Irreducible Module Algebras over Quantum Groupoids and semiprime Smash products

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Following Linchenko and Montgomery's arguments we show that the smash product of a semiprime module algebra, satisfying a polynomial identity and an involutive quantum groupoid is semiprime. Furthermore we extend a result by Bergen et al. and Cohen at al. on irreducible module algebras to certain operator algebras and show that self-injective hulls of Ore domains are finite dimensional over their subring of invariants.

1 Introduction

Group actions, Lie algebras acting as derivations and finite group gradings are typical examples of Hopf algebra actions which have been studied for many years. Several generalizations of Hopf algebras have emerged in recent years, like quantum groupoids (or quantum groupoids) introduced by Böhm et al. [5]. The action of such objects on algebras, as given by quantum groupoids acting on C^*-algebras, [20] or quantum groupoids arising from Jones towers [11] are particularly interesting. New examples of quantum groupoids arose from double groupoids [3], which were also used to find new actions of quantum groupoids (see [20]).

We will fist consider actions of quantum groupoids H on algebras A such that A is an irreducible left $A\#H$-module. We generalize a theorem by Bergen et al. says that $dim(A_{A^H})$ is bound by the uniform dimension of A as a left A-module and the number of generators of $A\#H/\mathrm{Ann}_{A\#H}(A)$ as A-module and obtain a similar theorem for quantum groupoids acting on Ore domains using Zelmanowitz' version of Jacobson's Density Theorem. The final section is devoted to a long-standing open problem in the theory of Hopf action which is the question of $A\#H$ being semiprime provided A is semiprime and H semisimple (see [7]). The most recent partial answer to this problem has been given by Linchenko and Montgomery in [14] where they prove the semiprimness of $A\#H$ under the condition of A satisfying a polynomial identity. We will see that their result carries over to actions of quantum groupoids. We reach more generality by considering actions of linear operators that satisfy certain intertwining relations with the regular multiplications on the algebra.

2 Irreducible actions of linear Operators.

The left uniform dimension of A is denoted by $\mathrm{udim}(A)$ and is the supremum of the cardinalities of the index sets of direct sums of left ideals contained in A. A result [4, Theorem 2.2] by Bergen, Cohen and Fishman says that if a Hopf algebra H acts finitely on a module algebra A with finite uniform dimension, such that A is a simple $A\#H$-module, then A has finite dimension over A^H.

Let k be a field and let A be an associative unital k-algebra. For any $a \in A$ define two linear operators L_a and R_a in $\mathrm{End}_k(A)$ given by $\langle L(a), x\rangle = ax$ and $\langle R(a), x\rangle = xa$ for all $x \in A$. We identify A with the subalgebra $L(A)$ of $\mathrm{End}_k(A)$ generated by all left multiplications $L(a)$ and denote the subalgebra generated by all operators $L(a)$ and $R(a)$ by $M(A)$, which is also sometimes referred to as the *multiplication algebra* of A. As a left $L(A)$-module, A is isomorphic to $L(A)$ since we assume A to be unital. Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ be any intermediate algebra. Then A becomes a cyclic left B-module by the evaluation pairing $\langle\ ,\ \rangle : \mathrm{End}_k(A) \times A \to A$ with $\psi \cdot a := \langle \psi, a\rangle = \psi(a)$. A subset I of A is called B-stable if $B \cdot I \subseteq I$. The B-stable left ideals are precisely the (left) B-submodules of A.

Proposition 2.1 *Suppose $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that A is a simple left B-module. If ${}_AB$ is generated by n elements then $dim(A_{A^B}) \leq n \cdot \mathrm{udim}(A)$.*

Proof: Since A is a simple left B-module, $\mathrm{End}_B(A)$ and hence A^B is a division ring by Schur's Lemma. Let $x_1, \ldots, x_k$ be linearly independent elements of A_{A^B}. Then by the Jacobson Density theorem there exist elements $\psi_1, \ldots, \psi_k$ such that $\psi_j \cdot x_i = \delta_{ij}1$. Since B is n-generated over A, there exist an epimorphism of left A-modules $\varphi : A^n \to B$ therefore elements $t_i \in A^n$ such that $(t_i)\varphi = \psi_i$ for all i. If $\sum_{i=1}^k a_i t_i = 0$ for some $a_i \in A$, then

$$0 = \left(\sum_{i=1}^n a_i t_i\right) \varphi \cdot x_j = \sum_{i=1}^n a_i(\psi_i \cdot x_j) = L(a_j)$$

for all j. Thus $\bigoplus_{i=1}^k At_i$ is a direct sum in A^n and $k \leq \mathrm{udim}(L(A)^n) = n \cdot \mathrm{udim}(A)$. Hence $dim(A_{A^B}) \leq n \cdot \mathrm{udim}(A)$. $\square$

Before we apply this result to quantum groupoid actions, we first generalize another result from the theory of Hopf actions to a more general setting. Let A be an R-ring with k-algebra homomorphism $\eta : R \to A$. The existence of a left character $\chi : A \to R$ is equivalent to R being a left A-module, namely through the following action $a \cdot r := \chi(a\eta(r))$ for all $a \in A, r \in R$. It also allows to define invariants of left A-modules M as

$$M_\chi = \{m \in M \mid am = \chi(a)m\ \forall a \in A\} \simeq \mathrm{Hom}_A(R, M).$$

In particular $\mathrm{End}_A(R) \simeq R_\chi$ can be considered a subalgebra of R. A is called a *Galois* R-ring if the homomorphism $A \to \mathrm{End}(R_{R_\chi}), a \mapsto \chi(a\eta(-))$ that defines the A-module structure on R is an isomorphism.

The smash product $A\#H$ of a module algebra A and a quantum groupoid H is an A-ring with left character χ given by $a\#h \mapsto a\epsilon_t(h) \cdot 1_A$. So the next result will automatically apply to this case.

Proposition 2.2 *Let A be an R-ring with left character χ such that A is finitely generated as left and right R-module . Suppose R has finite left uniform dimension, R is simple as left A-module. The following statements are equivalent:*

(a) *A is a simple ring;*

(b) *A is a Galois R-ring and* R_{R_χ} *is finitely generated projective;*

(c) *R is a faithful A-module and* $A_\chi \neq 0$*;*

If A_R *is free of rank n, then* $(a-c)$ *are also equivalent to:*

(d) $dim(R_{R_\chi}) = n$ *and* $A_\chi \neq 0$*;*

Proof: To put us in the situation where we can apply Lemma 2.1 we consider the subalgebra B of $\mathrm{End}_k(R)$ generated by all operators $\lambda(a) := [r \mapsto r \cdot a = \chi(a\eta(r))]$. The left B-module structure of R is the same as the left A-module structure. Hence R is a simple (faithful) left B-module. Note that $R^B \simeq \mathrm{End}_B(R) = R_\chi$ is a division algebra. Furthermore, B is finitely generated as left $L(R)$-module since A is finitely generated as left R-module and $R \simeq L(R) \subseteq B$. By Lemma 2.1 $dim(R_{R_\chi})$ is finite. Since A_R is finitely generated, say by R^k, we have $k \cdot dim(R_{R_\chi}) \geq dim(A_{R_\chi})$. Hence A is finite dimensional over R_χ.

$(a) \Rightarrow (b)$ Suppose A is simple, then it is a matrix ring over a division ring, hence χ splits as left A-module by some map $\phi : R \to A$. Since A is simple, the trace ideal $(R)\phi A$ equals A, i.e. R is a generator in A-Mod. Hence by a standard module theoretic argument $A \simeq \mathrm{End}(R_{R_\chi})$ and R_{R_χ} is finitely generated projective (see [21, 5.5]).

$(b) \Rightarrow (c)$ is clear since R is a generator. Recall that $A_\chi \simeq \mathrm{Hom}_A(R, A)$.

$(c) \Rightarrow (a)$ $A_\chi \neq 0$ implies that R is isomorphic to a left ideal of A (since R is a simple A-module). Hence also B contains a copy of R as a left ideal and is therefore a left primitive ring having a faithful minimal left ideal, i.e. $B \simeq M_m(R_\chi)$ with $m = dim(R_{R_\chi})$. If R is a faithful left A-module, then $A \simeq B$ is simple.

If A_R is free of rank n, then $n \cdot dim(R_{R_\chi}) = dim(A_{R_\chi})$ holds. $(c) \Leftrightarrow (d)$ As seen in the last step $A_\chi \neq 0$ implies $B \simeq M_m(R_\chi)$ with $m = dim(R_{R_\chi})$. On the other hand $n \cdot m = dim(A_{R_\chi}) \geq dim(B_{R_\chi}) = m^2$ holds. Hence $n = m$ if and only if $A \simeq B$ if and only if R is a faithful left A-module.

□

Before we apply the results to actions of quantum groupoids, we first recall its definition as introduced by G. Böhm et al. in [5].

Definition 2.3 *An associative k-algebra H with multiplication m and unit* 1 *which is also a coassociative coalgebra with comultiplication* Δ *and counit* ϵ *is called a quantum groupoid or a weak Hopf algebra if it satisfies the following properties:*

1. *the comultiplication is multiplicative, i.e. for all* $g, h \in H$*:*

$$\Delta(gh) = \Delta(g)\Delta(h). \tag{1}$$

2. *the unit and counit satisfy:*

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) \tag{2}$$

$$\epsilon(fgh) = \epsilon(fg_1)\epsilon(g_2h) = \epsilon(fg_2)\epsilon(g_1h) \tag{3}$$

3. there exists a linear map $S : A \to A$, called antipode, such that

$$h_1 S(h_2) = (\epsilon \otimes id)(\Delta(1)(h \otimes 1)) =: \epsilon_t(h) \tag{4}$$

$$S(h_1)h_2 = (id \otimes \epsilon)((1 \otimes h)\Delta(1)) =: \epsilon_s(h) \tag{5}$$

$$S(h) = S(h_1)h_2 S(h_3) \tag{6}$$

Note that we will use Sweedler's notation for the comultiplication with suppressed summation symbol.

The image of ϵ_t and ϵ_s are subalgebras H_t and H_s of H which are separable over k ([19, 2.3.4]) and their images commute with each other. Those subalgebras are also characterized by $H_t = \{h \in H : \Delta(h) = 1_1 h \otimes 1_2\}$ respectively $H_s = \{h \in H : \Delta(h) = 1_1 \otimes 1_2 h\}$. A left H-module algebra A over a quantum groupoid H is an associative unital algebra A such that A is a left H-module and for all $a, b \in A, h \in H$:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b) \text{ and } h \cdot 1_A = \epsilon_t(h) \cdot 1_A \tag{7}$$

Let A be a left H-module algebra over a quantum groupoid H and let λ be the ring homomorphism from H to $\mathrm{End}_k(A)$ that defines the left module structure on A, i.e. $\langle \lambda(h), a\rangle := h \cdot a$ for all $h \in H, a \in A$. Property (7) of the definition above can be interpreted as an intertwining relation $\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2)$ of left multiplications $L(a)$ and left H-actions $\lambda(h)$. Recall that the smash product $A\#H$ of a left H-module algebra A and a quantum groupoid H is defined on the tensor product $A \otimes_{H_t} H$ where A is considered a right H_t-module by $a \cdot z = a(z \cdot 1_A)$ for $a \in A, z \in H_t$. Considering $A\#H$ as an A-ring with left character $\chi : A\#H \to A$ with $\chi(a\#h) = a\epsilon_t(h) \cdot 1_A$ and by considering the subalgebra of $\mathrm{End}_k(A)$ generated by the H-action and $L(A)$ we can apply Propositions 2.1 and 2.2 to obtain quantum groupoid analogs of results by Bergen et al. [4, Theorem 2.2] and Cohen et al. [8, Theorem 3.3].

Corollary 2.4 *Let A be a left H-module algebra over a finite dimensional quantum groupoid H. If A has finite left uniform dimension and is a simple left $A\#H$-module, then it is finite dimensional over A^H and satisfies $dim(A_{A^H}) \leq \dim(H) \cdot \mathrm{udim}(A)$. Moreover the following statements are equivalent:*

(a) $A\#H$ is simple;

(b) $A^H \subseteq A$ is an H^*-Galois extension;

(c) A is a faithful left $A\#H$-module;

If $A\#H_A$ is free of rank $dim(H)$, then $(a - c)$ are also equivalent to:

(d) $dim(A_{A^H}) = dim(H)$.

The associativity of A is not needed to prove Lemma 2.1. Non-associative examples of module algebras are given by module algebras over quasi-Hopf algebras. Let H be a quasi-Hopf algebra, that is H is an associative algebra which is a not necessarily coassociative coalgebra satisfying some compatibility conditions (see [9]). A left H-module algebra A is a unital algebra in the category of left H-modules (see [6]). In particular its multiplication satisfies $(ab)c = \sum (x^1 \cdot a)[(x^2 \cdot b)(x^3 \cdot c)]$ where $\phi^{-1} = x^1 \otimes x^2 \otimes x^3$ is the inverse of the Drinfeld reassociator of H. By the proof of 2.1, substituting A by $L(A)$ we get the following

Corollary 2.5 *Let A be a left H-module algebra over a quasi-Hopf algebra H which acts finitely on it. If A has finite left uniform dimension and is a simple left $A\#H$-module, then it is finite dimensional over A^H.*

This applies in particular to finite quasi-Hopf action on non-associative division rings, which are now seen to be finite extensions of their (associative) subring of invariants. Quasi-Hopf actions on non-associative division rings were considered for example by Albuquerque and Majid in [1].

3 Quantum groupoids acting on Ore domains

The argument of Proposition 2.1 uses Jacobson Density Theorem which had been generalized by J.Zelmanowitz in [22] using monoform modules instead of simple modules. A left R-module M is called **monoform** if any non-zero partial endomorphism $f : N \to M$ from a submodule N to M is injective. Zelmanowitz proved the following Theorem:

Theorem 3.1 (Zelmanowitz, [22, Proposition 2.1]) *Let M be a left R-module with self-injective hull $\widehat{M}$ and $\Delta = \mathrm{End}_R(\widehat{M})$. If M is a faithful monoform left R-module, then R is an m-dense ring of linear transformations on $\widehat{M}_\Delta$ i.e. for all $v_1, \ldots, v_k \in \widehat{M}$, linearly independent over Δ and $u_1, \ldots, u_k \in \widehat{M}$ with $u_1 \neq 0$, there exist $r, s \in R$ with $ru_j = sv_j$ for $j = 1, \ldots, k$ and $ru_1 \neq 0$.*

The self-injective hull of a left R-module M is the submodule $\widehat{M} = \sum\{(M)f \mid f : M \to E(M)\}$, where $E(M)$ is the injective hull of M. Hence any element of $\widehat{M}$ can be written as a finite sum $\sum_{i=1}^n (m_i)f_i$ of images of homomorphisms $f_i : M \to E(M)$. Using the self-injective hull of A as a B-module, we are able of extending Proposition 2.1.

Proposition 3.2 *Suppose $L(A) \subseteq B \subseteq End_k(A)$ such that A is a domain and a monoform left B-module. If ${}_AB$ is generated by n elements then $dim(\hat{A}_\Delta) \leq n \cdot udim(A)$, where $\hat{A}$ is the self-injective hull of A and $\Delta = End_B(\hat{A})$.*

Proof: Since $B \subseteq End_k A$, $Ann_B A = 0$ and hence A is a faithful left B-module. By [22, Proposition 2.1], B is an m-dense ring of linear transformations on $\widehat{A}_\Delta$. We want to show that the dimension of $\widehat{A}_\Delta$ is bound by n and $udim(A)$. Since $\widehat{A}$ is generated as Δ-vector space by the elements of A (any element is a linear combination $\sum_{i=1}^n (a_i)f_i$ with $f_i \in \Delta$ and $a_i \in A$, it is enough to consider linearly independent elements in A. Let $v_1, \ldots, v_k \in A$ be linearly independent over Δ. To apply [22, Proposition 2.1] we choose for each $1 \leq i \leq k$ elements u_{ij} with $u_{ij} = \delta_{ij} 1_A$ for $1 \leq j \leq k$. By [22, Proposition 2.1], for each $1 \leq i \leq k$ there exist elements $r_i, s_i \in B$ such that for all $1 \leq j \leq k : r_i \cdot u_{ji} = s_i \cdot v_j$ and $r_i \cdot u_{ii} = r_i \cdot 1_A \neq 0$. By the definition of u_{ji} we have

$$\forall 1 \leq i, j \leq k : \;\; s_i v_j = \delta_{ij} r_i.$$

Since B is $n-$generated over A, there exist an epimorphism of left A-modules $\varphi : A^n \to B$ therefore elements $t_i \in A^n$ such that $(t_i)\varphi = s_i$ for all $1 \leq i \leq k$. We will show that $\sum_{i=1}^k At_i$ is direct. Suppose that exist elements $a_1, \ldots, a_k \in A$ such that $\sum_{i=1}^k a_i t_i = 0$, then for all $1 \leq j \leq k$:

$$0 = \left(\sum_{i=1}^n a_i t_i\right) \varphi \cdot v_j = \sum_{i=1}^n a_i \, (t_i) \, \varphi \cdot v_j = \sum_{i=1}^n a_i \, (s_i \cdot v_j) = \sum_{i=1}^n a_i \, (r_i \delta_{ij}) = a_j (r_j \cdot 1_A).$$

Since A is a domain and $r_j \cdot 1_A \neq 0$, we have $a_j = 0$. Thus $\oplus_{i=1}^n At_i$ is a direct sum in A^n. It follows $dim(\hat{A}_\Delta) \leq udim(A^n) = n \cdot udim(A)$. $\square$

Note that Proposition 3.2 makes no statement if the uniform dimension of A is infinite. However a domain with finite uniform dimension has to have uniform dimension 1 (see [10]). Domains which are left uniform are called left Ore domains. If R is a left Ore domain and $f : I \to R$ is a partial non-trivial left R-linear endomorphism, then there exists $x \in I$ such that $(x)f \neq 0$. For any $y = rx \in \mathrm{Ker}(f) \cap Rx$ we have $r(x)f = (y)f = 0$. Since R is a domain and $(x)f \neq 0$, $y = 0$ and $\mathrm{Ker}(f) \cap Rx = 0$. Since ${}_RR$ is uniform $\mathrm{Ker}(f) = 0$, i.e. f is injective. Hence any left Ore domain R is itself a monoform left R-module. On the other hand if R is a monform left R-module, then it is also a uniform module by [21, 11.3]. Moreover for any $a \in R$, the endomorphism $R(a)$ is injective. Hence $ba = 0$ implies $b = 0$ or $a = 0$, i.e. R is a domain. This shows the following Lemma:

Lemma 3.3 *Let A be k-algebra. Then A is a left Ore domain if and only if A is a monoform left A-module. In this case A is a monoform left B-module, for all $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$.*

The last statement is clear, since whenever $f : I \to A$ is a B-linear partial endomorphism, then it is also A-linear and hence injective if A was monoform as left A-module. Note that if A is a monoform B-module, then it is non-A-singular (see [21, Chapter 10] and hence $\mathrm{Hom}_B(\widehat{A}/A, \widehat{A}) = 0$ as A is essential in $\widehat{A}$. Thus the submodule of B-invariants of $\widehat{A}$ is actually a ring, namely $\widehat{A}^B \simeq \mathrm{Hom}_B(A, \widehat{A}) = \mathrm{End}_B(\widehat{A})$. And as a ring it is a divison ring (see [21, 11.3]).

Hence we can state Propositon 3.2 as follows:

Corollary 3.4 *Let A be a left Ore domain. Then for any $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ such that ${}_AB$ is generated by n elements, then $[\widehat{A}, \widehat{A}^B] \leq n$, where $\widehat{A}$ is the self-injective hull of A as left B-module. In particular if a finite dimensional quantum groupoid acts on A, then $dim(\widehat{A}_{\widehat{A}^H}) \leq dim(H)$.*

4 Semiprime Smash products

The first technical Lemma generalizes a corresponding result of Linchenko [13, Theorem 3.1] for Hopf actions and Nikshych [18, Theorem 6.1.3] for quantum groupoid actions. Recall that an ideal whose elements are nilpotent is called a nil ideal.

Lemma 4.1 *Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ and suppose that for all $\psi \in B$ there exist $m \geq 1$ and elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ such that*

$$L(\langle \psi, a \rangle) = \sum_{i=1}^{m} \psi_i^1 \circ L(a) \circ \psi_i^2 \quad \text{and} \quad \sum_{i=1}^{n} \psi_i^2 \circ \psi_i^1 \in L(A) \tag{8}$$

for any $a \in A$. If A is finite dimensional over a field of characteristic 0 and if I is a nil ideal, then $B \cdot I$ is nil. In particular the Jacobson radical of A is B-stable.

Proof: Denote the trace of a k-linear endomorphism f of A by $\mathrm{Tr}(f)$. Let $\psi \in B$, $a \in A$. Using $\mathrm{Tr}(fgh) = \mathrm{Tr}(hfg)$ and the hypotheses we get:

$$\mathrm{Tr}(L(\langle \psi, a \rangle)) = \mathrm{Tr}\left(\left(\sum_{i=1}^{n} \psi_i^2 \circ \psi_i^1 \right) \circ L(a) \right) = \mathrm{Tr}(L(y) \circ L(a)) = \mathrm{Tr}(L(ya))$$

for some $y \in A$. Suppose that $a \in I$ with I a nil ideal, then $ya \in I$ is nilpotent, hence $\mathrm{Tr}(L(\langle\psi, a\rangle)) = \mathrm{Tr}(L(ya)) = 0$. For any $k > 0$ set $z^k := \langle\psi, a\rangle^k$. Then

$$z^k = \langle L(\langle\psi, a\rangle), z^{k-1}\rangle = \sum_{i=1}^{n} \langle\psi_i^1, a_i\rangle$$

for $a_i = a\langle\psi_i^2, z^{k-1}\rangle$. Since I is an ideal, $a_i \in I$. Hence

$$\mathrm{Tr}\Big(L(z)^k\Big) = \mathrm{Tr}\Big(L(z^k)\Big) = \sum_{i=1}^{n} \mathrm{Tr}\big(L(\langle\psi_i^1, a_i\rangle)\big) = 0$$

Since A is finite dimensional, $char(k) = 0$ and the trace of all powers of $L(z)$ is zero, $L(z)$ is a nilpotent operator, i.e. $z = \langle\psi, a\rangle$ is nilpotent. Thus $B \cdot I$ is a nil ideal. Since the Jacobson radical of an Artinian ring is the largest nilpotent ideal, we have $B \cdot \mathrm{Jac}(A) = \mathrm{Jac}(A)$. □

The last Lemma, which had been proven first by Linchenko for Hopf actions and then by Nikshych for quantum groupoid actions allows us to show the stability of the Jacobson radical of an algebra A which satisfies a polynomial identity and on which act some operator algebra B which is finitely generated over A. The hypotheses of the following Theorem allow the reduction to finite dimensional factors.

Theorem 4.2 *Let $L(A) \subseteq B \subseteq \mathrm{End}_k(A)$ over some field k of characteristic 0 with B being finitely generated as right A-module. Suppose that for all $\psi \in B$ there exist $n \geq 1$ and elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ satisfying*

$$L(\langle\psi, a\rangle) = \sum_{i=1}^{m} \psi_i^1 \circ L(a) \circ \psi_i^2 \quad \textit{and} \quad \sum_{i=1}^{m} \psi_i^2 \circ \psi_i^1 \in L(A) \tag{9}$$

for any $a \in A$. If A satisfies a polynomial identity or if k is an uncountable algebraically closed field, A countably generated and all left primitive factor rings of A are Artinian, then $B \cdot I \subseteq \mathrm{Jac}(A)$ for all nil ideals I of A.

Proof: Let I be a nil ideal. It is enough to show that $(B \cdot I)V = 0$ for all simple left A-modules V, then $B \cdot I \subseteq \mathrm{Jac}(A)$. Let V be a simple left A-module and $P = \mathrm{Ann}_A(V)$ be its annihilator. If k is an uncountable algebraically closed field and A is countably generated, then it satisfies the Nullstellensatz, hence $\mathrm{End}_k(V) = k$ (see [15, 9.1.8]). If primitive factors of A are Artinian, then by the Weddeburn-Artin Theorem $A/P \simeq M_n(k)$ for some n, hence V is a finite dimensional simple left A-module. On the other hand, if A satisfies a polynomial identity, then $A/P \simeq M_n(D)$ where D is a finite dimensional division algebra over $F = Z(A/P)$ by Kaplansky's Theorem [15, 13.3.8]. Tensoring A by F yields an F-algebra $\widetilde{A} := A \otimes F$ with F-action on the right. Then

$$L(\widetilde{A}) = L(A) \otimes F \subseteq B \otimes F \subseteq \mathrm{End}_k(A) \otimes F \subseteq \mathrm{End}_F(A \otimes F) = \mathrm{End}_F(\widetilde{A}).$$

Moreover V is a finite dimensional simple left $\widetilde{A}$-module since $\mathrm{Ann}_{\widetilde{A}}(V) = P \otimes F =: \widetilde{P}$ and $\widetilde{A}/\widetilde{P} \simeq M_n(D)$ is finite dimensional over F. Note also that the nil ideal I extends to a nil ideal $\widetilde{I} := I \otimes F$ since by [12, Theorem 5] I is a locally nilpotent algebra and hence any element $\sum_{i=1}^m a_i \otimes f_i$ lies in a nilpotent finitely generated subalgebra generated by the a_i's and F.

To summarize, our hypothesis on A allows us to consider V to be a finite dimensional simple left A-module, where A and B are algebras over some field k of characteristic 0. Denote by $W = B \otimes_A V$ the induced left B-module. Since B_A is finitely generated and V is finite dimensional, W is finite dimensional. Note that the left B-action on W is given by $\psi \cdot (\phi \otimes v) := (\psi \circ \phi) \otimes v$. Let $Q = \mathrm{Ann}_A(W)$. Then Q is B-stable, because if $a \in Q$ and $\psi \in B$, then by hypothesis there exist elements $\psi_1^1, \ldots, \psi_m^1, \psi_1^2, \ldots, \psi_m^2 \in B$ satisfying equation (9). Thus for any $w = \phi \otimes v \in W$ we have

$$\langle \psi, a \rangle \cdot (\phi \otimes v) = (L(\langle \psi, a \rangle) \circ \phi) \otimes v = \sum_{i=1}^m \psi_i^1 \circ L(a) \circ \psi_i^2 \circ \phi \otimes v = \sum_{i=1}^m \psi_i^1 \cdot (a \cdot (\psi_i^2 \cdot w)) = 0,$$

since $\psi_i^2 \cdot w \in W$ and $a \cdot W = 0$. Hence $B \cdot Q \subseteq Q$. Let $Q_B = \mathrm{Ann}_B(A/Q)$. Then

$$A/Q \simeq L(A/Q) \subseteq B/Q_B \subseteq \mathrm{End}_k(A/Q).$$

Since W is finite dimensional, A/Q is finite dimensional. Note that V is a simple left A/Q-module. Any nil ideal I of A yields a nil ideal $(I+Q)/Q$ of A/Q. Moreover every element $\psi + Q_B \in B/Q_B$ satisfies (9). By Lemma 4.1, $((B \cdot I) + Q)/Q = B/Q_B \cdot (I+Q)/Q$ is included in $\mathrm{Jac}(A/Q)$. Thus

$$(B \cdot I)V = (B/Q_B \cdot (I+Q)/Q) \cdot V \subseteq \mathrm{Jac}(A/Q) \cdot V = 0.$$

Hence $B \cdot I \subseteq \mathrm{Jac}(A)$ for any nil ideal I of A. $\square$

To apply our results to the action of quantum groipid we record the following properties, which are deduced from the axioms:

Lemma 4.3 *Let A be a left H-module algebra over a quantum groupoid H. Then*

1. *$\forall z \in H_t : \lambda(z) = L(z \cdot 1_A)$ and $\forall z \in H_s : \lambda(z) = R(z \cdot 1_A)$;*
2. *$\forall h \in H, a \in A : L(h \cdot a) = \lambda(h_1) \circ L(a) \circ \lambda(S(h_2))$*
3. *If $S^2 = id$, then $\lambda(S(h_2)) \circ \lambda(h_1) \in L(A)$ for all $h \in H$.*

Proof: (1) Let $z \in H_t$. Since $\Delta(z) = 1_1 z \otimes 1_2$, we have for all $a \in A$:

$$z \cdot a = z \cdot (1_A a) = (1_1 z \cdot 1_A)(1_2 \cdot a) = 1_H \cdot ((z \cdot 1_A)a) = (z \cdot 1_A)a$$

The proof of the second statement is analogous.

(2) For $h \in H, a, x \in A$ we have

$$\begin{aligned}(h \cdot a)x &= (h_1 \cdot a)(\epsilon_t(h_2) \cdot 1_A)x \\ &= (h_1 \cdot a)(\epsilon_t(h_2) \cdot x) = (h_1 \cdot a)(h_2 S(h_3) \cdot x) = h_1 \cdot (a(S(h_3) \cdot x))\end{aligned}$$

(3) Suppose $S^2 = id$, then $S(\epsilon_s(h)) = S(h_2)h_1$ and as $S(H_s) \subseteq H_t$, we have using (1):

$$\lambda(S(h_2)) \circ \lambda(h_1) = \lambda(S(\epsilon_s(h))) = L(S(\epsilon_s(h)) \cdot 1_A) \in L(A).$$

$\square$

We say that a quantum groupoid H is involutive if its antipode is an involution.

Say that H acts finitely on a left H-module algebra A if the image of $\lambda : H \to \mathrm{End}_k(A)$ is finite dimensional. The following statement follows from the last Lemma and Theorem 4.2

Theorem 4.4 *Let H be an involutive quantum groupoid over a field k of characteristic zero acting finitely on a left H-module algebra A. If A satisfies a polynomial identity or if k is an uncountable algebraically closed field, A is countably generated and all left primitive factor rings of A are Artinian, then the Jacobson radical of A is H-stable.*

Proof: Let $\lambda : H \to \mathrm{End}_k(A)$ be the ring homomorphism inducing the left H-module structure on A. Denote by B the subalgebra of $\mathrm{End}_k(A)$ generated by $L(A)$ and $\lambda(H)$. Let $h_1, \ldots, h_m$ be elements of H such that $\{\lambda(h_1), \ldots, \lambda(h_m)\}$ forms a basis of $\lambda(H)$. We claim that any element of B is of the form $\sum_{i=1}^m \lambda(h_i) \circ L(a_i)$ for some $a_i \in A$. It is enough to show $L(A)\lambda(H) \subseteq \lambda(H)L(A)$. So take elements $h \in H$ and $a, b \in A$. Then using Lemma 4.3(2), $S^{-1} = S$ and $S(H_t) = H_s$ we have

$$\begin{aligned} h_2 \cdot ((S(h_1) \cdot a)b) &= (h_2 S(h_1) \cdot a)(h_3 \cdot b) \\ &= (S(\epsilon_t(h_1)) \cdot a)(h_2 \cdot b) = a(\epsilon_t(h_1) \cdot 1_A)(h_2 \cdot b) = a(h \cdot b). \end{aligned} \tag{10}$$

This shows the intertwining relation $L(a) \circ \lambda(h) = \lambda(h_2) \circ L(S(h_1) \cdot a)$ in B which yields that B is finitely generated as a right A-module. By the definition of module algebras, we also have that $\lambda(h) \circ L(a) = L(h_1 \cdot a) \circ \lambda(h_2)$. Hence $\lambda(H)L(A) = L(A)\lambda(H)$. For any $a \in A$ and $\psi = \sum_{i=1}^m L(a_i) \circ \lambda(h_i) \in B$ we have by Lemma 4.3(3) and by equation (10):

$$\begin{aligned} L(\langle \psi, a \rangle) &= \sum L(a_i) \circ L(\langle h_i, a \rangle) \\ &= \sum L(a_i) \circ \lambda(h_{i1}) \circ L(a) \circ \lambda(S(h_{i2})) \\ &= \sum \lambda(h_{i2}) \circ L((S(h_{i1}) \cdot a_i)) \circ L(a) \circ \lambda(S(h_{i3})) = \sum_j \psi_j^1 \circ L(a) \circ \psi_j^2. \end{aligned}$$

for $\psi_j^1 = \lambda(h_{i2}) \circ L(S(h_{i1}) \cdot a_i)$, $\psi_j^2 = \lambda(S(h_{i3})$ and some appropriate choice of indices j. Moreover

$$\sum \psi_j^2 \circ \psi_j^1 = \sum \lambda(S(h_{i3})) \circ \lambda(h_{i2}) \circ L(S(h_{i1}) \cdot a_i) = \sum L(y_{h_{i2}}) \circ L(S(h_{i1}) \cdot a_i) \in L(A)$$

for some elements $y_{h_{i2}} \in A$ that exist by Lemma 4.3(4). Therefore the hypotheses of Theorem 4.2 are fulfilled and the statement follows.
□

The (k-linear) dual H^* of H becomes also a quantum groupoid and acts on $A\#H$ by $\phi \cdot (a\#h) := a\#\phi(\rightharpoonup h)$, where $\phi \rightharpoonup h = h_1\langle \phi, h_2 \rangle$. Using the Montgomery-Blattner duality theorem for quantum groupoids proven by Nikshych we have:

Lemma 4.5 *Let H be a finite dimensional quantum groupoid and A a left H-module algebra. Then $A\#H$ is a finitely generated projective right A-module and $A\#H\#H^* \simeq eM_n(A)e$ for some idempotent $e \in M_n(A)$ where $M_n(A)$ denotes the ring of $n \times n$-matrixes for some number $n > 0$.*

Proof: By [16, Theorem 3.3] $A\#H\#H^* \simeq \mathrm{End}(A\#H_A)$. Since H_t is a separable k-algebra, it is semisimple Artinian. Hence H is a (finitely generated) projective right H_t-module and H is a direct summand of H_t^n for some $n > 0$. Moreover it follows from the proof of Lemma 4.3 that $A\#H = (1\#H)(A\#1)$. Thus $H \otimes_{H_t} A \simeq A\#H$ as right A-modules by $h \otimes a \mapsto (1\#h)(a\#1)$.

On the other hand $H \otimes_{H_t} A$ is a direct summand of $H_t^n \otimes_{H_t} A \simeq A^n$ as right A-module. Hence $A\#H$ is a projective right A-module of rank n and $\mathrm{End}(A\#H)_A \simeq eM_n(A)e$ for some idempotent $e \in M_n(A)$. $\square$

We can now transfer Linchenko and Montgomery's result [14, Theorem 3.4] on the semiprimness of smash products to quantum groupoid actions.

Theorem 4.6 *Let A be a left H-module algebra over a finite dimensional involutive quantum groupoid H over a field of characteristic zero. If A is semiprime and satisfies a polynomial identity, then $A\#H$ is semiprime.*

Proof: Set $B = A\#H\#H^*$. Note that H^* is also involutive since its antipode is defined by $\langle S^*(\phi), h\rangle := \langle \phi, S(h)\rangle$ for all $\phi \in H^*, h \in H$. By [17, Corollary 6.5] H^* is semisimple and by [5, 3.13] there exists a normalized left integral $\Lambda \in H^*$. This implies that $A\#H$ is a projective left B-module as the left B-linear map $A\#H \to B$ with $a\#h \mapsto a\#h\#\Lambda$ splits the projection $B \to A\#H$ given by $a\#h\#\phi \mapsto a\#h(\phi \rightharpoonup 1)$.

First suppose that $\mathrm{Jac}(A) = 0$. By Lemma 4.5, $\mathrm{Jac}(B) \simeq eM_n(\mathrm{Jac}(A))e = 0$ for some idempotent e. This implies also that $\mathrm{Rad}_B(A\#H) = 0$ as well, since $A\#H$ is supposed to be a projective left B-module. Recall that the radical $\mathrm{Rad}(M)$ of a module M is the intersection of all maximal submodules of M or equivalently the sum of all small submodules, i.e. of those submodules N of M such that $N + L \neq M$ for all $L \neq M$.

Since $A\#H$ is a finite extension of A, also $A\#H$ satisfies a polynomial identity and since H^* is finite dimensional it acts finitely on $A\#H$. Thus 4.4 applies and for any nil ideal I of $A\#H$ we have $B \cdot I \subseteq \mathrm{Jac}(A\#H)$. On the other hand any B-submodule N of $\mathrm{Jac}(A\#H)$ is contained in $\mathrm{Rad}_B(A\#H)$, which is zero. Hence $I = 0$ and $A\#H$ is semiprime.

In general, if A is semiprime, we can extend the H-action of A to the polynomial ring $A[x]$ by identifying $A[x]$ with $A \otimes_{H_t} H_t[x]$, which is a left H-module algebra, where H acts on x by $h \cdot x = (\epsilon_t(h) \cdot 1_A)x$. Since A is semiprime, satisfying a polynomial identity, $\mathrm{Jac}(A[x]) = 0$ by [2]. Moreover $A[x]$ also satisfies a polynomial identity and by the argument above $A[x]\#H$ is semiprime. As any ideal I of $A\#H$ can be extended to an ideal $I[x]$ of $A\#H[x] = A[x]\#H$, also $A\#H$ is semiprime. $\square$

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