto f_{(1)}|_R \otimes f_{(2)} g \tag{24}$$

$(f, g \in H^*)$ by noting the following from standard pairing,

$$\langle \beta^*(f \otimes g), r \otimes h \rangle = \langle f \otimes g, rh_{(1)} \otimes h_{(2)} \rangle$$

$$= \langle f, rh_{(1)} \rangle \langle g, h_{(2)} \rangle = \langle (f \leftharpoonup r) g, h \rangle$$

$$= \langle f_{(1)}|_R, r \rangle \langle f_{(2)} g, h \rangle = \langle f_{(1)}|_R \otimes f_{(2)} g, r \otimes h \rangle$$

for each $r \in R, h \in H$. As the dual of a monic, $\beta^*$ is epi.

Note that ${}^{co\,R^*}H^* = V^*$ follows from the computation given in [31]:

$${}^{co\,R^*}H^* = \{ f \in H^* | f_{(1)}|_R \otimes f_{(2)} = \varepsilon_R \otimes f \}$$

$$= \{ f \in H^* | \forall r \in R, f \leftharpoonup r = \varepsilon(r) f \} = \{ f \in H^* | \forall r \in R, f \leftharpoonup (r - \varepsilon(r) 1_H)$$

$$= 0 \} = \{ f \in H^* | f|_{R^+ H} = 0 \} = (H/R^+ H)^*.$$

Now by a Kreimer-Takeuchi theorem the epi $\beta^*$ is an isomorphism [26, 8.3.1].

The left normal basis property (cf. [26, 3.3]) follows from Eq. (22), which is equivalently an isomorphism of right $V^*$-modules and left $R^*$-comodules. □

Note that the proof shows that $\beta$ in Eq. (23) is an isomorphism, so that the epi $\pi : H \to V$ is a Galois coextension of left $R$-module coalgebras [31]. The next corollary follows from the left version of [26, 8.2.5].

**Corollary 7.3.** *Given a Hopf subalgebra $R \subseteq H$, the algebra $H^*$ is isomorphic to a crossed product of $V^*$ and $R^*$, i.e., $H^* \cong V^* \#_\sigma R^*$ for some cocycle $\sigma : R^* \otimes R^* \to V^*$ (cf.* [26, Ch. 7]*).*

Let $\{h_i\}_{i=1}^q$ be a left $R$-module basis of ${}_R H$. Then $H \xrightarrow{\cong} R \otimes V$ is determined from $h = \sum_{i=1}^q r_i h_i$ as follows: $h \mapsto \sum_{i=1}^q r_i \otimes \overline{h_i}$, which is a left $R$-module and right $V$-comodule isomorphism. Then $\phi : V^* \otimes R^* \xrightarrow{\cong} H^*$ is given by $\phi(v^* \otimes r^*)(h) = \sum_{i=1}^q r^*(r_i) v^*(\overline{h_i})$.

The mapping $\gamma : R^* \to H^*$ given by $\gamma = \phi(\varepsilon_V \otimes -)$ is convolution-invertible by [26, Theorem 8.2.4]. Then by an application of [26, Proposition 7.2.3], the cocycle $\sigma : R^* \otimes R^* \to V^*$ is given by $\sigma(r^* \otimes s^*) = \gamma(r^*_{(1)})\gamma(s^*_{(1)})\gamma^{-1}(r^*_{(2)}s^*_{(2)})$.

**Example 7.4.** Let $H$ be the Taft Hopf algebra (generated by a grouplike $g$ and $(g, 1)$-skew primitive and nilpotent element $x$) of dimension $n^2$ and $R$ the cyclic group algebra in $H$ of dimensiona $n$. Since $H \cong H^*$ and $R \cong R^*$ as Hopf algebras, it follows from a computation that $V^* \cong \mathbb{C}[x]$, where $x^n = 0$, a Frobenius algebra. Indeed it is easy to compute from the standard basis $\{x^i g^j\}$ and Taft's anticommutation relation $gx = qxg$ that $H \cong \mathbb{C}[x]\#\mathbb{C}[\mathbb{Z}_n]$ (expressing a strongly graded $\mathbb{Z}_n$-algebra as a smash product of its group with its coinvariants).

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