EXTREME VALUE LAWS FOR DYNAMICAL SYSTEMS WITH COUNTABLE EXTREMAL SETS

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Abstract. We consider stationary stochastic processes arising from dynamical systems by evaluating a given observable along the orbits of the system. We focus on the extremal behaviour of the process, which is related to the entrance in certain regions of the phase space, which correspond to neighbourhoods of the maximal set $\mathcal{M}$, *i.e.*, the set of points where the observable is maximised. The main novelty here is the fact that we consider that the set $\mathcal{M}$ may have a countable number of points, which are associated by belonging to the orbit of a certain point, and may have accumulation points. In order to prove the existence of distributional limits and study the intensity of clustering, given by the Extremal Index, we generalise the conditions previously introduced in [FFT12, FFT15].

1. Introduction

The study of Extreme Value Laws (EVL) for dynamical systems has received a lot of attention in the past few years. We refer to the recently published book [LFF$^+$16], which makes an introduction to the subject and gives the latest developments in the field.

The main goal of this theory is to study the extremal properties of stochastic processes generated by the orbits of a dynamical system. To be more specific, we consider an observable function φ, choose an initial condition, let the system evolve and keep record of the values of this observable evaluated on the successive states that the system presents, along the evolution of this particular orbit. Then, we analyse such realisations of the stochastic process, which depend on the initial condition, in terms of their extremal behaviour, where we are particularly interested in the occurrence of abnormally high observations, exceeding high thresholds.

For chaotic systems, the sensitivity to initial conditions lends the stochastic processes just described an erratic behaviour that resembles the pure randomness of independent and identically distributed (iid) sequences of random variables studied in classical Extreme Value Theory.

However, the existence of periodic points was seen to create a strong dependence that turned out to be responsible for the appearance of clustering of exceedances of high thresholds. In this setting the rare events corresponding to the exceedances of high thresholds correspond to the entrances of the orbit in certain regions of the phase space, where the observable function φ is maximised. This connection between excedances and visits to certain target sets of the phase space is the base of the formal link established in [FFT10, FFT11] between the existence of EVL and Hitting Time Statistics (HTS). By EVL we mean the distributional limit of the

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Date: December 23, 2016.
2000 *Mathematics Subject Classification.* 37A50, 60G70, 37B20, 60G10, 37C25.

partial maxima of the considered stochastic processes and by HTS we mean the distributional limit for waiting time before hitting a certain target of the phase space.

From the application point of view, one can think that the observable function φ detects the possibly hazardous states of the system in the phase space. When the orbits of the system hit small neighbourhoods of such areas (where φ is maximised), one observes abnormally high values of the stochastic process that typically correspond to unwanted events, such as: storms, rogue waves, draughts, financial crisis, etc.

In the literature of both EVL and HTS for dynamical systems, in most situations φ is maximised on a certain point ζ so that exceedances of increasingly high thresholds correspond to hits to shrinking neighbourhoods around a single point ζ of a low dimensional phase space.

However, in application scenarios, the systems are setup on high dimensional spaces and the critical areas (corresponding to the maximal sets of certain observables φ) have an intricate geometric/topological nature. This type of situation arises easily in practical situations like, the examples mentioned in the introduction of [MP16]. In particular, we recall the "sniffing" robots in search of odor sources ([VVS07]) or landmine detecting robots, where the sources of odor and mines play the role of the maximal sets.

Of course, there is still a gap between the complex high dimensional models appearing in many applications and the examples that the theory can treat. Yet, some of the complex features can be captured by simpler mathematical models, whose performance allows some sort of inference of the expected behaviour of more general and complex systems. As in [MP16], the main goal of this paper is to provide another step to close such gap.

In order to put in perspective the progress provided here we make a brief summary of the state of the art.

As mentioned earlier, most of the times the maximal set is reduced to a point, ζ, and, at first, the neighbourhoods considered were dynamical cylinders, while, more recently, they have been taken, more generally, as metric balls.

For hyperbolic and non-uniformly hyperbolic systems (including intermittent maps, multimodal quadratic maps, Hénon maps, billiards) and for typical points ζ, *i.e.*, for almost all ζ with respect to an invariant measure Sinai-Ruelle-Bowen (SRB) measure, the limiting laws that apply for both EVL and HTS have been proved to be the standard exponential distribution (with mean 1), see for example [HSV99, Col01, BSTV03, HNT12, CC13, PS16].

When ζ is a periodic point, then periodicity creates a short recurrence dependence structure which is responsible for the appearance of clustering of exceedances or hits to the neighbourhoods of ζ (usually, metric balls around ζ). This means that the limiting law is exponential with a parameter $0 \leq \theta < 1$ (with mean θ^{-1}). This parameter is called the Extremal Index (EI) and measures the intensity of clustering since, in most situations, θ^{-1} coincides with the average size of the cluster, *i.e.*, the average number of exceedances within a cluster. The existence of an EI was proved for hyperbolic systems, ϕ-mixing systems, intermittent maps, Benedicks-Carleson quadratic maps, see for example: [Hir93, Aba04, HV09, FFT12, FFT13].

Moreover, in [FFT12, FP12], for uniformly expanding systems such as the doubling map, a dichotomy was shown which states that either ζ is periodic and we have an EI with a very precise formula depending on the expansion rate at ζ, or for every non-periodic ζ, we have an EI equal to 1 (which means no clustering). The dichotomy was obtained for more general systems such as: conformal repellers [FP12], systems with spectral gaps for the Perron-Frobenius operator [Kel12], mixing countable alphabet shifts [KR14], systems with strong decay of correlations [AFV15] and intermittent maps [FFTV16].

Very recently, in [AFFR16], the authors considered the possibility of having a finite number of maximal points of φ. Moreover, it was shown that when these maximal points are correlated, in the sense of belonging to the same orbit of some point (not necessarily periodic), then clustering of exceedances is created by what turned out to be a mechanism that emulates some sort of fake periodic effect. The fact that the maximal points lay on the same orbit makes the occurrence of an excedance, which corresponds to an entrance in a neighbourhood of one of such points, followed by another exceedance, in a very short time period, a very likely event, which was the same effect observed at maximal periodic points. In [Kel12, Remark 8], the author also gives some insight into computing the EI for a particular case of this sort of phenomenon, but, again, for finite maximal sets.

In this paper, we develop the theory further by letting the number of points where φ is maximised to be infinite. Namely, we will consider countably many maximal points, which, in compact phase spaces as we consider here, have accumulation points which complicate the analysis. The tools used in [AFFR16] were originally developed in [FFT12] and later refined in [FFT15]. They are based on some conditions on the dependence structure of the stochastic processes. These conditions are ultimately verified on account of the rates of decay of correlations of the systems. However, the conditions used there build upon the fact that the periodic or fake periodic effect creating the clustering has finite range. This derives from the fact that the period of both the periodic or fake periodic effect is finite since, in the first case, each periodic point has a finite period and, in the second case, the number of maximal points used to emulate periodicity is finite. This means we need to make some adjustments to the conditions devised in [FFT12, FFT15] to cope with infinite "periods". Another technical problem arises in the use of decay of correlations to prove the dependence conditions. Typically, the test functions plugged into the decay of correlations statements have finitely many connected components, which is not necessarily the case here. Hence, we need to play with adapted truncated function approximations that have to be chosen to balance the estimates and obtain the result.

In the recent paper, [MP16], the authors provide a very insightful numerical study that shows evidence that EVL can be proved for dynamical systems with observable functions maximised on Cantor sets. The techniques we introduce here can be used to provide a theoretical proof of some of the statements in [MP16]. This is an ongoing work of the last three authors.

2. The setting

Take a system $(\mathcal{X}, \mathcal{B}, \mu, f)$, where $\mathcal{X}$ is a Riemannian manifold, $\mathcal{B}$ is the Borel σ-algebra, $f : \mathcal{X} \to \mathcal{X}$ is a measurable map and μ an f-invariant probability measure. Suppose that the time series $X_0, X_1, \ldots$ arises from such a system simply by evaluating a given observable $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ along the orbits of the system, or in other words, the time evolution given by successive iterations by f:

$$X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

Clearly, $X_0, X_1, \ldots$ defined in this way is not necessarily an independent sequence. However, f-invariance of μ guarantees that this stochastic process is stationary.

In this paper the novelty of the approach resides in the fact that instead of considering observables $\varphi : \mathcal{X} \to \mathbb{R}$ achieving a global maximum at a single point $\zeta \in \mathcal{X}$, as in the great majority of the literature about EVL and HTS for dynamical systems, or at a finite number of points $\xi_1, \ldots, \xi_k \in \mathcal{X}$, with $k \in \mathbb{N}$, as in [HNT12, AFFR16], we assume that the maximum is achieved on a countable set $\mathcal{M} = \{\xi_i\}_{i \in \mathbb{N}_0}$. This means that the target sets, corresponding to neighbourhoods of the maximal set (see (7) below), which are the regions of the phase space where the observable reaches high values, may have an infinite number of connected components, which contrasts with the previous situations. To handle this more complex structure of the target sets and possibly of their recurrence properties, in Section 3, we generalise the existing theory in order to cope with infinitely many connected components and the respective underlying periodic behaviour of possibly arbitrarily large period. This result may be applied in a much more general setting then we consider, here, in Section 4.

As we have seen before ([FFT12, AFFR16]), with finite maximal sets, the recurrence properties of the maximal set $\mathcal{M}$ to itself play an important role in the determination of the EI and consequently of the limiting law. In Section 4, in order to illustrate possible applications of the general result of Section 3 and to guarantee the persistence of recurrence of $\mathcal{M}$ to itself, along the time line, we consider that $\mathcal{M}$ is the closure of a subset of the orbit of some chosen point $\zeta \in \mathcal{X}$. More precisely, for a certain point $\zeta \in \mathcal{X}$, we have that $\mathcal{M} := \overline{\{f^{m_i}(\zeta) : i \in \mathbb{N}\}}$. For simplicity, we assume that $\xi_i = f^{m_i}(\zeta)$, for each $i \in \mathbb{N}$ and the sequence $\{\xi_i\}_{i \in \mathbb{N}}$ has only one accumulation point that we denote by ξ_0. Hence, for each $i \in \mathbb{N}_0$ we have that ξ_i is an isolated point of $\mathcal{M}$. However, the important facts are that $\mathcal{M}$ is closed and countable. In case there are more accumulation points in $\mathcal{M}$ of the orbit of some chosen point $\zeta \in \mathcal{X}$, as long as they are finitely many, then each such accumulation point would have to be analysed as we will do here for ξ_0. Having more general accumulation sets, such as Cantor sets, raises new technical challenges, which we are leaving for an already mentioned ongoing work.

We remark that the general results presented in Section 4 do not assume any restriction to the sequence $\{m_i\}_{i \in \mathbb{N}}$, although, in the particular examples given afterwards, this sequence is quite sparse so that the technical conditions are more easily verified.

The worked-out examples that we present below reveal that the accumulation point plays a secondary role in the determination of the EI. In fact, we picked an accumulation point that happens to be a fixed point of the dynamics and when the maximal set is reduced to it, we get intense clustering. On the other hand, when this fixed point is an accumulation point of the maximal set, no only we may have no clustering at all, as when clustering is observed it is created by the first points in the orbit we used to define the maximal set. (See Remark 4.8). In some sense, the EI is determined by the parts of $\mathcal{M}$ that recur fast to $\mathcal{M}$, rather than the slowly recurrent parts.

We assume that the observable φ also satisfies

$$\varphi(x) = h_i(\mathrm{dist}(x, \xi_i)),\ \forall x \in B_{\varepsilon_i}(\xi_i), \quad i \in \{1, 2, \ldots\} \tag{2}$$

where $B_{\varepsilon_i}(\xi_i) \cap B_{\varepsilon_j}(\xi_j) = \emptyset$, for all $i \neq j$, dist denotes some metric in $\mathcal{X}$ and the function $h_i : [0, +\infty) \to \mathbb{R} \cup \{\infty\}$ is such that 0 is a global maximum ($h_i(0)$ may be $+\infty$), h_i is a strictly decreasing continuous bijection $h_i : V \to W$ in a neighbourhood V of 0; and has one of the following three types of behaviour:

(1) Type 1: there exists some strictly positive function $g : W \to \mathbb{R}$ such that for all $y \in \mathbb{R}$

$$\lim_{s \to h_i(0)} \frac{h_i^{-1}(s + yg(s))}{h_i^{-1}(s)} = e^{-y};$$

(2) Type 2: $h_i(0) = +\infty$ and there exists $\beta > 0$ such that for all $y > 0$

$$\lim_{s \to \infty} \frac{h_i^{-1}(sy)}{h_i^{-1}(s)} = y^{-\beta};$$

(3) Type 3: $h_i(0) = D < +\infty$ and there exists $\gamma > 0$ such that for all $y > 0$

$$\lim_{s \to 0} \frac{h_i^{-1}(D - sy)}{h_i^{-1}(D - s)} = y^{\gamma}.$$

We assume, of course, that $h_i(0) = h_j(0)$ for all $i, j \in \mathbb{N}_0$. Now, at ξ_0, we may have different types of behaviour. We may have, for example that φ is continuous at ξ_0 or not. We will see examples of application of both types. One particular case of study, in which we have continuity of φ at ξ_0, is when we take:

$$\varphi(x) = h(\mathrm{dist}(x, \mathcal{M})), \tag{3}$$

where h is of one of the three types above and $\mathrm{dist}(x, \mathcal{M}) = \inf\{\mathrm{dist}(x, y) : y \in \mathcal{M}\}$. Note that in this case we may take $h_i = h$ for all $i \in \mathbb{N}$.

In order to study the extremal behaviour of the systems, we consider the random variables $M_1, M_2, \ldots$ given by

$$M_n = \max\{X_0, \ldots, X_{n-1}\}. \tag{4}$$

We say that we have an *Extreme Value Law* (EVL) for M_n if there is a non-degenerate d.f. $H : \mathbb{R} \to [0, 1]$ with $H(0) = 0$ and, for every $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$, $n = 1, 2, \ldots$, such that

$$n\mathbb{P}(X_0 > u_n) \to \tau, \quad \text{as } n \to \infty, \tag{5}$$

and for which the following holds:

$$\mathbb{P}(M_n \leq u_n) \to \bar{H}(\tau) = 1 - H(\tau), \quad \text{as } n \to \infty. \tag{6}$$

where the convergence is meant at the continuity points of $H(\tau)$.

As described in [Fre13, LFF+16] the study of the distributional limit for M_n is tied to the occurrence of excedances, *i.e.*, the occurrence of events such as $\{X_j > u\}$, for some high threshold u, close to $u_F = \sup\{x : F(x) < 1\} = \varphi(\xi_i)$, the right end of the support of the distribution function, F, of X_0. Note that the exceedances of u correspond to hits of the orbits to the target set on $\mathcal{X}$ defined by:

$$U(u) := \{x \in \mathcal{X} : \ \varphi(x) > u\} = \{X_0 > u\}. \tag{7}$$

Observe that by the assumptions on φ then $U(u)$ has possibly countably many connected components. Namely, we may write that

$$U(u) = \bigcup_{j=1}^{\infty} B_{\varepsilon_j(u)}(\xi_j) \cup \{\xi_0\}, \tag{8}$$

where $B_{\varepsilon_j(u)}(\xi_j)$ denotes a ball of radius $\varepsilon_j(u) > 0$ centred at ξ_j. Note that each $\varepsilon_j(u)$ is determined by the function h_j that applies to each ξ_j in equation (2). These balls may overlap, such as when (3) holds, in which case we can write that

$$U(u) = \bigcup_{j=1}^{N(u)} B_{\varepsilon_j(u)}(\xi_j) \cup B_{\varepsilon_0(u)}(\xi_0), \tag{9}$$

for some positive integer $N(u)$, which goes to ∞ as u gets closer to u_F.

As usual, in order to avoid a non-degenerate limit for M_n we assume:

(R) The quantity $\mu(U(u))$, as a function of u, varies continuously on a neighbourhood of u_F.

Remark 2.1. Note that as long as the invariant measure has no atoms, then under the assumptions above on the observable we have that condition (R) is easily satisfied.

3. EVL with clustering caused by arbitrarily large periods

In the study of extremes for dynamical systems, the appearance of clustering has been associated with the periodicity of a unique maximum of φ, as in [FFT12], or, more recently, with the fake periodicity borrowed by the existence of multiple correlated maxima, as in [AFFR16]. In both cases, the main idea to handle the short recurrence created by the periodic phenomena is to replace the events $U(u)$ by

$$\mathcal{A}_q(u) := U(u) \cap \bigcap_{i=1}^{q} f^{-i}(U(u)^c) = \{X_0 > u, X_1 \leq u, \ldots, X_q \leq u\}, \tag{10}$$

where we use the notation $A^c := \mathcal{X} \setminus A$ for the complement of A in $\mathcal{X}$, for all $A \in \mathcal{B}$, and q plays the role of the "period" . In fact, in the case of a single maximum at a periodic point ζ, then q is actually the period of ζ. This idea appeared first in [FFT12, Proposition 1] and was further elaborated in [FFT15, Proposition 2.7].

However, the applications made so far always assumed that q was a fixed positive integer. This is no longer compatible with the persistence of recurrence of the neighbourhoods of the maximal set $\mathcal{M}$, as it happens in the case of a countable infinite number of maximal points of φ, which lay on the same orbit of some point. This means we need to adjust the conditions and arguments in order to consider the possibility of arbitrarily large q. Hence, we start by considering the sequence $(q_n)_{n\in\mathbb{N}}$ to be such that

$$\lim_{n\to\infty} q_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{q_n}{n} = 0. \tag{11}$$

Let $(u_n)_{n\in\mathbb{N}}$ satisfy condition (5) and set $U_n := U(u_n)$ and $\mathcal{A}_{q_n,n} := \mathcal{A}_{q_n}(u_n)$, for all $n \in \mathbb{N}$. Also, let

$$\theta_n := \frac{\mu\left(\mathcal{A}_{q_n,n}\right)}{\mu(U_n)}. \tag{12}$$

Let $B \in \mathcal{B}$ be an event. For some $s \geq 0$ and $\ell \geq 0$, we define:

$$\mathscr{W}_{s,\ell}(B) = \bigcap_{i=\lfloor s \rfloor}^{\lfloor s \rfloor + \max\{\lfloor \ell \rfloor - 1,\ 0\}} f^{-i}(B^c). \tag{13}$$

The notation f^{-i} is used for the preimage by f^i. We will write $\mathscr{W}^c_{s,\ell}(B) := (\mathscr{W}_{s,\ell}(B))^c$. Whenever is clear or unimportant which event $B \in \mathcal{B}$ applies, we will drop the B and write just $\mathscr{W}_{s,\ell}$ or $\mathscr{W}^c_{s,\ell}$. Observe that

$$\mathscr{W}_{0,n}(U(u)) = \{M_n \leq u\}. \tag{14}$$

Here we adapt the two conditions $Д(u_n)$ and $Д'_q(u_n)$ of [FFT15] for q_n satisfying (11).

Condition ($Д_{q_n}(u_n)$)**.** We say that $Д_{q_n}(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if for every $\ell, t, n \in \mathbb{N}$

$$\left|\mu\left(\mathcal{A}_{q_n,n} \cap \mathscr{W}_{t,\ell}\left(\mathcal{A}_{q_n,n}\right)\right) - \mu\left(\mathcal{A}_{q_n,n}\right)\mu\left(\mathscr{W}_{0,\ell}\left(\mathcal{A}_{q_n,n}\right)\right)\right| \leq \gamma(n,t), \tag{15}$$

where $\gamma(n,t)$ is decreasing in t for each n and, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(n)$ and $n\gamma(n,t_n) \to 0$ when $n \to \infty$.

Consider the sequence $(t_n)_{n\in\mathbb{N}}$, given by condition $Д_{q_n}(u_n)$ and let $(k_n)_{n\in\mathbb{N}}$ be another sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{16}$$

Condition ($Д'_{q_n}(u_n)$)**.** We say that $Д'_{q_n}(u_n)$ holds for the sequence $X_0, X_1, X_2, \ldots$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (16) and such that

$$\lim_{n\to\infty} n \sum_{j=q_n+1}^{\lfloor n/k_n \rfloor - 1} \mu\left(\mathcal{A}_{q_n,n} \cap f^{-j}\left(\mathcal{A}_{q_n,n}\right)\right) = 0. \tag{17}$$

We are now ready to state a result that gives us the existence of EVL under conditions $Д_{q_n}$ and $Д'_{q_n}$.

Theorem 3.1. *Let $X_0, X_1, \ldots$ be a stationary stochastic process and $(u_n)_{n\in\mathbb{N}}$ a sequence satisfying* (5)*, for some $\tau > 0$. Assume that conditions $Д_{q_n}(u_n)$ and $Д'_q(u_n)$ hold for some sequence $(q_n)_{n\in\mathbb{N}_0}$ satisfying* (11)*, and sequences $(t_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ as in the statement of those conditions. Moreover assume that the limit $\lim_{n\to\infty}\theta_n =: \theta$ exists. Then*

$$\lim_{n\to\infty}\mu(M_n \leq u_n) = \lim_{n\to\infty}\mu(\mathscr{W}_{0,n}(U_n)) = \lim_{n\to\infty}\mu(\mathscr{W}_{0,n}(\mathcal{A}_{q_n,n})) = \mathrm{e}^{-\theta\tau}.$$

Proof. The first equality follows trivially from (14). The second equality follows from applying [FFT15, Proposition 2.7] and stationarity to obtain that

$$|\mu(\mathscr{W}_{0,n}(U_n)) - \mu(\mathscr{W}_{0,n}(\mathcal{A}_{q_n,n}))| \leq q_n\mu(U_n \setminus \mathcal{A}_{q_n,n})$$

and then observe that the term on the right vanishes as $n \to \infty$ because of the defining properties of q_n and u_n. For the third equality we only need to use [FFT15, Proposition 2.10] and adapt the proofs of [FFT15, Theorem 2.3 and Corollary 2.4], by replacing q by q_n, satisfying (11), to obtain:

$$\left|\mu(\mathscr{W}_{0,n}(\mathcal{A}_{q_n,n})) - \mathrm{e}^{-\theta\tau}\right| \leq C\left[k_n t_n \frac{\tau}{n} + n\gamma(n,t_n) + n\sum_{j=q_n+1}^{\lfloor n/k_n \rfloor} \mu\left(\mathcal{A}_{q_n,n} \cap f^{-j}\left(\mathcal{A}_{q_n,n}\right)\right)\right.$$
$$\left. + \mathrm{e}^{-\theta\tau}\left(|\tau - n\mu\left(U_n\right)| + \frac{\tau^2}{k_n} + |\theta_n - \theta|\,\tau\right)\right], \tag{18}$$

for some $C > 0$ and then use the properties of k_n, t_n, u_n, plus the convergence of θ_n to θ and the new conditions $Д_{q_n}(u_n)$ and $Д'_{q_n}(u_n)$ in order to show that all terms on the right vanish as $n \to \infty$. □

4. Applications to systems with countable maximal sets

4.1. **Assumptions on the system and examples of application.** We assume that the system admits a first return time induced map with decay of correlations against L^1 observables. In order to clarify what is meant by the latter we define:

Definition 4.1 (Decay of correlations)**.** Let $\mathcal{C}_1, \mathcal{C}_2$ denote Banach spaces of real valued measurable functions defined on $\mathcal{X}$. We denote the *correlation* of non-zero functions $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ w.r.t. a measure $\mathbb{P}$ as

$$\mathrm{Cor}_{\mathbb{P}}(\phi,\psi,n) := \frac{1}{\|\phi\|_{\mathcal{C}_1}\|\psi\|_{\mathcal{C}_2}}\left|\int \phi\,(\psi\circ f^n)\,\mathrm{d}\mathbb{P} - \int \phi\,\mathrm{d}\mathbb{P}\int \psi\,\mathrm{d}\mathbb{P}\right|.$$

We say that we have *decay of correlations*, w.r.t. the measure $\mathbb{P}$, for observables in $\mathcal{C}_1$ *against* observables in $\mathcal{C}_2$ if, for every $\phi \in \mathcal{C}_1$ and every $\psi \in \mathcal{C}_2$ we have

$$\mathrm{Cor}_{\mathbb{P}}(\phi,\psi,n) \to 0, \quad \text{ as } n \to \infty.$$

We say that we have *decay of correlations against L^1 observables* whenever this holds for $\mathcal{C}_2 = L^1(\mathbb{P})$ and $\|\psi\|_{\mathcal{C}_2} = \|\psi\|_1 = \int |\psi|\,\mathrm{d}\mathbb{P}$.

If a system already has decay of correlations against L^1 observables, then by taking the whole set $\mathcal{X}$ as the base for the first return time induced map, which coincides with the original system, then the assumption we impose on the system is trivially satisfied. Examples of systems with such property include:

- Uniformly expanding maps on the circle/interval (see [BG97]);
- Markov maps (see [BG97]);
- Piecewise expanding maps of the interval with countably many branches like Rychlik maps (see [Ryc83]);
- Higher dimensional piecewise expanding maps studied by Saussol in [Sau00].

Remark 4.2. In the first three examples above the Banach space $\mathcal{C}_1$ for the decay of correlations can be taken as the space of functions of bounded variation. In the fourth example the Banach space $\mathcal{C}_1$ is the space of functions with finite quasi-Hölder norm studied in [Sau00]. We refer to [BG97, Sau00] or [AFV15] for precise definitions but mention that if $J \subset \mathbb{R}$ is an interval then $\mathbf{1}_J$ is of bounded variation and its BV-norm is equal to 2, *i.e.*, $\|\mathbf{1}_J\|_{BV} = 2$ and if A denotes a ball or an annulus then $\mathbf{1}_A$ has a finite quasi-Hölder norm.

Although the examples above are all in some sense uniformly hyperbolic, we can consider non-uniformly hyperbolic systems, such as intermittent maps, which admit a 'nice' first return time induced map over some subset $Y \subset \mathcal{X}$, called the base of the induced map. To be more precise, consider the usual original system as $f : \mathcal{X} \to \mathcal{X}$ with an ergodic f-invariant probability measure μ, choose a subset $Y \subset \mathcal{X}$ and consider $F_Y : Y \to Y$ to be the first return map f^{r_Y} to Y (note that F may be undefined at a zero Lebesgue measure set of points which do not return to Y, but most of these points are not important, so we will abuse notation here). Let $\mu_Y(\cdot) = \frac{\mu(\cdot\cap Y)}{\mu(Y)}$ be the conditional measure on Y. By Kac's Theorem μ_Y is F_Y-invariant.

From [BSTV03, HWZ14], we know that the Hitting Times Statistics (which can be put in terms of EVL by [FFT10, FFT11]) of the first return induced system coincide with that of the original system. So, as long as the maximal set $\mathcal{M}$ is contained in the base of the induced system, Y, then the induced and original system share the same EVL. Hence, in order to cover all these examples of systems with 'nice' first return time induced maps we are reduced to proving the existence of EVL for systems with decay of correlations against L^1 observables. This fact motivates the following:

Assumption A *Let $f : \mathcal{X} \to \mathcal{X}$ be a system with summable decay of correlations against L^1 observables,* i.e., *for all $\varphi \in \mathcal{C}_1$ and $\psi \in L^1$, then $Cor(\varphi, \psi, n) \le \rho_n$, with $\sum_{n\ge\mathbb{N}} \rho_n < \infty$.*

Among the examples of systems with these 'nice' induced maps we mention the *Manneville-Pomeau* (MP) map equipped with an absolutely continuous invariant probability measure (see for example [LSV99, BSTV03]) and Misiurewicz quadratic maps (see [MvS93]).

4.2. **Main results.**

Theorem 4.3. *Assume that $f : \mathcal{X} \to \mathcal{X}$ satisfies Assumption A. Let $X_0, X_1, \dots$ be given by* (1)*, where φ achieves a global maximum on the compact set $\mathcal{M} = \{\xi_i\}_{i\in\mathbb{N}_0}$, where $\xi_i = f^{m_i}(\zeta)$, for some $\zeta \in \mathcal{X}$, and ξ_0 is the only accumulation point of the sequence $\{\xi_i\}_{i\in\mathbb{N}}$. Let $(u_n)_{n\in\mathbb{N}}$ be as sequence of thresholds as in* (5)*. Assume further that φ is continuous at ξ_0 (as when* (3) *holds) and there exist $N(n) \in \mathbb{N}$ and $\varepsilon_0(n), \dots, \varepsilon_{N(n)}(n)$ so that $\lim_{n\to\infty} N(n) = \infty$, $\lim_{n\to\infty} N(n)/n = 0$ and*

$$U_n = U(u_n) = \bigcup_{j=1}^{N(n)} B_{\varepsilon_j(n)}(\xi_j) \cup B_{\varepsilon_0(n)}(\xi_0),$$

where for each $i \ne j$, we have $B_{\varepsilon_i(n)}(\xi_i) \cap B_{\varepsilon_j(n)}(\xi_j) = \emptyset$, for sufficiently large n. Let $q_n = N(n)$, set $\mathcal{A}_{q_n,n} := \mathcal{A}_{q_n}(u_n)$ as given by (10)*. If the conditions:*

(1) $\lim_{n\to\infty} \|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{\mathcal{C}_1} n\rho_{t_n} = 0$, *for some sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(n)$*

(2) $\lim_{n\to\infty} \|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{\mathcal{C}_1} \sum_{j=N(n)}^{\infty} \rho_j = 0$

hold and the limit $\lim_{n\to\infty} \theta_n := \theta$ exists, where θ_n is given by (12)*, then we have that*

$$\lim_{n\to\infty} \mu(M_n \le u_n) = \mathrm{e}^{-\theta\tau}.$$

Proof. By Theorem 3.1, we need to check that $X_0, X_1, \ldots$ satisfies conditions $Д_{q_n}$ and $Д'_{q_n}$.

Verification of condition $Д_{q_n}(u_n)$.

Taking $\phi = \mathbf{1}_{\mathcal{A}_{q_n,n}}$ and $\psi = \mathbf{1}_{\mathscr{W}_{t,\ell}(\mathcal{A}_{q_n,n})}$ in Definition 4.1, there exists $C > 0$, so that for any positive numbers ℓ and t we have

$$\begin{aligned}
&|\mu(\mathcal{A}_{q_n,n} \cap \mathscr{W}_{t,\ell}(\mathcal{A}_{q_n,n})) - \mu(\mathcal{A}_{q_n,n})\mu(\mathscr{W}_{0,\ell}(\mathcal{A}_{q_n,n}))| \\
&\qquad = \left| \int_{\mathcal{X}} \mathbf{1}_{\mathcal{A}_{q_n,n}} \cdot (\mathbf{1}_{\mathscr{W}_{0,\ell}(\mathcal{A}_{q_n,n})} \circ f^t) d\mu - \int_{\mathcal{X}} \mathbf{1}_{\mathcal{A}_{q_n,n}} d\mu \int_{\mathcal{X}} \mathbf{1}_{\mathscr{W}_{0,\ell}(\mathcal{A}_{q_n,n})} d\mu \right| \\
&\qquad \leq C \|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{\mathcal{C}_1} \rho(t).
\end{aligned}$$

Then, Condition $Д_{q_n}(u_n)$ follows if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(n)$ and $\lim_{n\to\infty} \|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{\mathcal{C}_1} n\rho_{t_n} = 0$, which is precisely the content of hypothesis (1).

Verification of condition $Д'_{q_n,}(u_n)$ Taking $\phi = \psi = \mathbf{1}_{\mathcal{A}_{q_n,n}}$ in Definition 4.1 we obtain

$$\mu\left(\mathcal{A}_{q_n,n} \cap f^{-j}(\mathcal{A}_{q_n,n})\right) = \int_Y \phi \cdot (\phi \circ f^j) d\mu \leq (\mu(\mathcal{A}_{q_n,n}))^2 + \left\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\right\|_{\mathcal{C}_1} \mu\left(\mathcal{A}_{q_n,n}\right) \rho_j. \tag{19}$$

Let t_n be as above and take $(k_n)_{n\in\mathbb{N}}$ as in (16). Recalling that $\lim_{n\to\infty} n\mu(U_n) = \tau$ it follows that

$$\begin{aligned}
n \sum_{j=q_n+1}^{\lfloor n/k_n \rfloor} \mu\left(\mathcal{A}_{q_n,n} \cap f^{-j}(\mathcal{A}_{q_n,n})\right) &= n \sum_{j=N(n)+1}^{\lfloor n/k_n \rfloor} \mu\left(\mathcal{A}_{q_n,n} \cap f^{-j}(\mathcal{A}_{q_n,n})\right) \\
&\leq n \lfloor \tfrac{n}{k_n} \rfloor \mu\left(\mathcal{A}_{q_n,n}\right)^2 + n \left\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\right\|_{\mathcal{C}_1} \mu\left(\mathcal{A}_{q_n,n}\right) \sum_{j=N(n)+1}^{\lfloor n/k_n \rfloor} \rho_j \\
&\leq \frac{(n\mu(\mathcal{A}_{q_n,n}))^2}{k_n} + n \left\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\right\|_{\mathcal{C}_1} \mu\left(\mathcal{A}_{q_n,n}\right) \sum_{j=N(n)}^{\infty} \rho_j \\
&\leq \frac{\tau^2}{k_n} + \tau \left\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\right\|_{\mathcal{C}_1} \sum_{j=N(n)}^{\infty} \rho_j \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

by choice of k_n and hypothesis (2). □

We observe that in the previous theorem since for each n the number of connected components of both U_n and $\mathcal{A}_{q_n,n}$ is finite then the function $\mathbf{1}_{\mathcal{A}_{q_n,n}}$ in principle belongs to the Banach space $\mathcal{C}_1$, when $\mathcal{C}_1$ is the space of functions of bounded variation or with finite quasi-Hölder norm considered in [Sau00]. However, if the observable φ is discontinuous at 0 then both U_n and $\mathcal{A}_{q_n,n}$ may have an infinite number of connected components, which means that $\mathbf{1}_{\mathcal{A}_{q_n,n}}$ does not belong to any of the Banach spaces mentioned. In such cases we must be more careful and introduce some suitable truncated versions of U_n and $\mathcal{A}_{q_n,n}$ as we will see in the following theorem.

Theorem 4.4. *Assume that $f : \mathcal{X} \to \mathcal{X}$ satisfies Assumption A. Let $X_0, X_1, \ldots$ be given by (1), where φ achieves a global maximum on the compact set $\mathcal{M} = \{\xi_i\}_{i\in\mathbb{N}_0}$, where $\xi_i = f^{m_i}(\zeta)$, for some $\zeta \in \mathcal{X}$, and ξ_0 is the only accumulation point of the sequence $\{\xi_i\}_{i\in\mathbb{N}}$. Let $(u_n)_{n\in\mathbb{N}}$*

be as sequence of thresholds as in (5). *Assume further that* φ *is discontinuous at* ξ_0 *and there exist* $\varepsilon_1(n), \varepsilon_2(n), \ldots,$ *such that*

$$U_n = U(u_n) = \bigcup_{j=1}^{\infty} B_{\varepsilon_j(n)}(\xi_j) \cup \{\xi_0\}.$$

Moreover, assume that and there exists $N(n) \in \mathbb{N}$ *such that* $\lim_{n\to\infty} N(n) = \infty$, $\lim_{n\to\infty} \frac{N(n)}{n} = 0$ *and*

$$\lim_{n\to\infty} \frac{\mu(U_n \setminus \tilde{U}_n)}{\mu(U_n)} = 0, \quad \textit{where} \quad \tilde{U}_n = \bigcup_{j=1}^{N(n)} B_{\varepsilon_j(n)}(\xi_j).$$

Let $q_n = N(n)$, *set* $\tilde{\mathcal{A}}_{q_n,n} := \tilde{U}_n \cap \bigcap_{i=1}^{q_n} f^{-i}(\tilde{U}_n^c)$. *If the conditions:*

(1) $\lim_{n\to\infty} \|\mathbf{1}_{\tilde{\mathcal{A}}_{q_n,n}}\|_{\mathcal{C}_1} n\rho_{t_n} = 0$, *for some sequence* $(t_n)_{n\in\mathbb{N}}$ *such that* $t_n = o(n)$

(2) $\lim_{n\to\infty} \|\mathbf{1}_{\tilde{\mathcal{A}}_{q_n,n}}\|_{\mathcal{C}_1} \sum_{j=N(n)}^{\infty} \rho_j = 0$

hold and the limit $\lim_{n\to\infty} \theta_n := \theta$ *exists, where* $\theta_n = \frac{\mu(\tilde{\mathcal{A}}_{q_n,n})}{\mu(\tilde{U}_n)}$, *then we have that*

$$\lim_{n\to\infty} \mu(M_n \le u_n) = \mathrm{e}^{-\theta\tau}.$$

Proof. Using the exact same argument as in the proof of the previous theorem one can check that conditions $Д_{q_n}$ and $Д'_{q_n}$ hold, here, when the role of $\mathcal{A}_{q_n,n}$ is replaced by that of $\tilde{\mathcal{A}}_{q_n,n}$. Then an application of the second and third equalities of the conclusion of Theorem 3.1 allows us to obtain that $\lim_{n\to\infty} \mu(\mathscr{W}_{0,n}(\tilde{U}_n)) = \mathrm{e}^{-\theta\tau}$. The missing step is to show that

$$\lim_{n\to\infty} \mu(M_n \le u_n) = \lim_{n\to\infty} \mu(\mathscr{W}_{0,n}(\tilde{U}_n)).$$

To see this we observe first that, by (14), we can replace $\{M_n \le u_n\} = \mathscr{W}_{0,n}(U_n)$ and then, by stationarity,

$$\mu(\mathscr{W}_{0,n}(\tilde{U}_n) \setminus \mathscr{W}_{0,n}(U_n)) \le \sum_{i=0}^{n-1} \mu(f^{-i}(U_n \setminus \tilde{U}_n)) = n\mu(U_n \setminus \tilde{U}_n).$$

Recalling that $\lim_{n\to\infty} n\mu(U_n) = \tau$ and by hypothesis $\lim_{n\to\infty} \frac{\mu(U_n \setminus \tilde{U}_n)}{\mu(U_n)} = 0$, then we realise that the last term on the right vanishes as $n \to \infty$ and the result follows. $\square$

Example 4.5. Let $(\mathbb{S}^1, f, \mathrm{Leb})$ be the system where $f(x) = 3x \mod 1$ and Leb is the Lebesgue measure. Let $(\xi_j)_{j\in\mathbb{N}}$ be the sequence defined as $\xi_j = f^{3^j}(z)$, where $z = \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{3^i}$. Letting $\mathcal{M} = \{0, z, \xi_1, \xi_2, \ldots\}$, we notice that $\mathcal{M}$ is closed and 0 is the unique accumulation point of $\mathcal{M}$ with $\lim_{j\to\infty} \xi_j = 0$.

Notice that $z > \xi_1 > \xi_2 > \ldots$. We set $I_0 := \left[\frac{z+\xi_1}{2}, z\right]$, $I_1 := \left[\frac{\xi_2+\xi_1}{2}, \frac{\xi_1+z}{2}\right]$ and $I_j := \left[\frac{\xi_{j+1}+\xi_j}{2}, \frac{\xi_{j-1}+\xi_j}{2}\right]$, j=2, 3, ...

Consider the observable $\varphi : \mathbb{S}^1 \to \mathbb{R}$ given by

$$\varphi|_{[\frac{z+\xi_1}{2}, z]}(x) = \frac{2x}{z-\xi_1} - \frac{z+\xi_1}{z-\xi_1}, \;\; \varphi|_{[z,1)}(x) = 0, \;\; \varphi(1) = 1,$$

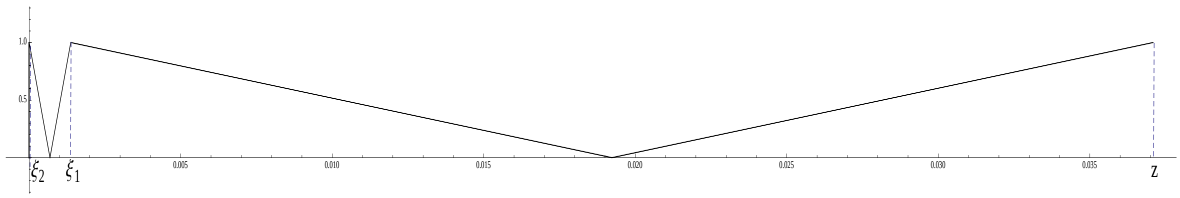

FIGURE 1. The picture is a graph of the observable of Example 4.5 in the interval $[0, z]$.

and

$$\varphi|_{I_j}(x) = \begin{cases} \frac{2x}{\xi_j - \xi_{j+1}} - \frac{\xi_{j+1}+\xi_j}{\xi_j-\xi_{j+1}}, & \text{for } x \in [\frac{\xi_{j+1}+\xi_j}{2}, \xi_j] \\ -\frac{2x}{\xi_{j-1}-\xi_j} + \frac{\xi_{j-1}+\xi_j}{\xi_{j-1}-\xi_j}, & \text{for } x \in [\xi_j, \frac{\xi_{j-1}+\xi_j}{2}] \end{cases},$$

for $j \in \mathbb{N}$ if we set $\xi_0 := z$.

The maximum of φ is 1 and occurs on $\mathcal{M}$. Given $(u_n)_{n\in\mathbb{N}}$ a sequence of tresholds as in (5),

$$\varphi|_{I_0}(x) > u_n \Leftrightarrow x \in \left(\frac{\xi_1 + z}{2} + \frac{z - \xi_1}{2}u_n, z\right) := I_{0,n},$$

$$\varphi|_{I_1}(x) > u_n \Leftrightarrow x \in \left(\frac{\xi_2 + \xi_1}{2} + \frac{\xi_1 - \xi_2}{2}u_n, \frac{\xi_1 + z}{2} - \frac{z - \xi_1}{2}u_n\right) := I_{1,n},$$

and, for $j \geq 2$,

$$\varphi|_{I_j}(x) > u_n \Leftrightarrow x \in \left(\frac{\xi_{j+1} + \xi_j}{2} + \frac{\xi_j - \xi_{j+1}}{2}u_n, \frac{\xi_j + \xi_{j-1}}{2} - \frac{\xi_{j-1} - \xi_j}{2}u_n\right) := I_{j,n}.$$

As in the hypotheses of Theorem 4.4[1], $U_n = \bigcup_{j=0}^{\infty} I_{j,n} \cup \{0\}$, with $|I_{j,n}| = (1 - u_n)|I_j|$. For each $n \in \mathbb{N}$, set $N(n) = \lceil \log_3(\log_3 n) + 1\rceil$. Let $\tilde{U}_n = \bigcup_{j\leq N(n)} I_{j,n}$ and notice that, as

$$\left(\frac{1}{3}\right)^{2\cdot 3^j} \leq \xi_j \leq \left(\frac{1}{3}\right)^{2\cdot 3^j} + \frac{9}{8}\left(\frac{1}{3}\right)^{8\cdot 3^j}, \tag{20}$$

then

$$\frac{1}{2}\left(\frac{1}{3}\right)^{2\cdot 3^{j-1}} \leq \frac{\xi_j + \xi_{j-1}}{2} \leq \left(\frac{1}{3}\right)^{2\cdot 3^{j-1}} \tag{21}$$

So, based on the last inequality, we obtain

$$\bigcup_{j>N(n)} I_{j,n} \subset \left[0, \frac{1}{n^2}\right) \Rightarrow \mu\left(\bigcup_{j>N(n)} I_{j,n}\right) \leq \frac{1}{n^2} \Rightarrow \frac{\mu\left(U_n \backslash \tilde{U}_n\right)}{\mu(U_n)} = \frac{\mu\left(\bigcup_{j>N(n)} I_{j,n}\right)}{\mu(U_n)} \lesssim \frac{1}{n},$$

[1]We note that the sets $I_j, I_{j,n}$ are not symmetric with respect to the centre ξ_j but by changing the metric we can still identify the sets $I_j, I_{j,n}$ as balls around ξ_j.

since $\mu(U_n) \sim \tau/n$. Letting $q_n = N(n)$ and $\tilde{\mathcal{A}}_{q_n,n}$ as in Theorem 4.4, we have that

$$\left\| \mathbf{1}_{\tilde{\mathcal{A}}_{q_n,n}} \right\|_{BV} \le 4N(n) + 1.$$

By [BG97], the system $(\mathbb{S}^1, f, \mathrm{Leb})$ has exponential decay of correlations against L^∞ observables with $\mathcal{C}_1 = BV$, *i.e.*, there exist $C > 0$ and $r \in (0,1)$ so that for $\phi \in BV$ and $\psi \in L^\infty$

$$\mathrm{Cor}_{\mathrm{Leb}}(\phi, \psi, n) \le C r^n.$$

For $t_n = \sqrt{n}$ we get that

$$\lim_{n\to\infty} \left\| \mathbf{1}_{\tilde{\mathcal{A}}_{q_n,n}} \right\|_{BV} n\, r^{t_n} \le \lim_{n\to\infty} \frac{(4N(n)+1)}{n} \frac{n^2}{\mathrm{e}^{\sqrt{n}\log\frac{1}{r}}} = 0,$$

and for some $C' > 0$,

$$\lim_{n\to\infty} \left\| \mathbf{1}_{\tilde{\mathcal{A}}_{q_n,n}} \right\|_{BV} \sum_{j=N(n)}^{\infty} r^j \le C' \lim_{n\to\infty} (4N(n)+1) r^{N(n)} = 0.$$

Now we observe that

$$\mu(\tilde{U}_n) = (1-u_n)\left(z - \frac{\xi_{N(n)} + \xi_{N(n)+1}}{2}\right)$$

and, letting $J(n) = \max\{j : 3^j \le N(n)\}$,

$$\begin{aligned}
\mu\left(\tilde{\mathcal{A}}_{q_n,n}\right) &= (1-u_n)\left(\frac{z-\xi_1}{2} - 3^{-3}\frac{\xi_1-\xi_2}{2} + \sum_{j=1}^{J(n)}\left(|I_j| - 3^{-3^{(j+1)}+3^j}|I_{j+1}|\right) + \sum_{j=J(n)+1}^{N(n)} |I_j|\right) \\
&= (1-u_n)\left(\frac{z-\xi_1}{2} + \sum_{j=1}^{J(n)} |I_j| - 3^{-3}\frac{\xi_1-\xi_2}{2} - \sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}|I_{j+1}|\right) + \sum_{j=J(n)+1}^{N(n)} |I_j|\right) \\
&= (1-u_n)\left(\sum_{j=0}^{J(n)} |I_j| - 3^{-3}\frac{\xi_1-\xi_2}{2} - \sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}|I_{j+1}|\right) + \sum_{j=J(n)+1}^{N(n)} |I_j|\right).
\end{aligned}$$

Note that, since $\lim_{j\to\infty} \xi_j = 0$, we have that

$$\lim_{n\to\infty} \frac{(1-u_n)\sum_{j=J(n)+1}^{N(n)} |I_j|}{\mu(\tilde{U}_n)} = \lim_{n\to\infty} \frac{\frac{\xi_{J(n)}+\xi_{J(n)+1}-(\xi_{N(n)}+\xi_{N(n)+1})}{2}}{z - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}} = 0.$$

Moreover

$$\lim_{n\to\infty} \frac{(1-u_n)\sum_{j=0}^{J(n)} |I_j|}{\mu(\tilde{U}_n)} = \lim_{n\to\infty} \frac{z - \frac{\xi_{J(n)}+\xi_{J(n)+1}}{2}}{z - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}} = 1.$$

Consequently, by Theorem by 4.4, the extremal index is given by

$$\theta = \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{\mu\left(\tilde{\mathcal{A}}_{q_n,n}\right)}{\mu(\tilde{U}_n)} = 1 - \lim_{n\to\infty} \frac{(1-u_n)\left(3^{-3}\frac{\xi_1-\xi_2}{2} + \sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}|I_{j+1}|\right)\right)}{(1-u_n)\left(z - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}\right)}$$
$$= 1 - \frac{1}{z}\left(3^{-3}\cdot\frac{\xi_1-\xi_2}{2} + \lim_{n\to\infty}\sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}\cdot\frac{\xi_j-\xi_{j+2}}{2}\right)\right)$$

and so, $\lim_{n\to\infty}\mu(M_n\le u_n) = \mathrm{e}^{-\theta\tau}$. A numerical approximation for θ to the 12th digit gives:

$$\theta \approx 0.999289701946552$$

The value of the EI is very close to 1 because it takes a very long time for the maximal set to recur to itself and the major contributions for reducing the EI come from the points that recur faster.

Example 4.6. In the previous example we saw that although the EI was smaller than 1, its value was very close to 1 and the major contribution for it came from the first points of the orbit of z. The value of the EI can be easily reduced simply by putting more weight on these fast recurrent points. We illustrate this by making a small change to the previous example, which consists in adding the point $f(z)$ to $\mathcal{M}$ and make the necessary adjustments to the potential.

Let $(\mathbb{S}^1, f, \mathrm{Leb})$ be the system defined in Example 4.5. Let $(\xi_j)_{j\in\mathbb{N}}$ be the sequence defined as $\xi_j = f^{3^j}(z)$, where $z = \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{3^i}$. Letting $\mathcal{M} = \{0, z, f(z), \xi_1, \xi_2, \dots\}$, we notice that $\mathcal{M}$ is closed and 0 is the unique accumulation point of $\mathcal{M}$ with $\lim_{j\to\infty}\xi_j = 0$.

The points in $\mathcal{M}$ satisfies the following $f(z) > z > \xi_1, \xi_2 > \dots$. We set $I_{-1} = \left[\frac{f(z)+z}{2}, f(z)\right]$, $I_0 := \left[\frac{z+\xi_1}{2}, \frac{f(z)+z}{2}\right]$, $I_1 := \left[\frac{\xi_2+\xi_1}{2}, \frac{\xi_1+z}{2}\right]$ and $I_j := \left[\frac{\xi_{j+1}+\xi_j}{2}, \frac{\xi_{j-1}+\xi_j}{2}\right]$, j=2, 3, ...

Consider the observable $\varphi : \mathbb{S}^1 \to \mathbb{R}$ given by

$$\varphi|_{[\frac{f(z)+z}{2}, z]}(x) = \frac{2x}{f(z)-z} - \frac{f(z)+z}{f(z)-z},\ \ \varphi|_{[f(z),1)}(x) = 0,\ \varphi(1) = 1,$$

and

$$\varphi|_{I_j}(x) = \begin{cases} \frac{2x}{\xi_j-\xi_{j+1}} - \frac{\xi_{j+1}+\xi_j}{\xi_j-\xi_{j+1}}, & \text{for } x\in[\frac{\xi_{j+1}+\xi_j}{2}, \xi_j] \\ -\frac{2x}{\xi_{j-1}-\xi_j} + \frac{\xi_{j-1}+\xi_j}{\xi_{j-1}-\xi_j}, & \text{for } x\in[\xi_j, \frac{\xi_{j-1}+\xi_j}{2}] \end{cases},$$

for $j\in\mathbb{N}\cup\{-1,0\}$, if we set $\xi_{-1} := f(z)$ and $\xi_0 := z$. By an analogous argument as the one

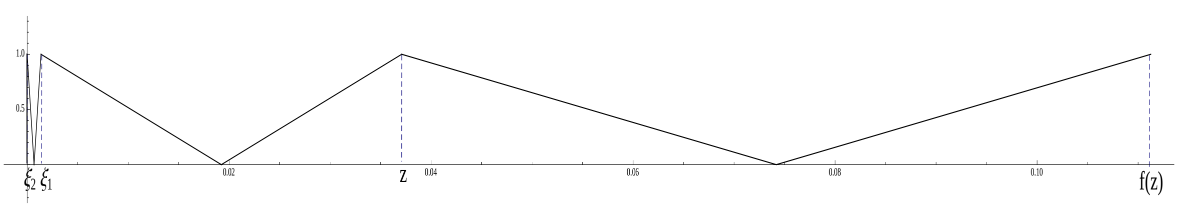

FIGURE 2. The picture is a graph of the observable of Example 4.6 in the interval $[0, f(z)]$.

presented in Example 4.5 we can see that the hypotheses of Theorem 4.3 are satisfied and so $\lim_{n\to\infty}\mathrm{Leb}(M_n \le u_n) = \mathrm{e}^{-\theta\tau}$.

Our next step is to compute the extremal index θ. Applying Theorem 4.4 again, the extremal index is given by

$$
\begin{aligned}
\theta &= \lim_{n\to\infty} \frac{\mu\left(\tilde{A}_{q_n,n}\right)}{\mu(\tilde{U}_n)}\\
&= \lim_{n\to\infty} \frac{\left(\left(\frac{f(z)-z}{2} - 3^{-2}\frac{\xi_1-\xi_2}{2}\right) + \left(\frac{z-\xi_1}{2} - 3^{-1}\frac{f(z)-z}{2}\right) + \left(\frac{f(z)-z}{2} - 3^{-3}\frac{z-\xi_1}{2}\right)\right.}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&+ \frac{\sum_{j=1}^{J(n)}\left(|I_j| - 3^{-3^{(j+1)}+3^j}|I_{j+1}|\right) + \left.\sum_{j=J(n)+1}^{N(n)}|I_j|\right)}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&= \lim_{n\to\infty} \frac{\left(\left(\frac{f(z)-z}{2}\right) + \left(\frac{f(z)-\xi_1}{2}\right) + \sum_{j=1}^{J(n)}|I_j| - \left(\frac{1}{3^2}\frac{\xi_1-\xi_2}{2}\right) - \left(\frac{1}{3}\frac{f(z)-z}{2}\right) + \left(\frac{1}{3^3}\frac{z-\xi_1}{2}\right)\right.}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&- \frac{\sum_{j=1}^{J(n)} 3^{-3^{(j+1)}+3^j}|I_{j+1}| + \left.\sum_{j=J(n)+1}^{N(n)}|I_j|\right)}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&= \lim_{n\to\infty} \frac{\left(\frac{f(z)-z}{2}\right) + \left(\frac{f(z)-\xi_1}{2}\right) + \sum_{j=1}^{J(n)}|I_j|}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&- \lim_{n\to\infty} \frac{\left(\frac{1}{3^2}\frac{\xi_1-\xi_2}{2}\right) + \left(\frac{1}{3}\frac{f(z)-z}{2}\right) + \left(\frac{1}{3^3}\frac{z-\xi_1}{2}\right) + \sum_{j=1}^{J(n)} 3^{-3^{(j+1)}+3^j}|I_{j+1}|}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&+ \lim_{n\to\infty} \frac{\sum_{j=J(n)+1}^{N(n)}|I_j|}{f(z) - \frac{\xi_{N(n)}+\xi_{N(n)+1}}{2}}\\
&= 1 - \frac{1}{f(z)}\left(\left(\frac{1}{3^2}\frac{\xi_1-\xi_2}{2}\right) + \left(\frac{1}{3}\frac{f(z)-z}{2}\right) + \left(\frac{1}{3^3}\frac{z-\xi_1}{2}\right)\right)\\
&- \frac{1}{f(z)}\lim_{n\to\infty}\sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}\cdot\frac{\xi_j-\xi_{j+2}}{2}\right)\\
&= 1 - \frac{f(z)}{6f(z)} + \frac{z}{3f(z)} - \frac{1}{f(z)}\left(\left(\frac{1}{3^2}\frac{\xi_1-\xi_2}{2}\right) + \left(\frac{1}{3^3}\frac{z-\xi_1}{2}\right)\right)\\
&- \frac{1}{f(z)}\lim_{n\to\infty}\sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}\cdot\frac{\xi_j-\xi_{j+2}}{2}\right)\\
&= \frac{8}{9} - \frac{1}{f(z)}\left(\left(\frac{1}{3^2}\frac{\xi_1-\xi_2}{2}\right) + \left(\frac{1}{3^3}\frac{z-\xi_1}{2}\right) - \lim_{n\to\infty}\sum_{j=1}^{J(n)}\left(3^{-3^{(j+1)}+3^j}\cdot\frac{\xi_j-\xi_{j+2}}{2}\right)\right)
\end{aligned}
$$

and so, $\theta < \frac{8}{9}$. A numerical approximation for θ to the 12th digit gives: $\theta \approx 0.882250973721$.

Example 4.7. In this example we consider the same system and maximal set, but we change the potential. As a consequence of that change we get that the set U_n is given by a finite union of open intervals and, in this case, it is not necessary to build approximations for the set U_n.

Let $(\mathbb{S}^1, f, \text{Leb})$ and $\mathcal{M}$ be the system and the set defined in Example 4.5. Define $\varphi : \mathbb{S}^1 \to \mathbb{R} \cup \{\infty\}$ by $\varphi(x) = -\log d(x, \mathcal{M})$, which attains its maximum at $\mathcal{M}$ and is continuous in 0, the only accumulation point of $\mathcal{M}$.

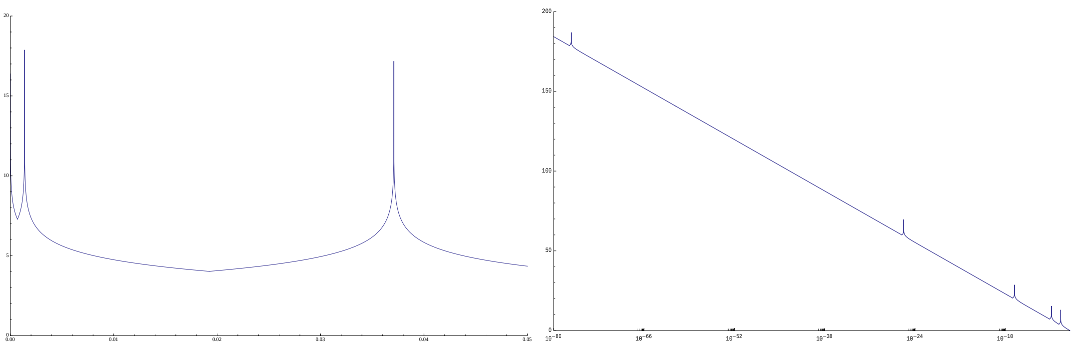

FIGURE 3. The picture on the left is a graph of the observable of Example 4.7 and the picture on the right is the same graph with a logarithmic scale on the x-axis. The spikes correspond to the vertical asymptotes at ξ_j.

In this example, we denote $I_j := [\xi_{j+1}, \xi_j]$, for $j = 0, 1, 2, \ldots$, with $\xi_0 := z$, and ξ_j, $j = 1, 2, \ldots$, as in Example 4.5. In each interval I_j we have that the minimum of φ occurs in the medium point, *i.e.*, in $\frac{\xi_{j+1}+\xi_j}{2}$. So, if $\varphi\left(\frac{\xi_{j+1}+\xi_j}{2}\right) > u_n$, then $\varphi|_{I_j} > u_n$. Now we observe that $|\xi_j - \xi_{j+1}| > |\xi_{j+1} - \xi_{j+2}|$ for all $j = 0, 1, \ldots$. Then

$$\varphi\left(\frac{\xi_{j+1}+\xi_{j+2}}{2}\right) > \varphi\left(\frac{\xi_{j+1}+\xi_j}{2}\right) \text{ for all } j = 0, 1, \ldots$$

For each $n \in \mathbb{N}$ let $N(n) = \max\{j \in \mathbb{N} : \xi_j - \xi_{j+1} \geq 2\mathrm{e}^{-u_n}\}$. So,

$$j > N(n) \Rightarrow \varphi\left(\frac{\xi_{j+1}+\xi_j}{2}\right) > u_n$$

and $B_{\mathrm{e}^{-u_n}}(\xi_{j+1}) \cap B_{\mathrm{e}^{-u_n}}(\xi_j) = \emptyset$, for $j \in \{1, \ldots, N(n)\}$. Therefore,

$$U_n = \left(0, \xi_{N(n)+1} + \mathrm{e}^{-u_n}\right) \cup \left(\cup_{j=1}^{N(n)} B_{\mathrm{e}^{-u_n}}(\xi_j)\right),$$

and then U_n has finitely many connected components, for each $n \in \mathbb{N}$. Let $q_n := N(n)$ and consider $\mathcal{A}_{q_n,n} = \left\{x \in U_n : f^i(x) \notin U_n,\ i = 1, 2, \ldots, q_n\right\}$. By the definition of $N(n)$ and the sequence $(\xi_j)_{j\in\mathbb{N}}$ we have that

$$\xi_{N(n)+1} + \mathrm{e}^{-u_n} < \xi_{N(n)} \Rightarrow 3^{N(n)}(\xi_{N(n)+1} + \mathrm{e}^{-u_n}) < 3^{N(n)}\xi_{N(n)} < \xi_{N(n)-1}.$$

It follows that

$$\left(\bigcup_{j=1}^{N(n)} f^j\left(\left(0, \xi_{N(n)+1} + \mathrm{e}^{-u_n}\right)\right)\right) \cap \left(\bigcup_{j=1}^{N(n)-2} B_{\mathrm{e}^{-u_n}}(\xi_j)\right) = \emptyset,$$

and then, $\mathcal{A}_{q_n,n} \cap \left(0, \xi_{N(n)+1} + \mathrm{e}^{-u_n}\right)$ has at most 3 connected components, which implies that $\mathcal{A}_{q_n,n}$ has at most $2N(n)+3$ and so, $\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{BV} \leq 2(2N(n)+3) < 5N(n)$.

Now we observe that $\frac{\xi_j - \xi_{j+1}}{2} \leq \frac{\xi_j}{2} \leq \left(\frac{1}{3}\right)^{2\cdot 3^j}$. So $N(n)$ is less or equal to $\left\lceil \log_3\left(\frac{u_n}{2\ln 3}\right)\right\rceil$ and $\lim_{n\to\infty} \frac{N(n)}{n} = 0$. Moreover, for $t_n = \sqrt{n}$,

$$\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{BV} n r^{t_n} \leq \left\lceil \log_3\left(\frac{u_n}{2\cdot\ln 3}\right)\right\rceil^2 n r^{t_n},$$

$$\|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{BV} \sum_{j=N(n)}^{\infty} r^j \leq 5\left\lceil \log_3\left(\frac{u_n}{2\cdot\ln 3}\right)\right\rceil^2 \sum_{j=N(n)}^{\infty} r^j. \tag{22}$$

By the definition of u_n and by (22) we have that $\lim_{n\to\infty} \|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{BV} n r^{t_n} = 0$ and $\lim_{n\to\infty} \|\mathbf{1}_{\mathcal{A}_{q_n,n}}\|_{BV} \sum_{j=N(n)}^{\infty} r^j = 0$.

Now, let $J(n) = \max\{j : 3^j \leq N(n)\}$ and notice that

$$\begin{aligned}\theta_n &= \frac{\mu\left(\mathcal{A}_{q_n,n}\right)}{\mu(U_n)} = \frac{\mu\left(U_n \cap \mathcal{A}_{q_n,n}\right)}{\mu(U_n)} = \frac{\mu\left(U_n\right) - \mu\left(U_n \setminus \mathcal{A}_{q_n,n}\right)}{\mu(U_n)} \\ &= 1 - \frac{\mu\left(U_n \setminus \mathcal{A}_{q_n,n}\right)}{\mu(U_n)} \\ &= 1 - \frac{3^{-1}(\xi_{N(n)+1} + \mathrm{e}^{-u_n}) + 2\mathrm{e}^{-u_n}\sum_{j=0}^{J(n)} 3^{3^j - 3^{j+1}}}{(\xi_{N(n)+1} + \mathrm{e}^{-u_n}) + 2\mathrm{e}^{-u_n}(N(n)+1)}.\end{aligned}$$

Note now that $\xi_{N(n)+1} \leq \mathrm{e}^{-u_n}$ and $\lim_{n\to\infty} \mathrm{e}^{-u_n} = 0$. Moreover $J(n) \leq \log_3 N(n)$. So by Theorem 4.3, the extremal index is given by $\theta = \lim_{n\to\infty} \theta_n = 1$, that is, $\lim_{n\to\infty} \mu(M_n \leq u_n) = \mathrm{e}^{-\tau}$.

In this case, we note that the set U_n has an increasing number $(N(n))$ of connected components being that all of them have the same measure except for the utmost left one, which is at most 3 times larger. Again, as in the previous example, the maximal set takes a long time to recur, which means that the pieces that are extracted from U_n to obtain $A_{q_n,n}$ become superexponentially small. Moreover, since all components have almost the same size, then the relative weight of the largest extractions, which occur in the fastest recurrent components, becomes more and more negligible when compared to the components that up to time $N(n)$ have still not suffered any extraction because they have not recurred, yet. Hence, in the end, the increasing weight of the components with no extractions leads to an EI equal to 1.

Remark 4.8. We note that, in both examples, the accumulation point $\xi_0 = 0$ plays a secondary role on the computation of the EI. In fact, we can easily conclude that if there was only one maximal point at 0, which is a fixed point, then we would obtain an EI equal to $\theta = 1 - \frac{1}{2} = \frac{1}{2}$, by the formula in [FFT12, Theorem 3], for example. This contrasts with the values obtained here for the EI.

Remark 4.9. In [MP16], the authors observed that the Minkowski dimension and capacity of the maximal set $\mathcal{M}$ played an important role in the calculation of the normalising sequences to obtain the limiting laws. In the cases treated here, the Minkowski dimension is zero and does not affect the liming laws nor, in particular, the EI, which, as we have seen, is mostly determined by the parts of the maximal set that return fast to $\mathcal{M}$.

Acknowledgements

DA was partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. ACMF and JMF were partially supported by FCT projects FAPESP/19805/2014 and PTDC/MAT-CAL/3884/2014. FBR was supported by BREUDS, a Brazilian-European partnership of the FP7-PEOPLE-2012-IRSES program, with project number 318999, which is supported by an FP7 International International Research Staff Exchange Scheme (IRSES) grant of the European Union. All authors were partially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

JMF would like to thank Mike Todd for careful reading and useful suggestions.

References

[Aba04] Miguel Abadi, *Sharp error terms and necessary conditions for exponential hitting times in mixing processes*, Ann. Probab. **32** (2004), no. 1A, 243–264. MR MR2040782 (2004m:60042)

[AFFR16] Davide Azevedo, Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Fagner B. Rodrigues, *Clustering of extreme events created by multiple correlated maxima*, Phys. D **315** (2016), 33–48. MR 3426918

[AFV15] Hale Aytaç, Jorge Milhazes Freitas, and Sandro Vaienti, *Laws of rare events for deterministic and random dynamical systems*, Trans. Amer. Math. Soc. **367** (2015), no. 11, 8229–8278. MR 3391915

[BG97] Abraham Boyarsky and Pawel Góra, *Laws of chaos*, Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, 1997, Invariant measures and dynamical systems in one dimension. MR 1461536 (99a:58102)

[BSTV03] H. Bruin, B. Saussol, S. Troubetzkoy, and S. Vaienti, *Return time statistics via inducing*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 991–1013. MR MR1997964 (2005a:37004)

[CC13] J.-R. Chazottes and P. Collet, *Poisson approximation for the number of visits to balls in non-uniformly hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **33** (2013), no. 1, 49–80. MR 3009103

[Col01] P. Collet, *Statistics of closest return for some non-uniformly hyperbolic systems*, Ergodic Theory Dynam. Systems **21** (2001), no. 2, 401–420. MR MR1827111 (2002a:37038)

[FFT10] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Mike Todd, *Hitting time statistics and extreme value theory*, Probab. Theory Related Fields **147** (2010), no. 3-4, 675–710. MR 2639719 (2011g:37015)

[FFT11] ———, *Extreme value laws in dynamical systems for non-smooth observations*, J. Stat. Phys. **142** (2011), no. 1, 108–126. MR 2749711 (2012a:60149)

[FFT12] ———, *The extremal index, hitting time statistics and periodicity*, Adv. Math. **231** (2012), no. 5, 2626–2665. MR 2970462

[FFT13] ———, *The compound Poisson limit ruling periodic extreme behaviour of non-uniformly hyperbolic dynamics*, Comm. Math. Phys. **321** (2013), no. 2, 483–527. MR 3063917

[FFT15] ———, *Speed of convergence for laws of rare events and escape rates*, Stochastic Process. Appl. **125** (2015), no. 4, 1653–1687.

[FFTV16] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, Mike Todd, and Sandro Vaienti, *Rare events for the Manneville–Pomeau map*, Stochastic Process. Appl. **126** (2016), no. 11, 3463–3479. MR 3549714

[FP12] Andrew Ferguson and Mark Pollicott, *Escape rates for Gibbs measures*, Ergod. Theory Dynam. Systems **32** (2012), 961–988.

[Fre13] Jorge Milhazes Freitas, *Extremal behaviour of chaotic dynamics*, Dyn. Syst. **28** (2013), no. 3, 302–332.

[Hir93] Masaki Hirata, *Poisson law for Axiom A diffeomorphisms*, Ergodic Theory Dynam. Systems **13** (1993), no. 3, 533–556. MR MR1245828 (94m:58137)

[HNT12] Mark Holland, Matthew Nicol, and Andrei Török, *Extreme value theory for non-uniformly expanding dynamical systems*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 661–688. MR 2846347 (2012k:37064)

[HSV99] Masaki Hirata, Benoît Saussol, and Sandro Vaienti, *Statistics of return times: a general framework and new applications*, Comm. Math. Phys. **206** (1999), no. 1, 33–55. MR MR1736991 (2001c:37007)

[HV09] Nicolai Haydn and Sandro Vaienti, *The compound Poisson distribution and return times in dynamical systems*, Probab. Theory Related Fields **144** (2009), no. 3-4, 517–542. MR MR2496441

[HWZ14] Nicolai Haydn, Nicole Winterberg, and Roland Zweimüller, *Return-time statistics, hitting-time statistics and inducing*, Ergodic theory, open dynamics, and coherent structures, Springer Proc. Math. Stat., vol. 70, Springer, New York, 2014, pp. 217–227. MR 3213501

[Kel12] Gerhard Keller, *Rare events, exponential hitting times and extremal indices via spectral perturbation*, Dyn. Syst. **27** (2012), no. 1, 11–27. MR 2903242

[KR14] Yuri Kifer and Ariel Rapaport, *Poisson and compound Poisson approximations in conventional and nonconventional setups*, Probab. Theory Related Fields **160** (2014), no. 3-4, 797–831. MR 3278921

[LFF+16] Valerio Lucarini, Davide Faranda, Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, Mark Holland, Tobias Kuna, Matthew Nicol, and Sandro Vaienti, *Extremes and recurrence in dynamical systems*, Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts, Wiley, Hoboken, NJ, 2016.

[LSV99] Carlangelo Liverani, Benoît Saussol, and Sandro Vaienti, *A probabilistic approach to intermittency*, Ergodic Theory Dynam. Systems **19** (1999), no. 3, 671–685. MR MR1695915 (2000d:37029)

[MP16] Giorgio Mantica and Luca Perotti, *Extreme value laws for fractal intensity functions in dynamical systems: Minkowski analysis*, J. Phys. A **49** (2016), no. 37, 374001, 21. MR 3544737

[MvS93] Welington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR MR1239171 (95a:58035)

[PS16] Françoise Pène and Benoît Saussol, *Poisson law for some non-uniformly hyperbolic dynamical systems with polynomial rate of mixing*, Ergodic Theory Dynam. Systems **36** (2016), no. 8, 2602–2626. MR 3570026

[Ryc83] Marek Rychlik, *Bounded variation and invariant measures*, Studia Math. **76** (1983), no. 1, 69–80. MR MR728198 (85h:28019)

[Sau00] Benoît Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248. MR 1759406 (2001e:37037)

[VVS07] M. Vergassola, E. Villermaux, and B.I. Shraiman, *'infotaxis' as a strategy for searching without gradients*, Nature **445** (2007), no. 7126, 406–409, cited By 244.

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