THE BRAID AND THE SHI ARRANGEMENTS AND THE PAK-STANLEY LABELLING

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Dedicated to the memory of Michel Las Vergnas

Abstract. In this article we study a construction, due to Pak and Stanley, with which every region R of the Shi arrangement is (bijectively) labelled with a parking function $\lambda(R)$. In particular, we construct an algorithm that returns R out of $\lambda(R)$. This is done by relating λ to another bijection, that labels every region S of the braid arrangement with $r(S)$, the unique central parking function f such that $\lambda^{-1}(f) \subseteq S$. We also prove that λ maps the bounded regions of the Shi arrangement bijectively onto the prime parking functions. Finally, we introduce a variant (that we call "s-parking") of the parking algorithm that is in the very origin of the term "parking function". S-parking may be efficiently used in the context of our new algorithm, but we show that in some (well defined) cases it may even replace it.

1. Introduction

Let n be a natural number and $[n] := \{1, \ldots, n\}$. A *parking function (of* n *cars* or *of size* n*)* is a function $f\colon [n] \to [n]$ such that $|f^{-1}([i])| \geq i$ for every $i \in [n]$.

Suppose that n cars, one after another, enter a one-way street, and the driver of car i wants to park in position $f(i)$. Suppose also that each driver goes directly to his preferred position, and parks either there, if it is free, or in the first free position thereafter. Then f is a parking function exactly when all cars are thus parked in the first n places. Note that the parking functions are exactly the functions of form $f = c \circ \pi$ for some central parking function c and some permutation π of $[n]$. In other words, $f : [n] \to [n]$ is a parking function exactly when $f \preceq \pi$ for some $\pi \in \mathfrak{S}_n$, where as usual $\mathfrak{S}_n$ stands for the set of permutations of $[n]$ and $f \preceq g$ means that $f(i) \leq g(i)$ for every $i \in [n]$.

The *prime parking functions* are those for which $|f^{-1}([i])| > i$ for every $i \in [n-1]$ and a parking function is *central* if $f(i) \leq i$ for every $i \in [n]$.

Now, let $\mathcal{A}$ be an *arrangement* (i.e., a finite set) *of hyperplanes of* $\mathbb{R}^n$ and $\mathcal{R}(\mathcal{A})$ be the set of *regions* of $\mathcal{A}$, the connected components of the complement in $\mathbb{R}^n$ of the union of the hyperplanes. The central parking functions may be used to label the regions of the *braid arrangement (on* n *strands)*, the Coxeter arrangement of hyperplanes in $\mathbb{R}^n$ given by

$$\mathcal{B}_n = \big\{ x_i - x_j = 0 \;\big|\; 1 \leq i < j \leq n \big\}$$

while the parking functions may be used to label the regions of the *Shi arrangement (of size* n*)*, given by

$$\mathcal{S}_n = \mathcal{B}_n \cup \big\{ x_i - x_j = 1 \;\big|\; 1 \leq i < j \leq n \big\}.$$

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The main purpose of this paper is to further deepen the study of the relation between the two (bijective) labellings, which we show to coincide after suitable transformations, and, by doing that, to define an algorithm for the evaluation of the Pak-Stanley bijection from parking functions to regions [9], namely Algorithm 4.9.

In fact, whereas the number of regions of $\mathcal{B}_n$, $n!$, is equal to the number of central parking functions, the number of regions of $\mathcal{S}_n$, $(n+1)^{n-1}$, is equal to the number of parking functions. Let CF_n be the set of central parking functions of size n and PF_n be the set of all parking functions of the same size. Stanley [8] considers explicitly two bijections, the labellings $r : \mathcal{R}(\mathcal{B}_n) \to \mathrm{CF}_n$ and $\lambda : \mathcal{R}(\mathcal{S}_n) \to \mathrm{PF}_n$, the latter being due to Pak and Stanley [9]. We now know proofs of the bijectivity of λ that are not difficult (see, for example, the work of Rincón [6]). It is also easy to evaluate directly, from regions to parking functions, but so far we did not know how to easily evaluate its inverse.

It is perhaps worth mentioning that we know two other bijections between the regions of the Shi arrangement and parking functions, due to Athanasiadis and Linusson [2], for which proving that they are bijections and for which evaluating in both directions is also not difficult. But neither of them can replace the bijection of Pak and Stanley as to the important properties that we indicate briefly now, and more precisely in Proposition 2.2 and Proposition 2.4. Let us fix two given regions, still following Stanley [8], T_0 and R_0 ($T_0 \supseteq R_0$), in $\mathcal{B}_n$ and $\mathcal{S}_n$, respectively. Then, in the image of a different region we may count, for example, the number of hyperplanes that separate the region from T_0, or from R_0, respectively.

In Section 2, we recall, essentially from Stanley [8], these and other properties that relate the bijections, and study r in more detail. We end this section by defining explicitly $\lambda^{-1}(f)$ for a central parking function f. This definition is an important tool in Algorithm 4.9.

In Section 3, we introduce Algorithm 3.1, a new variant, called *s-parking*, of the parking algorithm that is in the very origin of the term "parking function". It provides an easy algorithmic way of evaluating $S \in \mathcal{R}(\mathcal{B}_n)$ out of $r(S) \in \mathrm{PF}_n$.

Section 4 is the main section of this paper. We continue the study of the relations between the two bijections and show that, given a parking function f, we may consider adequate "sections of f" where λ and r coincide (cf. Proposition 4.8 and the previous lemmas). Then, we introduce Algorithm 4.9 for the evaluation of the Pak-Stanley bijection, from parking functions to regions, where, for efficiency sake, the use the s-parking algorithm is recommended. This is one of the main results of this article.

The other main result is Proposition 4.11, where we prove that the Pak-Stanley bijection shares a property with one of the bijections of Athanasiadis and Linusson (cf. [2, Theorem 2.4]). Namely, we prove that λ maps bijectively the bounded regions of the Shi arrangement onto the *prime* parking functions on $[n]$. We do not know if this property was studied before, but it seems likely that it was at least considered, since the Pak-Stanley bijection was known before the article of Athanasiadis and Linusson (which it possibly inspired). We believe that it was the difficulty in handling this bijection, which is mentioned by Athanasiadis and Linusson [2, p.29] and which we hope to lessen significantly here, that prevented earlier proofs of this fact.

Finally, in Section 5, we show that, even if f is not central, in some cases the s-parking algorithm still produces $\lambda^{-1}(f)$ when applied directly to a parking function f, and we explain at the end of the section when does this happen (see Remark 5.4 and the final note).

2. Preliminaries

Recall that, for $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$ and let $[0] := \varnothing$. Given $i, j \in \mathbb{N}$, set $[i, j] := [j] \setminus [i - i]$, so that $[i, j] = \{i, i + 1, \ldots, j\}$ if $i \leq j$ and $[i, j] := \varnothing$ if $i > j$. As usual, given $w : [n] \to \mathbb{N}$, we let w_i denote $w(i)$ and use the one-line notation, writing $w = w_1 \cdots w_n$, which we may view as a word. But we may also view w as the element $(w_1, \ldots, w_n) \in \mathbb{R}^n$. When f is defined in a subset of $[n]$, we also use a variant of Cauchy's two-line notation where we write x over $f(x)$, as in $f = \overset{1\,2\,4\,6}{3411}$.

2.1. **Braid arrangement.** Every region in the braid arrangement is of form

$$T_w = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{w_1} < x_{w_2} < \cdots < x_{w_n}\big\},$$

for a given, clearly unique, permutation $w \in \mathcal{S}_n$. Note that $w^{-1} \in T_w$ as an element of $\mathbb{R}^n$. Let $T_0 = T_{n \cdots 21}$.

The following definition is used by Stanley [8, p. 476] in the construction of a distance enumerator for the braid arrangement. However, for reasons that will become clear later, we add 1 to all coordinates and reverse w in the definition of T_w. Given $w \in \mathfrak{S}_n$, let

$$f_{w_i} = \big|\big\{k \in [i] \mid w_k \leq w_i\big\}\big| \ , \ i = 1, \ldots, n \ ,$$
$$t(w) = (f_1, \ldots, f_n) \in \mathbb{R}^n \ .$$

Example 2.1. Consider $n = 9$ and $w = 843967125$, so that $T = T_w$ is the region of $\mathcal{B}_9$ defined by

$$x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5 \, .$$

For example, 6 is the fifth element of w and, within the first five elements of w, three (4, 3 and 6) are less than or equal to 6. Thus, $f_6 = 3$. Throughout, $t(w) = 121153414$.

We note that $t(w) \preceq \mathrm{Id}$ and that $t(w) \preceq w^{-1}$, where $\mathrm{Id} = 12 \cdots n \in \mathfrak{S}_n$. In fact, the ith component of $w^{-1} - t(w)$ is the number of integers greater than i to the left of i in w. Hence, $w^{-1} - t(w)$ is the *inversion vector of* w. Remember that $\mathrm{CF}_n = \big\{f \in [n]^n \mid f \preceq \mathrm{Id}\,\big\}$. We recall the following important theorem, where we define

$$r(T_w) := t(w) \, .$$

Proposition 2.2 ([8, Proposition 6.1.9, ad.]). *The function* $r \colon \mathcal{R}(\mathcal{B}_n) \to \mathrm{CF}_n$ *is characterized by the following properties:*

- $r(T_0) = 11 \cdots 1$*;*
- *if the regions* T *and* T' *are separated by a unique hyperplane* H *of equation* $x_i - x_j = 0$ *with* $i < j$ *and* T_0 *and* T *are in the same side of* H*, then* $r(T') = r(T) + e_j$*, where* e_j *is the* j*th unit coordinate vector of* $\mathbb{R}^n$*.*

Lemma 2.3. *The function* $t \colon \mathfrak{S}_n \to \mathrm{CF}_n$ *is a bijection, and hence so is* $r \colon R(\mathcal{B}_n) \to \mathrm{CF}_n$*.*

Proof. Clearly, $|\,\mathrm{CF}_n\,| = n!$. Let us prove that t is injective. Consider distinct $w, \tilde{w} \in \mathfrak{S}_n$, let k be the least integer such that $w_k \neq \tilde{w}_k$ and suppose without loss of generality that $w_k > \tilde{w}_k$. Let $a := w_k$, $b := \tilde{w}_k$, $f := t(w)$ and $\tilde{f} := t(\tilde{w})$. Note that $a = \tilde{w}_\ell$ for some integer $\ell > k$. Now, by definition, $f_a = |A|$ and $\tilde{f}_a = |\tilde{A}|$, where

$$A := \big\{i \in [k] \mid w_i \leq w_k = a\big\}$$

and

$$\tilde{A} := \big\{i \in [\ell] \mid \tilde{w}_i \leq \tilde{w}_\ell = a\big\} \, .$$

Since $A \subseteq \tilde{A}$ and $\ell \in \tilde{A} \setminus A$, $f_a < \tilde{f}_a$. ☐

2.2. **Shi arrangement and parking functions.** Shi [7] introduced the arrangement of hyperplanes that now has his name, and counted the number of regions it determines. This number is $(n+1)^{n-1}$, which is, in particular, also the number of parking functions of n cars introduced in literature by Konheim and Weiss [5]. Three different explicit bijections are known between these two sets: two of them are due to Athanasiadis and Linusson [2], and a third one, the first in time, is due to Pak and Stanley [9]. The latter is a bijection $\lambda\colon \mathcal{R}(\mathcal{S}_n) \to \mathrm{PF}_n$ characterized as follows (compare to Proposition 2.2):

Proposition 2.4 ([8, Proposition 6.1.9, ad.]). *Let R_0 be the region defined by*

$$x_n < x_{n-1} < \cdots < x_2 < x_1 < x_n + 1\,,$$

(in other words, $x \in R_0$ if and only if $0 < x_i - x_j < 1$ for all $1 \le i < j \le n$). Then

- $\lambda(R_0) = 11\cdots 1$*;*
- *if the regions R and R' are separated by a unique hyperplane H of equation $x_i = x_j$ with $i < j$ and if R and R_0 lie on the same side of H, then $\lambda(R') = \lambda(R) + e_j$;*
- *if the regions R and R' are separated by a unique hyperplane H of equation $x_i = x_j + 1$ with $i < j$ and if R and R_0 lie on the same side of H, then $\lambda(R') = \lambda(R) + e_i$.*

Before defining λ according to Stanley [1] [8], we need to introduce the concept of valid pairs. Every region $R \in \mathcal{R}(\mathcal{S}_n)$ corresponds bijectively to a pair $(w, \mathfrak{I})$, called a *valid pair*, where

- $w \in \mathfrak{S}_n$;
- $\mathfrak{I}$ is an *anti-chain of proper intervals*, meaning that $\mathfrak{I}$ is a collection of intervals $[i,j]$ with $1 \le i < j \le n$ such that if $I, I' \in \mathfrak{I}$ and $I \ne I'$ then $I \nsubseteq I'$ (and $I' \nsubseteq I$);
- for every $I = [i,j] \in \mathfrak{I}$, $w_i > w_j$.

In fact, given a valid pair $(w, \mathfrak{I})$, the elements $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ of the corresponding region R are characterized by both the set of signs of $x_j - x_k$, for $1 \le j < k \le n$, and by the set of signs of $x_j - x_k - 1$. The first set of signs characterize the points of the region of the braid arrangement that contains R, given by

$$T_w = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \;\mid\; x_{w_1} < x_{w_2} < \cdots < x_{w_n}\big\} \supseteq R\,,$$

and the second set of signs can be codified as follows: for every $i < \ell$ such that $w_\ell < w_i$ (otherwise $x_{w_\ell} - x_{w_i} < 0$), if there exist $I \in \mathfrak{I}$ such that $i, \ell \in I$, then $x_{w_\ell} - x_{w_i} < 1$; if they do not exist, then $x_{w_\ell} - x_{w_i} > 1$.

Still following Stanley, we represent the valid pair $(w, \mathfrak{I})$ where $\mathfrak{I} = \{[i_1, \ell_1], \ldots, [i_k, \ell_k]\}$ by decorating w with k arcs above the elements of w, starting in position i_j and ending in position ℓ_j, for $j \in [k]$. These are the *dual diagrams* of Athanasiadis and Linusson [2]. We will call them *arc diagrams*. See Example 2.6, below. Note that by definition $\ell_j \ge i_j + 1$.

Now, in order to define the parking function $f = \lambda(R)$ associated with R, we consider $d = d_1 \cdots d_n$, where

$$d_{w_i} = \begin{cases} 0, & \text{if } i \notin \bigcup\limits_{I \in \mathfrak{I}} I\,; \\ \big|\{j < i \mid w_j > w_i \ \text{ and } \ j, i \in I \ \text{ for some } \ I \in \mathfrak{I}\}\big|\,, & \text{otherwise.} \end{cases} \tag{2.1}$$

Then, as a vector, $\lambda(R) = w^{-1} - d$.

[1]Although Stanley's w and ours are again reversed from each other.

Our definition of λ is equivalent to the definition of Pak and Stanley [8, Equation 56, ad.], since $i = \lambda(R)_{w_i} + d_{w_i}$, where

$$\begin{aligned}\lambda(R)_{w_i} = 1 + &\left|\left\{j < i \mid w_j < w_i\right\}\right| \\ &+ \left|\left\{j < i \mid w_j > w_i,\ \text{no } I \in \mathfrak{I} \text{ satisfies } j, i \in I\right\}\right| .\end{aligned}$$

If R is the region associated with the valid pair $(w, \mathfrak{I})$ and $f = \lambda(R)$, then we simply write $f = \lambda(w, \mathfrak{I})$. We also denote the permutation w of the valid pair by $w(f)$.

Note that we may subsume the definitions of t and λ as follows.

Definition 2.5. Let $w \in \mathfrak{S}_n$, and let $j = w_i$ for some $i, j \in [n]$.

- Suppose that $f = t(w)$ for a (central) parking function f. Then
$$f_j = 1 + |w([1, i]) \cap [j - 1]| .$$
- Suppose that $(w, \mathfrak{I})$ is a valid pair, $f = \lambda(w, \mathfrak{I})$ and k is either i, if $i \notin I$ for every $I \in \mathfrak{I}$, or is the least element for which there exists $\ell \in [n]$ with $i \in [k, \ell] \in \mathfrak{I}$ (in other words, w_k is the leftmost left-endpoint of an arc that contains j in the corresponding arc diagram). Then
$$f_j = k + |w([k, i]) \cap [j - 1]| .$$

We now consider two examples that will be used throughout the paper.

Example 2.6 ([8, example p. 484, ad.]). Let $w = 843967125$ and $\mathfrak{I} = \left\{[1, 6], [3, 8], [6, 9]\right\}$. The arc diagram of the valid pair $(w, \mathfrak{I})$ is

$$8\,4\,3\,9\,6\,7\,1\,2\,5\,,$$

and is in bijection with the region R of $\mathcal{R}(\mathcal{S}_9)$ defined by

$$\begin{aligned}&x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5\,; \\ &x_8 + 1 > x_7,\ x_3 + 1 > x_2,\ x_7 + 1 > x_5\,; \\ &x_4 + 1 < x_1,\ x_3 + 1 < x_5,\ x_6 + 1 < x_5\,.\end{aligned}$$
[2]

Then

$$f = \lambda(R) = 341183414\,.$$

For instance, $f_1 = 3$ since, of the left-endpoints of the two arcs that "cover" 1, the leftmost starts at position 3 (with a 3) and there are no elements less than 1 between 3 and 1 (among $3, 9, 6, 7, 1$). Between 3 and 2, one element (1) is less than 2, and so $f_2 = 4$. There are no elements less than 3 between 8 and 3 ($f_3 = 1$), etc.

Example 2.7. Let $n = 4$ and consider the arc diagram $4\,2\,3\,1$. The points $(x, y, z, t) \in \mathbb{R}^4$ that belong to the associated region R verify the inequalities $0 < x - y < 1$, $0 < x - z < 1$, $x - t > 1$, $y - z < 0$, $0 < y - t < 1$ and $z - t > 1$. Note that $d = 2100$ and $\lambda(R) = 2131$.

We have seen how to evaluate the Pak-Stanley bijection directly, from valid pairs to parking functions. However, in general, the problem of finding $\lambda^{-1}(f)$ for a given parking function f is considered to be difficult. Stanley [8] presented the draft of an algorithm for this purpose, which is based on the very definition of λ; but the algorithm seems hard to analyse. In the next section, we evaluate $\lambda^{-1}(f)$ when f is a central parking function, and afterwards, based on this, we consider a new general algorithm for this purpose.

[2]Hence, also $x_8 + 1 > x_6$ and $x_8 + 1 < x_1$, for example.

Two parking functions f and g such that $w(f) = w(g)$ are *braid equivalent*. For example, the parking functions of Example 2.1 and Example 2.6 are braid equivalent. The equivalence classes for this relation obviously correspond to the regions of the braid arrangement. In particular, it is not difficult to see that the union of the closure of the regions associated with the dominant parking functions is indeed the closure of T_0, being the number of dominant parking functions the Catalan number of order n. The case $n = 3$ is depicted in Figure 1, where we can see, for each region R of $\mathcal{S}_3$, the corresponding arc diagram below the parking function $\lambda(R)$. Note that we added a line at infinity.

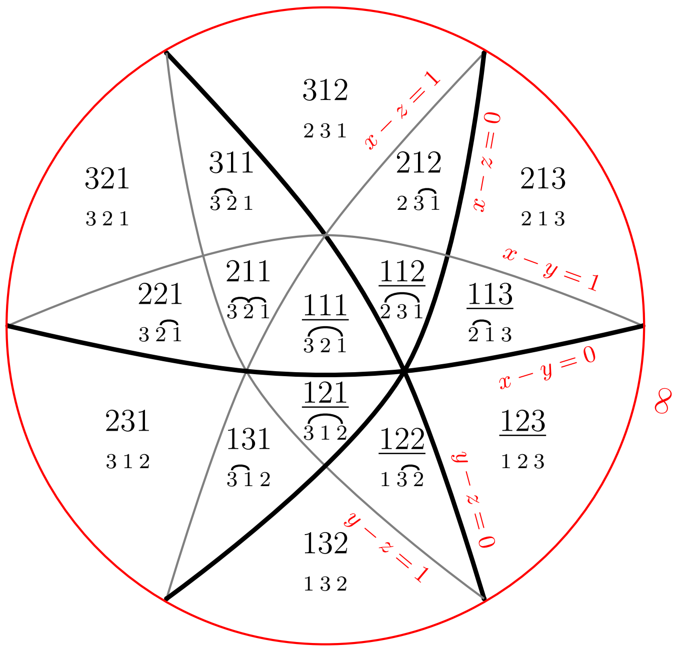

FIGURE 1. Parking functions of 3 cars and the bijection of Pak and Stanley

Remark 2.8. Given f and $w = w(f)$, and hence d, we may also obtain $\mathfrak{I}$ by drawing from right to left an arc over w, starting in j and encompassing exactly d_j elements greater than j, for each $j \in [n]$, and then by deleting all arcs that are contained in other arcs (cf. [2]). For example, if again $f = 341183414$ and $w = w(f) = 843967125$, $d = 442142200$ since $w^{-1} = 783295614$. Then, we may draw

(2.2) $$8\,4\,3\,9\,6\,7\,1\,2\,5 \quad \text{that gives us} \quad 8\,4\,3\,9\,6\,7\,1\,2\,5.$$

and so $\mathfrak{I}(f) = \{[1,6],[3,8],[6,9]\}$.

2.3. **Inclusion of $\mathcal{B}_n$ in $\mathcal{S}_n$ via parking functions.** Since the elements of CF_n are parking functions, $\iota = \lambda^{-1} \circ r \colon \mathcal{R}(\mathcal{B}_n) \to \mathcal{R}(\mathcal{S}_n)$ is well-defined. In fact, $\iota(T) \subseteq T$ for any region T of the braid arrangement since $w(f) = t^{-1}(f)$ for any central parking function f (see below). Hence, by Propositions 2.2 and 2.4, the hyperplanes that separate $\iota(T)$ and R_0 are all of form $x_i = x_j$. Hence, $\overline{\iota(T)}$ is a (closed) region of $\mathcal{S}_n$ that contains the line defined by $x_1 = \cdots = x_n$ as a face. This is why we call "central" to these parking functions, which are underlined in Figure 1.

Let $w = t^{-1}(f)$. If $w_1 > w_n$, let $\mathfrak{I} = \{[1,n]\}$. Then clearly $\lambda(w,\mathfrak{I}) = f$ (see Definition 2.5), which shows that $w(f) = t^{-1}(f)$.

In general, consider the set of *inversions of* w,

$$\mathrm{inv}(w) := \{(i,j) \mid i < j, w_i > w_j\}$$

and order this set as follows:

$$(i,j) \leq (k,\ell) \text{ if and only if } [i,j] \subseteq [k,\ell]\,.$$

Then clearly $\lambda(w,\mathfrak{I}) = f$ if $\mathfrak{I}$ is *the set* maxinv(w) *of maximal elements* of this poset. For example, the arc diagram of $f = 1132$ is 2413 : $w(f) = 2413$, $\mathrm{inv}(2413) = \big\{\{1,3\},\{2,3\},\{2,4\}\big\}$ and $\mathfrak{I}(f) = \mathrm{maxinv}(2413) = \big\{\{1,3\},\{2,4\}\big\}$.

3. S-parking as the inverse of t

In this section we introduce an algorithm that produces a permutation w out of any parking function f (in fact, it produces a permutation if and only if $f : [n] \to [n]$ is a parking function — see Lemma 5.2, below). But, for now, we will only see that when f is a *central* parking function then $f = t(w)$.

Evaluating $t^{-1}(f)$, in itself, is not difficult, as we can see in the example that follows. Consider again Example 2.1, where $T = T_w$ is the region of $\mathcal{B}_9$ defined by

$$x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5$$

and consider the lattice points of form (j,i), where $i = 1,\ldots,9$ and $1 \leq j \leq w_i$. Below, in Table A, we mark these points. Note that $v := w^{-1} = 783295614$. In the row below the table, in column j we count the number of "free" cells before $i = v_j$, from bottom to top. This is the number of pairs (w_k,k) with $w_k \leq w_i$ and $k \leq i$, added by 1, and so it is f_j. Hence, we have written the vector $f = t(w) = 121153414$, by Definition 2.5. But, obviously, given the row below the table, the full table can be reconstructed, from right to left, thus obtaining $w = t^{-1}(f)$ out of f.

Table A

•	•	•	•	9				
•	8							
7								
•	•	•	•	•	•	6		
•	•	•	•	•	5			
•	•	•	•	•	•	•	•	4
•	•	3						
•	•	•	2					
•	•	•	•	•	•	•	1	
1	2	1	1	5	3	4	1	4

Table B

				5				
	2			~~5~~				
1	~~2~~			~~5~~				
~~1~~	~~2~~			~~5~~		7		
~~1~~	~~2~~			~~5~~	6	~~7~~		
~~1~~	~~2~~				~~6~~	~~7~~		9
~~1~~	~~2~~	3			~~6~~			
~~1~~	~~2~~	~~3~~	4					
~~1~~		~~3~~	~~4~~				8	
1	2	1	1	5	3	4	1	4

Table C

				5				○
			9				○	
			7			○		
		6			○			
	2			○				
8			○					
4		○						
3	○							
①								
8	4	3	9	6	7	1	2	5

Figure 2. Three tables

Table B was built with s-parking.

Konheim and Weiss [5], in the very introduction of parking functions, considered a setting that is now widely used and that explains the present name "parking functions" and the use of the term "cars". In short, given $f : [n] \to [n]$, view each $i \in [n]$ as a *car* and $f(i)$ as its driver's preferred *place* to park, in a one-way street with a sufficient number of parking places. Suppose the drivers follow the rule that each one enters the street after the previous car has parked, drives to its favourite place and parks if it is free, and if not free it parks in the first free place after that. Then we may see that $f \in \mathrm{PF}_n$ exactly if all cars can be parked in the first n places.

We introduce here different parking rules: in our case, any car, when entering the street and finding his favourite place occupied, has the power (and the will) to shift the parked cars to the next empty space and to park (temporarily!) where he wants. In other words, it parks in the way we insert books in a non-empty shelf.

Algorithm 3.1 (*s-parking*)**.** Let $A = \{a_1, \dots, a_m\} \subseteq [n]$ with $a_1 < \cdots < a_m$ and $f\colon A \to [n]$ be an injective function. Set $q = q(f)\colon A \to [n]$ recursively as follows:

- Define $q(a_1) = f(a_1)$;
- Suppose that $q(i)$ is defined for $i \le j$, and consider the least integer $k \ge f(a_{j+1})$ such that $k \notin q([\{a_1, \dots, a_j\}])$. If $k > f(a_{j+1})$, redefine $q(a_i)$ as $q(a_i) + 1$ for every $i \le j$ such that $f(a_{j+1}) \le q(a_i) < k$. Finally, define $q(a_{j+1}) = f(a_{j+1})$.

The application of this algorithm to the example above (where $n = 9$ and $A = [n]$) is shown in Table B of Figure 2. According to the parking function f, below the table, we place (car) 1 in column 1, row 1, and 2 in column 2, row 2 (row 1 is "occupied"). Then, we place 3 in row 1, thus displacing 1, to next row, 2; this displaces 2 to row 3. Now, 4 displaces 3, 1 and 2, and so on.

The obvious similarity between the tables A and B is not a coincidence, of course. In fact, we have the following lemma.

Lemma 3.2. *In the notation of Algorithm 3.1, if $f(a_i) \le i$ for every integer i between 1 and m, then, for any such integer i, the number of elements a_j such that $j > i$ and $q(j) < q(i)$ is $q(i) - f(i)$. In particular, if $A = [n]$ then $q \circ t(w) = w^{-1}$ for every $w \in \mathfrak{S}_n$.*

Proof. It is easy to see that, under the conditions above, we may (and will) assume without any loss of generality that $A = [m]$. The function q is defined by iterating $m-1$ times the second clause of Algorithm 3.1; let $q^{(1)} : \{1\} \to \{1\}$ and $q^{(j)} : [j] \to [m]$ be the $(j-1)$th iterate, for $j > 1$. Note that, by the condition imposed on f, $q^{(j)}([j]) \subseteq [j]$ (and so $q^{(j)}([j]) = [j]$) for every $j \in [m]$. In other words, $q = q^{(m)}$ and in this case, for $j > i$,

$$q^{(j)}(i) = \begin{cases} q^{(j-1)}(i), & \text{if } i < j \text{ and } q^{(j-1)}(i) < f(j); \\ q^{(j-1)}(i) + 1, & \text{if } i < j \text{ and } q^{(j-1)}(i) \ge f(j); \\ f(j), & \text{if } i = j. \end{cases} \tag{3.3}$$

We prove the lemma by proving by induction on $j = 1, \dots, n$ that

$$q^{(j)}(i) = f(i) + \left|\left\{\ell \le j \mid \ell > i\,,\ q^{(j)}(\ell) < q^{(j)}(i)\right\}\right|\,, \quad \text{for every } 1 \le i \le j\,.$$

But either

$$\left|\left\{\ell \le j-1 \mid \ell > i\,,\ q^{(j-1)}(\ell) < q^{(j-1)}(i)\right\}\right| = \left|\left\{\ell \le j \mid \ell > i\,,\ q^{(j)}(\ell) < q^{(j)}(i)\right\}\right|,$$

which is the case when $q^{(j)}(i) = q^{(j-1)}(i) < f(j)$, or the second set differs from the first in that it contains j, otherwise. □

We will see in Lemma 5.2 that, again, $f \in \mathrm{PF}_n$ exactly if all cars can be parked *with this algorithm* in the first n places.

We now consider general parking functions, were we do not recommend the use of a table such as Table B. Instead, we recall the representation proposed by Garsia and Haiman [3], exemplified in Table C, that we adopt with minor changes. Counting from bottom to top and from left to right, column i is formed by the elements of $f^{-1}(\{i\})$ in decreasing descending order. In the representation of Garsia and Haiman, the bottom element of the first column is on the first line and the bottom element of any other non-empty column is placed immediately above the top element of the last non-empty column to the left. The fact that all elements are (weakly) above the main diagonal proves that $f \in \mathrm{PF}_n$. For completeness sake, we have drawn over the table the least Dyck path that covers the representation of f. Clearly, if f is *decreasing*, that is, if $f_1 \geq f_2 \geq \cdots \geq f_n$, then it is in one-to-one correspondence with its Dyck path. On the other hand, by Definition 2.5, f is decreasing if and only if $w(f) = n \cdots 21$, and hence $(w(f), \mathfrak{I})$ is a valid pair if and only if the partition of $[n]$ induced by $\mathfrak{I}$ is nonnesting. In other words, we obtain through Pak-Stanley bijection a proof of the well-known fact that the number of nonnesting partitions of $[n]$ is the nth Catalan number [1]. Later, we obtain $\mathfrak{I}$ explicitly in this case, as an example of the use of Algorithm 4.9.

We placed w below the table so as to note that $t(w) \preceq w^{-1}$, or, in other words, so as to note that every integer j moves left from its position w_j^{-1} in w to the position f_j in the table above. In fact, by definition (cf. (2.1)), it moves exactly d_j positions.

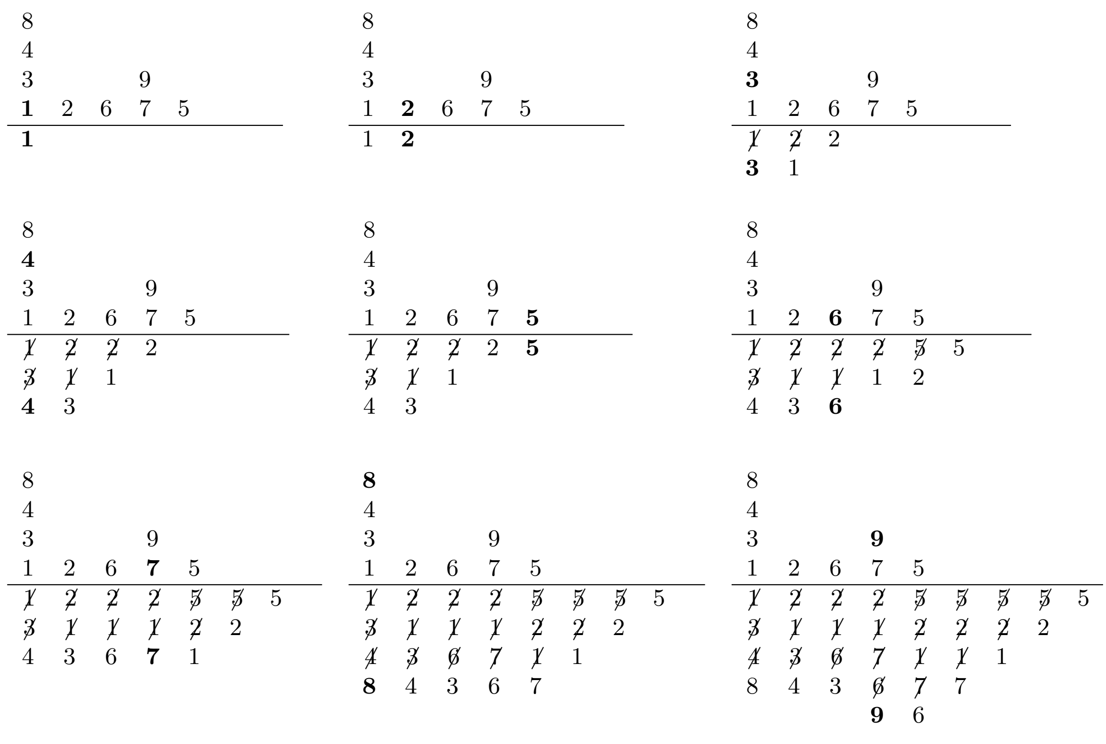

Figure 3. S-parking

We end this section with a direct application of this algorithm for finding $t^{-1}(f)$ for $f = 121153414$, as in Example 2.1 again, but in an easier form. We start by introducing our representation of parking functions. Whereas we still write in column i, in decreasing descending order, the elements of $f^{-1}(\{i\})$, for $i = 1, \ldots, n$, the bottom elements of all the columns are now horizontally aligned. This representation is the starting point. Then we park directly car 1 and car 2 in their favourite places, directly below, hence. Car 3 finds its favourite place occupied and shifts right car 1, that shifts car 2. Now, when car 4 enters, all three previous cars are shifted right. Car 5 parks directly but car 6 shifts cars 1, 2 and 5; and so forth. Step by step, we obtain the tables of Figure 3. Note that the last table encodes the full process, and remember that $T = r(f)$ is the region of $\mathcal{B}_9$ defined by

$$x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5 \,.$$

4. Decomposition

In this section we obtain the main results of the paper. What we have in mind, in the first place, can be roughly described with the parking function of Example 2.6 through the diagram

843967125 "="	843967 "+"	396712 "+"	7125
$\overset{\lambda}{\mapsto}$ 341183414	$\overset{\text{“}t\text{”}\ 346789}{\mapsto}$ 113414	$\overset{\text{“}t\text{”}\ 123679}{\mapsto}$ 121232	$\overset{\text{“}t\text{”}\ 1257}{\mapsto}$ 1231

8			9					
4		6	7					
3	␣	1	2	␣	␣	␣	5	␣

8			
4			9
3	␣	6	7

	9	
3	6	
1	2	7

7		
1	2	5

We have "decomposed" the arc diagram of the *non-central* parking function $f = 341183414$ in three components, each one associated with a *central* parking function. We want to generalize this idea, that shows the bijection of Pak and Stanley not only as a generalization of the bijection t of Proposition 2.2 but as an iteration of a similar bijection. Our approach consists in the construction of the "parking functions" as represented below. Then, s-parking shows to be a very convenient way of recovering the arcs above. But let us be more precise.

Definition 4.1. Let $f : [n] \to [n]$ be a parking function.

- Given a subset X of $[n]$ with m elements, let $x = x_1 \cdots x_m$ be the *increasing* bijection $x \colon [m] \to X$ (in other words, $X = \{x_1, \ldots, x_m\}$, where $x_i < x_j$ whenever $i < j$); we write $x = X_<$.
- We say that X *is* f*-central* if
$$f(x_i) \le i\,, \quad i = 1, \ldots, m\,.$$
The centre of f is the (unique) maximal f-central subset $X(f)$ of $[n]$.
- If $(f \circ x)([m]) \subseteq [m]$ and $\tilde{f} := f \circ x : [m] \to [m]$ is *also* a parking function, define
$$(\tilde{w}, \tilde{\mathfrak{I}}) = (\tilde{w}(X), \tilde{\mathfrak{I}}(X)) \quad \text{by} \quad \tilde{\lambda}\big(x^{-1} \circ \tilde{w}, \tilde{\mathfrak{I}}\big) = \tilde{f}\,,$$
where $\tilde{\lambda} : \mathcal{R}(\mathcal{S}_m) \to \mathrm{PF}_m$ is the Pak-Stanley bijection.

Remark 4.2. By definition, X is f-central for a subset $X \subseteq [n]$ if and only if there are at least $f(x)$ elements in X less than or equal to x, for every $x \in X$. Hence, if for some $Y \subseteq [n]$, X and Y are f-central, then $X \cup Y$ is f-central. This shows that $X(f)$ is well defined above. Note that $j \in X(f)$ if and only if $f_j \le 1 + |X \cap [j-1]|$. Moreover, *if* X *is the centre of* f*, then* $(f \circ x)([m]) \subseteq [m]$ *and* $f \circ x : [m] \to [m]$ *is a central parking function*, and so $\tilde{w}(X) = x \circ t\big(\tilde{f}\big)$ (cf. Definition 2.5). For example, if $f = 341183414$ then $X = \{3, 4, 6, 7, 8, 9\}$, $\tilde{f} = 113414$, $\tilde{\lambda}^{-1}(113414) = \big(521634, \{[1, 6]\}\big)$ and so $\tilde{\mathfrak{I}} = \{[1, 6]\}$ and $\tilde{w} = x \circ 521634 = 843967$.

We are now in a position where we can explain how the "decomposition" works. The example we presented in the beginning of this section, for motivation, can be a little misleading, in that the decomposition is not necessarily in arcs, but rather in central parking functions, and a central parking function may or may not correspond to a unique arc, as we have seen before. The main lemma, presented below, explains the importance of the detection of the centre of a parking function f. In particular, with it we know the "beginning" of the valid pair associated with f, from left to right. Then, by removing some specific points and by adapting f accordingly, we obtain a new parking function, g, with a new centre. We repeat this procedure recursively, and at the end we put together the various "beginnings" obtained in the lemma, which show to be the different sections of the decomposition.

Lemma 4.3. *Let* $f = \lambda(w, \mathfrak{I}) \in \mathrm{PF}_n$ *for a valid pair* $(w, \mathfrak{I})$*,* $X = X(f)$ *and* $m = |X| < n$*. Then*

$$\tilde{w} = w_1 \cdots w_m \quad \textit{and} \quad \tilde{\mathfrak{I}} = \big\{[i,j] \in \mathfrak{I} \;\mid\; j \leq m\big\}\,.$$

Proof. Note that the second statement is an obvious consequence of the first one. Now, suppose, contrary to the hypothesis, that $w_1 \cdots w_m \neq \tilde{w}$, and let $k \leq m$ be the least element such that $w_k \neq \tilde{w}_k$, let $a := w_k$ and $b := \tilde{w}_k \in X$, and note that $b = w_p$ for some $p > k$.

Let $B = \{i \in [p] \mid w_i \leq w_p = b\}$ and $\tilde{B} = \{i \in [k] \mid \tilde{w}_i \leq \tilde{w}_k = b\}$, so that, in particular, $\tilde{B} \subseteq B$. Moreover, by definition of $\tilde{w}$ and by Definition 2.5, since X if f-central, $f_b = |\tilde{B}|$.

$$\begin{aligned} |\tilde{B}| &= f_b = x + \big|w([x,p]) \cap [b-1]\big| \text{ for some } x \in [p] \\ &\geq 1 + \big|w([p]) \cap [b-1]\big| = |B|\,. \end{aligned}$$

Hence, since $\tilde{w}([k-1]) = w([k-1])$,

$$|\tilde{B}| = 1 + |w([k-1]) \cap [b-1]| = 1 + |w([p]) \cap [b-1]| = |B|$$

and, in particular, $a > b$.

Now, suppose that $a \in X$, so that $a = \tilde{w}_\ell$ for some $\ell \leq m$. Then $f_a > f_b$, since $\{b\} \cup \tilde{B} \subseteq \tilde{A} := \{i \in [\ell] \mid \tilde{w}_i \leq \tilde{w}_\ell = a\}$. On the other hand, if $a \notin X$ then $f_a > 1 + |X \cap [a-1]|$. In any case, given $k_a < k$ such that

$$f_a = k_a + \big|w([k_a, k]) \cap [a-1]\big|\,,$$

there exists $i < k_a$ such that $w_i > a > b$. This means than there exist $I, I' \in \mathfrak{I}$, corresponding to an arc over $a = w_k$ and to an arc over $b = w_p$, respectively, such that $I \subsetneq I'$, contradicting the fact that $\mathfrak{I}$ is an anti-chain. ☐

The previous lemma shows that, starting with the centre of f, the inverse of t determines the way in which the set $\mathfrak{I}$, and w, viewed as a word, start. From here a new parking function, g, of smaller size, is constructed, and the same is recursively done. This is the core of Algorithm 4.9. For the definition of g we must previously evaluate various parameters, as defined below. In particular, a is the first element of the Garsia-Haiman representation of f, from top to bottom and from left to right, that is not in the centre. Then, b is the order of its column. See Figure 4.6 for examples.

Definition 4.4. Given a parking function $f : [n] \to [n]$, $X := X(f)$, $m := |X|$ and $w := x \circ t(f \circ x)$ $(= \tilde{w}(X)$, see Remark 4.2$)$,

- let $b = b(f) := \min f\big([n] \setminus X\big)$ and let $a = a(f) := \max\big(f^{-1}(\{b\}) \setminus X\big)$;
- if $b > m$, let $c = c(f) := b$;
  if $b \leq m$, let $c = c(f) > 1$ be the greatest element $j \in [m]$ such that

$$j + |w([j,m]) \cap [a-1]| = b$$

(see Remark 4.5);
- let $Y = Y(f) := w([1, c-1])$ and let $Z = Z(f) := [n] \setminus Y$;
- let $g = g(f)\colon$

$$\begin{array}{rcl} Z & \to & [n-c+1] \\ i & \mapsto & \begin{cases} f_i - |Y \cap [i-1]|\,, & \text{if } i \in X \setminus Y\,; \\ f_i - c + 1\,, & \text{if } i \in Z \setminus X\,. \end{cases} \end{array}$$

Remark 4.5. Let us show that c is well defined above and that $c > 1$. Suppose that $b \le m$, let $h\colon [m] \to \mathbb{N}$ be such that $h_i = i + |w([i, m]) \cap [a-1]|$ and note that h_{i+1} is either h_i or $h_i + 1$, depending on whether w_i is less than a or greater than a. Since $h_m = m$ if $w_m \ge a$, or else $h_m = m + 1$, and so in any case h_m is not less than b, all we must prove is that $h_1 < b$ or, equivalently, that $f_a > |X \cap [a-1]| + 1$. But this happens since $a \notin X$ (see Remark 4.2). If $b > m$ then naturally $c = b > 1$.

Example 4.6 (Examples 2.6 and 2.7, continued). We consider again the parking function of Example 2.6, $f = 341183414$, for which $x = 346789$, $w = 843967$, $b = 3$, $a = 1$, $Y = \{4, 8\}$ since $c = 3$, and $g = \overset{1\,2\,3\,5\,6\,7\,9}{1216232}$. Starting with this "parking function" [3], we obtain $x = 123679$, $w = 396712$, $b = 6$, $a = 5$, $c = 4$, $Y = \{3, 6, 9\}$ and now $g = \overset{1\,2\,5\,7}{1231}$. For the parking function of Example 2.7, $f = 2131$, we obtain $x = 24$, $w = 42$, $b = 2$, $a = 1$, $c = 2$, $Y = \{4\}$, and $g = 112$. See Figure 4, where the elements of X are in italic type and a is in boldface type.

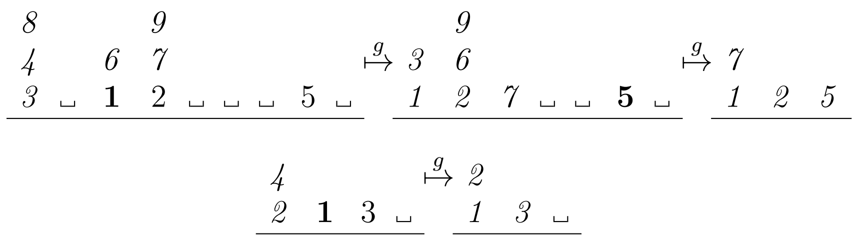

FIGURE 4. From f to $g(f)$: three examples

Lemma 4.7. *Let, for a parking function $f\colon [n] \to [n]$, $f = \lambda(w, \mathfrak{I})$ for a valid pair $(w, \mathfrak{I})$ and let m, $\tilde{w}$ and $\tilde{\mathfrak{I}}$ be defined as in Definition 4.1, b, c, Z and g as in Definition 4.4 and, finally, let $U := X \setminus Y$, $z := Z_<$ and $n' := |Z|$.*

(1) *$f' := g \circ z : [n'] \to [n']$ is a parking function.*
(2) *The following conditions are equivalent*
   - *$b > m$;*
   - *$U = \varnothing$;*
   - *$X = f^{-1}([m])$*
   - *$b = m + 1$;*
(3) *If none of the conditions of (2) holds, let $u := U_<$. Then*
   - *$g \circ u = t(u^{-1} \circ w_c \cdots w_m)$;*
   - *$U \subseteq X(f')$ and $a \in X(f') \setminus X(f)$.*

[3] The true parking function is $g \circ 1235679 = \overset{1\,2\,3\,4\,5\,6\,7}{1216232}$.

Proof. By definition of $Y \subseteq X$, $b > m$ if and only if $X = Y$. So, the first two conditions are equivalent. Now, by definition of b, if $b > m$ then $f^{-1}([m]) \subseteq X$, and of course $X \subseteq f^{-1}([m])$. Thus, $m = \left|f^{-1}([m])\right|$, and since f is a parking function, $|f^{-1}([m+1])| \geq m+1$. This implies that $b = m+1 > m$. Hence, the first condition implies the third condition, which implies the last condition, which in its turn is obviously stronger than the first one. Finally, note that, under these conditions, for $i \geq c$, $f^{-1}([i]) = g^{-1}([i-c+1]) \cup X$, which proves (1) in this case.

For proving (3), first note that, for every $i \in U$,

$$\text{whereas } g_i = f_i - |Y \cap [i-1]|\,, \text{ also } u_i = i - |Y \cap [i-1]|. \tag{4.4}$$

From here it follows that $g \circ u = t(u^{-1} \circ w_c \cdots w_m)$ and also that if $\left|g^{-1}([i])\right| < i$ for some $i \in [n]$, then $i > \max g(X \setminus Y)$ and thus $g^{-1}([i]) = f^{-1}([i+c-1]) \setminus Y$, which is at least i since f is a parking function. This is absurd. We have proven (1) in this case too.

Finally, (4.4) implies that $U \subseteq X(f')$. This means, in particular, that

$$w([c,m]) \cap [a-1] \subseteq X(f') \cap [a-1]$$

which implies, successively, that

$$b = c + \left|w([c,m]) \cap [a-1]\right| \leq (c-1) + \left(1 + \left|X(f') \cap [a-1]\right|\right),$$
$$g_a \leq 1 + \left|X(f') \cap [a-1]\right|,$$

and thus $a \in X(f')$. ☐

Proposition 4.8. *With the notation of Lemma 4.7, suppose that $f' = \lambda'(w', \mathfrak{I}')$ for a valid pair $(w', \mathfrak{I}')$, where $\lambda' : \mathcal{R}(\mathcal{S}_{n'}) \to \mathrm{PF}_{n'}$ is the Pak-Stanley bijection, and let $\mathfrak{I}^* = \left\{[i+c-1, \ell+c-1] \mid [i,\ell] \in \mathfrak{I}\right\}$. Then*

$$w = \tilde{w} \oplus z \circ w'$$
$$\mathfrak{I} = \tilde{\mathfrak{I}} \uplus \mathfrak{I}^*$$

where the operation considered in the first equality is (word) concatenation with possible overlapping and the operation in the second one is union with deletion of possible intervals contained in other intervals.

Proof. All we have to prove is that $f = \lambda(\tilde{w} \oplus z \circ w', \tilde{\mathfrak{I}} \uplus \mathfrak{I}^*)$. But this is immediate from Lemma 4.3 and Lemma 4.7. ☐

The problem of finding, for a parking function f, the valid pair $(w, \mathfrak{I})$ such that $f = \lambda(w, \mathfrak{I})$ now can be solved through Proposition 4.8, in the following algorithm.

Algorithm 4.9. Let $f : [n] \to [n]$ be a parking function, let w be the empty word and let $\mathfrak{I} = \varnothing$.

1. Set $k \leftarrow 0$, $z \leftarrow \mathrm{Id}$, $F^0 \leftarrow f$ and $N^0 \leftarrow n$.

2. Let $X = X(f)$, $m = |X|$ and $\tilde{w}$ and $\tilde{\mathfrak{I}}$ be as in Definition 4.1 for $f = F^k$ and $n = N^k$.
Set
$$w \leftarrow w \oplus z \circ \tilde{w}$$
$$\mathfrak{I} \leftarrow \mathfrak{I} \uplus \tilde{\mathfrak{I}}$$
If $m = N^k$ then stop.

3. Let Z and g be as in Definition 4.4 for $f = F^k$ and $n = N^k$, and let $z := Z_<$ and $n' := |Z|$. Set $k \leftarrow k+1$, $F^k \leftarrow g \circ z$ and $N^k \leftarrow n'$. Go to **2.**

Note that, according to the proof of Lemma 4.3, $N^{i+1} < N^i$ (and thus, in particular, $F^{i+1} \neq F^i$). Hence, the algorithm always comes to to an end. Also note that, given a parking function $f : [n] \to [n]$, $X := X(f)$ and $m := |X|$, we may obtain $w = \tilde{w}(X) = x \circ t(f \circ x)$ through s-parking without explicitly composing with x. See the following example.

Example 4.10 (Examples 2.6 and 2.7, continued). In Figure 5 we complete Example 4.6 by applying both Algorithm 4.9 *and* s-parking. Note that, in each step, we park only the elements of the centre of f, in italic.

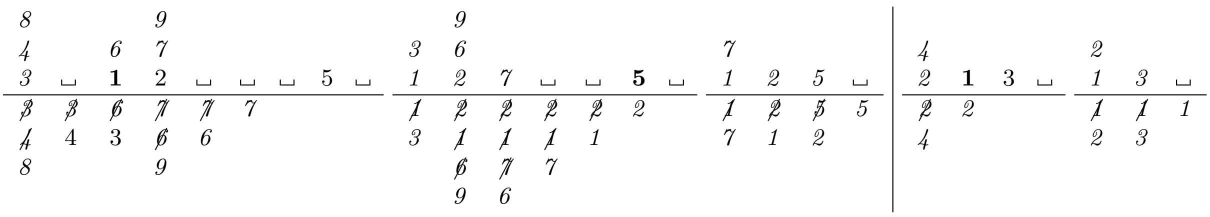

FIGURE 5. Algorithm 4.9 together with s-parking

For another example, consider again the *dominant* or *decreasing* parking functions, the functions $f : [n] \to [n]$ such that $f_1 \geq f_2 \geq \cdots \geq f_n$; suppose, more precisely, that $k_0 = 1$ and

$$f_1 = \cdots = f_{k_1-1} > f_{k_1} = \cdots = f_{k_2-1} > \cdots > f_{k_\ell} = \cdots = f_n = 1\,.$$

Then clearly $\lambda^{-1}(f) = (w, \mathfrak{I})$, where w is the identity reversed, $w = n \cdots 21$ ($\succeq f$), and

$$\mathfrak{I} = \left\{ [f_{k_i}, n+1-k_i] \mid 0 \leq i < \ell\,, f_{k_i} \neq n+1-k_i \right\}.$$

This can be directly checked, or it can be obtained through Algorithm 4.9. In fact, for example, $X(f) = [k_\ell, n]$, $a = k_\ell - 1$, $c = f_a$, $Z = [c, n]$ and $g_i = 1$ if $i \in [c, a-1]$ and $g_i = f_i + 1 - c$ if $i \in [a, n]$.

The decomposition can also be used in relation with a question raised by Athanasiadis and Linusson [2, p. 39]. Remember from the introduction that a *prime parking function* is a function $f : [n] \to [n]$ such that $\left|f^{-1}([i])\right| > i$ for every $i \in [n-1]$. Paraphrasing Athanasiadis and Linusson [2, Theorem 2.4] and the proof therein, we have

Proposition 4.11. *The map λ is a bijection between the bounded regions of $\mathcal{S}_n$ and prime parking functions on $[n]$.*

Proof. A region R of $\mathcal{S}_n$ is unbounded if and only if the following property is valid:

(P) There is $j \in [n-1]$ such that no arc is directed from the first j integers to the last $n-j$ integers in its arc diagram.

Let us see that this means that the cardinality of $f^{-1}([j])$ is exactly j.

If f is central, this is obvious, since necessarily $[j] \subseteq f^{-1}([j])$ and so Property (P) holds if and only if $\left|f^{-1}([j])\right| = j$.

Now, suppose that f is not central, and, with the notation introduced above, consider the first $k \geq 0$ such that $j+1$ is central in F^k.

We end the proof by noting that the number of elements $x \in X(F^k)$ such that $F^k(x) < F^k(j+1)$ is exactly $F^k(j+1) - 1$, whereas the number of elements withdrawn from the domain of F^i to the domain of F^{i+1} is exactly the difference between $F^i(j+1)$ and $F^{i+1}(j+1)$. $\square$

5. S-parking directly

Like the parking algorithm, the s-parking algorithm can be used for defining parking functions, in the same sense as before. More precisely, consider the following definition:

Definition 5.1. Define, for $1 \le k \le \ell \le n$, the following cycle, which is the identity if $k = \ell$.

$$\pi_{[k\,\ell]} := (k\ k+1\ \cdots\ \ell-1\ \ell).$$

Lemma 5.2. *Consider a function* $f\colon [n] \to [n]$ *and let* $p = p(f)$ *and* $q = q(f)$*. Then*

$$q = \pi_{[f(n)\,p(n)]} \circ \cdots \circ \pi_{[f(1)\,p(1)]} \circ p\,;$$

Furthermore, f *is a parking function if and only if* $q([n]) \subseteq [n]$ *or, equivalently, if* $q \in \mathfrak{S}_n$*.*

Proof. Let $p^{(i)} = p_{|[i]}$ be the restriction of p to $[i]$ and $q^{(i)}$ be defined as in the proof of Lemma 3.2 (although we do not require here that $f(i) \le i$ and so (3.3) may not be valid), for $1 \le i \le n$. Then

$$q^{(1)} = \pi_{[f(1)\,p(1)]} \circ p^{(1)}\,,$$

since $p(1) = f(1) = q(1)$. We end the proof of the first statement of the lemma by proving by induction on j that

$$\begin{aligned} p([j]) &= q([j])\,; \\ q^{(j)}_{|[j-1]} &= \pi_{[f(j)\,p(j)]} \circ q^{(j-1)}\,; \\ q^{(j)}(j) &= \pi_{[f(j)\,p(j)]} \circ \cdots \circ \pi_{[f(1)\,p(1)]}(p(j))\,. \end{aligned}$$

Note that, by definition, $p(j)$ is exactly the least integer $k \ge f(j)$ such that $k \notin p([j-1])$, which equals $q([j-1])$, by induction. On the other hand, $f([j]) \subseteq p([j])$ by definition. Hence, the identities we want to prove just rephrase the second clause of Algorithm 3.1.

The second statement is an obvious consequence of the first one. ☐

We end this section with an example of application of the last lemma, to $f = 121153414$ as before, in Figure 6. We apply in Figure 6 the s-parking algorithm for obtaining $w^{-1} = q(f)$. For the second part, remember that $p(f) = 123456789$.

8								
4								
3			9					
1	2	6	7	5				
~~1~~	~~2~~	~~2~~	~~2~~	~~5~~	~~5~~	~~5~~	~~5~~	5
~~3~~	~~1~~	~~1~~	~~1~~	~~2~~	~~2~~	~~2~~	2	
~~4~~	~~3~~	~~6~~	~~7~~	~~1~~	~~1~~	1		
8	4	3	~~6~~	~~7~~	7			
			9	6				

$$\begin{aligned} &\pi_{[4\,9]}\pi_{[1\,8]}\pi_{[4\,7]}\pi_{[3\,6]}\pi_{[1\,4]}\pi_{[1\,3]} \\ &= (4\ 5\ 6\ 7\ 8\ 9) \circ (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8) \\ &\quad \circ (4\ 5\ 6\ 7) \circ (3\ 4\ 5\ 6) \\ &\quad \circ (1\ 2\ 3\ 4) \circ (1\ 2\ 3) \\ &= 7\,8\,3\,2\,9\,5\,6\,1\,4 \\ &= (8\,4\,3\,9\,6\,7\,1\,2\,5)^{-1} \end{aligned}$$

FIGURE 6. Illustration of Lemma 5.2

Example 5.3 (Example2.6, conclusion — compare to [8, example p. 484]). Let us now use s-parking with input $f = 341183414$ of Example 2.6, that is *not* central, in Figure 7. We obtain at the end the sequence $\beta = 843967125 = q^{-1}(f)$.

$$
\begin{array}{ccccccccc}
8 & & & 9 & & & & & \\
4 & & 6 & 7 & & & & & \\
3 & \sqcup & 1 & 2 & \sqcup & \sqcup & \sqcup & 5 & \sqcup \\
\hline
\not 3 & \not 3 & \not 1 & \not 2 & \not 2 & \not 2 & \not 2 & \not 5 & 5 \\
\not 4 & 4 & \not 6 & \not 1 & \not 1 & \not 1 & 1 & 2 & \\
8 & & 3 & \not 7 & \not 7 & 7 & & & \\
 & & & \not 6 & 6 & & & & \\
 & & & 9 & & & & &
\end{array}
$$

FIGURE 7. S-parking a non-central parking function

We note that f is not central and yet $\beta = w(f)$. This is not always the case, as we shall see. However, the result is always a permutation, by Lemma 5.2. Moreover, we note the following:

Remark 5.4. Let f be a parking function, q be defined by the Algorithm 3.1 and suppose that $\lambda(w, \mathfrak{I}) = f$ for some valid pair $(w, \mathfrak{I})$. Then

$$q^{-1}(f) = w(f)$$

if, while applying the construction of the arc diagram described in Remark 2.8, where two nested arcs simplify to the outer arc, these arcs are of form either

$$\overparen{a \cdots b \cdots c} \;\; (a > b, c) \qquad \text{or} \qquad \overparen{a \cdots \overparen{b \cdots c} \cdots d} \;\; (a > d,\, b > c)\ ,$$

where, in the second case, *all elements between a and b, including a and excluding b, are less than c*. Note that this is what happened in the example of Remark 2.8. In fact, if this is the case, at the end a valid pair $(\beta, \mathfrak{I})$ is formed for which f is the image by λ, by definition, and since a situation cannot occur where, for $\alpha := \beta^{-1}$, $f_i = \alpha_i$ for some $j < i < k$ such that $[j, k] \in \mathfrak{I}$ and such that $\beta_j > \beta_i$.

It is perhaps noteworthy that for all of the parking functions of three or less cars, and for 124 out of the 125 parking functions of 4 cars, the s-parking defines directly λ^{-1}. The exception is $f = 2131$ of Example 2.7, for which $w(f) = 4231$, as we have seen. In fact, we have

$$
\begin{array}{cccc}
4 & & & \\
2 & 1 & 3 & \sqcup \\
\hline
\not 2 & \not 1 & \not 3 & 3 \\
4 & 2 & 1 &
\end{array}\ , \qquad 4213^{-1} = 3241\,, \qquad 3241 - 2131 = 1110 \rightsquigarrow \overparen{\overparen{4\,2}\overparen{\,1}\,3}\,.
$$

For $n = 5$, in 36 (out of the 1296) cases s-parking does *not* give the proper w, and for $n = 6$ this happens in 1015 out of the 16807 cases.

Lemma 5.2 explains this situation. For example, if we consider the sequence of parking functions starting with $f = 2131$ of Example 2.7 and ending with $p(f)^{-1} = 2134$,

$$2131 \qquad 2132 \qquad 2133 \qquad 2134\,,$$

and the corresponding regions, *it is not true* that each region is separated from the previous one by a hyperplane of form $x_i = x_j$. In fact, $w(2131) = 4231$ and $w(2132) = 2413$. More precisely, the arc diagrams of the regions are, respectively, $\overparen{4\,2}\overparen{\,3\,1}$ and $2\,\overparen{4\,1\,3}$ and the corresponding regions are separated by the hyperplanes of equations $x_2 = x_4$ and $x_1 = x_3$ (and by those of equations $x_1 = x_2 + 1$, $x_1 = x_4 + 1$ and $x_3 = x_4 + 1$).

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