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$\mathrm{SO}(p,q)$-HIGGS BUNDLES AND HIGHER TEICHMÜLLER COMPONENTS

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Abstract. Some connected components of a moduli space are mundane in the sense that they are distinguished only by obvious topological invariants or have no special characteristics. Others are more alluring and unusual either because they are not detected by primary invariants, or because they have special geometric significance, or both. In this paper we describe new examples of such 'exotic' components in moduli spaces of $\mathrm{SO}(p,q)$-Higgs bundles on closed Riemann surfaces or, equivalently, moduli spaces of surface group representations into the Lie group $\mathrm{SO}(p,q)$. Furthermore, we discuss how these exotic components are related to the notion of positive Anosov representations recently developed by Guichard and Wienhard. We also provide a complete count of the connected components of these moduli spaces (except for $\mathrm{SO}(2,q)$, with $q \geqslant 4$).

Contents

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1. Introduction

For a closed surface S and a Lie group G, the representation variety $\mathcal{R}(S,\mathrm{G})$ parameterizes conjugacy classes of group homomorphisms from the fundamental group of S into G. For each Riemann surface structure X on S, the Non-Abelian Hodge (NAH) correspondence defines a homeomorphism between $\mathcal{R}(S,\mathrm{G})$ and $\mathcal{M}(X,\mathrm{G})$, the moduli space of polystable G-Higgs bundles on X. In general these moduli spaces have multiple connected components. Some of the components are mundane in the sense that they are distinguished by obvious topological invariants and have no known special characteristics. Others are more alluring and unusual, either because they are not detected by the primary invariants or because they parameterize objects of special significance, or both.

Instances of such 'exotic' components are well understood in two situations. The first is the case where G is the split real form of a complex semisimple Lie group, in which case the exotic components are known as Hitchin components (see [28]). The second occurs when G is the isometry group of a non-compact Hermitian symmetric space, in which case the subspace with so-called maximal Toledo invariant has exotic components (see [7]). In [11], both of these classes of exotic components of representation varieties have been called *higher Teichmüller components* since they enjoy many of the geometric features of Teichmüller space.

One distinguishing feature common to all higher Teichmüller components is that every representation in them is an Anosov representation, a concept introduced by Labourie [31]. Anosov representations have many interesting dynamical and geometric properties which generalize convex cocompact representations into rank one Lie groups. In particular, higher Teichmüller components consist entirely of discrete and faithful representations [31] which are holonomies of geometric structures on certain closed manifolds [24]. In general, the Anosov condition is open in the representation variety and so does not by itself distinguish connected components. More recently, in [25], Guichard and Wienhard defined a notion of positivity which refines the Anosov property and is still an open condition. They conjecture that such positivity for Anosov representations is also a closed condition, and hence should detect connected components of a representation variety. They showed, moreover, that apart from the split real forms and the real forms of Hermitian type, the only other non-exceptional groups which allow positive representations are the disconnected groups $\mathrm{SO}(p,q)$ for $1 < p < q$, i.e. the special orthogonal groups with signature (p,q). This leads directly to the conjecture that $\mathcal{R}(S,\mathrm{SO}(p,q))$ and hence $\mathcal{M}(X,\mathrm{SO}(p,q))$ should have 'exotic' connected components.

In this paper we establish the existence of such exotic components, count them, and show that each exotic component contains positive Anosov representations. Our methods lie on the Higgs bundle side of the NAH correspondence, so our results actually address the connected components of $\mathcal{M}(\mathrm{SO}(p,q))$ (where we drop the X from the notation unless explicitly needed for clarity or emphasis). Except for the special cases $p = 2$, $q = p$ or $q = p+1$, the group $\mathrm{SO}(p,q)$ is neither split nor of Hermitian type, so the relation between topological invariants and connected components in the representation varieties or related moduli spaces cannot be inferred from previously known mechanisms.

Our main theorem[1] has two parts — one is an existence result and one is a non-existence result. Namely we prove

(1) the existence of a class of explicitly described exotic components of $\mathcal{M}(\mathrm{SO}(p,q))$ for $1 < p \leqslant q$, and
(2) the non-existence of any other exotic components of $\mathcal{M}(\mathrm{SO}(p,q))$ for both $p = 1$ and $2 < p \leqslant q$.

Combining these two results and including the 2^{2g+2} 'mundane' components yields a complete count of the connected components for the moduli spaces of $\mathrm{SO}(p,q)$-Higgs bundles $\mathcal{M}(X, \mathrm{SO}(p,q))$ or, equivalently, the representation varieties $\mathcal{R}(S, \mathrm{SO}(p,q))$, for $2 < p \leqslant q$.

Theorem 6.1. *Let X be a compact Riemann surface of genus $g \geqslant 2$ and denote the moduli space of $\mathrm{SO}(p,q)$-Higgs bundles on X by $\mathcal{M}(\mathrm{SO}(p,q))$. For $2 < p \leqslant q$, we have*

$$|\pi_0(\mathcal{M}(\mathrm{SO}(p,q)))| = 2^{2g+2} + \begin{cases} 2^{2g} & \text{if } q = p \\ 2^{2g+1} + 2p(g-1) - 1 & \text{if } q = p+1 \\ 2^{2g+1} & \text{if } q > p+1 \ . \end{cases}$$

Remark 1.2. In fact, our methods also show that $\mathcal{M}(\mathrm{SO}(1,q))$ does not have exotic components for $q > 2$, yielding 2^{2g+1} connected components. We also give a precise count of the components of $\mathcal{M}(\mathrm{SO}(2,2))$ and $\mathcal{M}(\mathrm{SO}(2,3))$ (the latter case basically follows from previously known results), but for $q \geqslant 4$ our techniques fall short of a component count of $\mathcal{M}(\mathrm{SO}(2,q))$. However, we expect no new exotic components to exist (see Section 6.2 for details).

The primary topological invariants are apparent from the structure of the Higgs bundles. In the case of $\mathrm{SO}(p,q)$-Higgs bundles on X, the objects are described by a triple (V, W, η), where V and W are holomorphic orthogonal bundles of rank p and q respectively, such that $\Lambda^p V \cong \Lambda^q W$, and η is a holomorphic section of the bundle $\mathrm{Hom}(W, V) \otimes K$, where K is the canonical bundle of X. The topological invariants are then the first and second Stiefel-Whitney classes of V and W subject to the constraint that $sw_1(V) = sw_1(W)$. These invariants provide a primary decomposition of the moduli space $\mathcal{M}(\mathrm{SO}(p,q))$ into (not necessarily connected) components labeled by triples $(a,b,c) \in H^1(S,\mathbb{Z}_2) \times H^2(S,\mathbb{Z}_2) \times H^2(S,\mathbb{Z}_2)$. Using the notation $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$ to denote the union of components labeled by (a,b,c), we can thus write

$$\mathcal{M}(\mathrm{SO}(p,q)) = \coprod_{(a,b,c) \in \mathbb{Z}_2^{2g} \times \mathbb{Z}_2 \times \mathbb{Z}_2} \mathcal{M}^{a,b,c}(\mathrm{SO}(p,q)) \ . \tag{1.1}$$

Each space $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$ has one connected component characterized entirely by the topological invariants (a,b,c). This is the connected component which contains the moduli space of polystable orthogonal bundles with these invariants, which correspond to Higgs bundles for the maximal compact subgroup of $\mathrm{SO}(p,q)$. Denoted by $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))_{\mathrm{top}}$, these comprise the 2^{2g+2} 'mundane' components for $2 < p \leqslant q$. Our existence result identifies additional components disjoint from the $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))_{\mathrm{top}}$ components. Identifying the topological invariants of each component of Theorem 6.1 gives the following precise component count.

Corollary 6.3. *For $2 < p < q-1$ and $(a,b,c) \in H^1(S,\mathbb{Z}_2) \times H^2(S,\mathbb{Z}_2) \times H^2(S,\mathbb{Z}_2)$*

$$|\pi_0(\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q)))| = \begin{cases} 2 & \text{if } p \text{ is odd and } b = 0 \\ 2^{2g}+1 & \text{if } p \text{ is even, } a = 0 \text{ and } b = 0 \\ 1 & \text{otherwise} \ . \end{cases}$$

Remark 1.4. For $p = 1$ and $p = 2$, the primary topological invariants are slightly different. For $p = q$ and $p = q-1$, the connected component count of $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$ is different (see Corollaries 6.4 and

[1]This result was announced, without details, in [1]. We now provide the details of the proof.

6.5). For $p = q$ and $p = q-1$, all components had been previously detected in [28] and [12] respectively. Nevertheless, the nonexistence of additional components is new.

Another advantage of working on the Higgs bundle side of the NAH correspondence is that the Higgs bundles and their moduli spaces possess a rich structure that provides tools which are not readily available in the representation varieties. Two of these tools, which we exploit, are a real-valued proper function defined by the L^2-norm of the Higgs field, called the Hitchin function, and a natural holomorphic $\mathbb{C}^*$-action. These two tools are related since the critical points of the Hitchin function occur at fixed points of the $\mathbb{C}^*$-action. When the moduli space is smooth the Hitchin function is a perfect Morse-Bott function. While this is not the case in general, the properness of the Hitchin function nevertheless allows one to extract useful information about π_0 from the loci of local minima which, in turn, can be described using information about the corresponding $\mathbb{C}^*$-fixed points.

For many groups G the Hitchin function has no local minima other than those defining the mundane components (see for example [19, 32, 18]). In these cases, this approach yields enough information to completely count the components of $\mathcal{M}(\mathrm{G})$. Interestingly, this is not the case for $\mathrm{SO}(p,q)$; nevertheless, we are able to classify all the local minima. Even though the singularities in the space render this insufficient for completely determining the number of connected components of $\mathcal{M}(\mathrm{SO}(p,q))$, the classification of local minima plays a crucial role in the non-existence part of our main result, and the $\mathbb{C}^*$-fixed points are helpful in the proof of the main existence theorem. The new exotic components are detected by a more direct approach.

To show that the components exist, we first describe a model for the supposed components. We then construct a map from the model to $\mathcal{M}(\mathrm{SO}(p,q))$ and show that the map is open and closed. The description of the model invokes a variant of Higgs bundles in which the canonical bundle K is replaced by the p^{th} power of K.

Theorem 4.1. *Let X be a compact Riemann surface with genus $g \geqslant 2$ and canonical bundle K. Denote the moduli space of K^p-twisted $\mathrm{SO}(1,q-p+1)$-Higgs bundles on X by $\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))$ and the moduli space of K-twisted $\mathrm{SO}(p,q)$-Higgs bundles on X by $\mathcal{M}(\mathrm{SO}(p,q))$. For $1 \leqslant p \leqslant q$, there is a well defined map*

$$\Psi : \mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(X, K^{2j}) \longrightarrow \mathcal{M}(\mathrm{SO}(p,q)) \tag{1.2}$$

which is an isomorphism onto its image and has an open and closed image. Furthermore, if $p > 1$, then every Higgs bundle in the image of Ψ has a nowhere vanishing Higgs field.

In the case $p = 2$, the model described in this theorem coincides exactly with the description of the 'exotic' maximal components of $\mathcal{M}(\mathrm{SO}(2,q))$ (see [7, 5]), where the objects parameterized by the components are described by K^2-twisted Higgs bundles referred to as Cayley partners. In that setting, the emergence of the Cayley partners is a consequence of the fact that $\mathrm{SO}(2,q)$ is a group of Hermitian type; our new results for $\mathrm{SO}(p,q)$ with $p \geqslant 2$ show that the phenomenon has a more fundamental origin. In this regard, we note that our new components generalize both the afore-mentioned Cayley partners in the Hermitian case (i.e. for $p = 2$) and also the Hitchin components for the split real forms $\mathrm{SO}(p,p)$ and $\mathrm{SO}(p,p+1)$ (see Section 7.3 for more details).

A key technical detail required to show that the map (1.2) is open, is the fact that the spaces (both the model and its image under the map) are essentially smooth. This means that all points are either smooth points or mildly singular, thus allowing the use of Kuranishi's methods to describe open neighborhoods of all points. The proof of this key technical detail uses the relation between the tangent spaces for points in $\mathcal{M}(\mathrm{SO}(p,q))$ and hypercohomology spaces computed from a deformation complex. The deformation complex has three terms, with the first term coming from infinitesimal automorphisms and the third term encoding integrability obstructions. The crucial lemma establishes the vanishing of the second hypercohomology, i.e. of integrability obstructions for infinitesimal deformations. This is the first place where we exploit the natural $\mathbb{C}^*$-action on the moduli space. More precisely, it is the special structure of the fixed points of the action which allows us to prove the vanishing results for the deformation complexes at those points. We then use an upper-semicontinuity argument to extend the result to all points where it is needed. To show that the image of the map (1.2) is closed, the properness of the Hitchin fibration is exploited.

The non-existence result also relies heavily on the fixed points of the $\mathbb{C}^*$-action and on their relation to the critical points of the Hitchin function on $\mathcal{M}(\mathrm{SO}(p,q))$. The properness of this function implies that it attains its minimum on each connected component. The non-existence result thus follows from a careful analysis of all the $\mathbb{C}^*$-fixed points, most of which is devoted to identifying which fixed points correspond to local minima of the function. We show that these are of two types, namely those where the Higgs field is identically zero, and those which lie in the new exotic components. Since the former label the known 'mundane' components, this proves that we have not missed any components.

We now discuss a few consequences of our work for the $\mathrm{SO}(p,q)$-representation variety $\mathcal{R}(S,\mathrm{SO}(p,q))$. Recall that a representation $\rho:\pi_1(S)\to\mathrm{SO}_0(2,1)$ is called *Fuchsian* if it is discrete and faithful and that, since $\mathrm{SO}_0(p-1,p)$ is a split group of adjoint type, there is a unique principal embedding

$$\iota:\mathrm{SO}_0(2,1)\to\mathrm{SO}_0(p-1,p)\ . \tag{1.3}$$

One consequence of our techniques is a dichotomy for polystable $\mathrm{SO}(p,q)$-Higgs bundles (see Corollary 6.2). Translating this statement across the NAH correspondence leads to the following dichotomy for surface group representations into $\mathrm{SO}(p,q)$.

Theorem 7.6. *Let S be a closed surface of genus $g\geqslant 2$. For $2<p<q-1$, the representation variety $\mathcal{R}(\mathrm{SO}(p,q))$ of S is a disjoint union of two sets,*

$$\mathcal{R}(\mathrm{SO}(p,q))\ =\ \mathcal{R}^{cpt}(\mathrm{SO}(p,q))\ \sqcup\ \mathcal{R}^{ex}(\mathrm{SO}(p,q))\ , \tag{1.4}$$

where

- $[\rho]\in\mathcal{R}^{cpt}(\mathrm{SO}(p,q))$ *if and only if ρ can be continuously deformed to a compact representation,*
- $[\rho]\in\mathcal{R}^{ex}(\mathrm{SO}(p,q))$ *if and only if ρ can be continuously deformed to a representation*

$$\rho'=\alpha\oplus(\iota\circ\rho_{\mathrm{Fuch}})\otimes\det(\alpha)\ , \tag{1.5}$$

where α is a representation of $\pi_1(S)$ into the compact group $\mathrm{O}(q-p+1)$, ρ_{Fuch} is a Fuchsian representation of $\pi_1(S)$ into $\mathrm{SO}_0(2,1)$, and ι is the principal embedding from (1.3).

Remark 1.7. For $2<p=q-1$, the above theorem does not hold. Namely, there are exactly $2p(g-1)$ exotic components of $\mathcal{R}(S,\mathrm{SO}(p,p+1))$ for which the result fails. With the exception of the Hitchin component, in [12] it is conjectured that all representations in these components are Zariski dense.

It is Theorem 7.6 which connects our work on the Higgs bundle side of the NAH correspondence to the theory of Anosov representations. For a fixed parabolic subgroup $\mathrm{P}\subset\mathrm{G}$, a representation $\rho:\pi_1(S)\to\mathrm{G}$ is P-Anosov if there is an equivariant boundary curve

$$\xi_\rho:\partial_\infty\pi_1(S)\to\mathrm{G}/\mathrm{P}$$

from the Gromov boundary of $\pi_1(S)$ to the flag variety G/P with certain dynamical properties (see Definition 7.7). The set of P-Anosov representations defines an open set in the representation variety consisting of representations with desirable dynamic and geometric properties. In [25], Guichard and Wienhard show that for certain pairs (G,P), triples of transverse points in G/P admit a notion of being positively ordered. For such pairs (G,P), an Anosov representation is called *positive* if the boundary curve ξ_ρ takes positively oriented triples in $\partial_\infty\pi_1(S)$ to positively ordered triples in G/P.

The set of positive Anosov representations is open in $\mathcal{R}(S,\mathrm{G})$ and conjectured by Guichard and Wienhard to also be closed [25]. For the classical groups, the pairs (G,P) which admit a notion of positivity come in three families: G a split real form and P is the minimal parabolic subgroup, G a Hermitian group of tube type and G/P the Shilov boundary of the symmetric space, and $\mathrm{G}=\mathrm{SO}(p,q)$ with $p<q$ and P the stabilizer of the partial flag $V_1\subset V_2\subset\cdots\subset V_{p-1}$, where $V_j\subset\mathbb{R}^{p+q}$ is an isotropic j-plane. For the first two families the set of positive Anosov representations corresponds exactly to the connected components of Hitchin representations and maximal representations respectively; thus, for these families, positivity is indeed a closed condition. For the group $\mathrm{SO}(p,q)$, the conjecture is open.

However, it follows from the work of Guichard and Wienhard that the model representation (1.5) is a positive Anosov representation. Thus, as a corollary to Theorem 7.6 we have:

Proposition 7.13. *Let* $\mathrm{P} \subset \mathrm{SO}(p,q)$ *be the stabilizer of the partial flag* $V_1 \subset V_2 \subset \cdots \subset V_{p-1}$*, where* $V_j \subset \mathbb{R}^{p+q}$ *is an isotropic* j*-plane. If* $2 < p < q-1$*, then each connected component of* $\mathcal{R}^{ex}(\mathrm{SO}(p,q))$ *from* (1.4) *contains a nonempty open set of positive* P*-Anosov representations.*

For the group $\mathrm{SO}(p,q)$, we expect the exotic components described in this paper to correspond exactly to the positive Anosov $\mathrm{SO}(p,q)$-representations. Indeed, this would follow from Proposition 7.13 and a positive answer to the conjecture of Guichard and Wienhard.

Though our main results prove the existence of the first exotic components outside the realm of higher Teichmüller theory, evidence has been building for some time. The first indication came from the local minima of the Hitchin function described above. While the absolute minimum, i.e. the zero level, is attained on the components $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))_{\mathrm{top}}$, in [3] the first author described additional smooth local minima at non-zero values, thus opening up the possibility that further components exist.

The special case $q = p+1$ provided a further early indication of the phenomenon which we see more generally for any $q \geqslant p$, i.e. for the existence of additional exotic components. Hitchin components were known to exist in $\mathcal{M}(\mathrm{SO}(p,p+1))$ by virtue of the fact that the group $\mathrm{SO}(p,p+1)$ is the split real form of $\mathrm{SO}(2p+1,\mathbb{C})$. The results in [12] show that these are not the only exotic components. With the luxury of hindsight, we now see that the additional components in $\mathcal{M}(\mathrm{SO}(p,p+1))$ coincide exactly with the exotic components described by our main results for the case $q = p+1$.

We note finally that additional features of the connected components of $\mathcal{M}(\mathrm{SO}(p,q))$ have recently been detected by Baraglia and Schaposnik (in [4]) by examining spectral data on generic fibers of the Hitchin fibration for $\mathcal{M}(\mathrm{SO}(p+q,\mathbb{C}))$. Their methods cannot distinguish connected components because of the genericity assumption on the fibers, but, where they apply, their methods provide an intriguing alternative perspective.

2. Higgs bundle background

In this section we recall the necessary background on G-Higgs bundles on a compact Riemann surface and their deformation theory. Special attention is then placed on the group $\mathrm{SO}(p,q)$. Higgs bundles were introduced by Hitchin in [26] and Simpson in [38], and have been studied extensively by many authors. For real groups we will mostly follow [17]. For the rest of the paper, let X be a compact Riemann surface of genus $g \geqslant 2$ and with canonical bundle $K \to X$.

2.1. General Definitions.

Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$ and choose a maximal compact subgroup $\mathrm{H} \subset \mathrm{G}$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Fix a Cartan splitting $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$ with respect to a nondegenerate $\mathrm{Ad}(\mathrm{G})$-invariant bilinear form (which is taken to be the Killing form when G is semisimple). In particular, $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, thus such a splitting is preserved by the adjoint action of H on $\mathfrak{g}$, giving a linear representation $\mathrm{H} \to \mathrm{GL}(\mathfrak{m})$. Complexifying everything yields an $\mathrm{Ad}(\mathrm{H}^{\mathbb{C}})$-invariant splitting $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$.

For any group G, if P is a principal G-bundle and $\alpha : \mathrm{G} \to \mathrm{GL}(V)$ is a linear representation, denote the associated vector bundle $P \times_{\mathrm{G}} V$ by $P[V]$.

Definition 2.1. *Fix a* C^∞ *principal* $\mathrm{H}^{\mathbb{C}}$*-bundle* $P \to X$ *and a holomorphic line bundle* $L \to X$*. An* L-twisted G-Higgs bundle structure *on* P *is a pair* $(\mathcal{E},\varphi)$ *where* $\mathcal{E}$ *is a holomorphic principal* $\mathrm{H}^{\mathbb{C}}$*-bundle with underlying smooth bundle* P *and* $\varphi \in H^0(X, \mathcal{E}[\mathfrak{m}^{\mathbb{C}}] \otimes L)$ *is a holomorphic section of the associated* $\mathfrak{m}^{\mathbb{C}}$*-bundle twisted by* L*. The section* φ *is called the* Higgs field*.*

Remark 2.2. As usual, when the line bundle L is the canonical bundle K of the Riemann surface, we refer to a K-twisted Higgs bundle as a *Higgs bundle*. We are mainly interested in the case $L = K$, however, taking $L = K^p$ will also play an important role.

Example 2.3. Note that when G is a compact group, we have $\mathrm{G}^{\mathbb{C}} = \mathrm{H}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}} = 0$, so a G-Higgs bundle is just a holomorphic $\mathrm{G}^{\mathbb{C}}$-bundle on X. When G is a complex group, we have $\mathrm{G} = \mathrm{H}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}} \cong \mathfrak{g}$. In this case, the Higgs field is just an L-twisted section of the adjoint bundle.

Recall that a holomorphic structure on a C^∞ principal $\mathrm{H}^{\mathbb{C}}$-bundle P is equivalent to a Dolbeault operator $\bar{\partial}_P$, and that the space of such operators is an affine space modelled on $\Omega^{0,1}(X, P[\mathfrak{h}^{\mathbb{C}}])$. We denote the space of L-twisted Higgs bundle structures on P by

$$\mathcal{H}_L(\mathrm{G}, P) = \{(\bar{\partial}_P, \varphi) \mid \bar{\partial}_P \varphi = 0\}, \tag{2.1}$$

where $\varphi \in \Omega^0(X, P[\mathfrak{m}^{\mathbb{C}}] \otimes L)$ is the Higgs field. The set of Dolbeault operators is an affine space modelled on $\Omega^{0,1}(X, P[\mathfrak{h}^{\mathbb{C}}])$ so $\mathcal{H}_L(\mathrm{G}, P)$ can be identified with a subvariety of the vector space $\Omega^{0,1}(X, P[\mathfrak{h}^{\mathbb{C}}]) \times \Omega^0(X, P[\mathfrak{m}^{\mathbb{C}}] \otimes L)$.

Since we are concerned with classical groups, rather than dealing with principal bundles, we will use a linear representation $\alpha : \mathrm{H}^{\mathbb{C}} \to \mathrm{GL}(V)$ and work with vector bundles and sections of associated bundles. The standard representations of $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$ on $\mathbb{C}^n$ give the following vector bundle definitions, which are of course equivalent to their corresponding principal bundle formulations given by Definition 2.1.

Definition 2.4. *An* L-twisted $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle *on* X *is a pair* (E, Φ) *where* $E \to X$ *is a rank* n *holomorphic vector bundle and* $\Phi : E \to E \otimes L$ *is a holomorphic* L*-twisted endomorphism. If* $\Lambda^n E \cong \mathcal{O}$ *and* $\mathrm{tr}(\Phi) = 0$*, then* (E, Φ) *is an* L-twisted $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle*.*

Definition 2.5. *An* L-twisted $\mathrm{O}(n, \mathbb{C})$-Higgs bundle *is a triple* (E, Q, Φ) *where* (E, Φ) *is an* L*-twisted* $\mathrm{GL}(n, \mathbb{C})$*-Higgs bundle,* Q *is an everywhere nondegenerate holomorphic section of* $\mathrm{Sym}^2 E^*$ *such that* $\Phi^T Q + Q\Phi = 0$*, where we are considering* Q *as a symmetric isomorphism* $Q : E \to E^*$*. If* $\Lambda^n E \cong \mathcal{O}$*, then* (E, Q, Φ) *defines an* L-twisted $\mathrm{SO}(n, \mathbb{C})$-Higgs bundle*.*

The group $\mathrm{O}(p, q)$ is the group of linear automorphisms of $\mathbb{R}^{p+q}$ which preserve a nondegenerate symmetric quadratic form of signature (p, q). We are mainly interested in the subgroup $\mathrm{G} = \mathrm{SO}(p, q)$ of $\mathrm{O}(p, q)$ which also preserves an orientation of $\mathbb{R}^{p+q}$. The group $\mathrm{SO}(p, q)$ has two connected components and the connected component of the identity is denoted by $\mathrm{SO}_0(p, q)$.

If Q_p and Q_q are positive definite symmetric $p \times p$ and $q \times q$ matrices, then the Lie algebra $\mathfrak{so}(p, q)$ is defined by the matrices

$$\mathfrak{so}(p, q) \cong \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} Q_p & \\ & -Q_q \end{pmatrix} + \begin{pmatrix} Q_p & \\ & -Q_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \right\},$$

where A is a $p \times p$ matrix, B is a $p \times q$ matrix, C is a $q \times p$ matrix and D is a $q \times q$ matrix. Thus,

$$A^T Q_p + Q_p A = 0, \quad D^T Q_q + Q_q D = 0 \quad \text{and} \quad C = -Q_q^{-1} B^T Q_p\ . \tag{2.2}$$

The maximal compact subgroup of $\mathrm{O}(p, q)$ is $\mathrm{O}(p) \times \mathrm{O}(q)$ and the maximal compact subgroup of $\mathrm{SO}(p, q)$ is $\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$. Using (2.2), the complexified Cartan decomposition of $\mathfrak{so}(p, q)$ is

$$\mathfrak{so}(p + q, \mathbb{C}) \cong (\mathfrak{so}(p, \mathbb{C}) \oplus \mathfrak{so}(q, \mathbb{C})) \oplus \mathrm{Hom}(W, V),$$

where V and W are the standard representations of $\mathrm{O}(p, \mathbb{C})$ and $\mathrm{O}(q, \mathbb{C})$. Using these representations, we have the following vector bundle definition of an $\mathrm{SO}(p, q)$-Higgs bundle.

Definition 2.6. *An* L-twisted $\mathrm{O}(p, q)$-Higgs bundle *on* X *is a tuple* (V, Q_V, W, Q_W, η) *where*

- V *and* W *are respectively rank* p *and* q *holomorphic vector bundles on* X*,* Q_V *and* Q_W *are respectively everywhere nondegenerate holomorphic section of* $\mathrm{Sym}^2 V^*$ *and* $\mathrm{Sym}^2 W^*$*,*
- $\eta : W \to V \otimes L$ *is a holomorphic section of* $\mathrm{Hom}(W, V) \otimes L$*.*

An L-twisted $\mathrm{SO}(p, q)$-Higgs bundle *is an* L*-twisted* $\mathrm{O}(p, q)$*-Higgs bundle* (V, Q_V, W, Q_W, η) *with the extra condition* $\Lambda^p(V) \cong \Lambda^q(W)$*. Finally, an* L-twisted $\mathrm{SO}_0(p, q)$-Higgs bundle *is an* L*-twisted* $\mathrm{SO}(p, q)$*-Higgs bundle* (V, Q_V, W, Q_W, η) *such that* $\Lambda^p(V) \cong \mathcal{O} \cong \Lambda^q(W)$*.*

Remark 2.7. We will usually interpret the orthogonal structures as symmetric isomorphisms:

$$Q_V : V \xrightarrow{\cong} V^* \qquad \text{and} \qquad Q_W : W \xrightarrow{\cong} W^* .$$

Moreover, when the orthogonal structures are clear, we will omit them from the notation.

Example 2.8. For $p = 1$, an L-twisted $\mathrm{SO}(1, q)$-Higgs bundle is a tuple (V, W, η) where V is a holomorphic line bundle $I \cong \Lambda^q W$ with $I^2 \cong \mathcal{O}$ and $\eta \in H^0(\mathrm{Hom}(W, I) \otimes L)$. In particular, an L-twisted $\mathrm{SO}(1, n)$-Higgs bundle is determined by (W, η).

Given an $\mathrm{SO}(p, q)$-Higgs bundle (V, Q_V, W, Q_W, η), let

$$\eta^* = Q_W^{-1} \eta^T Q_V.$$

The L-twisted $\mathrm{SO}(p + q, \mathbb{C})$-Higgs bundle associated to (V, Q_V, W, Q_W, η) is given by

$$(E, Q, \Phi) = \left(V \oplus W, \begin{pmatrix} Q_V & 0 \\ 0 & -Q_W \end{pmatrix}, \begin{pmatrix} 0 & \eta \\ \eta^* & 0 \end{pmatrix} \right) . \tag{2.3}$$

In subsequent sections, we will also need to the notions of $\mathrm{U}(p, q)$-Higgs bundles and $\mathrm{GL}(n, \mathbb{R})$-Higgs bundles. The complexified Cartan decompositions for these groups are given by

$$\mathfrak{u}(p, q)^{\mathbb{C}} \cong (\mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C})) \oplus (\mathrm{Hom}(E, F) \oplus \mathrm{Hom}(F, E))$$
$$\mathfrak{gl}(n, \mathbb{R})^{\mathbb{C}} \cong \mathfrak{o}(n, \mathbb{C}) \oplus \mathrm{sym}(\mathbb{C}^n),$$

where E and F are respectively the standard representations of $\mathrm{GL}(p, \mathbb{C})$ and $\mathrm{GL}(q, \mathbb{C})$ and $\mathrm{sym}(\mathbb{C}^n)$ denotes the set of symmetric endomorphisms of $\mathbb{C}^n$. As above, we have the following vector bundle definitions of the associated Higgs bundles.

Definition 2.9. *An* L-twisted $\mathrm{U}(p, q)$-Higgs bundle *on* X *is a tuple* (E, F, β, γ) *where*

- E *and* F *are holomorphic vector bundles on* X*, of rank* p *and* q *respectively;*
- $\beta \in H^0(\mathrm{Hom}(F, E) \otimes L)$ *and* $\gamma \in H^0(\mathrm{Hom}(E, F) \otimes L)$*.*

An L-twisted $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle *on* X *is a tuple* (E, Q, Φ) *where*

- E *is a rank* n *holomorphic vector bundle on* X *and* Q *is a everywhere nondegenerate holomorphic section of* $\mathrm{Sym}^2 E^*$*;*
- $\Phi \in H^0(\mathrm{End}(E) \otimes L)$ *such that* $\Phi^T Q = Q\Phi$*.*

If $\Lambda^n E \cong \mathcal{O}$ *and* $\mathrm{tr}(\Phi) = 0$*, then* (E, Q, Φ) *is an* L-twisted $\mathrm{SL}(n, \mathbb{R})$-Higgs bundle*.*

2.2. **The Higgs bundle moduli space and deformation theory.** To form a moduli space of G-Higgs bundles we need a notion of stability for these objects. In general, these stability notions involve the interaction of the Higgs field with certain parabolic reductions of structure group (see [17]). For the above groups stability can be simplified and expressed in vector bundle terms in the following way (see [17]).

Proposition 2.10. *An* L*-twisted* $\mathrm{SL}(n, \mathbb{C})$*-Higgs bundle* (E, Φ) *is*

- *semistable if for every holomorphic subbundle* $F \subset E$ *with* $\Phi(F) \subset F \otimes L$ *we have* $\deg(F) \leqslant 0$*,*
- *stable if for every proper holomorphic subbundle* $F \subset E$ *with* $\Phi(F) \subset F \otimes L$ *we have* $\deg(F) < 0$*,*
- *polystable if it is semistable and for every degree zero subbundle* $F \subset E$ *with* $\Phi(F) \subset F \otimes L$*, there is a subbundle* F' *with* $\Phi(F') \subset F' \otimes L$ *so that* $E = F \oplus F'$*. That is,*

$$(E, \Phi) = \left(F \oplus F', \begin{pmatrix} \Phi_F & 0 \\ 0 & \Phi_{F'} \end{pmatrix} \right) .$$

Remark 2.11. For the notions of stability, semistability and polystability for an L-twisted $\mathrm{O}(n, \mathbb{C})$-Higgs bundles (E, Q, Φ), one only needs to consider *isotropic* subbundles $F \subset E$ with $\Phi(F) \subset F \otimes K$ (see for example [17]). Here a subbundle $F \subset E$ is isotropic if $F \subset F^{\perp}$ where $F^{\perp}$ is the perpendicular subbundle defined by Q. For a polystable L-twisted $\mathrm{O}(n, \mathbb{C})$-Higgs bundle, if $F \subset E$ is a zero degree isotropic subbundle with $\Phi(F) \subset F \otimes L$, then $E \cong F \oplus F'$ where F' is a degree zero coisotropic subbundle satisfying $\Phi(F') \subset F' \otimes L$. We note also that the polystability of (E, Q) as an orthogonal vector bundle is equivalent to the polystability of W as a vector bundle [34].

For real groups, the notions of semistability, stability and polystability are a bit more involved. However, to define the moduli spaces we are interested in we may use the following result of [17].

Proposition 2.12. *Let* G *be a real form of a simple subgroup of* $\mathrm{SL}(n,\mathbb{C})$. *An* L*-twisted* G*-Higgs bundle* $(\mathcal{E},\varphi)$ *is polystable if and only if the induced* $\mathrm{SL}(n,\mathbb{C})$*-Higgs bundle is polystable in the sense of Proposition 2.10.*

The gauge group $\mathcal{G}_{\mathrm{H}^\mathbb{C}}$ of C^∞ bundle automorphisms of a smooth $\mathrm{H}^\mathbb{C}$-bundle $P_{\mathrm{H}^\mathbb{C}}$ acts on the space $\mathcal{H}_L(\mathrm{G},P)$ of L-twisted Higgs bundle structures from (2.1). Moreover, this action preserves the subspace $\mathcal{H}_L(\mathrm{G},P)^{ps}\subset\mathcal{H}_L(\mathrm{G},P)$ of polystable L-twisted Higgs bundles and the orbits of the $\mathcal{G}_{\mathrm{H}^\mathbb{C}}$-action on $\mathcal{H}_L(\mathrm{G},P)^{ps}$ are closed.

If (V,Q_V) and (W,Q_W) are respectively rank p and rank q orthogonal vector bundles with $\Lambda^pV\cong\Lambda^qW$, then the $\mathrm{S}(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C}))$-gauge group consists of pairs (g_V,g_W), where g_V and g_W are smooth automorphisms of V and W such that

$$g_V^TQ_Vg_V=Q_V\ ,\qquad g_W^TQ_Wg_W=Q_W\qquad\text{and}\qquad\det(g_V)\otimes\det(g_W)=\mathrm{Id}\ .$$

Such a gauge transformation acts on the data (V,W,η) by

$$(g_V,g_W)\cdot(\bar\partial_V,\bar\partial_W,\eta)=(g_V\bar\partial_Vg_V^{-1},g_W\bar\partial_Wg_W^{-1},g_V\eta g_W^{-1})\ .$$

Definition 2.13. *Fix a smooth principal* $\mathrm{H}^\mathbb{C}$*-bundle* $P_{\mathrm{H}^\mathbb{C}}$ *and a holomorphic line bundle* L *on* X. *The* moduli space $\mathcal{M}_L(P_{\mathrm{H}^\mathbb{C}},\mathrm{G})$ of L-twisted G-Higgs bundle structures on $P_{\mathrm{H}^\mathbb{C}}$ *consists of isomorphism classes of polystable* L*-twisted Higgs bundles with underlying smooth bundle* $P_{\mathrm{H}^\mathbb{C}}$,

$$\mathcal{M}_L(P_{\mathrm{H}^\mathbb{C}},\mathrm{G})=\mathcal{H}_L(P_{\mathrm{H}^\mathbb{C}},\mathrm{G})^{ps}/\mathcal{G}_{\mathrm{H}^\mathbb{C}}\ .$$

The union over the set of isomorphism classes of smooth principal $\mathrm{H}^\mathbb{C}$*-bundles on* X *of the spaces* $\mathcal{M}_L(P_{\mathrm{H}^\mathbb{C}},\mathrm{G})$ *will be referred to as the* moduli space of L-twisted G-Higgs bundles *and denoted by* $\mathcal{M}_L(\mathrm{G})$.

In the case $L=K$, we shall denote the corresponding moduli space just by $\mathcal{M}(\mathrm{G})$.

Remark 2.14. The moduli space $\mathcal{M}_L(\mathrm{G})$ of L-twisted G-Higgs bundles can also be constructed as the set of $\mathcal{S}$-equivalence classes of semistable G-Higgs bundles. Such a construction is a particular case of a construction of Schmitt [36] using geometric invariant theory. In particular, $\mathcal{M}_L(\mathrm{G})$ is naturally a complex algebraic variety. Suppose G is such that its maximal compact subgroup $\mathrm{H}\subset\mathrm{G}$ is semisimple. Then, for $L=K$, the expected dimension of $\mathcal{M}(\mathrm{G})$ is $\dim(\mathrm{G})(g-1)$, while for L such that $\deg(L)>2g-2$, the expected dimension of $\mathcal{M}_L(\mathrm{G})$ is $\dim(\mathfrak{h})(g-1)+\dim(\mathfrak{m})(\deg(L)+1-g)$, where we recall that $\mathfrak{h}\oplus\mathfrak{m}$ is a Cartan decomposition of the Lie algebra of G. If the maximal compact $\mathrm{H}\subset\mathrm{G}$ is only reductive, then the expected dimension is obtained by adding $\dim(Z(\mathfrak{h})\cap\ker\mathrm{ad})$ to both formulas, where $\mathrm{ad}:\mathfrak{h}\to\mathrm{End}(\mathfrak{m})$ is the map induced by the linear representation $\mathrm{Ad}:\mathrm{H}\to\mathrm{GL}(\mathfrak{m})$. See [26, 39, 36, 33, 17].

Remark 2.15. The moduli space of L-twisted G-Higgs bundles is homeomorphic to the moduli space of solutions to the Hitchin self-duality equations. These are equations for a reduction of structure group of the bundle to a maximal compact subgroup (see [17]). When $L=K$, this solution is sometimes referred as a harmonic metric.

The automorphism group $\mathrm{Aut}(\mathcal{E},\varphi)$ of a G-Higgs bundle $(\mathcal{E},\varphi)$ is defined by

$$\mathrm{Aut}(\mathcal{E},\varphi)=\{g\in\mathcal{G}_{\mathrm{H}^\mathbb{C}}|\ (\mathrm{Ad}_g\,\bar\partial_{\mathcal{E}},\mathrm{Ad}_g\,\varphi)=(\bar\partial_{\mathcal{E}},\varphi)\}.\tag{2.4}$$

The center $\mathcal{Z}(\mathrm{G}^\mathbb{C})$ of $\mathrm{G}^\mathbb{C}$ is the intersection of the center of $\mathrm{H}^\mathbb{C}$ and the kernel of the representation $\mathrm{Ad}:\mathrm{H}^\mathbb{C}\to\mathrm{GL}(\mathfrak{m}^\mathbb{C})$. Thus, we always have $\mathcal{Z}(\mathrm{G}^\mathbb{C})\subset\mathrm{Aut}(\bar\partial_{\mathcal{E}},\varphi)$.

Remark 2.16. If $\mathrm{G}^\mathbb{C}$ is semisimple, then a G-Higgs bundle is *stable* if it is polystable with finite automorphism group. In particular, if $\mathcal{H}_L^s(P,\mathrm{G})\subset\mathcal{H}_L^{ps}(P,\mathrm{G})$ denotes the subset of stable Higgs bundle structures, then $\mathcal{H}_L^s(P,\mathrm{G})$ is *open* in $\mathcal{H}_L^{ps}(P,\mathrm{G})$.

Given a polystable G-Higgs bundle $(\mathcal{E},\varphi)$, consider the complex of sheaves

$$C^\bullet(\mathcal{E},\varphi):\ \mathcal{E}[\mathfrak{h}^\mathbb{C}]\xrightarrow{\mathrm{ad}_\varphi}\mathcal{E}[\mathfrak{m}^\mathbb{C}]\otimes L. \tag{2.5}$$

This gives a long exact sequence in hypercohomology:

$$\begin{aligned}0\longrightarrow\mathbb{H}^0(C^\bullet(\mathcal{E},\varphi))\longrightarrow H^0(\mathcal{E}[\mathfrak{h}^\mathbb{C}])\xrightarrow{\mathrm{ad}_\varphi}H^0(\mathcal{E}[\mathfrak{m}^\mathbb{C}]\otimes L)\longrightarrow\mathbb{H}^1(C^\bullet(\mathcal{E},\varphi))\\ \longrightarrow H^1(\mathcal{E}[\mathfrak{h}^\mathbb{C}])\xrightarrow{\mathrm{ad}_\varphi}H^1(\mathcal{E}[\mathfrak{m}^\mathbb{C}]\otimes L)\longrightarrow\mathbb{H}^2(C^\bullet(\mathcal{E},\varphi))\longrightarrow 0\ .\end{aligned} \tag{2.6}$$

Remark 2.17. When the group G is complex, Serre duality implies that the second hypercohomology group in this deformation complex is isomorphic to the dual of the zeroth hypercohomology group [17, Proposition 3.17]. In particular, this implies that for, complex semisimple groups, $\mathbb{H}^2(C^\bullet(\mathcal{E},\varphi))$ vanishes if and only if the Higgs bundle $(\mathcal{E},\varphi)$ is stable.

Note that the automorphism group $\mathrm{Aut}(\mathcal{E},\varphi)$ acts on $\mathbb{H}^1(C^\bullet(\mathcal{E},\varphi))$. Using standard slice methods of Kuranishi (see [30, Chapter 7.3] for details for the moduli space of holomorphic bundles), a neighborhood of the isomorphism class of a polystable Higgs bundle $(\mathcal{E},\varphi)$ in $\mathcal{M}_L(\mathrm{G})$ is given by

$$\kappa^{-1}(0)\,/\!\!/\,\mathrm{Aut}(\mathcal{E},\varphi)$$

where $\kappa:\mathbb{H}^1(C^\bullet(\mathcal{E},\varphi))\to\mathbb{H}^2(C^\bullet(\mathcal{E},\varphi))$ is the so called Kuranishi map.

When $\mathbb{H}^2(C^\bullet(\mathcal{E},\varphi))=0$, this simplifies considerably. Namely, in this case, a neighborhood of the isomorphism class of a polystable Higgs bundle $(\mathcal{E},\varphi)$ in $\mathcal{M}_L(\mathrm{G})$ is isomorphic to

$$\mathbb{H}^1(C^\bullet(\mathcal{E},\varphi))\,/\!\!/\,\mathrm{Aut}(\mathcal{E},\varphi)\ .$$

When the automorphism group $\mathrm{Aut}(\mathcal{E},\varphi)$ is finite, the GIT quotient above simplifies to a regular quotient, and the isomorphism class $(\mathcal{E},\varphi)$ defines at most an orbifold point of $\mathcal{M}_L(\mathrm{G})$.

Remark 2.18. For all of the $\mathrm{SO}(p,q)$-Higgs bundles considered in the subsequent sections we will prove that the relevant $\mathbb{H}^2$ always vanishes. Thus, we will not recall the construction of the Kuranishi map.

2.3. **Stability and deformation complex for** $\mathrm{G}=\mathrm{SO}(p,q)$**.** We shall need the precise notion of stability $\mathrm{SO}(p,q)$-Higgs bundles. The derivation of the following simplification of the stability notion for $\mathrm{SO}(p,q)$-Higgs bundles is very similar to many cases treated in the literature. For example, see [19] for the case $\mathrm{G}=\mathrm{Sp}(2p,2q)$.

Proposition 2.19. *Let (V,Q_V,W,Q_W,η) be an L-twisted $\mathrm{SO}(p,q)$-Higgs bundle and let $\eta^*=Q_W^{-1}\eta^TQ_V$. Then it is*

- *semistable if for any pair of isotropic subbundles $V_1\subset V$ and $W_1\subset W$ such that $\eta(W_1)\subset V_1\otimes L$ and $\eta^*(V_1)\subset W_1\otimes L$ we have $\deg(V_1)+\deg(W_1)\leqslant 0$,*
- *stable if for any pair of isotropic subbundles $V_1\subset V$ and $W_1\subset W$ such that $\eta(W_1)\subset V_1\otimes L$ and $\eta^*(V_1)\subset W_1\otimes L$ we have $\deg(V_1)+\deg(W_1)<0$,*
- *polystable if it is semistable and whenever $V_1\subset V$ and $W_1\subset W$ are isotropic subbundles of with $\eta(W_1)\subset V_1\otimes L$, $\eta^*(V_1)\subset W_1\otimes L$ and $\deg(V_1)+\deg(W_1)=0$, there are coisotropic bundles $V_2\subset V$ and $W_2\subset W$ so that $\eta(W_2)\subset V_2\otimes L$ and $\eta^*(V_2)\subset W_2\otimes L$. That is,*

$$(V,W,\eta)=\Big(V_1\oplus V_2,W_1\oplus W_2,\left(\begin{smallmatrix}\eta_1&0\\0&\eta_2\end{smallmatrix}\right)\Big)\ .$$

We now give a recursive classification of strictly polystable $\mathrm{SO}(p,q)$-Higgs bundles, which will be important in the following sections of the paper.

Given a $\mathrm{U}(p,q)$-Higgs bundle (E,F,β,γ) with $\deg(E\oplus F)=0$, consider the associated $\mathrm{SO}(2p,2q)$-Higgs bundle

$$(V,Q_V,W,Q_W,\eta)=\Big(E\oplus E^*,\left(\begin{smallmatrix}0&\mathrm{Id}\\\mathrm{Id}&0\end{smallmatrix}\right),F\oplus F^*,\left(\begin{smallmatrix}0&\mathrm{Id}\\\mathrm{Id}&0\end{smallmatrix}\right),\left(\begin{smallmatrix}\beta&0\\0&\gamma^T\end{smallmatrix}\right)\Big)\ .$$

If (E,F,β,γ) is a polystable $\mathrm{U}(p,q)$-Higgs bundle, then this $\mathrm{SO}(2p,2q)$-Higgs bundle is strictly polystable. Indeed, E, E^*, F and F^* are all isotropic subbundles with $\deg(E)+\deg(F)=0$ and

$$\eta(F)\subset E\otimes K,\qquad \eta(F^*)\subset E^*\otimes K,\qquad \eta^*(E)\subset F\otimes K,\qquad\text{and}\qquad \eta^*(E^*)\subset F^*\otimes K.$$

Proposition 2.20. *An* $\mathrm{SO}(p,q)$*-Higgs bundle* (V,Q_V,W,Q_W,η) *is polystable if and only if it is isomorphic to*

$$\left(E\oplus E^*\oplus V_0,\begin{pmatrix}0&\mathrm{Id}&0\\ \mathrm{Id}&0&0\\ 0&0&Q_{V_0}\end{pmatrix},F\oplus F^*\oplus W_0,\begin{pmatrix}0&\mathrm{Id}&0\\ \mathrm{Id}&0&0\\ 0&0&Q_{W_0}\end{pmatrix},\begin{pmatrix}\beta&0&0\\ 0&\gamma^T&0\\ 0&0&\eta_0\end{pmatrix}\right), \tag{2.7}$$

where (E,F,β,γ) *is a polystable* $\mathrm{U}(p_1,q_1)$*-Higgs bundle and* $(V_0,Q_{V_0},W_0,Q_{W_0},\eta_0)$ *is a stable* $\mathrm{SO}(p-2p_1,q-2q_1)$*-Higgs bundle.*

Proof. Let (V,W,η) be a strictly polystable $\mathrm{SO}(p,q)$-Higgs bundle and suppose $E\subset V$ and $F\subset W$ are isotropic subbundles of rank p_1 and q_1 respectively, such that $\deg(E)+\deg(F)=0$ and

$$\eta(F)\subset E\otimes K\qquad\text{and}\qquad \eta^*(E)\subset F\otimes K\ .$$

Since (V,W,η) is polystable, the bundles V and W split as $V=E\oplus V'$ and $W=F\oplus W'$ where V' and W' are both coisotropic subbundles with the property

$$\eta(W')\subset V'\otimes K\qquad\text{and}\qquad \eta^*(V')\subset W'\otimes K\ .$$

Since the bundles E and F are isotropic, the bundles V' and W' are extensions of the form:

$$0\longrightarrow E^\perp/E\longrightarrow V'\longrightarrow E^*\longrightarrow 0\qquad\text{and}\qquad 0\longrightarrow F^\perp/F\longrightarrow W'\longrightarrow F^*\longrightarrow 0\ .$$

We claim that the above extension classes vanish. For the bundle V we have a holomorphic splitting $E\oplus V'$ and a smooth splitting $E\oplus E^\perp/E\oplus E^*$. In this smooth splitting, the orthogonal structure Q_V and the $\bar\partial$-operator on V are isomorphic to

$$Q_V\cong\begin{pmatrix}0&0&\mathrm{Id}\\ 0&Q_{E^\perp/E}&0\\ \mathrm{Id}&0&0\end{pmatrix}\qquad\text{and}\qquad \bar\partial_V\cong\begin{pmatrix}\bar\partial_E&0&0\\ 0&\bar\partial_{E^\perp/E}&\alpha\\ 0&0&\bar\partial_{E^*}\end{pmatrix},$$

where $\alpha\in\Omega^{0,1}(\mathrm{Hom}(E^*,E^\perp/E))$. However, since the orthogonal structure Q_V is holomorphic, we have $\alpha=0$. By applying the same argument to the bundle W, we have the following holomorphic splitting

$$(W,Q_W)\cong\left(F\oplus F^\perp/F\oplus F^*,\begin{pmatrix}0&0&\mathrm{Id}\\ 0&Q_{F^\perp/F}&0\\ \mathrm{Id}&0&0\end{pmatrix}\right).$$

The conditions $\eta(F)\subset E\otimes K$, $\eta^*(E)\subset F\otimes K$ and $\eta(W')\subset V'\otimes K$ imply that η is given by

$$\eta=\begin{pmatrix}\beta&0&0\\ 0&\eta_0&0\\ 0&0&\gamma^T\end{pmatrix}:F\oplus F^\perp/F\oplus F^*\longrightarrow E\oplus E^\perp/E\oplus E^*.$$

The tuple (E,F,β,γ) defines a polystable $\mathrm{U}(p_1,q_1)$-Higgs bundle and

$$(V_0,Q_{V_0},W_0,Q_{W_0},\eta_0)=(E^\perp/E,Q_{E^\perp/E},F^\perp/F,Q_{F^\perp/F},\eta_0)$$

defines a polystable $\mathrm{SO}(p-2p_1,q-2q_1)$-Higgs bundle. By iterating this process if necessary, we may assume (V_0,W_0,η_0) is a stable $\mathrm{SO}(p-2p_1,q-2q_1)$-Higgs bundle. □

For the group $\mathrm{SO}(p,q)$ we have that the complexified Lie algebra of its maximal compact subgroup is $\mathfrak{h}^{\mathbb{C}}\cong\mathfrak{so}(p,\mathbb{C})\oplus\mathfrak{so}(q,\mathbb{C})$. If $(\mathcal{E},\varphi)$ is an L-twisted $\mathrm{SO}(p,q)$-Higgs bundle in the sense of Definition 2.1, let (V,Q_V,W,Q_W,η) denote the associated L-twisted $\mathrm{SO}(p,q)$ in the sense of Definition 2.6. Write

$$\mathfrak{so}(V)=\{\alpha\in\mathrm{End}(V)\mid\alpha^TQ_V+Q_V\alpha=0\}\qquad\text{and}\qquad\mathfrak{so}(W)=\{\beta\in\mathrm{End}(W)\mid\beta^TQ_W+Q_W\beta=0\}.$$

Then the bundles $\mathcal{E}[\mathfrak{h}^{\mathbb{C}}]$ and $\mathcal{E}[\mathfrak{m}^{\mathbb{C}}]\otimes L$ are given by

$$\mathcal{E}[\mathfrak{so}(p,\mathbb{C})\oplus\mathfrak{so}(q,\mathbb{C})]\cong\mathfrak{so}(V)\oplus\mathfrak{so}(W)\qquad\text{and}\qquad\mathcal{E}[\mathfrak{m}^{\mathbb{C}}]\otimes L\cong\mathrm{Hom}(W,V)\otimes L.$$

The deformation complex (2.5) is given by

(2.8)
$$C^\bullet(V,W,\eta):\ \mathfrak{so}(V)\oplus\mathfrak{so}(W)\xrightarrow{\ \mathrm{ad}_\eta\ }\mathrm{Hom}(W,V)\otimes L\ ,\qquad (\alpha,\beta)\longmapsto \eta\otimes\beta-(\alpha\otimes\mathrm{Id}_L)\otimes\eta$$

and the long exact sequence (2.6) is given by

(2.9)
$$0\longrightarrow\mathbb{H}^0(C^\bullet(V,W,\eta))\longrightarrow H^0(\mathfrak{so}(V)\oplus\mathfrak{so}(W))\xrightarrow{\mathrm{ad}_\eta}H^0(\mathrm{Hom}(W,V)\otimes L)\to\mathbb{H}^1(C^\bullet(V,W,\eta))$$
$$\to H^1(\mathfrak{so}(V)\oplus\mathfrak{so}(W))\xrightarrow{\mathrm{ad}_\eta}H^1(\mathrm{Hom}(W,V)\otimes L)\longrightarrow\mathbb{H}^2(C^\bullet(V,W,\eta))\longrightarrow 0\ .$$

We will use the above complex and long exact sequence extensively throughout the paper.

2.4. **The Hitchin fibration and Hitchin component.** Let $\mathrm{G}^\mathbb{C}$ be a complex semisimple Lie group of rank ℓ and let $p_1,\dots,p_\ell$ be a basis of $\mathrm{G}^\mathbb{C}$-invariant homogeneous polynomials on $\mathfrak{g}^\mathbb{C}$ with $\deg(p_j)=m_j+1$. Given an L-twisted $\mathrm{G}^\mathbb{C}$-Higgs bundle $(\mathcal{E},\varphi)$, the tensor $p_j(\varphi)$ is a holomorphic section of L^{m_j+1}. The map $(\mathcal{E},\varphi)\mapsto(p_1(\varphi),\dots,p_\ell(\varphi))$ descends to a map

(2.10)
$$h:\ \mathcal{M}_L(\mathrm{G}^\mathbb{C})\longrightarrow\bigoplus_{j=1}^{\ell}H^0(L^{m_j+1})$$

known as the Hitchin fibration. In [27], Hitchin showed that h is a *proper* map for $L=K$, and for general L properness was shown by Nitsure in [33]. The properness of the Hitchin fibration will play a key role in Section 4.

Another important aspect of the Hitchin fibration for this paper is the Hitchin section.

Theorem 2.21. *(Hitchin [28]) Let* G *be the split real form of a complex semisimple Lie group* $\mathrm{G}^\mathbb{C}$ *of rank* ℓ*. There is a section* s_H *of the fibration (2.10) with* $L=K$*, such that the image of* s_H *consists of* G*-Higgs bundles. Moreover, the map*

$$s_H:\bigoplus_{j=1}^{\ell}H^0(K^{m_j+1})\to\mathcal{M}(\mathrm{G})$$

maps the vector space $\bigoplus_{j=1}^{\ell}H^0(K^{m_j+1})$ *homeomorphically onto a connected component of* $\mathcal{M}(\mathrm{G})$*.*

Remark 2.22. For a split real group G, a connected component of $\mathcal{M}_K(\mathrm{G})$ described by Theorem 2.21 is called a Hitchin component. When $\mathrm{G}^\mathbb{C}$ is an adjoint group, their is exactly one Hitchin component. Since the Hitchin component is smooth, the automorphism group of a Higgs bundle in a Hitchin component is as small as possible. For $\mathrm{O}(p,p-1)$, it is given by $\pm(\mathrm{Id}_V,\mathrm{Id}_W)$.

We now describe an explicit construction of such sections for $\mathrm{G}^\mathbb{C}=\mathrm{O}(2p-1,\mathbb{C})$. This construction will be used in Section 4. We will construct one such section s_H^I for each choice of a holomorphic line bundle I with $I^2\cong\mathcal{O}$. In this case, the rank is $p-1$, the integers m_j+1 equal to $2j$ and the split real form is isomorphic to $\mathrm{O}(p,p-1)$. Therefore the Hitchin section is given by

$$s_H^I:\bigoplus_{j=1}^{p-1}H^0(K^{2j})\to\mathcal{M}(\mathrm{O}(2p-1,\mathbb{C})).$$

For each n, consider the holomorphic orthogonal bundle

(2.11)
$$(\mathcal{K}_n,Q_n)=\left(K^n\oplus K^{n-2}\oplus\cdots\oplus K^{2-n}\oplus K^{-n},\begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}\right).$$

For $(q_2,\ldots,q_{2p-2})\in\bigoplus_{j=1}^{p-1}H^0(K^{2j})$, the $\mathrm{O}(p,p-1)$-Higgs bundle (V,Q_V,W,Q_W,η) in the image of a Hitchin section s_H^I is given by

$$s_H^I(q_2,\ldots,q_{2p-2})=(I\otimes\mathcal{K}_{p-1},Q_{p-1},I\otimes\mathcal{K}_{p-2},Q_{p-2},\eta(q_2,\ldots,q_{2p-2})), \tag{2.12}$$

where $\eta(q_2,\ldots,q_{2p-2})$ depends on a choice of the basis of invariant polynomials. Notice that, in particular, the holomorphic structures on $V=I\otimes\mathcal{K}_{p-1}$ and $W=I\otimes\mathcal{K}_{p-2}$ are fixed. One choice for $\eta(q_2,\ldots,q_{2p-2})$ is given by

$$\eta(q_2,\ldots,q_{2p-2})=\begin{pmatrix} q_2 & q_4 & \cdots & q_{2p-2}\\ 1 & q_2 & \cdots & q_{2p-4}\\ & \ddots & \ddots & \\ & & 1 & q_2\\ & & & 1\end{pmatrix}:\ I\otimes\mathcal{K}_{p-2}\longrightarrow I\otimes\mathcal{K}_{p-1}\otimes K\ . \tag{2.13}$$

For example, when $p=3$ we have

$$(V,Q_V,W,Q_W,\eta(q_2,q_4))=\left(IK^2\oplus I\oplus IK^{-2},\left(\begin{smallmatrix} & & 1\\ & 1 & \\ 1 & & \end{smallmatrix}\right),IK\oplus IK^{-1},\left(\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right),\left(\begin{smallmatrix} q_2 & q_4\\ 1 & q_2\\ 0 & 1\end{smallmatrix}\right)\right)\ .$$

Using (2.3), the associated $\mathrm{O}(5,\mathbb{C})$-Higgs bundle is given by

$$(E,Q,\Phi)=\left(IK^2\oplus I\oplus IK^{-2}\oplus IK\oplus IK^{-1},\left(\begin{smallmatrix} & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ & & & & -1\\ & & & -1 & \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 & 0 & 0 & q_2 & q_4\\ 0 & 0 & 0 & 1 & q_2\\ 0 & 0 & 0 & 0 & 1\\ 1 & q_2 & q_4 & 0 & 0\\ 0 & 1 & q_2 & 0 & 0\end{smallmatrix}\right)\right)\ .$$

One computes that $\mathrm{tr}(\Phi^2)=8q_2$ and $\mathrm{tr}(\Phi^4)=20q_2^2+8q_4$, so the above description describes the Hitchin section for the basis $p_1(\Phi)=\frac{1}{8}\mathrm{tr}(\Phi^2)$ and $p_2=\frac{1}{8}\mathrm{tr}(\Phi^4)-\frac{20}{64}(\mathrm{tr}(\Phi^2))^2$.

2.5. **Topological invariants.** Since $\mathrm{H}^{\mathbb{C}}$ and G are both homotopy equivalent to H, the set of equivalence classes of topological $\mathrm{H}^{\mathbb{C}}$-bundles on X is the same as the set of equivalence classes of topological G-bundles on X. Denote this set by $\mathrm{Bun}_X(\mathrm{G})$. This gives a decomposition of the Higgs bundle moduli space:

$$\mathcal{M}_L(\mathrm{G})=\coprod_{a\in\mathrm{Bun}_X(\mathrm{G})}\mathcal{M}_L^a(\mathrm{G})\ ,$$

where $a\in\mathrm{Bun}_X(\mathrm{G})$ is the topological type of the underlying $\mathrm{H}^{\mathbb{C}}$-bundle of the Higgs bundles in $\mathcal{M}_L^a(\mathrm{G})$.

In general, the number of connected components of the moduli space of K-twisted G-Higgs for a simple Lie group G has not been established. However, there have been many partial results. For instance, when G is *compact* and semisimple, the spaces $\mathcal{M}^a(\mathrm{G})$ are connected and nonempty [35]. Using Example 2.3, this implies the following proposition.

Proposition 2.23. *If* G *is a connected real semisimple Lie group such that the maximal compact subgroup* H *is semisimple, then, for each* $a\in\mathrm{Bun}_X(\mathrm{G})$, *the space* $\mathcal{M}^a(\mathrm{G})$ *is nonempty. Moreover, each component* $\mathcal{M}^a(\mathrm{G})$ *contains a unique connected component with the property that every Higgs bundle in it can be continuously deformed to a Higgs bundle with zero Higgs field.*

The above proposition implies that, when G is a semisimple *complex* Lie group, the space $\mathcal{M}^a(\mathrm{G})$ is nonempty for each $a\in\mathrm{Bun}_X(\mathrm{G})$. In fact, each of the spaces $\mathcal{M}^a(\mathrm{G})$ is connected. This was proven for connected groups by Li [32] and in general in [18]. In particular, we have the following:

Corollary 2.24. *If* G *is a semisimple complex Lie group, then every Higgs bundle* $(\mathcal{E},\varphi)\in\mathcal{M}_K(\mathrm{G})$ *can be continuously deformed to Higgs bundle with vanishing Higgs field. In particular,*

$$|\pi_0(\mathcal{M}(\mathrm{G}))|=|\mathrm{Bun}_X(\mathrm{G})|.$$

A semisimple Lie group G whose maximal compact subgroup is not semisimple but only reductive is called *a group of Hermitian type.* We will discuss this case in more detail in Section 6.2.

For $p = 1$, $\mathrm{O}(1) \cong \mathbb{Z}_2$ and O(1)-bundles are classified by their first Stiefel-Whitney class $sw_1 \in H^1(X, \mathbb{Z}_2)$. For $p \geqslant 2$, topological $\mathrm{O}(p)$-bundles have two characteristic classes, a first Stiefel-Whitney class and a second Stiefel-Whitney class $sw_2 \in H^2(X, \mathbb{Z}_2)$. When the first Stiefel-Whitney class vanishes, the structure group can be reduced to $\mathrm{SO}(p)$. Since SO(2) is a circle, the second Stiefel-Whitney class of an O(2)-bundle lifts to the degree of a circle bundle when $sw_1 = 0$. However, as an O(2)-bundle, it is only the absolute value of the degree which is a topological invariant. For $p > 2$, the Steifel-Whitney classes classify topological $\mathrm{O}(p)$-bundles over X.

We will be particularly interested in the case of K^p-twisted $\mathrm{SO}(1, n)$-Higgs bundles and K-twisted $\mathrm{SO}(p, q)$-Higgs bundles. Since the maximal compact subgroup of $\mathrm{SO}(p, q)$ is $\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$, the Higgs bundles are determined by two orthogonal bundles which have the same first Stiefel-Whitney class. Let $\mathcal{M}_L^{a,b,c}(\mathrm{SO}(p, q))$ denote the subset of $\mathrm{SO}(p, q)$-Higgs bundles (V, Q_V, W, Q_V, η) so that

$$a = sw_1(V, Q_V) = sw_1(W, Q_W) \qquad b = sw_2(V, Q_V) \qquad \text{and} \qquad c = sw_2(W, Q_W).$$

These invariants are constant on connected components, thus we have a decomposition

$$\mathcal{M}_L(\mathrm{SO}(p, q)) = \coprod \mathcal{M}_L^{a,b,c}(\mathrm{SO}(p, q)) \ . \tag{2.14}$$

Note that when $p = 1$ the invariant b is zero and when $q = 1$, the invariant $c = 0$.

We now focus on the special case of K^p-twisted Higgs bundles, with $p \geqslant 1$, for the group $\mathrm{SO}(2, q)$ with $q \geqslant 1$, and with vanishing first Stiefel-Whitney class. Let (V, W, η) be a polystable K^p-twisted $\mathrm{SO}(2, q)$-Higgs bundle with $sw_1(V) = 0$. Then there is a line bundle L so that the $\mathrm{SO}(2, \mathbb{C})$-bundle (V, Q_V) is isomorphic to

$$(V, Q_V) \cong (L \oplus L^{-1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)) \ .$$

With respect to this splitting, the Higgs field $\eta : W \to V \otimes K^p$ decomposes as

$$\eta = \left(\begin{smallmatrix} \gamma \\ \beta \end{smallmatrix}\right) : W \to (L \oplus L^{-1}) \otimes K^p.$$

When $q = 2$ then W also splits as $W \cong M \oplus M^{-1}$. With respect to these splittings we have

$$\eta = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) : M \oplus M^{-1} \to (L \oplus L^{-1}) \otimes K^p.$$

Moreover, as described in Section 6.2, polystability puts constraints on the degree of L (and also on the degree of M if $q = 2$).

3. The $\mathbb{C}^*$-action and its fixed points

In this section we recall the definition of the $\mathbb{C}^*$-action on the Higgs bundle moduli space and discuss its importance. The action of $\mathbb{C}^*$ on the L-twisted Higgs bundle moduli space is defined by scaling the Higgs field. Namely, if $(\mathcal{E}, \varphi)$ is an L-twisted G-Higgs bundle, then, for $\lambda \in \mathbb{C}^*$, $(\mathcal{E}, \lambda \cdot \varphi)$ is also an L-twisted G-Higgs bundle. Since this action clearly preserves notions of (poly)stability, we have a holomorphic action on the moduli space. Using the properness of the Hitchin fibration, it can be shown that if $(\mathcal{E}, \varphi)$ is the isomorphism class of a polystable L-twisted G-Higgs bundle, then, for $\lambda \in \mathbb{C}^*$, the limit $\lim\limits_{\lambda \to 0} (\mathcal{E}, \lambda \cdot \varphi)$ exists and is a polystable fixed point of the $\mathbb{C}^*$-action [38].

Notation 3.1. Note that we have denoted the isomorphism class of a Higgs bundle and the Higgs bundle itself with the same symbol. The context will always clarify which object we are referring to.

Consider the function on the moduli space of K-twisted G-Higgs bundles which assigns the L^2-norm of the Higgs field with respect to the harmonic metric solving the self-duality equations:

$$f : \mathcal{M}(\mathrm{G}) \to \mathbb{R}, \quad (\mathcal{E}, \varphi) \mapsto \int_X ||\varphi||^2. \tag{3.1}$$

Note that f is non-negative and zero if and only if $\varphi = 0$. Using Uhlenbeck compactness, Hitchin showed that the map f is proper [27]. Moreover, the critical points of f correspond exactly to the fixed points of the $\mathbb{C}^*$-action.

We will refer to the function f as the *Hitchin function*. Since it is a proper function, it attains its local minima on each closed subset of $\mathcal{M}(\mathrm{G})$. In particular, if $\mathrm{Min}(\mathcal{M}(\mathrm{G})) \subset \mathcal{M}(\mathrm{G})$ denotes the subset where f attains a local minimum, we have

$$|\pi_0(\mathcal{M}(\mathrm{G}))| \leqslant |\pi_0(\mathrm{Min}(\mathcal{M}(\mathrm{G})))|.$$

Thus, the Hitchin function can be used to study the connected components of the moduli space of G-Higgs bundles.

We now describe the structure of the L-twisted Higgs bundles at the fixed points of the $\mathbb{C}^*$-action. A detailed understanding of this structure is used extensively in the proofs of our main results. If $(\mathcal{E}, \varphi) \in \mathcal{M}_L(\mathrm{G})$ is such a fixed point, there is a one parameter family g_λ of holomorphic gauge transformations of $\mathcal{E}$ which realize the $\mathbb{C}^*$-action: $\mathrm{Ad}_{g_\lambda} \cdot \varphi = \lambda\varphi$. For each point $x \in X$ the gauge transformation g_λ gives a weight space grading on the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{j=-M}^{M} \mathfrak{g}_j^{\mathbb{C}}$, where $g_\lambda(x)$ acts on $\mathfrak{g}_j^{\mathbb{C}}$ by $\lambda^j \cdot \mathrm{Id}$. Since $g_\lambda(x) \in \mathrm{H}^{\mathbb{C}}$, this grading respects the Cartan decomposition, namely, $\mathfrak{g}_j^{\mathbb{C}} = \mathfrak{h}_j^{\mathbb{C}} \oplus \mathfrak{m}_j^{\mathbb{C}}$.

The holomorphicity of g_λ defines a weight space splitting of the Lie algebra bundles

$$\mathcal{E}[\mathfrak{h}^{\mathbb{C}}] = \bigoplus_{j=-M}^{M} \mathcal{E}[\mathfrak{h}_j^{\mathbb{C}}] \qquad \text{and} \qquad \mathcal{E}[\mathfrak{m}^{\mathbb{C}}] \otimes L = \bigoplus_{j=-M}^{M} \mathcal{E}[\mathfrak{m}_j^{\mathbb{C}}] \otimes L\ .$$

Moreover, the Higgs field takes values in the weight one space: $\varphi \in H^0(\mathcal{E}[\mathfrak{m}_1^{\mathbb{C}}] \otimes L)$. Thus, for such a fixed point $(\mathcal{E}, \varphi)$, the complex $C^\bullet = C^\bullet(\mathcal{E}, \varphi) : \ \mathcal{E}[\mathfrak{h}^{\mathbb{C}}] \xrightarrow{\mathrm{ad}_\varphi} \mathcal{E}[\mathfrak{m}^{\mathbb{C}}] \otimes L$ defined in (2.5) splits as

$$C_k^\bullet : \ \mathcal{E}[\mathfrak{h}_k^{\mathbb{C}}] \xrightarrow{\mathrm{ad}_\varphi} \mathcal{E}[\mathfrak{m}_{k+1}^{\mathbb{C}}] \otimes L, \tag{3.2}$$

yielding a corresponding splitting of the long exact sequence in cohomology from (2.6):

$$\begin{array}{l} 0 \longrightarrow \mathbb{H}^0(C_k^\bullet) \longrightarrow H^0(\mathcal{E}[\mathfrak{h}_k^{\mathbb{C}}]) \xrightarrow{\mathrm{ad}_\varphi} H^0(\mathcal{E}[\mathfrak{m}_{k+1}^{\mathbb{C}}] \otimes L) \longrightarrow \mathbb{H}^1(C_k^\bullet) \\ \longrightarrow H^1(\mathcal{E}[\mathfrak{h}_k^{\mathbb{C}}]) \xrightarrow{\mathrm{ad}_\varphi} H^1(\mathcal{E}[\mathfrak{m}_{k+1}^{\mathbb{C}}] \otimes L) \longrightarrow \mathbb{H}^2(C_k^\bullet) \longrightarrow 0. \end{array} \tag{3.3}$$

When $L = K$ and $(\mathcal{E}, \varphi)$ is a smooth or orbifold point of $\mathcal{M}(\mathrm{G})$, the spaces $\mathbb{H}^1(C_k^\bullet)$ can be interpreted as the eigendirections (for eigenvalue $-k$) of the Hessian of the Hitchin function f. In particular, when such a fixed point is a local minimum of f, we have $\mathbb{H}^1(C_k^\bullet) = 0$ for all $k > 0$. In fact, we have the following criterion for such local minima of f (see [8, Section 3.4]).

Proposition 3.2. *If $(\mathcal{E}, \varphi)$ is a K-twisted G-Higgs bundle which is a fixed point of the $\mathbb{C}^*$-action such that $\mathbb{H}^0(C^\bullet) = 0$ and $\mathbb{H}^2(C^\bullet) = 0$, then $(\mathcal{E}, \varphi)$ is a local minimum of the Hitchin function f if and only if either $\varphi = 0$ or the map (3.2) is an isomorphism of sheaves for every $k > 0$.*

The following result will help us show the vanishing $\mathbb{H}^2(C^\bullet)$ for relevant Higgs bundles.

Lemma 3.3. *If $(\mathcal{E}, \varphi)$ is a polystable L-twisted Higgs bundle and $(\mathcal{E}', \varphi') = \lim_{\lambda \to 0} (\mathcal{E}, \lambda\varphi)$, then*

$$\dim\left(\mathbb{H}^2(C^\bullet(\mathcal{E}, \varphi))\right) \leqslant \dim\left(\mathbb{H}^2(C^\bullet(\mathcal{E}', \varphi'))\right).$$

Proof. If $(\mathcal{E}, \varphi)$ is fixed by the $\mathbb{C}^*$-action then we are done. If $(\mathcal{E}, \varphi)$ is not fixed by $\mathbb{C}^*$, then consider the $\mathbb{C}^*$-family $(\mathcal{E}, \lambda\varphi)$. Since $\lim_{\lambda \to 0} (\mathcal{E}, \lambda\eta)$ exists, we can extend this to a family over $\mathbb{A}^1$, hence the result follows by semi-continuity of $\mathbb{H}^2$. ∎

Example 3.4. The above minima criterion was used in [28] to classify all local minima for the group $\mathrm{SL}(n,\mathbb{R})$ and in [6] to classify all local minima for the group $\mathrm{U}(p,q)$ (cf. Definition 2.9). For $\mathrm{SL}(n,\mathbb{R})$, the only local minima (E,Q,Φ) with nonzero Higgs field are the ones defining the Hitchin components. More precisely, they are given by

$$E = K^{(n-1)/2} \oplus \cdots \oplus K^{(1-n)/2}, \quad Q = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

For the group $\mathrm{U}(p,q)$, all minima (E,F,β,γ) have either $\beta = 0$ or $\gamma = 0$.

3.1. $\mathrm{SO}(p,q)$**-fixed points.** We now focus on the details of fixed points of the $\mathbb{C}^*$-action on the L-twisted $\mathrm{SO}(p,q)$-Higgs bundle moduli space. Let (V,W,η) be a polystable $\mathrm{SO}(p,q)$-Higgs bundle with $(V,W,\eta) \cong (V,W,\lambda\eta)$ for all $\lambda \in \mathbb{C}^*$. If $\eta \neq 0$, then for each λ there are holomorphic orthogonal gauge transformations g_λ^V and g_λ^W of V and W such that $(g_\lambda^V)^{-1} \cdot \eta \cdot g_\lambda^W = \lambda\eta$.

Let $V = \bigoplus\limits_{\nu\in\mathbb{R}} V_\nu$ and $W = \bigoplus\limits_{\mu\in\mathbb{R}} W_\mu$ denote the eigenbundle decompositions of g_λ^V and g_λ^W respectively, so that $g_\lambda^V|_{V_\nu} = \lambda^\nu \cdot \mathrm{Id}_{V_\nu}$ and $g_\lambda^W|_{W_\mu} = \lambda^\mu \cdot \mathrm{Id}_{W_\mu}$. Since the gauge transformations g_λ^V and g_λ^W are orthogonal, two eigenbundles V_ν and $V_{\nu'}$ or W_μ and $W_{\mu'}$ are orthogonal if $\nu + \nu' \neq 0$ or $\mu + \mu' \neq 0$.

For all weights μ and ν, we have $\eta(W_\mu) \subset V_{\mu+1} \otimes L$ and $\eta^*(V_\nu) \subset W_{\nu+1} \otimes L$. Thus, $\eta = \sum \eta_\mu$ and $\eta^* = \sum \eta^*_\nu$, where

$$\eta_\mu = \eta|_{W_\mu} : W_\mu \longrightarrow V_{\mu+1} \otimes L \qquad \text{and} \qquad \eta^*_\nu = \eta^*|_{V_\nu} : V_\nu \longrightarrow W_{\nu+1} \otimes L\,. \tag{3.4}$$

In particular, the eigenvalues of g_λ^V and g_λ^W are related via η and η^*, and each set of eigenvalues is of the form $\{\lambda^{-x}, \lambda^{1-x}, \ldots, \lambda^{x-1}, \lambda^x\}$. Thus the eigenbundle decompositions of V and W are of the form

$$V = V_{-r} \oplus V_{1-r} \oplus \cdots \oplus V_{r-1} \oplus V_r \qquad \text{and} \qquad W_{-s} \oplus W_{1-s} \oplus \cdots \oplus W_{s-1} \oplus W_s \tag{3.5}$$

for some half-integers r and s. Notice that $Q_W\eta = -\eta^T Q_V$ and (3.4) imply that $2r$ and $2s$ have the same parity, i.e. the number of summands in (3.5) are either both even or both odd.

We summarize the above characterization of $\mathbb{C}^*$-fixed points in the following proposition.

Proposition 3.5. *If (V,W,η) is a polystable L-twisted $\mathrm{SO}(p,q)$-Higgs bundle which is a fixed point of the $\mathbb{C}^*$-action with $\eta \neq 0$, then there are half-integers r and s with $2r = 2s \pmod 2$ such that*

$$V = \bigoplus_{j=-r}^{r} V_j \qquad \textit{and} \qquad W = \bigoplus_{j=-s}^{s} W_j\,.$$

Moreover, the corresponding quadratic forms define isomorphisms $V_j \cong V^*_{-j}$ and $W_j \cong W^*_{-j}$ and the Higgs field $\eta : W \to V \otimes K$ splits as a sum $\eta = \sum \eta_j$ with $\eta_j : W_j \to V_{j+1} \otimes L$.

Notation 3.6. By the preceding proposition, a Higgs bundle (V,W,η) which is a $\mathbb{C}^*$-fixed point can be represented by one of the following holomorphic chains:

$$\begin{array}{c} \cdots \xrightarrow{\eta_{-3}} V_{-2} \xrightarrow{\eta^*_1} W_{-1} \xrightarrow{\eta_{-1}} V_0 \xrightarrow{\eta^*_{-1}} W_1 \xrightarrow{\eta_1} V_2 \xrightarrow{\eta^*_{-3}} \cdots \\ \oplus \\ \cdots \xrightarrow{\eta^*_2} W_{-2} \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\eta^*_0} W_0 \xrightarrow{\eta_0} V_1 \xrightarrow{\eta^*_{-2}} W_2 \xrightarrow{\eta_2} \cdots \end{array} \tag{3.6}$$

or

$$\begin{array}{c} \cdots \xrightarrow{\eta_{-5/2}} V_{-3/2} \xrightarrow{\eta^*_{1/2}} W_{-1/2} \xrightarrow{\eta_{-1/2}} V_{1/2} \xrightarrow{\eta^*_{-3/2}} W_{3/2} \xrightarrow{\eta_{3/2}} \cdots \\ \oplus \\ \cdots \xrightarrow{\eta^*_{3/2}} W_{-3/2} \xrightarrow{\eta_{-3/2}} V_{-1/2} \xrightarrow{\eta^*_{-1/2}} W_{1/2} \xrightarrow{\eta_{1/2}} V_{3/2} \xrightarrow{\eta^*_{-5/2}} \cdots \end{array} \tag{3.7}$$

where each chain ends with a subbundle of V or W depending on the parity of r and s. For simplicity of notation, we have suppressed the twisting by L from the Higgs field. This will be done every time we use these chain representations.

Proposition 3.5 provides a characterization of polystable $\mathbb{C}^*$-fixed points with non-vanishing Higgs field. The next result shows that stability imposes further conditions on such fixed points.

Proposition 3.7. *If (V, W, η) is a stable L-twisted $\mathrm{SO}(p,q)$-Higgs bundle which is a $\mathbb{C}^*$-fixed point, then each component of the Higgs field is nonzero and the maximal weights r and s in Proposition 3.5 are integers (i.e. it is represented by a chain of type (3.6)).*

Proof. Suppose (V, W, η) is represented by (3.7), i.e., $2r = 2s = 1 \pmod 2$. Consider the subbundles

$$V' = \cdots \oplus V_{-3/2} \oplus V_{1/2} \oplus \cdots \subset V \qquad \text{and} \qquad W' = \cdots \oplus W_{-1/2} \oplus W_{3/2} \oplus \cdots \subset W\ .$$

The bundles V' and V'^* define non-trivial isotropic subbundles of V, and the bundles W' and W'^* define non-trivial isotropic subbundles of W. Since $\deg(V') + \deg(W') = -\deg(V'^*) - \deg(W'^*)$, such an $\mathrm{SO}(p,q)$-Higgs bundle is not stable. If one of the maps in (3.6) is identically zero, then the chain splits as $A \oplus B \oplus A^*$ where A is a subchain consisting of isotropic subbundles and A^* is the dual chain. This again contradicts stability. $\square$

3.2. **Fixed points on $\mathcal{M}(\mathrm{SO}(2,2))$.** Fixed points of the $\mathbb{C}^*$-action in $\mathcal{M}(\mathrm{SO}(2,2))$ are particularly easy to describe using (3.6) and (3.7). Let (V, W, η) be an $\mathrm{SO}(2,2)$-Higgs bundle. If $sw_1(V) = sw_1(W) \neq 0$, then neither V nor W have holomorphic isotropic subbundles, thus (V, W, η) is a fixed point if and only if $\eta = 0$. If $sw_1(V) = sw_1(W) = 0$, then $V = L \oplus L^{-1}$ and $W = M \oplus M^{-1}$ where L and M are isotropic line bundles. Up to switching the roles of L, M, L^{-1} and M^{-1}, the holomorphic chains are given by

$$M \xrightarrow{\binom{a}{b}} L \oplus L^{-1} \xrightarrow{(b\ a)} M^{-1}\ . \tag{3.8}$$

Polystability of the associated Higgs bundle puts certain constraints on the degrees of L and M, depending on the shape of $\eta : M \oplus M^{-1} \to LK \oplus L^{-1}K$.

Proposition 3.8. *Every fixed point in $\mathcal{M}(\mathrm{SO}(2,2))$ is a local minimum.*

Proof. Take a fixed point (V, W, η) in $\mathcal{M}(\mathrm{SO}(2,2))$ with non-vanishing Higgs field. Up to switching the roles of V and W, it must be of the form (3.8). Hence we see that $V = L \oplus L^{-1}$ has weight 0, while $M \subset W$ has weight -1 and $M^{-1} \subset W$ has weight 1. It follows that the corresponding positive weight subcomplexes $C_1^\bullet$ and $C_2^\bullet$ are both zero, and hence so are $\mathbb{H}^1(C_1^\bullet)$ and $\mathbb{H}^1(C_2^\bullet)$. This implies that (3.8) is a local minimum because any non-trivial deformation of (V, W, η) in $\mathcal{M}(\mathrm{SO}(2,2))$ which decreases the value of the Hitchin function must correspond to a non-trivial direction in $\mathbb{H}^1(C_1^\bullet) \oplus \mathbb{H}^1(C_2^\bullet)$. $\square$

3.3. **$\mathrm{SO}(1,n)$-fixed points and local structure of $\mathcal{M}_{K^p}(\mathrm{SO}(1,n))$.** We now focus on K^p-twisted $\mathrm{SO}(1,n)$-Higgs bundles which are fixed by the $\mathbb{C}^*$-action. These results will be used in the next section to describe the exotic connected components of $\mathcal{M}(\mathrm{SO}(p,q))$. Recall from Example 2.8 that a K^p-twisted $\mathrm{SO}(1,n)$-Higgs bundle is a triple (I, W, η) where W is a rank n holomorphic vector bundle with an orthogonal structure Q_W, $I = \Lambda^n W$ and $\eta \in H^0(\mathrm{Hom}(W, I) \otimes K^p)$.

Lemma 3.9. *If (I, W, η) is a polystable K^p-twisted $\mathrm{SO}(1,n)$-Higgs bundle which is a $\mathbb{C}^*$-action fixed point with $\eta \neq 0$, then it decomposes as*

$$(I, W, \eta) \cong \Big(I, W_{-1} \oplus W_0 \oplus W_1,\ \begin{pmatrix}\eta_{-1} & 0 & 0\end{pmatrix}\Big)\ ,$$

where W_0 is a polystable orthogonal bundle and $W_1 \cong W_{-1}^*$. Furthermore, $\big(I, W_{-1} \oplus W_1, \begin{pmatrix}\eta_{-1} & 0\end{pmatrix}\big)$ is a stable K^p-twisted $\mathrm{O}(1,n')$-Higgs bundle which is stable as a K^p-twisted $\mathrm{O}(n'+1, \mathbb{C})$-Higgs bundle. In the notation of (3.6), such a (I, W, η) is given by the chain

$$\begin{array}{ccccc} W_{-1} & \xrightarrow{\eta_{-1}} & I & \xrightarrow{\eta_{-1}^*} & W_1\ . \\ & & \oplus & & \\ & & W_0 & & \end{array}$$

Proof. The first part of the statement follows directly from Proposition 3.5. Since the bundles W_1 and W_{-1} are isotropic, if W_1 has a degree zero subbundle U, then W_{-1} has U^* as a subbundle contained in the kernel of η_{-1} by polystability. We may thus assume that the invariant polystable orthogonal subbundle $U^*\oplus U$ is a summand of W_0. Now since $(W_{-1}\oplus W_1, I, (\eta_{-1}\ 0))$ is a stable $\mathrm{O}(1,n')$-Higgs bundle, the associated $\mathrm{O}(n'+1,\mathbb{C})$-Higgs bundle is stable by Proposition 2.7 of [2]. □

At a $\mathbb{C}^*$-fixed point, we have $\mathfrak{so}(I)=0$ and $\mathrm{End}(W_{-1}\oplus W_0\oplus W_1)=\bigoplus_{j=-2}^{2}\mathrm{End}_j(W)$, where

$$\begin{aligned}\mathrm{End}_2(W)^*=\mathrm{End}_{-2}(W)&=\mathrm{Hom}(W_1,W_{-1}),\\ \mathrm{End}_1(W)^*=\mathrm{End}_{-1}(W)&=\mathrm{Hom}(W_1,W_0)\oplus\mathrm{Hom}(W_0,W_{-1}),\\ \mathrm{End}_0(W)&=\mathrm{End}(W_{-1})\oplus\mathrm{End}(W_0)\oplus\mathrm{End}(W_1).\end{aligned}$$

This gives a grading on $\mathfrak{so}(W)=\bigoplus_{j=-2}^{2}\mathfrak{so}_j(W)$, where

$$\begin{aligned}\mathfrak{so}_2(W)^*=\mathfrak{so}_{-2}(W)&=\{\beta\in\mathrm{Hom}(W_1,W_{-1})\mid \beta+\beta^*=0\},\\ \mathfrak{so}_1(W)^*=\mathfrak{so}_{-1}(W)&=\{(\beta,-\beta^*)\in\mathrm{End}_{-1}(W)\},\\ \mathfrak{so}_0(W)&=\{(\beta_{-1},\beta_0,-\beta_{-1}^*)\in\mathrm{End}_0(W)\mid \beta_0+\beta_0^*=0\}.\end{aligned}$$

Also, $\mathrm{Hom}(W,I)\otimes K^p=\mathrm{Hom}_{-1}(W,I)\otimes K^p\oplus\mathrm{Hom}_0(W,I)\otimes K^p\oplus\mathrm{Hom}_1(W,I)\otimes K^p$, where

$$\mathrm{Hom}_{\pm1}(W,I)\otimes K^p=\mathrm{Hom}(W_{\mp1},I)\otimes K^p\qquad\text{and}\qquad \mathrm{Hom}_0(W,I)\otimes K^p=\mathrm{Hom}(W_0,I)\otimes K^p.$$

Corresponding to each subcomplex $C_k^\bullet$, the above splittings give $\mathrm{ad}_\eta:\mathfrak{so}_k(W)\to\mathrm{Hom}_{k+1}(W,I)\otimes K^p$ for $k=-2,\dots,2$. Note that $C_k^\bullet$ is defined by composing with η_{-1}. For each such k, this yields the long exact sequence in cohomology

$$0\longrightarrow\mathbb{H}^0(C_k^\bullet)\longrightarrow H^0(\mathfrak{so}_k(W))\xrightarrow{\eta_{-1}}H^0(\mathrm{Hom}_{k+1}(W,IK^p))\longrightarrow\mathbb{H}^1(C_k^\bullet)\longrightarrow H^1(\mathfrak{so}_k(W))\xrightarrow{\eta_{-1}}H^1(\mathrm{Hom}_{k+1}(W,IK^p))\longrightarrow\mathbb{H}^2(C_k^\bullet)\longrightarrow 0\,. \tag{3.9}$$

Lemma 3.10. *For $p>1$, if (I,W,η) is a polystable K^p-twisted $\mathrm{SO}(1,n)$-Higgs bundle, then the second hypercohomology group $\mathbb{H}^2(C^\bullet(I,W,\eta))$ vanishes.*

Proof. By Lemma 3.3, to show that $\mathbb{H}^2(C^\bullet(I,W,\eta))$ vanishes it suffices to show the vanishing of each graded piece of (3.9) at a fixed point of the $\mathbb{C}^*$-action. Such fixed points are given by Lemma 3.9.

First note that $\mathbb{H}^2(C_k^\bullet)=0$ for $k\geqslant1$ since $\mathrm{Hom}_{k+1}(W,I)=0$ for $k\geqslant1$. Stability implies W_1 and W_0 have no positive degree subbundles, and, by Serre duality, we have

$$H^1(\mathrm{Hom}_{k+1}(W,IK^p))\cong\begin{cases}H^0(\mathrm{Hom}(IK^{p-1},W_1))^* & k=-2\\ H^0(\mathrm{Hom}(IK^{p-1},W_0))^* & k=-1\,.\end{cases}$$

Thus, since $p>1$, $H^1(\mathrm{Hom}_{k+1}(W,IK^p))=0$ for $k\leqslant-1$.

Finally, the form of the Higgs field implies the kernel of $\mathrm{ad}_\eta:\mathfrak{so}_0(W)\to\mathrm{Hom}_1(W,I)\otimes K^p$ is $\mathfrak{so}(W_0)$. Hence, $\mathbb{H}^2(C_0^\bullet)$ injects into the second hypercohomology group of the stable $\mathrm{O}(1,n')$-Higgs bundle $\big(I,W_{-1}\oplus W_1,\big(\eta_{-1}\ \ 0\big)\big)$. The associated $\mathrm{O}(n'+1,\mathbb{C})$-Higgs bundle is stable by Lemma 3.9, so this hypercohomology group vanishes by Remark 2.17. □

Lemma 3.11. *If $p>1$ and $(I,W,\eta)=\big(I,W_{-1}\oplus W_0\oplus W_1,\big(\eta_{-1}\ \ 0\ \ 0\big)\big)$ is a polystable K^p-twisted $\mathrm{SO}(1,n)$-Higgs bundle which is fixed by the $\mathbb{C}^*$-action, then*

$$\mathbb{H}^0(C^\bullet)\cong H^0(\mathfrak{so}(W_0))\quad\text{and}\quad\mathbb{H}^1(C^\bullet)=\bigoplus_{k=-2}^{2}\mathbb{H}^1(C_k^\bullet),$$

where $\mathfrak{so}(W_0)$ is the bundle of skew-symmetric endomorphisms of W_0 with respect to Q_0. Moreover,

- $\mathbb{H}^1(C_2^\bullet) \cong H^1(\mathfrak{so}_2(W)) = \{\beta \in \mathrm{Hom}(W_{-1}, W_1) | \beta + \beta^* = 0\}$,
- $\mathbb{H}^1(C_1^\bullet) \cong H^1(\mathrm{Hom}(W_{-1}, W_0))$,
- $\mathbb{H}^1(C_0^\bullet) \cong H^1(\mathfrak{so}(W_0)) \oplus \mathbb{H}_0^1$, *where* $\mathbb{H}_0^1$ *is defined by the sequence*

$$0 \longrightarrow H^0(\mathrm{End}(W_{-1})) \xrightarrow{\eta_{-1}} H^0(\mathrm{Hom}(W_{-1}, IK^p)) \longrightarrow \mathbb{H}_0^1 \longrightarrow H^1(\mathrm{End}(W_{-1})) \xrightarrow{\eta_{-1}} H^1(\mathrm{Hom}(W_{-1}, IK^p)) \longrightarrow 0\ ,$$

- $\mathbb{H}^1(C_{-1}^\bullet)$ *is defined by the sequence*

$$0 \longrightarrow H^0(\mathrm{Hom}(W_0, W_{-1})) \xrightarrow{\eta_{-1}} H^0(\mathrm{Hom}(W_0, IK^p)) \longrightarrow \mathbb{H}^1(C_{-1}^\bullet) \longrightarrow H^1(\mathrm{Hom}(W_0, W_{-1})) \longrightarrow 0\ ,$$

- $\mathbb{H}^1(C_{-2}^\bullet)$ *is defined by the sequence*

$$0 \longrightarrow H^0(\mathfrak{so}_{-2}(W)) \xrightarrow{\eta_{-1}} H^0(\mathrm{Hom}(W_1, IK^p)) \longrightarrow \mathbb{H}^1(C_{-2}^\bullet) \longrightarrow H^1(\mathfrak{so}_{-2}(W)) \longrightarrow 0\ ,$$

where $\mathfrak{so}_{-2}(W) = \{\beta \in \mathrm{Hom}(W_1, W_{-1}) | \beta + \beta^* = 0\}$.

Proof. By Lemma 3.9, a $\mathbb{C}^*$-fixed point is given by $(I, W, \eta) = \big(I, W_{-1} \oplus W_0 \oplus W_1, \big(\eta_{-1}\ \ 0\ \ 0\big)\big)$, where W_0 is a polystable orthogonal bundle and $(I, W_{-1} \oplus W_1, (\eta_{-1}\ 0))$ is a stable $\mathrm{O}(1, n')$-Higgs bundle such that the associated $\mathrm{O}(n'+1, \mathbb{C})$-Higgs bundle is also stable. In particular, W_1 has no non-negative degree subbundles and W_0 has no positive degree subbundles. Recall that in the proof of Lemma 3.10 it was shown that $H^1(\mathrm{Hom}_{k+1}(W, IK^p)) = 0$ for $k \leqslant -1$.

For $k = 2$, we have $C_2^\bullet : \mathfrak{so}_2(W) \to 0$, thus, $\mathbb{H}^0(C_2^\bullet) = H^0(\mathfrak{so}_2(W))$ and $\mathbb{H}^1(C_2^\bullet) = H^1(\mathfrak{so}_2(W))$. In particular, $\mathbb{H}^0(C_2^\bullet)$ injects into the zeroth hypercohomology group of the deformation complex of the $\mathrm{O}(1, n')$-Higgs bundle $(I, W_{-1} \oplus W_1, (\eta_{-1}\ 0))$, which vanishes by stability.

For $k = 1$, $\mathfrak{so}_1(W) \cong \mathrm{Hom}(W_{-1}, W_0)$ and $C_1^\bullet : \mathfrak{so}_1(W) \to 0$ imply $\mathbb{H}^0(C_1^\bullet) = H^0(\mathrm{Hom}(W_{-1}, W_0))$ and $\mathbb{H}^1(C_1^\bullet) = H^1(\mathrm{Hom}(W_{-1}, W_0))$. The vanishing of $H^0(\mathrm{Hom}(W_{-1}, W_0)) \cong H^0(\mathrm{Hom}(W_0, W_1))$ follows from stability. Namely, any nonzero homomorphism $f : W_0 \to W_1$ defines a non-negative degree subbundle of W_1, contradicting the stability of $(I, W_{-1} \oplus W_1, (\eta_{-1}\ 0))$.

For $k = 0$, $C_0^\bullet : \mathfrak{so}_0(W) \to \mathrm{Hom}_1(W, I) \otimes K^p$ is given by

$$C_0^\bullet : \mathrm{End}(W_{-1}) \oplus \mathfrak{so}(W_0) \to \mathrm{Hom}(W_{-1}, I) \otimes K^p, \qquad (\beta_{-1}, \beta_0) \mapsto \eta_{-1}\beta_{-1}.$$

Thus, we can split $C_0^\bullet$ as $C_0^\bullet = C_0^{\bullet,\prime} \oplus C_0^{\bullet,\prime\prime}$ with $C_0^{\bullet,\prime} : \mathrm{End}(W_{-1}) \xrightarrow{\eta_{-1}} \mathrm{Hom}(W_{-1}, I) \otimes K^p$ and $C_0^{\bullet,\prime\prime} : \mathfrak{so}(W_0) \to 0$. The hypercohomology groups split accordingly, hence

$$\mathbb{H}^0(C_0^{\bullet,\prime\prime}) = H^0(\mathfrak{so}(W_0)) \qquad \text{and} \qquad \mathbb{H}^1(C_0^{\bullet,\prime\prime}) \cong H^1(\mathfrak{so}(W_0))\ .$$

For $C_0^{\bullet,\prime}$, $\mathbb{H}^0(C_0^{\bullet,\prime}) = 0$ by stability of $\big(I, W_{-1} \oplus W_1, \big(\eta_{-1}\ \ 0\big)\big)$. Thus, if $\mathbb{H}_0^1 = \mathbb{H}^1(C_0^{\bullet,\prime})$, we have

$$0 \longrightarrow H^0(\mathrm{End}(W_{-1})) \xrightarrow{\eta_{-1}} H^0(\mathrm{Hom}(W_{-1}, IK^p)) \longrightarrow \mathbb{H}_0^1 \longrightarrow H^1(\mathrm{End}(W_{-1})) \xrightarrow{\eta_{-1}} H^1(\mathrm{Hom}(W_{-1}, IK^p)) \longrightarrow 0\ .$$

For $k = -1$, we have $H^1(\mathrm{Hom}_0(W, IK^p)) = 0$ and $C_{-1}^\bullet : \mathrm{Hom}(W_0, W_{-1}) \xrightarrow{\eta_{-1}} \mathrm{Hom}(W_0, I) \otimes K^p$. Thus,

$$0 \longrightarrow \mathbb{H}^0(C_{-1}^\bullet) \longrightarrow H^0(\mathrm{Hom}(W_0, W_{-1})) \xrightarrow{\eta_{-1}} H^0(\mathrm{Hom}(W_0, IK^p)) \longrightarrow \mathbb{H}^1(C_{-1}^\bullet) \longrightarrow H^1(\mathrm{Hom}(W_0, W_{-1})) \longrightarrow 0\ .$$

It remains to show that $\mathbb{H}^0(C_{-1}^\bullet) = 0$. If N is the kernel of $\eta_{-1} : W_{-1} \to IK^p$, then $\mathbb{H}^0(C_{-1}^\bullet) \cong H^0(\mathrm{Hom}(W_0, N))$. If $N = 0$ we are done so suppose $N \neq 0$. Stability of $\big(I, W_{-1} \oplus W_1, \big(\eta_{-1}\ \ 0\big)\big)$ implies $\deg(N) < 0$ and moreover N has no non-negative degree subbundles. A non-zero section

$\beta \in H^0(\mathrm{Hom}(W_0, N))$ must have a non-trivial kernel since otherwise $\beta(W_0) \subset N$ would define a non-negative degree subbundle. However, this implies that $\deg(\ker(\beta)) > 0$, contradicting the polystability of W_0. We conclude that $H^0(\mathrm{Hom}(W_0, N)) = 0$, and thus $\mathbb{H}^0(C^\bullet_{-1}) = 0$.

Finally consider the case of $C^\bullet_{-2} : \mathfrak{so}_{-2}(W) \xrightarrow{\mathrm{ad}_\eta} \mathrm{Hom}(W_1, I) \otimes K^p$. As in the case $k = 2$, stability of the $\mathrm{O}(1, n')$-Higgs bundle $(I, W_{-1} \oplus W_1, (\begin{smallmatrix}\eta_{-1} & 0\end{smallmatrix}))$ implies $\mathbb{H}^0(C^\bullet_{-2}) = 0$. The group $\mathbb{H}^1(C^\bullet_{-2})$ is defined by the exact sequence in the statement of the lemma since $H^1(\mathrm{Hom}(W_1, IK^p)) = 0$. $\square$

4. Existence of exotic components of $\mathcal{M}(\mathrm{SO}(p, q))$

In this section we will prove the following theorem exhibiting connected components of $\mathcal{M}(\mathrm{SO}(p, q))$ which are not distinguished by primary characteristic classes for $p \geqslant 2$.

Theorem 4.1. *Let X be a compact Riemann surface with genus $g \geqslant 2$ and canonical bundle K. Denote the moduli space of K^p-twisted $\mathrm{SO}(1, q-p+1)$-Higgs bundles on X by $\mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1))$ and the moduli space of K-twisted $\mathrm{SO}(p, q)$-Higgs bundles on X by $\mathcal{M}(\mathrm{SO}(p, q))$. For $1 \leqslant p \leqslant q$, there is a well defined map*

$$\Psi : \ \mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \longrightarrow \mathcal{M}(\mathrm{SO}(p, q)) \tag{4.1}$$

which is an isomorphism onto its image and has an open and closed image. Furthermore, if $p \geqslant 2$, then every Higgs bundle in the image of Ψ has a nowhere vanishing Higgs field.

Remark 4.2. As a direct corollary of the above theorem, we have that, for $p > 2$,

$$\left|\pi_0\big(\mathcal{M}(\mathrm{SO}(p, q))\big)\right| \ \geqslant \ 2^{2g+2} + \left|\pi_0\big(\mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1))\big)\right|.$$

In particular, there are connected components of $\mathcal{M}(\mathrm{SO}(p, q))$ which are not distinguished by the Stiefel-Whitney classes of the underlying orthogonal bundles. In Theorem 6.1 we will show that the above inequality is in fact an equality.

Remark 4.3. The space of holomorphic differentials $H^0(K^{2j})$ can be identified with the moduli space $\mathcal{M}_{K^{2j}}(\mathrm{SO}_0(1, 1)) = \mathcal{M}_{K^{2j}}(\mathbb{R}^+)$. In Section 7.3, this identification will be used to interpret the components from Theorem 4.1 as a *generalized Cayley correspondence*.

4.1. **Defining the map Ψ.** Recall that a K^p-twisted $\mathrm{SO}(1, n)$-Higgs bundle is a triple $(I, \widehat{W}, \hat{\eta})$, where $\widehat{W}$ is a rank n vector bundle with an orthogonal structure $Q_{\widehat{W}}$, $I = \Lambda^n \widehat{W}$ and $\hat{\eta} \in H^0(\mathrm{Hom}(\widehat{W}, I) \otimes K^p)$.

Let $\mathcal{H}_{K^p}(\mathrm{SO}(1, q-p+1))$ denote the configuration space of all K^p-twisted $\mathrm{SO}(1, q-p+1)$-Higgs bundles and let $\mathcal{H}(\mathrm{SO}(p, q))$ denote the configuration space of all $\mathrm{SO}(p, q)$-Higgs bundles. That is, $\mathcal{H}_{K^p}(\mathrm{SO}(1, q-p+1))$ consists of pairs $(\bar\partial_{\widehat{W}}, \hat\eta)$ where $\bar\partial_{\widehat{W}}$ is a Dolbeault operator on $\widehat{W}$, $\hat\eta \in \Omega^{1,0}(\mathrm{Hom}(\widehat{W}, \Lambda^{q-p+1}\widehat{W}))$ such that $\bar\partial_{\widehat{W}} \hat\eta = 0$ and $\bar\partial_{\widehat{W}} Q_{\widehat{W}} = 0$. The space $\mathcal{H}(\mathrm{SO}(p, q))$ is defined analogously.

Recall that the Hitchin section $s_H^I : \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \to \mathcal{M}(\mathrm{SO}(p, p-1))$ is given by (2.12), and that

$$(I \otimes \mathcal{K}_n, Q_n) = \left(I \otimes (K^n \oplus K^{n-2} \oplus \cdots \oplus K^{2-n} \oplus K^{-n}), \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}\right).$$

Recall that the Higgs field in the image of s_H^I is given by $\eta(q_2, \ldots, q_{2p-2}) : I \otimes \mathcal{K}_{p-2} \to I \otimes \mathcal{K}_{p-1} \otimes K$, as in (2.13).

Define the map

$$\widetilde{\Psi} : \ \mathcal{H}_{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \longrightarrow \mathcal{H}(\mathrm{SO}(p, q)) \tag{4.2}$$

by

$$\widetilde{\Psi}((I,\widehat{W},\hat{\eta}),q_2,\dots,q_{2p-2}) = \Big(I\otimes\mathcal{K}_{p-1}, \widehat{W}\oplus I\otimes\mathcal{K}_{p-2}, \begin{pmatrix}\eta_{\widehat{W}} & \eta(q_2,\dots,q_{2p-2})\end{pmatrix}\Big) \tag{4.3}$$

where

$$\eta_{\widehat{W}} = \begin{pmatrix}\hat{\eta}\\0\\\vdots\\0\end{pmatrix}:\ \widehat{W}\longrightarrow I\otimes(K^p\oplus K^{p-2}\oplus\cdots\oplus K^{2-p}) = I\otimes\mathcal{K}_{p-1}\otimes K\ .$$

It is clear that the map $\widetilde{\Psi}$ is continuous.

Lemma 4.4. *For $(I,\widehat{W},\hat{\eta},q_2,\dots,q_{2p-2})\in\mathcal{H}_{K^p}(\mathrm{SO}(1,q-p+1))\times\bigoplus_{j=1}^{p-1}H^0(K^{2j})$, the* SO$(p,q)$*-Higgs bundle $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\dots,q_{2p-2})$ is (poly)stable if and only if the K^p-twisted* SO$(1,q-p+1)$*-Higgs bundle $(I,\widehat{W},\hat{\eta})$ is (poly)stable.*

Proof. Fix $(I,\widehat{W},\hat{\eta},q_2,\dots,q_{2p-2})\in\mathcal{H}_{K^p}(\mathrm{SO}(1,q-p+1))\times\bigoplus_{j=1}^{p-1}H^0(K^{2j})$. Recall that an SO$(p,q)$-Higgs bundle is polystable if and only if the associated SL$(p+q,\mathbb{C})$-Higgs bundle is polystable. Suppose first that $q_{2j}=0$ for all j. Then the SL$(p+q,\mathbb{C})$-Higgs bundle associated to the image of $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},0,\dots,0)$ is represented by

$$\begin{array}{c} IK^{p-1}\xrightarrow{1}IK^{p-2}\xrightarrow{1}\cdots\xrightarrow{1}IK^{2-p}\xrightarrow{1}IK^{1-p}\ .\\ \widehat{W}\xrightarrow{\hat{\eta}}IK^{p-1},\qquad IK^{1-p}\xrightarrow{\hat{\eta}^*}\widehat{W}\end{array}$$

To check (poly)stability for such a “cyclic” Higgs bundle, it suffices to show that each of the bundles in the above cycle do not contain an invariant destabilizing subbundle (see Proposition 6.3 of [37]). Thus $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},0,\dots,0)$ is polystable if and only if there are no destabilizing subbundles of $\widehat{W}$ in the kernel of $\hat{\eta}$, that is, if and only if $(I,\widehat{W},\hat{\eta})$ is polystable. Furthermore, since $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},0,\dots,0)$ is strictly polystable if and only if $\widehat{W}$ contains a degree zero isotropic subbundle in the kernel of $\hat{\eta}$, we conclude that $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},0,\dots,0)$ is stable if and only if $(I,\widehat{W},\hat{\eta})$ is stable.

Now suppose $(q_2,\dots,q_{2p-2})\neq(0,\dots,0)$ and let $(V,W,\eta)=\widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\dots,q_{2p-2})$ be given by (4.3). For $\lambda\in\mathbb{C}^*$, consider the following holomorphic orthogonal gauge transformations of V and W

$$g_V=\begin{pmatrix}\lambda^{1-p}&&&\\&\lambda^{3-p}&&\\&&\ddots&\\&&&\lambda^{p-1}\end{pmatrix}\quad\text{and}\quad g_W=\begin{pmatrix}\mathrm{Id}_{\widehat{W}}&&&&\\&\lambda^{2-p}&&&\\&&\lambda^{4-p}&&\\&&&\ddots&\\&&&&\lambda^{p-2}\end{pmatrix}.$$

Using the description of s_H^I from (2.12) and (2.13), a straightforward computation shows that

$$(g_V,g_W)\cdot(V,W,\lambda\eta)=\widetilde{\Psi}(I,\widehat{W},\lambda^p\hat{\eta},\lambda^2q_2,\lambda^4q_4,\dots,\lambda^{2p-2}q_{2p-2}). \tag{4.4}$$

Assume $(I,\widehat{W},\hat{\eta})$ is stable. In particular, $(I,\widehat{W},\lambda^p\hat{\eta})$ is a stable K^p-twisted SO$(1,q-p+1)$-Higgs bundle for all $\lambda\in\mathbb{C}^*$. By the above argument, $\widetilde{\Psi}(I,\widehat{W},\lambda^p\hat{\eta},0,\dots,0)$ is also stable for all $\lambda\in\mathbb{C}^*$. Hence, by the continuity of $\widetilde{\Psi}$ and since stability is an open condition (cf. Remark 2.16), there is a neighborhood U of $(0,\dots,0)\in\bigoplus_{j=1}^{p-1}H^0(K^{2j})$ such that $\widetilde{\Psi}(I,\widehat{W},\lambda^p\hat{\eta},\lambda^2q_2,\lambda^4q_4,\dots,\lambda^{2p-2}q_{2p-2})$ is stable for $(\lambda^2q_2,\dots,\lambda^{2p-2}q_{2p-2})\in U$ i.e. for small λ. From (4.4), $(V,W,\lambda\eta)$ is stable, and thus, (V,W,η) is also stable. This argument is reversible, so (V,W,η) is stable if and only if $(I,\widehat{W},\hat{\eta})$ is stable.

Assume now that $(I,\widehat{W},\hat{\eta})$ is strictly polystable. By Proposition 2.20, there is q' satisfying $p-1\leqslant q'<q$, such that

$$(I,\widehat{W},\hat{\eta})=\Big(\widehat{W}'\oplus\widehat{W}'',\begin{pmatrix}\hat{\eta}' & 0\end{pmatrix}\Big),$$

where $(I,\widehat{W}',\hat{\eta}')$ is a stable K^p-twisted $\mathrm{O}(1,q'-p+1)$-Higgs bundle and $\widehat{W}''$ is a polystable orthogonal bundle of rank $q-q'$. In this case, we have

$$\widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\ldots,q_{2p-2}) = \Big(V,\widehat{W}'\oplus\widehat{W}'',\big(\hat{\eta}' \quad 0\big)\Big)$$

where

$$(V,W',\hat{\eta}') = \widetilde{\Psi}(I,\widehat{W}',\hat{\eta}',q_2,\ldots,q_{2p-2}), \tag{4.5}$$

and the map $\widetilde{\Psi}$ in (4.5) is defined as in (4.2) and (4.3), but with q replaced by q'. By the above argument, $\widetilde{\Psi}(I,\widehat{W}',\hat{\eta}',q_2,\ldots,q_{2p-2})$ is a stable $\mathrm{O}(p,q')$-Higgs bundle. Since $\widehat{W}''$ is a polystable orthogonal bundle, we conclude that $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\ldots,q_{2p-2})$ is a strictly polystable $\mathrm{SO}(p,q)$-Higgs bundle. Again, the argument is reversible, hence the converse also holds. □

The next lemma shows that $\widetilde{\Psi}$ both respects isomorphism classes of the corresponding objects and is injective on such classes.

Lemma 4.5. *Two* $\mathrm{SO}(p,q)$*-Higgs bundles* $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\ldots,q_{2p-2})$ *and* $\widetilde{\Psi}(I',\widehat{W}',\hat{\eta}',q_2',\ldots,q_{2p-2}')$ *are in the same* $\mathrm{S}(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C}))$*-gauge orbit if and only if* $(I,\widehat{W},\hat{\eta})$ *and* $(I',\widehat{W}',\hat{\eta}')$ *are in the same* $\mathrm{S}(\mathrm{O}(1,\mathbb{C})\times\mathrm{O}(q-p+1,\mathbb{C}))$*-gauge orbit and* $q_{2j}=q_{2j}'$ *for all* $1\leqslant j\leqslant p-1$*. Furthermore, each* $\mathrm{S}(\mathrm{O}(1,\mathbb{C})\times\mathrm{O}(q-p+1,\mathbb{C}))$*-gauge transformation between* $(I,\widehat{W},\eta)$ *and* $(I',\widehat{W}',\hat{\eta}')$ *uniquely determines an* $\mathrm{S}(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C}))$*-gauge transformation between the Higgs bundles* $\widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\ldots,q_{2p-2})$ *and* $\widetilde{\Psi}(I',\widehat{W}',\hat{\eta}',q_2,\ldots,q_{2p-2})$*.*

Proof. Let $(I,\widehat{W},\hat{\eta})$ and $(I',\widehat{W}',\hat{\eta}')$ be two points in $\mathcal{H}_{K^p}(\mathrm{SO}(1,q-p+1))$, and $(q_2,\ldots,q_{2p-2})$ and $(q_2',\ldots,q_{2p-2}')$ be two points in $\bigoplus_{j=1}^{p-1} H^0(K^{2j})$. Denote the associated points in the image of the map $\widetilde{\Psi}$ from (4.3) by

$$(V,W,\eta) = \widetilde{\Psi}(I,\widehat{W},\hat{\eta},q_2,\ldots,q_{2p-2}) \qquad \text{and} \qquad (V',W',\eta') = \widetilde{\Psi}(I',\widehat{W}',\hat{\eta}',q_2',\ldots,q_{2p-2}'),$$

and recall that $V = I\otimes\mathcal{K}_{p-1}$ and $W = \widehat{W}\oplus I\otimes\mathcal{K}_{p-2}$.

First suppose $(\det(g_{\widehat{W}}),g_{\widehat{W}})$ is an $\mathrm{S}(\mathrm{O}(1,\mathbb{C})\times\mathrm{O}(q-p+1,\mathbb{C}))$-gauge transformation with

$$(\det(g_{\widehat{W}}),g_{\widehat{W}})\cdot(I,\widehat{W},\hat{\eta}) = (I',\widehat{W}',\hat{\eta}').$$

A straightforward computation shows that the $\mathrm{S}(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C}))$-gauge transformation

$$(g_V,g_W) = \left(\det(g_{\widehat{W}})\,\mathrm{Id}_V, \begin{pmatrix} g_{\widehat{W}} & 0\\ 0 & \det(g_{\widehat{W}})\,\mathrm{Id}_{\mathcal{K}_{p-2}}\end{pmatrix}\right) \tag{4.6}$$

acts on (V,W,η) as

$$(g_V,g_W)\cdot(V,W,\eta) = \widetilde{\Psi}(I',\widehat{W}',\hat{\eta}',q_2,\ldots,q_{2p-2}).$$

Thus, if (I,W,η) and (I',W',η') are in the same $\mathrm{S}(\mathrm{O}(1,\mathbb{C})\times\mathrm{O}(q-p+1,\mathbb{C}))$-gauge orbit, then $\widetilde{\Psi}(I,W,\eta,q_2,\ldots,q_{2p-2})$ and $\widetilde{\Psi}(I',W',\eta',q_2,\ldots,q_{2p-2})$ are in the same $\mathrm{S}(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C}))$-gauge orbit.

Now suppose (V,W,η) and (V',W',η') are in the same $\mathrm{S}(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C}))$-gauge orbit. The action of (g_V,g_W) on (V,W,η) is given by

$$(g_V,g_W)\cdot(\bar{\partial}_V,\bar{\partial}_W,\eta) = (g_V^{-1}\bar{\partial}_V g_V\ ,\ g_W^{-1}\bar{\partial}_W g_W\ ,\ g_V^{-1}\eta g_W)\ .$$

With respect to the decompositions $W = \widehat{W}\oplus I\otimes\mathcal{K}_{p-2}$ and $W' = \widehat{W}'\oplus I'\otimes\mathcal{K}_{p-2}$, write

$$g_W = \begin{pmatrix} g_{\widehat{W}} & A\\ B & g_{\mathcal{K}_{p-2}}\end{pmatrix} \qquad \text{and} \qquad \eta = \begin{pmatrix}\eta_{\widehat{W}} & \eta(q_2,\ldots,q_{2p-2})\end{pmatrix}.$$

The gauge transformation (g_V,g_W) acts on the Higgs field by

$$g_V^{-1}\eta g_W = g_V^{-1}\cdot\Big(\eta_{\widehat{W}}g_{\widehat{W}} + \eta(q_2,\ldots,q_{2p-2})B \quad \eta_{\widehat{W}}A + \eta(q_2,\ldots,q_{2p-2})g_{\mathcal{K}_{p-2}}\Big)\ ,$$

and hence

$$\begin{pmatrix}\eta'_{\widehat{W}} & \eta(q'_2,\ldots,q'_{2p-2})\end{pmatrix} = g_V^{-1}\cdot\begin{pmatrix}\eta_{\widehat{W}}g_{\widehat{W}}+\eta(q_2,\ldots,q_{2p-2})B & \eta_{\widehat{W}}A+\eta(q_2,\ldots,q_{2p-2})g_{\mathcal{K}_{p-2}}\end{pmatrix}. \tag{4.7}$$

We now use the description of $\eta(q_2,\ldots,q_{2p-2})$ from (2.13). Since g_V^{-1} is invertible and holomorphic, its matrix representation in the decompositions $V=I\otimes\mathcal{K}_{p-1}$ and $V'=I'\otimes\mathcal{K}_{p-1}$ is upper triangular with nonzero diagonal entries. A straightforward computation, using the form of $\eta(q'_2,\ldots,q'_{2p-2})$ and the fact that $g_V^{-1}\eta_{\widehat{W}}g_{\widehat{W}}$ has the form $\begin{pmatrix}*\\0\\\vdots\\0\end{pmatrix}$, shows that $B=0$. By orthogonality of g_W we conclude also that $A=0$, $g_{\widehat{W}}$ is an $Q_{\widehat{W}}$-orthogonal gauge transformation and $g_{\mathcal{K}_{p-2}}$ is a $Q_{\mathcal{K}_{p-2}}$-orthogonal gauge transformation.

We now have $\eta(q'_2,\ldots,q'_{2p-2})=g_V^{-1}\eta(q_2,\ldots,q_{2p-2})g_{\mathcal{K}_{p-2}}$. Since $(I\otimes\mathcal{K}_{p-1},I\otimes\mathcal{K}_{p-2},\eta(q_2,\ldots,q_{2p-2}))$ and $(I'\otimes\mathcal{K}_{p-1},I'\otimes\mathcal{K}_{p-2},\eta(q'_2,\ldots,q'_{2p-2}))$ define gauge equivalent Higgs bundle in an $\mathrm{O}(p,p-1)$-Hitchin component, we have $(q_2,\ldots,q_{2p-2})=(q'_2,\ldots,q'_{2p-2})$. By Remark 2.22, this implies

$$(g_V,g_{\mathcal{K}_{p-2}})=\pm(\mathrm{Id}_V,\mathrm{Id}_{\mathcal{K}_{p-2}})\ .$$

Finally, the determinant of $g_{\widehat{W}}$ determines the above sign since $\det(-\mathrm{Id}_V)\det(-\mathrm{Id}_{\mathcal{K}_{p-2}})=-1$ and

$$1=\det(g_V)\det(g_W)=\det(g_V)\det(g_{\mathcal{K}_{p-2}})\det(g_{\widehat{W}}).$$

Thus, the gauge transformation $g_{\widehat{W}}$ uniquely determines $g_{\mathcal{K}_{p-2}}$ and g_V. This shows that (g_V,g_W) is given by (4.6), completing the proof. □

As a consequence of the two previous lemmas, we have the following proposition.

Proposition 4.6. *The map $\widetilde{\Psi}$ from (4.3) descends to a continuous map of moduli spaces*

$$\Psi:\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))\times\bigoplus_{j=1}^{p-1}H^0(K^{2j})\longrightarrow\mathcal{M}(\mathrm{SO}(p,q)), \tag{4.8}$$

which is a homeomorphism onto its image.

Remark 4.7. From Remark 2.14, one can check that the dimension of $\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))\times\bigoplus_{j=1}^{p-1}H^0(K^{2j})$ is the expected dimension of $\mathcal{M}(\mathrm{SO}(p,q))$. In particular, the map Ψ is open on the smooth locus. Since the spaces $\mathcal{M}(\mathrm{SO}(p,q))$ and $\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))$ are singular, we have to examine the local structures of each space to prove openness of Ψ at singular points.

4.2. **Local structure of fixed points in the image of Ψ.** We will now analyze the local structure of fixed points of the $\mathbb{C}^*$-action in $\mathcal{M}(\mathrm{SO}(p,q))$ which lie in the image of the map Ψ. The following lemma follows immediately from Lemma 3.9 and Proposition 4.6.

Lemma 4.8. *An $\mathrm{SO}(p,q)$-Higgs bundle (V,W,η) in the image of Ψ is a fixed point of the $\mathbb{C}^*$-action if and only if $(V,W,\eta)=\Psi(I,\widehat{W},\hat{\eta},0,\ldots,0)$, where $(I,\widehat{W},\hat{\eta})$ is a fixed point of the $\mathbb{C}^*$-action in $\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))$. In particular, such a fixed point is given by* [2]

$$(I,\widehat{W},\hat{\eta})=\left(I,W_{-p}\oplus W'_0\oplus W_p,\begin{pmatrix}\eta_{-p} & 0 & 0\end{pmatrix}\right)\ ,$$

where W'_0 is a polystable orthogonal bundle of rank $q-p+1-2\,\mathrm{rk}(W_p)$ and $\det(W'_0)=I$, W_p is either zero or a negative degree vector bundle with no non-negative degree subbundles, $W_{-p}\cong W_p^*$ and η_{-p} is nonzero if W_{-p} is nonzero. The associated $\mathrm{SO}(p,q)$-Higgs bundle will be represented by

$$\begin{array}{c}W_{-p}\xrightarrow{\eta_{-p}}IK^{p-1}\xrightarrow{1}IK^{p-2}\xrightarrow{1}\cdots\xrightarrow{1}I\xrightarrow{1}\cdots\xrightarrow{1}IK^{2-p}\xrightarrow{1}IK^{1-p}\xrightarrow{\eta_{-p}^*}W_p\ .\\ \oplus\\ W'_0\end{array} \tag{4.9}$$

[2]The notation from Lemma 3.9 has changed slightly, (W_{-1},η_{-1},W_0) is now represented by (W_{-p},η_{-p},W'_0).

Let (V, W, η) be a polystable $\mathrm{SO}(p,q)$-Higgs bundle in the image of Ψ of the form (4.9). This will be fixed until the end of Section 4.2. If W_p is zero, some of the considerations below simplify.

We will consider a graded complex similar to (3.9) and repeatedly use the following bundle decompositions of V and W from (4.9):

$$\begin{gathered} V = V_{1-p} \oplus V_{3-p} \oplus \cdots \oplus V_{p-3} \oplus V_{p-1}, \\ W = W_{-p} \oplus W_{2-p} \oplus \cdots \oplus W_0 \oplus \cdots \oplus W_{p-2} \oplus W_p, \\ V_j = IK^{-j} \text{ for all } j, \quad W_j = IK^{-j} \text{ if } 0 < |j| < p, \quad \text{and} \quad W_0 = \begin{cases} W_0' & \text{if } p \text{ odd} \\ I \oplus W_0' & \text{if } p \text{ even.} \end{cases} \end{gathered} \tag{4.10}$$

In terms of the above splittings, we have $\mathrm{End}(V) = \bigoplus_{k=2-2p}^{2p-2} \mathrm{End}_k(V)$, where $\mathrm{End}_{2k+1}(V) = 0$ and

$$\mathrm{End}_{2k}(V) = \begin{cases} \bigoplus_{j=0}^{p-1-k} \mathrm{Hom}(V_{1-p+2j}, V_{1-p+2j+2k}) & k \geqslant 0 \\ \bigoplus_{j=0}^{p-1+k} \mathrm{Hom}(V_{p-1-2j}, V_{p-1-2j+2k}) & k < 0. \end{cases} \tag{4.11}$$

Similarly, $\mathrm{End}(W) = \bigoplus_{k=-2p}^{2p} \mathrm{End}_k(W)$, where

$$\mathrm{End}_{2k}(W) = \begin{cases} \mathrm{End}(W_0) \oplus \bigoplus_{j=0}^{p} \mathrm{End}(W_{p-2j}) & k = 0 \text{ and } p \text{ odd} \\ \bigoplus_{j=0}^{p-k} \mathrm{Hom}(W_{-p+2j}, W_{-p+2j+2k}) & k > 0 \text{ or } k = 0 \text{ and } p \text{ even} \\ \bigoplus_{j=0}^{p+k} \mathrm{Hom}(W_{p-2j}, W_{p-2j+2k}) & k < 0 \end{cases} \tag{4.12}$$

and

$$\mathrm{End}_{2k+1}(W) = \begin{cases} \mathrm{Hom}(W_{-2k-1}, W_0) \oplus \mathrm{Hom}(W_0, W_{2k+1}) & 2k+1 \leqslant p \text{ and } p \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

Finally, $\mathrm{Hom}(W, V) = \bigoplus_{k=1-2p}^{2p-1} \mathrm{Hom}_k(W, V)$, where

$$\mathrm{Hom}_{2k+1}(W, V) = \begin{cases} \bigoplus_{j=0}^{p-1-k} \mathrm{Hom}(W_{-p+2j}, V_{1-p+2j+2k}) & 2k+1 \geqslant 0 \\ \bigoplus_{j=0}^{p+k} \mathrm{Hom}(W_{p-2j}, V_{p-2j+1+2k}) & 2k+1 < 0, \end{cases} \tag{4.14}$$

and

$$\mathrm{Hom}_{2k}(W, V) = \begin{cases} \mathrm{Hom}(W_0, V_{2k}) & 1-p \leqslant 2k \leqslant p-1 \text{ and } p \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Note that the Higgs field η is a holomorphic section of $\mathrm{Hom}_1(W, V) \otimes K$.

The Lie algebra bundle $\mathfrak{so}(V) \oplus \mathfrak{so}(W) \subset \mathrm{End}(V) \oplus \mathrm{End}(W)$ with fiber $\mathfrak{so}(p, \mathbb{C}) \oplus \mathfrak{so}(q, \mathbb{C})$ consists of Q_V and $-Q_W$ skew symmetric endomorphisms of V and W respectively. The decompositions (4.11), (4.12) and (4.13) induce the following decomposition of $\mathfrak{so}(V)\oplus\mathfrak{so}(W)\subset \mathrm{End}(V)\oplus\mathrm{End}(V)$:

$$\mathfrak{so}(V)=\bigoplus_{k=2-2p}^{2p-2}\mathfrak{so}_k(V)\qquad\text{and}\qquad \mathfrak{so}(W)=\bigoplus_{k=-2p}^{2p}\mathfrak{so}_k(W).$$

Here $\mathfrak{so}_{2k+1}(V)=0$ and, using (4.11),

$$\mathfrak{so}_{2k}(V)=\begin{cases}\{(\alpha_{1-p},\alpha_{3-p},\dots,\alpha_{p-1-2k})\in\mathrm{End}_{2k}(V)\mid \alpha_i=-\alpha^*_{-2k-i}\} & k\geqslant 0\\ \{(\alpha_{p-1},\alpha_{p-3},\dots,\alpha_{1-p-2k})\in\mathrm{End}_{2k}(V)\mid \alpha_i=-\alpha^*_{-2k-i}\} & k<0,\end{cases}\tag{4.16}$$

where the index of each homomorphism corresponds to the index of its domain, i.e.,

$$\alpha_i:V_i\to V_{i+2k}.$$

For $\mathfrak{so}(W)$, using (4.12) we have
(4.17)

$$\mathfrak{so}_{2k}(W)=\begin{cases}\{(\beta',\beta_p,\beta_{p-2},\dots,\beta_{-p})\in\mathrm{End}_0(W)|\ \beta'=-(\beta')^*,\ \beta_i=-\beta^*_{-i}\} & k=0\text{ and }p\text{ odd}\\ \{(\beta_{-p},\beta_{2-p},\dots,\beta_{p-2k})\in\mathrm{End}_{2k}(W)\mid\beta_i=-\beta^*_{-2k-i}\} & k>0\text{ or }k=0\text{ and }p\text{ even}\\ \{(\beta_p,\beta_{p-2},\dots,\beta_{-p+2k})\in\mathrm{End}_{2k}(W)\mid\beta_i=-\beta^*_{-2k-i}\} & k<0,\end{cases}$$

where $\beta':W_0\to W_0$ and, as above, $\beta_i:W_i\to W_{i+2k}$. For odd weights, using (4.13) we have
(4.18)

$$\mathfrak{so}_{2k+1}(W)=\begin{cases}\{(\beta_{-2k-1},-\beta^*_{-2k-1})\in\mathrm{Hom}(W_{-2k-1},W_0)\oplus\mathrm{Hom}(W_0,W_{2k+1})\} & 2k+1\leqslant p\text{ and }p\text{ odd}\\ 0 & \text{otherwise.}\end{cases}$$

Since $\eta\in H^0(\mathrm{Hom}_1(W,V)\otimes K)$, the map ad_η restricts to $\mathfrak{so}_k(V)\oplus\mathfrak{so}_k(W)\to\mathrm{Hom}_{k+1}(W,V)\otimes K$, yielding the subcomplex $C^\bullet_k$ of $C^\bullet$ of weight k as in (3.2)

$$C^\bullet_k=C^\bullet(V,W,\eta)_k:\mathfrak{so}_k(V)\oplus\mathfrak{so}_k(W)\xrightarrow{\mathrm{ad}_\eta}\mathrm{Hom}_{k+1}(W,V)\otimes K,\qquad(\alpha,\beta)\mapsto\eta\circ\beta-\alpha\circ\eta.$$

This gives rise to a splitting of the hypercohomology sequence associated to $C^\bullet$:
(4.19)

$$0\longrightarrow\mathbb{H}^0(C^\bullet_k)\longrightarrow H^0(\mathfrak{so}_k(V)\oplus\mathfrak{so}_k(W))\xrightarrow{\mathrm{ad}_\eta}H^0(\mathrm{Hom}_{k+1}(W,V)\otimes K)\to\mathbb{H}^1(C^\bullet_k)$$
$$\hookrightarrow H^1(\mathfrak{so}(V)_{k+1}\oplus\mathfrak{so}_{k+1}(W))\xrightarrow{\mathrm{ad}_\eta}H^1(\mathrm{Hom}_{k+1}(W,V)\otimes K)\longrightarrow\mathbb{H}^2(C^\bullet_k)\longrightarrow 0.$$

For all k, we will compute $\mathbb{H}^1(C^\bullet_k)$ and show $\mathbb{H}^2(C^\bullet_k)$ vanishes in a series of lemmas. Using (4.10) and the decomposition of $\mathrm{Hom}_1(W,V)\otimes K$ from (4.14), we write

$$\eta=(\eta_{-p},\eta_{2-p},\dots,\eta_{p-2})\in\bigoplus_{j=0}^{p-1}H^0(\mathrm{Hom}(W_{-p+2j},V_{1-p+2j})\otimes K),\tag{4.20}$$

where

$$\begin{cases}\eta_{-p}:W_{-p}\to V_{1-p}\otimes K & \text{is defined in Lemma 4.8,}\\ \eta_0=\left(\begin{smallmatrix}1&0\end{smallmatrix}\right):I\oplus W_0'\to V_1\otimes K & \text{if }p\text{ even,}\\ \eta_i=1:W_i\to V_{i+1}\otimes K & \text{otherwise.}\end{cases}\tag{4.21}$$

Lemma 4.9. *The map $\mathrm{ad}_\eta:\mathfrak{so}_k(V)\oplus\mathfrak{so}_k(W)\to\mathrm{Hom}_{k+1}(W,V)\otimes K$ is an isomorphism for each positive weight $k\notin\{p,2p\}$. In particular,*

$$\mathbb{H}^0(C^\bullet_k)=0,\qquad\mathbb{H}^1(C^\bullet_k)=0\qquad\textit{and}\qquad\mathbb{H}^2(C^\bullet_k)=0\,.$$

Proof. We start by considering the case $C^\bullet_{2k+1}$ with $0<2k+1$ and $2k+1\neq p$. If p is even or $p<2k+1$, the result is immediate since $\mathfrak{so}_{2k+1}(V)$, $\mathfrak{so}_{2k+1}(W)$ and $\mathrm{Hom}_{2k+2}(W,V)\otimes K$ are all zero by (4.13) and (4.15). For p odd and $2k+1<p$, we have $\mathfrak{so}_{2k+1}(V)=0$, $\mathfrak{so}_{2k+1}(W)=\{(\beta_{-2k-1},-\beta^*_{-2k-1})\in$

$\mathrm{Hom}(W_{-2k-1}, W_0)\oplus\mathrm{Hom}(W_0, W_{2k+1})\}$ and $\mathrm{Hom}_{2k+2}(W,V)\otimes K = \mathrm{Hom}(W_0, V_{2k+2})\otimes K$. Using (4.20), the map ad_η is the isomorphism sending β_{-2k-1} to the composition of $-\beta^*_{-2k-1}$ with $1=\eta_{2k+1}$:

$$\begin{array}{ccc} W_0 & \longrightarrow & V_{2k+2}\otimes K \\ & {\scriptstyle -\beta^*_{-2k-1}}\searrow \quad W_{2k+1} \quad \nearrow{\scriptstyle 1} & \end{array}$$

Now consider the case $C^\bullet_{2k}$ with $0<2k$ and $2k\notin\{p,2p\}$. We first show $\mathfrak{so}_{2k}(V)\oplus\mathfrak{so}_{2k}(W)$ and $\mathrm{Hom}_{2k+1}(W,V)\otimes K$ are isomorphic. Using (4.10) and (4.14), we have

$$\mathrm{Hom}_{2k+1}(W,V)\otimes K \cong \begin{cases} \mathrm{Hom}(W_{-p}, IK^{p-2k})\oplus \underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{p-k-1\text{ times}} & 2k>p \text{ or } p \text{ odd},\\ \mathrm{Hom}(W_{-p}, IK^{p-2k})\oplus \mathrm{Hom}(W_0', IK^{-2k})\oplus \underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{p-k-1\text{ times}} & \text{otherwise.}\end{cases} \tag{4.22}$$

On the other hand, by (4.16) and since the weight is positive, we have

$$\mathfrak{so}_{2k}(V)\cong \bigoplus_{j=0}^{\lfloor\frac{p-k}{2}\rfloor-1}\mathrm{Hom}(V_{2j-p+1}, V_{2j-p+1+2k})\cong \underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{\lfloor\frac{p-k}{2}\rfloor\text{ times}}. \tag{4.23}$$

Similarly, by (4.17), $\mathfrak{so}_{2k}(W)\cong \bigoplus_{j=0}^{\lfloor\frac{p-k-1}{2}\rfloor}\mathrm{Hom}(W_{2j-p}, W_{2j-p+2k})$, and thus,

$$\mathfrak{so}_{2k}(W)\cong \begin{cases} \mathrm{Hom}(W_{-p}, IK^{p-2k})\oplus \underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{\lfloor\frac{p-k-1}{2}\rfloor\text{ times}} & 2k>p \text{ or } p \text{ odd}\\ \mathrm{Hom}(W_{-p}, IK^{p-2k})\oplus \mathrm{Hom}(W_0', IK^{-2k})\oplus \underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{\lfloor\frac{p-k-1}{2}\rfloor\text{ times}} & \text{otherwise.}\end{cases} \tag{4.24}$$

From (4.22), (4.23) and (4.24), we see that $\mathfrak{so}_{2k}(V)\oplus\mathfrak{so}_{2k}(W)$ is isomorphic to $\mathrm{Hom}_{2k+1}(W,V)\otimes K$.

Now we will show

$$C^\bullet_{2k}:\mathfrak{so}_{2k}(V)\oplus\mathfrak{so}_{2k}(W)\xrightarrow{\mathrm{ad}_\eta}\mathrm{Hom}_{2k+1}(W,V)\otimes K, \qquad \mathrm{ad}_\eta(\alpha,\beta)=\eta\circ\beta-\alpha\circ\eta$$

is an isomorphism. Using the notations of (4.16), (4.17) (for positive weight) and (4.20), if

$$\alpha=(\alpha_{1-p},\alpha_{3-p},\ldots,\alpha_{p-1-2k}),\quad \beta=(\beta_{-p},\beta_{2-p},\ldots,\beta_{p-2k})\quad\text{and}\quad \eta=(\eta_{-p},\eta_{2-p},\ldots,\eta_{p-2}),$$

then

$$\mathrm{ad}_\eta(\alpha,\beta)=(\eta_{-p+2k}\beta_{-p}-\alpha_{1-p}\eta_{-p},\eta_{2-p+2k}\beta_{2-p}-\alpha_{3-p}\eta_{2-p},\ldots,\eta_{p-2}\beta_{p-2-2k}-\alpha_{p-1-2k}\eta_{p-2-2k}).$$

First assume $p-k$ is even. In this case we have

$$\alpha=(\alpha_{1-p},\ldots,\alpha_{-k-1},-\alpha^*_{-k-1},\ldots,-\alpha^*_{1-p})\quad\text{and}\quad\beta=(\beta_{-p},\ldots,\beta_{-k-2},0,-\beta^*_{-k-2},\ldots,-\beta^*_{-p}).$$

For p odd or $2k>p$, we have $\eta_i=1$ for all $i\neq -p$ by (4.21). Hence $\mathrm{ad}_\eta(\alpha,\beta)$ is given by

$$(\beta_{-p}-\alpha_{1-p}\eta_{-p},\beta_{2-p}-\alpha_{3-p},\ldots,\beta_{-k-2}-\alpha_{-k-1},\alpha^*_{-k-1},-\beta^*_{-k-2}+\alpha^*_{-k-3},\ldots,-\beta^*_{2-p}+\alpha^*_{1-p}). \tag{4.25}$$

This vanishes if and only if α and β are both identically zero, so ad_η is an isomorphism. For p even and $2k\leqslant p$, the only difference is that $W_0=I\oplus W_0'$. Therefore, if we write

$$\beta_0=\begin{pmatrix}\beta_0^I & \beta_0'\end{pmatrix}: I\oplus W_0'\to W_{2k},$$

then the terms $W_0\to V_{2k+1}\otimes K$ and $W_{-2k}\to V_1\otimes K$ of ad_η are given by

$$\begin{pmatrix}\beta_0^I-\alpha_1 & \beta_0'\end{pmatrix}: I\oplus W_0'\to V_{2k+1}\otimes K\qquad\text{and}\qquad -\beta_0^{I*}+\alpha_1^*: W_{-2k}\to V_1\otimes K. \tag{4.26}$$

Again, ad_η vanishes if and only if α and β both vanish, and is therefore an isomorphism.

Now suppose $p-k$ is odd. In this case, (4.16) and (4.17) imply that

$$\alpha = (\alpha_{1-p}, \dots, \alpha_{-k-2}, 0, -\alpha^*_{-k-2}, \dots, -\alpha^*_{1-p}) \quad \text{and} \quad \beta \quad = (\beta_{-p}, \dots, \beta_{-k-1}, -\beta^*_{-k-1}, \dots, -\beta^*_{-p}).$$

For p odd or $2k > p$, $ad_\eta(\alpha, \beta)$ is given by

$$(\beta_{-p} - \alpha_{1-p}\eta_{-p}, \beta_{2-p} - \alpha_{3-p}, \dots, \beta_{-k-3} - \alpha_{-k-2}, \beta_{-k-1}, -\beta^*_{-k-1} + \alpha^*_{-k-2}, \dots, -\beta^*_{2-p} + \alpha^*_{1-p}).$$

Since this vanishes if and only if α and β both vanish, ad_η is an isomorphism. The case of p even and $2k \leqslant p$ follows from a similar calculation as the one done above.

Since $\mathfrak{so}_k(V) \oplus \mathfrak{so}_k(W) \xrightarrow{\mathrm{ad}_\eta} \mathrm{Hom}_{k+1}(W, V) \otimes K$ is an isomorphism for all positive weights k different than p and $2p$, we conclude that the hypercohomology groups $\mathbb{H}^*(C^\bullet_k)$ all vanish for such k. □

Next we consider the subcomplexes of weight p and $2p$.

Lemma 4.10. *The hypercohomology groups $\mathbb{H}^*(C^\bullet_p)$ and $\mathbb{H}^*(C^\bullet_{2p})$ are given by*

$$\mathbb{H}^0(C^\bullet_p) = 0, \qquad \mathbb{H}^1(C^\bullet_p) \cong H^1(\mathrm{Hom}(W_{-p}, W_0')) \qquad \textit{and} \qquad \mathbb{H}^2(C^\bullet_p) = 0,$$
$$\mathbb{H}^0(C^\bullet_{2p}) = 0, \qquad \mathbb{H}^1(C^\bullet_{2p}) \cong H^1(\mathfrak{so}_{2p}(W)) \qquad \textit{and} \qquad \mathbb{H}^2(C^\bullet_{2p}) = 0,$$

where $\mathfrak{so}_{2p}(W) = \{\beta \in \mathrm{Hom}(W_{-p}, W_p) | \beta + \beta^* = 0\}$.

Proof. First note that $\mathfrak{so}_{2p}(V) = 0$, $\mathfrak{so}_{2p}(W) = \{\beta \in \mathrm{Hom}(W_{-p}, W_p)|\beta+\beta^* = 0\}$ and $\mathrm{Hom}_{2p+1}(W, V) = 0$, hence

$$\mathbb{H}^0(C^\bullet_{2p}) \cong H^0(\mathfrak{so}_{2p}(W)), \qquad \mathbb{H}^1(C^\bullet_{2p}) \cong H^1(\mathfrak{so}_{2p}(W)) \qquad \text{and} \qquad \mathbb{H}^2(C^\bullet_{2p}) = 0\ .$$

If p is odd, then $W_0 = W_0'$, $\mathfrak{so}_p(W) \cong \mathrm{Hom}(W_{-p}, W_0')$, $\mathfrak{so}_p(V) = 0$ and $\mathrm{Hom}_{p+1}(W, V) = 0$, thus

$$\mathbb{H}^0(C^\bullet_p) \cong H^0(\mathrm{Hom}(W_{-p}, W_0')), \qquad \mathbb{H}^1(C^\bullet_p) \cong H^1(\mathrm{Hom}(W_{-p}, W_0')) \qquad \text{and} \qquad \mathbb{H}^2(C^\bullet_p) = 0.$$

Moreover, $H^0(\mathfrak{so}_{2p}(W))$ and $H^0(\mathrm{Hom}(W_{-p}, W_0'))$ were shown to vanish in the proof of Lemma 3.11, completing the proof for the case $2p$ and when p is odd.

Now suppose p is even, then $W_0 = I \oplus W_0'$ and, from (4.11), (4.12) and (4.14), we have

$$\mathfrak{so}_p(V) \cong \underbrace{K^{-p} \oplus \cdots \oplus K^{-p}}_{\lfloor \frac{p}{4} \rfloor \text{ times}}\ ,$$

$$\mathfrak{so}_p(W) \cong \mathrm{Hom}(W_{-p}, I) \oplus \mathrm{Hom}(W_{-p}, W_0') \oplus \underbrace{K^{-p} \oplus \cdots \oplus K^{-p}}_{\lfloor \frac{p-2}{4} \rfloor \text{ times}} \oplus \mathrm{Hom}(W_0', K^{-p})$$

and

$$\mathrm{Hom}_{p+1}(W, V) \otimes K \cong \mathrm{Hom}(W_{-p}, I) \oplus \underbrace{K^{-p} \oplus \cdots \oplus K^{-p}}_{\frac{p}{2}-1 \text{ times}}\ .$$

Thus, $\mathfrak{so}_p(V) \oplus \mathfrak{so}_p(W) \cong \mathrm{Hom}(W_{-p}, W_0') \oplus \mathrm{Hom}_{p+1}(W, V) \otimes K$.

If $\frac{p}{2}$ is even and $(\alpha, \beta) \in \mathfrak{so}_p(V) \oplus \mathfrak{so}_p(W)$, then

$$\alpha = (\alpha_{1-p}, \dots, \alpha_{-\frac{p}{2}-1}, -\alpha^*_{-\frac{p}{2}-1}, \dots, -\alpha^*_{1-p}) \quad \text{and} \quad \beta = (\beta_{-p}, \dots, \beta_{-\frac{p}{2}-2}, 0, -\beta^*_{-\frac{p}{2}-2}, \dots, -\beta^*_{-p})\ .$$

Using the decomposition of η from (4.20) and (4.21), we see that $\mathrm{ad}(\alpha, \beta)$ is given by

$$(\eta_0\beta_{-p} - \alpha_{1-p}\eta_{-p}, \beta_{2-p} - \alpha_{3-p}, \dots, \beta_{-\frac{p}{2}-2} - \alpha_{-\frac{p}{2}-1}, \alpha^*_{-\frac{p}{2}-1}, -\beta^*_{-\frac{p}{2}-2} + \alpha^*_{-\frac{p}{2}-3}, \dots, -\beta^*_{2-p} + \alpha^*_{1-p}).$$

If we write $\beta_{-p} = \begin{pmatrix} \beta^I_{-p} \\ \beta'_{-p} \end{pmatrix} : W_{-p} \to I \oplus W_0'$, then $\eta_0\beta_{-p} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \beta^I_{-p} \\ \beta'_{-p} \end{pmatrix} = \beta^I_{-p}$. Hence $\mathrm{Hom}(W_{-p}, W_0')$ is in the kernel of ad_η and $\eta_0\beta_{-p} - \alpha_{1-p}\eta_{-p} = \beta^I_{-p} - \alpha_{1-p}\eta_{-p}$. We conclude that the map induced by ad_η on $(\mathfrak{so}_p(V) \oplus \mathfrak{so}_p(W))/\mathrm{Hom}(W_{-p}, W_0') \to \mathrm{Hom}_{p+1}(W, V) \otimes K$ is given by

$$\mathrm{ad}_\eta : \mathrm{Hom}(W_{-p}, W_0') \oplus (\mathfrak{so}_p(V) \oplus \mathfrak{so}_p(W))/\mathrm{Hom}(W_{-p}, W_0') \xrightarrow{\begin{pmatrix} 0 & \delta \end{pmatrix}} \mathrm{Hom}_{p+1}(W, V) \otimes K$$

with δ an isomorphism. In particular, this implies that

$$\mathbb{H}^0(C_p^\bullet) \cong H^0(\mathrm{Hom}(W_{-p}, W_0')), \qquad \mathbb{H}^1(C_p^\bullet) \cong H^1(\mathrm{Hom}(W_{-p}, W_0')) \qquad \text{and} \qquad \mathbb{H}^2(C_p^\bullet) = 0.$$

Moreover, $H^0(\mathrm{Hom}(W_{-p}, W_0'))$ was shown to vanish in the proof of Lemma 3.11. The proof for $\frac{p}{2}$ odd follows from similar arguments. □

Now we consider negative odd weights different from $-p$.

Lemma 4.11. *The map* $\mathrm{ad}_\eta : \mathfrak{so}_{2k+1}(V) \oplus \mathfrak{so}_{2k+1}(W) \to \mathrm{Hom}_{2k+2}(W, V) \otimes K$ *is an isomorphism for* $2k+1 < 0$ *and* $2k+1 \neq -p$*. In particular,*

$$\mathbb{H}^0(C_{2k+1}^\bullet) = 0, \qquad \mathbb{H}^1(C_{2k+1}^\bullet) = 0 \qquad \textit{and} \qquad \mathbb{H}^2(C_{2k+1}^\bullet) = 0.$$

Proof. First, note that $\mathfrak{so}_{2k+1}(V) = 0$. Also, if p is even or $2k+1 < -p$, then $\mathfrak{so}_{2k+1}(W) = 0$ and $\mathrm{Hom}_{2k+2}(W, V) = 0$. For p odd and $2k+1 > -p$,

$$\mathfrak{so}_{2k+1}(W) = \{(\beta_{-2k-1}, -\beta_{-2k-1}^*) \in \mathrm{Hom}(W_{-2k-1}, W_0) \oplus \mathrm{Hom}(W_0, W_{2k+1})\}$$

and $\mathrm{Hom}_{2k+2}(W, V) \otimes K = \mathrm{Hom}(W_0, V_{2k+2}) \otimes K$. Moreover, $\mathrm{ad}_\eta : \mathfrak{so}_{2k+1}(W) \to \mathrm{Hom}_{2k+2}(W, V) \otimes K$ is given by

$$\begin{array}{ccc} W_0 & \xrightarrow{\hspace{3em}} & V_{2k+2} \otimes K \\ & {\scriptstyle -\beta_{-2k-1}^*} \searrow \quad W_{2k+1} \quad \nearrow {\scriptstyle 1} & \end{array}$$

which is an isomorphism. □

Next we deal with negative even weights different from $-p$ and $-2p$.

Lemma 4.12. *For* $2k < 0$ *and* $2k \notin \{-p, -2p\}$, $\mathrm{Hom}_{2k+1}(W, V) \otimes K \cong \mathfrak{so}_{2k}(W) \oplus \mathfrak{so}_{2k}(V) \oplus K^{-2k}$ *and* ad_η *decomposes as*

$$\mathrm{ad}_\eta = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) : \mathfrak{so}_{2k}(W) \oplus \mathfrak{so}_{2k}(V) \to \big(\mathfrak{so}_{2k}(W) \oplus \mathfrak{so}_{2k}(V)\big) \oplus K^{-2k},$$

where a is an isomorphism. In particular,

$$\mathbb{H}^0(C_{2k}^\bullet) = 0, \qquad \mathbb{H}^1(C_{2k}^\bullet) \cong H^0(K^{-2k}) \qquad \textit{and} \qquad \mathbb{H}^2(C_{2k}^\bullet) = 0.$$

Proof. Using (4.14), we have that

$$\mathrm{Hom}_{2k+1}(W, V) \otimes K \cong \begin{cases} \mathrm{Hom}(W_p, IK^{-p-2k}) \oplus \underbrace{K^{-2k} \oplus \cdots \oplus K^{-2k}}_{p+k \text{ times}} & \text{if } p \text{ odd or } 2k < -p \\ \mathrm{Hom}(W_p, IK^{-p-2k}) \oplus \mathrm{Hom}(W_0', IK^{-2k}) \oplus \underbrace{K^{-2k} \oplus \cdots \oplus K^{-2k}}_{p+k \text{ times}} & \text{otherwise} . \end{cases}$$

If $p+k$ is even, then by (4.16) and (4.17) we have

$$\mathfrak{so}_{2k}(V) \cong \Big\{(\alpha_{p-1}, \ldots, \alpha_{-k+1}, -\alpha_{-k+1}^*, \ldots, -\alpha_{p-1}^*) \in \bigoplus_{j=0}^{p-1+k} \mathrm{Hom}(V_{p-1-2j}, V_{p-1-2j+2k})\Big\}$$

$$\mathfrak{so}_{2k}(W) \cong \Big\{(\beta_p, \ldots, \beta_{-k+2}, 0, -\beta_{-k+2}^*, \ldots, -\beta_p^*) \in \bigoplus_{j=0}^{p+k} \mathrm{Hom}(W_{p-2j}, W_{p-2j+2k})\Big\}.$$

Thus,

$$\mathfrak{so}_{2k}(V) \cong \underbrace{K^{-2k} \oplus \cdots \oplus K^{-2k}}_{\frac{p+k}{2} \text{ times}}$$

and

$$\mathfrak{so}_{2k}(W)\cong\begin{cases}\mathrm{Hom}(W_p,IK^{-p-2k})\oplus\underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{\frac{p+k}{2}-1\text{ times}} & \text{if } p \text{ odd or } 2k<-p\\ \mathrm{Hom}(W_p,IK^{-p-2k})\oplus\mathrm{Hom}(W_0',IK^{-2k})\oplus\underbrace{K^{-2k}\oplus\cdots\oplus K^{-2k}}_{\frac{p+k}{2}-1\text{ times}} & \text{otherwise}\ .\end{cases}$$

Hence we conclude that $\mathrm{Hom}_{2k+1}(W,V)\otimes K\cong\mathfrak{so}_{2k}(W)\oplus\mathfrak{so}_{2k}(V)\oplus K^{-2k}$. By a similar argument, the conclusion also holds for the case $p+k$ odd.

For the form of ad_η in this splitting, first assume p is odd or $2k<-p$. If $p+k$ is even, then the map $\mathrm{ad}_\eta:\mathfrak{so}_{2k}(V)\oplus\mathfrak{so}_{2k}(W)\to\mathrm{Hom}_{2k+1}(W,V)\otimes K$ is given by

(4.27)
$$\mathrm{ad}_\eta(\alpha,\beta)=(\beta_p,\beta_{p-2}-\alpha_{p-1},\dots,\beta_{-k+2}-\alpha_{-k+3},-\alpha_{-k+1},-\beta^*_{-k+2}+\alpha^*_{-k+1},\dots,\alpha^*_{p-1}-\eta_{-p}\beta^*_p).$$

Consider the summand $K^{-2k}\cong\mathrm{Hom}(W_{-k},V_{k+1})\otimes K$ of $\mathrm{Hom}_{2k+1}(W,V)\otimes K$ and take the corresponding quotient $(\mathrm{Hom}_{2k+1}(W,V)\otimes K)/K^{-2k}$. Then $\mathrm{Hom}_{2k+1}(W,V)\otimes K=(\mathrm{Hom}_{2k+1}(W,V)\otimes K)/K^{-2k}\oplus K^{-2k}$ and, from (4.27), we conclude that ad_η can be written as

$$\mathrm{ad}_\eta=\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right):\mathfrak{so}_{2k}(V)\oplus\mathfrak{so}_{2k}(W)\to\big(\mathrm{Hom}_{2k+1}(W,V)\otimes K)/K^{-2k}\oplus K^{-2k}$$

where a is an isomorphism. If $p+k$ is odd, a similar conclusion holds.

If p is even and $-p<2k$, the only difference is that we have the following decompositions

$$\beta_0=\begin{pmatrix}\beta_0^I & \beta_0'\end{pmatrix}:I\oplus W_0'\to W_{2k}\qquad\text{and}\qquad\beta_0^*=\begin{pmatrix}(\beta_0^I)^*\\ (\beta_0')^*\end{pmatrix}:W_{-2k}\to I\oplus W_0'.$$

With these decompositions, the terms of ad_η which involve β_0 and β_0^* are given by

(4.28)
$$\begin{array}{ccccc} & & V_1\otimes K & & \\ & {\scriptstyle (1\ 0)}\nearrow & & \searrow{\scriptstyle \alpha_1\otimes\mathrm{Id}_K} & \\ I\oplus W_0' & & & & V_{2k+1}\otimes K\\ & {\scriptstyle \beta_0}\searrow & & \nearrow{\scriptstyle 1} & \\ & & W_{2k} & & \end{array}\qquad\text{and}\qquad\begin{array}{ccccc} & & V_{-2k+1}\otimes K & & \\ & {\scriptstyle 1}\nearrow & & \searrow{\scriptstyle -\alpha^*_{-1}\otimes\mathrm{Id}_K} & \\ W_{-2k} & & & & V_1\otimes K.\\ & {\scriptstyle -\beta_0^*}\searrow & & \nearrow{\scriptstyle (1\ 0)} & \\ & & I\oplus W_0' & & \end{array}$$

The map $I\oplus W_0'\to V_{2k+1}\otimes K$ is given by $\left(\begin{smallmatrix}\beta_0^I-\alpha_1 & \beta_0'\end{smallmatrix}\right)$ and the map $W_{-2k}\to W_1\otimes K$ is given by $-(\beta_0^I)^*+\alpha_{-1}$. In particular, we have $\mathrm{ad}_\eta=\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right):\mathfrak{so}_{2k}(W)\oplus\mathfrak{so}_{2k}(V)\to\big(\mathfrak{so}_{2k}(W)\oplus\mathfrak{so}_{2k}(V)\big)\oplus K^{-2k}$ with a an isomorphism.

This implies that in the long exact sequence (4.19), for $2k<0$ and $2k\notin\{-p,-2p\}$, we have

$$\mathbb{H}^0(C^\bullet_{2k})=0,\qquad\mathbb{H}^1(C^\bullet_{2k})\cong H^0(K^{-2k})\qquad\text{and}\qquad\mathbb{H}^2(C^\bullet_{2k})=H^1(K^{-2k})=0,$$

completing the proof. ∎

The next lemma deals with $\mathbb{H}^*(C^\bullet_{-p})$ and $\mathbb{H}^*(C^\bullet_{-2p})$.

Lemma 4.13. *In weight $-2p$ we have $\mathbb{H}^0(C^\bullet_{-2p})=0$, $\mathbb{H}^2(C^\bullet_{2k})=0$ and*

(4.29)
$$0\longrightarrow H^0(\mathfrak{so}_{-2p}(W))\xrightarrow{\eta_{-p}}H^0(\mathrm{Hom}(W_p,K^p))\longrightarrow\mathbb{H}^1(C^\bullet_{-2p})\longrightarrow H^1(\mathfrak{so}_{-2p}(W))\longrightarrow0,$$

where $\mathfrak{so}_{-2p}(W)=\{\beta\in\mathrm{Hom}(W_p,W_{-p})|\beta+\beta^*=0\}$. For p odd, we have

$$\mathbb{H}^0(C^\bullet_{-p})=0,\qquad\mathbb{H}^1(C^\bullet_{-p})\cong\mathbb{H}^1_{-p}\qquad\textit{and}\qquad\mathbb{H}^2(C^\bullet_{-p})=0,$$

where

(4.30)
$$0\longrightarrow H^0(\mathrm{Hom}(W_p,W_0'))\xrightarrow{\eta_{-p}}H^0(\mathrm{Hom}(W_0',K^p))\longrightarrow\mathbb{H}^1_{-p}\longrightarrow H^1(\mathrm{Hom}(W_p,W_0'))\longrightarrow0.$$

For p even,

$$\mathfrak{so}_{-p}(V)\oplus\mathfrak{so}_{-p}(W)\cong\mathrm{Hom}(W_p,W_0')\oplus A\quad\textit{and}\quad\mathrm{Hom}_{1-p}(W,V)\otimes K=K^p\oplus\mathrm{Hom}(W_0',K^p)\oplus A\ ,$$

and with respect to this splitting $\mathrm{ad}_\eta = \begin{pmatrix} 0 & b \\ \eta_{-p} & 0 \\ 0 & a \end{pmatrix}$, *where* $a : A \to A$ *is an isomorphism. In particular, for* p *even,*

$$\mathbb{H}^0(C^\bullet_{-p}) = 0, \qquad \mathbb{H}^1(C^\bullet_{-p}) \cong H^0(K^p) \oplus \mathbb{H}^1_{-p} \qquad \textit{and} \qquad \mathbb{H}^2(C^\bullet_{-p}) = 0.$$

Proof. For weight $-2p$ we have $\mathfrak{so}_{-2p}(V) = 0$, $\mathfrak{so}_{-2p}(W) = \{\beta \in \mathrm{Hom}(W_p, W_{-p}) | \beta + \beta^* = 0\}$ and $\mathrm{Hom}_{-2p+1}(W, V) \otimes K \cong \mathrm{Hom}(W_p, IK^p)$. The map $\mathrm{ad}_\eta : \mathfrak{so}_{-2p}(W) \to \mathrm{Hom}(W_p, K^p)$ is given by $\mathrm{ad}_\eta(\beta) = \eta_{-p}\beta$. The result now follows from Lemmas 3.10 and 3.11.

If p is odd, then by (4.18) and (4.15) we have

$$\mathfrak{so}_{-p}(V) = 0, \qquad \mathfrak{so}_{-p}(W) \cong \mathrm{Hom}(W_p, W_0') \qquad \text{and} \qquad \mathrm{Hom}_{1-p}(W, V) \otimes K = \mathrm{Hom}(W_0', IK^p).$$

The map $\mathrm{ad}_\eta : \mathrm{Hom}(W_p, W_0') \to \mathrm{Hom}(W_0', IK^p)$ is given by $\mathrm{ad}_\eta(\beta_p) = -\eta_{-p}\beta_p^*$. Again, the result now follows from Lemmas 3.10 and 3.11.

If p is even, then

$$\mathfrak{so}_{-p}(V) \cong \underbrace{K^p \oplus \cdots \oplus K^p}_{\lfloor \frac{p}{4} \rfloor \text{ times}},$$
$$\mathfrak{so}_{-p}(W) \cong \mathrm{Hom}(W_p, I) \oplus \mathrm{Hom}(W_p, W_0') \oplus \underbrace{K^p \oplus \cdots \oplus K^p}_{\lfloor \frac{p-2}{4} \rfloor \text{ times}},$$
$$\mathrm{Hom}_{1-p}(W, V) \otimes K \cong \mathrm{Hom}(W_p, I) \oplus \mathrm{Hom}(W_0', IK^p) \oplus \underbrace{K^p \oplus \cdots \oplus K^p}_{\frac{p}{2} \text{ times}}.$$

Setting $A = \mathrm{Hom}(W_p, I) \oplus \underbrace{K^p \oplus \cdots \oplus K^p}_{\frac{p}{2}-1 \text{ times}}$ we have $\mathfrak{so}_{-p}(V) \oplus \mathfrak{so}_{-p}(W) \cong \mathrm{Hom}(W_p, W_0') \oplus A$ and $\mathrm{Hom}_{-p}(W, V) \otimes K \cong K^p \oplus \mathrm{Hom}(W_0', IK^p) \oplus A$. The map ad_η is analogous to the one in the proof of Lemma 4.12 except that (4.28) is given by

$$\begin{array}{ccc} & V_1 \otimes K & \\ {\scriptstyle (1\ 0)} \nearrow & & \searrow {\scriptstyle \alpha_1 \otimes \mathrm{Id}_K} \\ I \oplus W_0' & & V_{-p+1} \otimes K \\ {\scriptstyle -\beta_p^*} \searrow & & \nearrow {\scriptstyle \eta_{-p}} \\ & W_{-p} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W_p & & V_1 \otimes K. \\ {\scriptstyle \beta_p} \searrow & & \nearrow {\scriptstyle (1\ 0)} \\ & I \oplus W_0' & \end{array}$$

Thus, ad_η restricted to $\mathrm{Hom}(W_p, W_0')$ is given by $\beta_p' \mapsto -\eta_{-p}\beta_p'^*$. Hence,

$$\mathrm{ad}_\eta = \begin{pmatrix} 0 & b \\ \eta_{-p} & 0 \\ 0 & a \end{pmatrix} : \mathrm{Hom}(W_p, W_0') \oplus A \longrightarrow K^p \oplus \mathrm{Hom}(W_0', K^p) \oplus A$$

where $a : A \to A$ is an isomorphism.

Since $H^1(K^p) = 0$, we have $\mathbb{H}^2(C^\bullet_{-p}) = 0$. As in the odd case, we also find that $\mathbb{H}^0(C^\bullet_{-p}) = 0$. Moreover, $\mathbb{H}^1(C^\bullet_{-p}) \cong H^0(K^p) \oplus \mathbb{H}^1_{-p}$ where $\mathbb{H}^1_{-p}$ is given by (4.30). □

The final case concerns the weight zero subcomplex.

Lemma 4.14. *There is a bundle* A *so that*

$$\mathfrak{so}_0(V) \oplus \mathfrak{so}_0(W) \cong \mathfrak{so}(W_0') \oplus \mathrm{End}(W_{-p}) \oplus A \qquad \textit{and} \qquad \mathrm{Hom}_1(W, V) \otimes K \cong \mathrm{Hom}(W_{-p}, IK^p) \oplus A\ ,$$

where $\mathfrak{so}(W_0')$ *is the bundle of skew-symmetric endomorphisms of* W_0' *(with respect to to* $Q_{W_0'}$*). With respect to this splitting,*

$$\mathrm{ad}_\eta = \begin{pmatrix} 0 & \eta_{-p} & 0 \\ 0 & b & a \end{pmatrix} : \ \mathfrak{so}(W_0') \oplus \mathrm{End}(W_{-p}) \oplus A \longrightarrow \mathrm{Hom}(W_{-p}, IK^p) \oplus A\ ,$$

where $a : A \to A$ *is an isomorphism. In particular,*

$$\mathbb{H}^2(C_0^\bullet) = 0, \qquad \mathbb{H}^0(C_0^\bullet) = H^0(\mathfrak{so}(W_0')) \qquad \textit{and} \qquad \mathbb{H}^1(C_0^\bullet) = H^1(\mathfrak{so}(W_0')) \oplus \mathbb{H}^1_{0,p}\ ,$$

where

$$0 \to H^0(\mathrm{End}(W_{-p})) \xrightarrow{\eta_{-p}} H^0(\mathrm{Hom}(W_{-p}, IK^p)) \to \mathbb{H}^1_{0,p} \to H^1(\mathrm{End}(W_{-p})) \to H^1(\mathrm{Hom}(W_{-p}, IK^p) \to 0.$$

Proof. By (4.14) we have

$$\mathrm{Hom}_1(W,V)\otimes K \cong \begin{cases} \mathrm{Hom}(W_{-p}, IK^p) \oplus \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{p-1 \text{ times}} & p \text{ odd} \\ \mathrm{Hom}(W_{-p}, IK^p) \oplus \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{p-1 \text{ times}} \oplus \mathrm{Hom}(W_0', I) & p \text{ even} \end{cases}$$

and by (4.16) and (4.17),

$$\mathfrak{so}_0(V)\oplus\mathfrak{so}_0(W) \cong \begin{cases} \mathrm{End}(W_{-p}) \oplus \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{p-1 \text{ times}} \oplus \mathfrak{so}(W_0') & p \text{ odd} \\ \mathrm{End}(W_{-p}) \oplus \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{p-1 \text{ times}} \oplus \mathrm{Hom}(W_0', I) \oplus \mathfrak{so}(W_0') & p \text{ even.} \end{cases}$$

Hence, setting A to be

$$A = \begin{cases} \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{p-1 \text{ times}} & p \text{ odd} \\ \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{p-1 \text{ times}} \oplus \mathrm{Hom}(W_0', I) & p \text{ even} \end{cases}$$

yields $\mathfrak{so}_0(V)\oplus\mathfrak{so}_0(W) = \mathfrak{so}(W_0') \oplus \mathrm{End}(W_{-p}) \oplus A$ and $\mathrm{Hom}(W,V)_1 \otimes K = \mathrm{Hom}(W_{-p}, IK^p) \oplus A$.

Since, W_0' is an invariant bundle, the restriction of the map $\mathrm{ad}_\eta : \mathfrak{so}_0(W)\oplus\mathfrak{so}_0(V) \to \mathrm{Hom}_1(W,V)\otimes K$ to $\mathfrak{so}(W_0')$ is identically zero. The restriction of the map ad_η to $\mathrm{End}(W_{-p})\oplus A$ is similar to (4.25) with the exception that the term $W_{-p} \to V_{1-p}\otimes K$ is given by

$$\begin{array}{ccccc} & & V_{1-p}\otimes K & & \\ & \overset{\eta_{-p}}{\nearrow} & & \overset{\alpha_{1-p}\otimes \mathrm{Id}_K}{\searrow} & \\ W_{-p} & & & & V_{1-p}\otimes K. \\ & \underset{\beta_{-p}}{\searrow} & & \underset{\eta_{-p}}{\nearrow} & \\ & & W_{-p} & & \end{array}$$

In particular, it is given by $\begin{pmatrix} \eta_{-p} & 0 \\ b & a \end{pmatrix} : \mathrm{End}(W_{-p})\oplus A \to \mathrm{Hom}(W_{-p}, IK^p)\oplus A$ where a is an isomorphism.

The hypercohomology complex for $C^\bullet$ splits as a direct sum of the following two complexes

$$0 \longrightarrow \mathbb{H}^0_{0,'} \longrightarrow H^0(\mathfrak{so}(W_0')) \longrightarrow 0 \longrightarrow \mathbb{H}^1_{0,'} \longrightarrow H^1(\mathfrak{so}(W_0')) \longrightarrow 0,$$

and

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0_{0,p} \longrightarrow H^0(\mathrm{End}(W_{-p})) \longrightarrow H^0(\mathrm{Hom}(W_{-p}, IK^p)) \longrightarrow \mathbb{H}^1_{0,p} \\ &\longrightarrow H^1(\mathrm{End}(W_{-p})) \longrightarrow H^1(\mathrm{Hom}(W_{-p}, IK^p)) \longrightarrow \mathbb{H}^2_{0,p} \longrightarrow 0. \end{aligned}$$

By Lemma 3.9, $\left(W_p \oplus I \oplus W_{-p}, \begin{pmatrix} 0 & 0 & 0 \\ \eta_{-p} & 0 & 0 \\ 0 & \eta_{-p}^* & 0 \end{pmatrix}\right)$ is a stable K^p-twisted $\mathrm{O}(2\,\mathrm{rk}(W_p)+1, \mathbb{C})$-Higgs bundle, so the hypercohomology groups $\mathbb{H}^0_{0,p}$ and $\mathbb{H}^2_{0,p}$ both vanish and $\mathbb{H}^1(C_0^\bullet) = H^1(\mathfrak{so}(W_0')) \oplus \mathbb{H}^1_{0,p}$. □

4.3. **Proof of Theorem 4.1.** We are now set up to prove Theorem 4.1. We start by describing a neighborhood of the image of the map Ψ which is open in $\mathcal{M}(\mathrm{SO}(p,q))$.

Proposition 4.15. *For each* $(I, \widehat{W}, \hat{\eta}, q_2, \ldots, q_{2p-2})$ *in* $\mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$*, the second hypercohomology group for the associated* $\mathrm{SO}(p,q)$*-Higgs bundle vanishes*

$$\mathbb{H}^2(C^\bullet(\Psi(I, \widehat{W}, \hat{\eta}, q_2, \ldots, q_{2p-2}))) = 0.$$

In particular, an open neighborhood of $\Psi(I, \widehat{W}, \hat{\eta}, q_2, \ldots, q_{2p-2})$ in $\mathcal{M}(\mathrm{SO}(p,q))$ is isomorphic to an open neighborhood of zero in

$$\mathbb{H}^1(C^\bullet(\Psi(I, \widehat{W}, \hat{\eta}, q_2, \ldots, q_{2p-2})) /\!\!/ \operatorname{Aut}(\Psi(I, \widehat{W}, \hat{\eta}, q_2, \ldots, q_{2p-2})).$$

Proof. By Lemma 3.3, it suffices to prove the above proposition at the fixed points of the $\mathbb{C}^*$-action in the image of Ψ. These are the Higgs bundles given in Lemma 4.8. In Lemmas 4.9, 4.10, 4.11, 4.12, 4.13 and 4.14 it is shown that if (W, V, η) is a fixed point of the $\mathbb{C}^*$-action in the image of Ψ, then each of the graded pieces of $\mathbb{H}^2(C^\bullet(W, V, \eta))$ vanish. □

Proposition 4.16. *For all $\Psi((I, \widehat{W}, \hat{\eta}), 0, \ldots, 0)$ which are fixed points of the $\mathbb{C}^*$-action we have*

$$\mathbb{H}^1(C^\bullet(\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0)) /\!\!/ \operatorname{Aut}(\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0)) \cong \Big(\mathbb{H}^1(C^\bullet(I, \widehat{W}, \hat{\eta})) /\!\!/ \operatorname{Aut}(\widehat{W})\Big) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}).$$

Proof. Let $\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0)$ be a fixed point of the $\mathbb{C}^*$-action. For the $\mathrm{SO}(1, q-p+1)$-Higgs bundle $(I, \widehat{W}, \hat{\eta})$, the first hypercohomology group $\mathbb{H}^1(C^\bullet(I, \widehat{W}, \hat{\eta}))$ of the deformation complex was computed in Lemma 3.11. In Lemmas 4.9, 4.10, 4.11, 4.12, 4.13 and 4.14 it was shown that the first hypercohomology group of the deformation complex of the $\mathrm{SO}(p,q)$-Higgs bundle is given by

$$\mathbb{H}^1(C^\bullet(\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0)) \cong \mathbb{H}^1(C^\bullet(I, \widehat{W}, \hat{\eta})) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}).$$

By Lemma 4.5, every $\mathrm{S}(\mathrm{O}(1,\mathbb{C}) \times \mathrm{O}(q-p+1, \mathbb{C}))$ automorphism $(\det(g_{\widehat{W}}), g_{\widehat{W}})$ of $(I, \widehat{W}, \hat{\eta})$ determines a unique automorphism of $\Psi(I, \widehat{W}, \hat{\eta}, 0 \ldots, 0)$

$$(g_V, g_W) = \left(\det(g_{\widehat{W}}) \,\mathrm{Id}_{\mathcal{K}_{p-1}}, \begin{pmatrix} g_{\widehat{W}} & 0 \\ 0 & \det(g_{\widehat{W}}) \,\mathrm{Id}_{\mathcal{K}_{p-2}} \end{pmatrix}\right).$$

Moreover, the action of such an automorphism on the holomorphic differentials in the above description of $\mathbb{H}^1(C^\bullet(\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0))$ is trivial. Thus,

$$\mathbb{H}^1(C^\bullet(\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0)) /\!\!/ \operatorname{Aut}(\Psi(I, \widehat{W}, \hat{\eta}, 0, \ldots, 0)) \cong \Big(\mathbb{H}^1(C^\bullet(I, \widehat{W}, \hat{\eta})) /\!\!/ \operatorname{Aut}(\widehat{W})\Big) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$$

as claimed. □

Theorem 4.17. *The image of the map Ψ from (4.1) is open and closed.*

Proof. By Proposition 4.15 and Proposition 4.16 the map Ψ is open at all fixed points of the $\mathbb{C}^*$-action. For (V, W, η) in the image of Ψ, there is λ sufficiently close to zero such that $(V, W, \lambda\eta)$ is in a sufficiently small open neighborhood of a fixed point of the $\mathbb{C}^*$-action. Thus, Ψ is open at all points.

To show the image of Ψ is closed, we use the properness of the Hitchin fibration. Namely, suppose $(I, \widehat{W}_i, \hat{\eta}_i, q_2^i, \ldots, q_{2p-2}^i)$ is a sequence of points in $\mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j})$ which diverges. Denote the associated Hitchin fibrations by

$$h_p : \mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1)) \to H^0(K^{2p}) \qquad \text{and} \qquad h : \mathcal{M}(\mathrm{SO}(p,q)) \to \bigoplus_{j=1}^{p} H^0(K^{2j}) .$$

By the properness of h_p, $(q_2^i, \ldots, q_{2p-2}^i, h_p(I, \widehat{W}_i, \hat{\eta}_i))$ diverges in $\bigoplus_{j=1}^{p} H^0(K^{2j})$. Moreover, by the definition of the map Ψ, applying the $\mathrm{SO}(p,q)$-Hitchin fibration to the image sequence yields

$$h(\Psi(I, \widehat{W}_i, \hat{\eta}_i, q_2^i, \ldots, q_{2p-2}^i)) = (q_2^i, \ldots, q_{2p-2}^i, h_p(I, \widehat{W}_i, \hat{\eta}_i)) .$$

Since h is proper, we conclude that $\Psi(I, \widehat{W}_i, \hat{\eta}_i, q_2^i, \ldots, q_{2p-2}^i)$ also diverges in $\mathcal{M}(\mathrm{SO}(p,q))$. □

5. Classification of local minima of the Hitchin function for $\mathcal{M}(\mathrm{SO}(p,q))$

In this section we will prove Theorem 5.9 which classifies all local minima of the Hitchin function (3.1) on $\mathcal{M}(\mathrm{SO}(p,q))$. The strategy of proof is to divide the objects into the following three families:

(1) stable $\mathrm{SO}(p,q)$-Higgs bundles with $\mathbb{H}^2(C^\bullet(V,W,\eta)) = 0$,
(2) stable $\mathrm{SO}(p,q)$-Higgs bundles whose corresponding $\mathrm{SO}(p+q,\mathbb{C})$-Higgs bundle is strictly polystable,
(3) strictly polystable $\mathrm{SO}(p,q)$-Higgs bundles.

The first family consists of points which are either smooth or orbifold points of $\mathcal{M}(\mathrm{SO}(p,q))$. For these points we can use Proposition 3.2 to classify such local minimum. The local minima in the other two families will be described by a direct study of their deformations.

Recall from (3.2) that the deformation complex (2.6) of an $\mathrm{SO}(p,q)$-Higgs bundle (V,W,η) which is a fixed point of the $\mathbb{C}^*$-action decomposes as

$$C_k^\bullet : \mathfrak{so}(V)_k \oplus \mathfrak{so}(W)_k \xrightarrow{\mathrm{ad}_\eta} \mathrm{Hom}(W,V)_{k+1} \otimes K. \tag{5.1}$$

Each graded piece gives rise to the long exact sequence (3.3) in hypercohomology.

5.1. **Stable minima with vanishing $\mathbb{H}^2(C^\bullet)$.** By Proposition 3.5, polystable fixed points of the $\mathbb{C}^*$-action are given by holomorphic chains of the form (3.6) or (3.7). The holomorphic chains (3.6) will be important for us, they are given by

$$\begin{array}{c}\cdots \xrightarrow{\eta_{-3}} V_{-2} \xrightarrow{\eta_1^*} W_{-1} \xrightarrow{\eta_{-1}} V_0 \xrightarrow{\eta_{-1}^*} W_1 \xrightarrow{\eta_1} V_2 \xrightarrow{\eta_{-3}^*} \cdots \\ \oplus \\ \cdots \xrightarrow{\eta_2^*} W_{-2} \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\eta_0^*} W_0 \xrightarrow{\eta_0} V_1 \xrightarrow{\eta_{-2}^*} W_2 \xrightarrow{\eta_2} \cdots\end{array}$$

We start by studying the constraints on these chains imposed by the local minima condition for stable $\mathrm{SO}(p,q)$-Higgs bundles with vanishing $\mathbb{H}^2(C^\bullet)$. This will be done by first proving two lemmas.

Lemma 5.1. *Let (V,W,η) be a stable $\mathrm{SO}(p,q)$-Higgs bundle with $\eta \neq 0$ and $\mathbb{H}^2(C^\bullet(V,W,\eta)) = 0$. If (V,W,η) is a local minimum of f, then the direct sum of holomorphic chains given by (3.6) must have one of the following forms:*

$$V_{-s} \xrightarrow{\eta_{s-1}^*} W_{1-s} \xrightarrow{\eta_{1-s}} \cdots \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\eta_0^*} W_0 \xrightarrow{\eta_0} V_1 \xrightarrow{\eta_{-2}^*} \cdots \xrightarrow{\eta_{1-s}^*} W_{s-1} \xrightarrow{\eta_{s-1}} V_s \tag{5.2}$$

$$W_{-r} \xrightarrow{\eta_{-r}} V_{1-r} \xrightarrow{\eta_{r-2}^*} \cdots \xrightarrow{\eta_1^*} W_{-1} \xrightarrow{\eta_{-1}} V_0 \xrightarrow{\eta_{-1}^*} W_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{r-2}} V_{r-1} \xrightarrow{\eta_{-r}^*} W_r \tag{5.3}$$

$$\begin{array}{c}V_{-r} \xrightarrow{\eta_{r-1}^*} W_{1-r} \xrightarrow{\eta_{1-r}} \cdots \xrightarrow{\eta_1^*} W_{-1} \xrightarrow{\eta_{-1}} V_0 \xrightarrow{\eta_{-1}^*} W_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{1-r}^*} W_{r-1} \xrightarrow{\eta_{r-1}} V_r, \\ \oplus \\ W_0\end{array} \tag{5.4}$$

$$\begin{array}{c}W_{-s} \xrightarrow{\eta_{-s}} V_{1-s} \xrightarrow{\eta_{s-2}^*} \cdots \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\eta_0^*} W_0 \xrightarrow{\eta_0} V_1 \xrightarrow{\eta_{-2}^*} \cdots \xrightarrow{\eta_{s-2}} V_{s-1} \xrightarrow{\eta_{-s}^*} W_s \\ \oplus \\ V_0\end{array} \tag{5.5}$$

Proof. Since (V,W,η) is stable and a fixed point of the $\mathbb{C}^*$-action, it is of the form (3.6) by Proposition 3.7. If one of the chains vanishes we are done, so assume there are two non-zero chains. Let $r \geqslant 0$ be the maximal weight of the first chain in (3.6) and $s \geqslant 0$ be the maximal weight of the second chain. We have $r > 0$ or $s > 0$ since $\eta \neq 0$. Since (V,W,η) is a stable local minimum of the Hitchin function with $\mathbb{H}^2(C^\bullet) = 0$, the subcomplexes from (5.1) are isomorphisms for $k \geqslant 1$ by Proposition 3.2.

If r and s have different parity, then both of the chains start and end with a summand of W if r is even and start and end with a summand of V if r is odd. In either case, $\mathrm{Hom}_{r+s+1}(W,V) \otimes K = 0$

but $\mathfrak{so}_{r+s}(W) \oplus \mathfrak{so}_{r+s}(V)$ is nonzero. Hence, the subcomplex $C^\bullet_{r+s}$ from (5.1) is not an isomorphism for $k = r+s$, contradicting (V, W, η) being a stable minima with $\mathbb{H}^2(C^\bullet) = 0$.

Now assume r and s have the same parity, so the first chain starts and ends with a summand of W if and only if r is odd and the second chain starts and ends with a summand of W if only only if s is even. If $r \geqslant s$, then $\mathrm{Hom}_{2r+1}(W, V) \otimes K = 0$ and $\mathfrak{so}_{2r}(V) \oplus \mathfrak{so}_{2r}(W) = \Lambda^2 V_r \oplus \Lambda^2 W_r$. So the isomorphism of $C^\bullet_{2r}$ implies $\mathrm{rk}(W_r) = 1$ or $\mathrm{rk}(V_r) = 1$. Since $r+s-1$ is odd, we have:

$$\mathfrak{so}_{r+s-1}(V) = \begin{cases} \{(\alpha, -\alpha^*) \in \mathrm{Hom}(V_{-s}, V_{r-1}) \oplus \mathrm{Hom}(V_{1-r}, V_s)\} & \text{if } r \text{ is odd} \\ \{(\alpha, -\alpha^*) \in \mathrm{Hom}(V_{-r}, V_{s-1}) \oplus \mathrm{Hom}(V_{1-s}, V_r)\} & \text{if } r \text{ is even} \end{cases}$$

$$\mathfrak{so}_{r+s-1}(W) = \begin{cases} \{(\beta, -\beta^*) \in \mathrm{Hom}(W_{-r}, W_{s-1}) \oplus \mathrm{Hom}(W_{1-s}, W_r)\} & \text{if } r \text{ is odd} \\ \{(\beta, -\beta^*) \in \mathrm{Hom}(W_{-s}, W_{r-1}) \oplus \mathrm{Hom}(W_{1-r}, W_s)\} & \text{if } r \text{ is even} \end{cases}$$

$$\mathrm{Hom}_{r+s}(W, V) \otimes K \cong \begin{cases} \mathrm{Hom}(W_r, V_{-s}) \otimes K & \text{if } r \text{ is odd} \\ \mathrm{Hom}(W_{-s}, V_r) \otimes K & \text{if } r \text{ is even .} \end{cases}$$

If $s > 0$, then $r+s-1 \geqslant 1$ and the isomorphism $C^\bullet_{r+s-1}$ gives

$$\begin{cases} \mathrm{rk}(V_s)\,\mathrm{rk}(V_{r-1}) + \mathrm{rk}(W_{s-1}) = \mathrm{rk}(V_s) & \text{if } r \text{ is odd} \\ \mathrm{rk}(W_s)\,\mathrm{rk}(W_{1-r}) + \mathrm{rk}(V_{s-1}) = \mathrm{rk}(W_s) & \text{if } r \text{ is even .} \end{cases}$$

This implies either $\mathrm{rk}(W_{1-s}) = 0$ or $\mathrm{rk}(V_{r-1}) = 0$, both of which contradict Proposition 3.7. Thus, we conclude that r is even and $s = 0$, so the holomorphic chain is given by (5.4). A similar argument shows that for $s > r$, the holomorphic chain is of the form (5.5). □

Lemma 5.2. *Let (V, W, η) be a stable $\mathrm{SO}(p,q)$-Higgs bundle which is a local minimum of the Hitchin function with $\eta \neq 0$ and $\mathbb{H}^2(C^\bullet(V, W, \eta)) = 0$; the associated holomorphic chain is given by (5.2), (5.3), (5.4) or (5.5). For all $j \neq 0$, we have $\mathrm{rk}(W_j) = 1$ and $\mathrm{rk}(V_j) = 1$. Moreover:*

- *In case (5.2), $V_j \cong V_{-1}K^{-j-1}$ and $W_j \cong V_{-1}K^{-j-1}$ for $0 < |j| < s$.*
- *In case (5.3), $V_j \cong W_{-1}K^{-j-1}$ and $W_j \cong W_{-1}K^{-j-1}$ for $0 < |j| < r$.*
- *In case (5.4), $\mathrm{rk}(V_0) = 1$, and $V_j \cong V_0K^{-j}$ and $W_j \cong V_0K^{-j}$ for $0 < |j| < r$.*
- *In case (5.5), $\mathrm{rk}(W_0) = 1$, and $V_j \cong V_0K^{-j}$ and $W_j \cong V_0K^{-j}$ for $0 < |j| < s$.*

Proof. The proof involves an inductive argument on the weights. We first consider the case where (V, W, η) is the holomorphic chain (5.4). We have the following decompositions

$$\mathrm{End}(V) = \bigoplus_{j=-2r}^{2r} \mathrm{End}_k(V)\ , \quad \mathrm{End}(W) = \bigoplus_{k=2-2r}^{2r-2} \mathrm{End}_k(W) \quad \text{and} \quad \mathrm{Hom}(W, V) = \bigoplus_{k=1-2r}^{2r-1} \mathrm{Hom}_k(W, V).$$

For $2k > 0$ we have $\mathrm{Hom}_{2k+1}(W, V) = \displaystyle\bigoplus_{j=0}^{r-k-1} \mathrm{Hom}(W_{1-r+2j}, W_{2-r+2j+2k})$,

(5.6)

$$\mathrm{End}_{2k}(V) = \bigoplus_{j=0}^{r-k} \mathrm{Hom}(V_{2j-r}, V_{2j+2k-r}) \quad \text{and} \quad \mathrm{End}_{2k}(W) = \bigoplus_{j=0}^{r-k-1} \mathrm{Hom}(W_{1-r-2j}, W_{1-r+2j+2k}).$$

With respect to these splittings, $\mathfrak{so}(V) = \bigoplus \mathfrak{so}_k(V)$ and $\mathfrak{so}(W) = \bigoplus \mathfrak{so}_k(W)$ where, for $k > 0$

$$\begin{aligned} \mathfrak{so}_{2k}(V) &= \{(\alpha_0, \dots, \alpha_{r-k}) \in \mathrm{End}_{2k}(V) \mid \alpha_i + \alpha^*_{r-k-i} = 0\}, \\ \mathfrak{so}_{2k}(W) &= \{(\beta_0, \dots, \beta_{r-k-1}) \in \mathrm{End}_{2k}(V) \mid \beta_i + \beta^*_{r-k-1-i} = 0\}\ . \end{aligned} \tag{5.7}$$

Since (V, W, η) is a stable minima of the Hitchin function with $\mathbb{H}^2(C^\bullet) = 0$, for all $k > 0$ we have $\mathfrak{so}_k(V) \oplus \mathfrak{so}_k(W) \cong \mathrm{Hom}_{2k+1}(W, V) \otimes K$. Note that r is even and nonzero. The isomorphism for $k = 2r$ implies $\Lambda^2 V_r \cong 0$, hence $\mathrm{rk}(V_r) = 1$.

The isomorphism for $k = 2r-2$ implies $\mathrm{Hom}(V_{-r}, V_{r-2}) \oplus \Lambda^2 W_{1-r} \cong \mathrm{Hom}(W_{1-r}, V_r) \otimes K$. Thus,

$$\mathrm{rk}(V_{r-2}) + \mathrm{rk}(\Lambda^2 W_{1-r}) = \mathrm{rk}(W_{1-r}),$$

which implies $\mathrm{rk}(W_{1-r})$ is either one or two. If $\mathrm{rk}(W_{1-r}) = 2$, taking the determinant of the isomorphism $C^\bullet_{2r-2}$ implies $V_r K^2 = V_{r-2}$. Also, the kernels of the maps $\eta_{r-1} : W_{r-1} \to V_r \otimes K$ and $\eta_{1-r} : W_{1-r} \to V_{2-r} \otimes K$ have negative degree by stability. Using $V_j^* \cong V_{-j}$ and $W_j^* \cong W_{-j}$, we have

$$\deg(V_{r-2}) - 2g + 2 < \deg(W_{r-1}) < \deg(V_r) + 2g - 2,$$

which contradicts $V_r K^2 = V_{r-2}$. So rank $W_{r-1} = 1$ and the isomorphism for $C^\bullet_{2r-2}$ gives the base case of our induction:

$$1 = \mathrm{rk}(V_{-r}) = \mathrm{rk}(W_{1-r}) = \mathrm{rk}(V_{2-r}) \qquad \text{and} \qquad W_{1-r} \cong V_{2-r} K \quad .$$

If $r = 2$ we are done, so assume $r \geqslant 4$ and that for an integer $k \in [1, \frac{r}{2} - 1]$ we have

$$W_{1-r} \cong V_{2-r} K \cong W_{3-r} K^2 \cong \cdots \cong W_{2k-1-r} K^{2k-2} \cong V_{2k-r} K^{2k-1} \ . \tag{5.8}$$

We will prove that $V_{2k-r} \cong W_{2k+1-r} K \cong V_{2k+2-r} K^2$.

The isomorphism $C^\bullet_{2r-2-2k}$ gives

$$\bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{Hom}(V_{2j-r}, V_{r+2j-2-2k}) \oplus \bigoplus_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \mathrm{Hom}(W_{2j+1-r}, W_{r+2j-1-2k}) \cong \bigoplus_{j=0}^{k} \mathrm{Hom}(W_{2j+1-r}, V_{r+2j-2k}) \otimes K. \tag{5.9}$$

since $\Lambda^2 V_{r-k-1} = 0$ for k odd and $\Lambda^2 W_{r-k-1} = 0$ for k even by (5.8). Using (5.8), computing the ranks of both sides gives $\mathrm{rk}(V_{2k+2-r}) + \lfloor \frac{k}{2} \rfloor + \mathrm{rk}(W_{2k+1-r}) + \lfloor \frac{k-1}{2} \rfloor = k + \mathrm{rk}(W_{2k+1-r})$. Thus,

$$\mathrm{rk}(V_{2k+2-r}) = 1.$$

The isomorphism $C^\bullet_{2r-2-4k}$ implies

$$\bigoplus_{j=0}^{k} \mathrm{Hom}(V_{2j-r}, V_{r+2j-2-4k}) \oplus \bigoplus_{j=0}^{k-1} \mathrm{Hom}(W_{2j+1-r}, W_{r+2j-1-4k}) \oplus \Lambda^2 W_{r-1-2k} \cong \bigoplus_{j=0}^{2k} \mathrm{Hom}(W_{2j+1-r}, V_{r+2j-4k}) \otimes K.$$

Using (5.8), this gives the following equality on ranks

$$\sum_{j=0}^{k} \mathrm{rk}(V_{r+2j-2-4k}) + \sum_{j=0}^{k-1} \mathrm{rk}(W_{r+2j-1-4k}) + \mathrm{rk}(\Lambda^2 W_{r-1-2k}) = \sum_{j=0}^{k-1} \mathrm{rk}(V_{r+2j-4k}) + \sum_{j=k}^{2k} \mathrm{rk}(W_{2j+1-r}).$$

Simplifying, yields $\mathrm{rk}(V_{4k+2-r}) + \mathrm{rk}(\Lambda^2 W_{2k+1-r}) = \mathrm{rk}(W_{2k+1-r})$. Thus, $\mathrm{rk}(W_{2k+1-r})$ is one or two.

If $\mathrm{rk}(W_{2k+1-r}) = 2$, then the determinant of the isomorphism in (5.9) gives

$$\bigotimes_{j=0}^{\lfloor \frac{k}{2} \rfloor} V_{r-2j} V_{r+2j-2-2k} \otimes W_{r-1}^2 \Lambda^2 W_{r-1-2k} \otimes \bigotimes_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} W_{r-2j-1} W_{r+2j-1-2k} \cong \bigotimes_{j=0}^{k-1} W_{r-2j-1} V_{r+2j-2k} K \otimes V_r^2 K^2 \otimes \Lambda^2 W_{r-1-2k} \ . \tag{5.10}$$

By (5.8), the above terms satisfy

$$V_{r-2k}^2K^{2-2k} \cong \begin{cases} V_{r-2j}V_{r+2j-2-2k}, & \text{for } j=1,\ldots,\lfloor\frac{k}{2}\rfloor \\ W_{r-2j-1}W_{r+2j-1-2k}, & \text{for } j=1,\ldots,\lfloor\frac{k-1}{2}\rfloor \\ W_{r-2j-1}V_{r+2j-2k}K, & \text{for } j=0,\ldots,k-1. \end{cases} \tag{5.11}$$

Hence, simplifying (5.10) yields $V_{r-2k-2} \cong V_rK^{2+2k}$. The Higgs field gives rise to nonzero maps $V_{r-2k-2} \to V_{r-2k}K^2$ and $V_{r-2k} \to V_rK^{2k}$ by Proposition 3.7. Thus, $\deg(V_{r-2k-2}) - \deg(V_{r-2k}) = 4g-4$. As in the base case, this leads to a contradiction of stability. Namely, stability implies that the kernels of $\eta_{2k+1-r} : W_{2k+1-r} \to V_{2k+2-r}K$ and of $\eta_{r-1-2k} : W_{r-1-2k} \to V_{r-2k}K$ have negative degree, so that $\deg(V_{2k-r}) - 2g + 2 < \deg(W_{2k+1-r}) < \deg(V_{2k+2-r}) + 2g - 2$. So $\mathrm{rk}(W_{2k+1-r}) = 1$.

Using $\mathrm{rk}(W_{2k+1-r}) = 1$, (5.8) and (5.11), the determinant of (5.9) gives

$$\begin{aligned} V_rV_{r-2k-2} \otes \bigotimes_{j=1}^{\lfloor\frac{k}{2}\rfloor}(V_{r-2k}^2K^{2-2k}) \otimes V_{r-2k}K^{1-2k}W_{r-1-2k} \otimes \bigotimes_{j=1}^{\lfloor\frac{k-1}{2}\rfloor}(V_{r-2k}^2K^{2-2k}) \\ \cong \quad \bigotimes_{j=0}^{k-1}(V_{r-2k}^2K^{2-2k}) \otimes W_{r-1-2k}V_rK \ , \end{aligned}$$

which simplifies to $V_{2k-r} \cong V_{2k+2-r}K^2$. The Higgs field defines a nonzero map $V_{2k-r} \to W_{2k+1-r}K \to V_{2k+2-r}K^2$. Thus,

$$V_{2k-r} \cong W_{2k+1-r}K \cong V_{2k+2-r}K^2 \ . \tag{5.12}$$

Recall that k was an integer between 1 and $\frac{r-2}{2}$. Since r is even, we can take $k = (r-2)/2$, and hence (5.12) gives $V_{-2} \cong W_{-1}K \cong V_0K^2$. This completes the proof for the chain (5.4).

The difference for the chain (5.3) is that r is odd and instead of (5.8) we must assume

$$V_{1-r} \cong W_{2-r}K \cong V_{3-r}K^2 \cong \cdots \cong V_{2k-1-r}K^{2k-2} \cong W_{2k-r}K^{2k-1},$$

where k is an integer satisfying $1 \leqslant k \leqslant (r-3)/2$. The same proof as above shows that $W_{2k-r} \cong V_{2k+1-r}K \cong W_{2k+2-r}K^2$. By taking $k = (r-3)/2$ we have $W_{-3} \cong V_{-2}K \cong W_{-1}K^2$, and no condition on V_0 is imposed. Switching the roles of V and W gives the proof for the chains (5.2) and (5.4). □

We can now complete the classification of the stable minima with with vanishing $\mathbb{H}^2(C^\bullet)$.

Theorem 5.3. *A stable $\mathrm{SO}(p,q)$-Higgs bundle (V,W,η) with $p \leqslant q$, $\eta \neq 0$ and $\mathbb{H}^2(C^\bullet(V,W,\eta)) = 0$ defines a local minimum of the Hitchin function if and only if it is a holomorphic chain of the form (5.2), (5.3), (5.4) or (5.5) which satisfies one of the following:*

(1) The chain is given by (5.2) with $p = 2$ and $0 < \deg(V_{-1}) < 2g-2$.
(2) The chain is given by (5.2) with $s = p-1$ and the bundle W_0 decomposes as $W_0 = I \oplus W_0'$, where W_0' is a stable $\mathrm{O}(q-p+1,\mathbb{C})$-bundle with $\det(W_0') = I$. Moreover, $V_j = IK^{-j}$ and $W_j = IK^{-j}$ for all $j \neq 0$, and with respect to the splitting of W_0, the chain is given by

$$V_{-s} \xrightarrow{\eta_{s-1}^*} W_{1-s} \xrightarrow{\eta_{1-s}} \cdots \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\begin{pmatrix}\eta_0^*\\0\end{pmatrix}} \begin{matrix} I \\ \oplus \\ W_0' \end{matrix} \xrightarrow{(\eta_0\ 0)} V_1 \xrightarrow{\eta_{-2}^*} \cdots \xrightarrow{\eta_{1-s}^*} W_{s-1} \xrightarrow{\eta_{s-1}} V_s \ , \tag{5.13}$$

(3) The chain is of the form (5.3) with $q = p+1$, $V_j = K^{-j}$ and $W_j = K^{-j}$ for all $|j| < p$ and W_{-p} is a line bundle satisfying $\deg(W_{-p}) \in (0, p(2g-2)]$.
(4) The chain is of the form (5.4) where W_0 is a stable $\mathrm{O}(q-p+1,\mathbb{C})$-bundle with $\det(W_0) = I$, and $V_j = IK^{-j}$ and $W_j = IK^{-j}$ for all $j \neq 0$.
(5) The chain is of the form (5.5) with $q = p+1$, $V_0 = 0$, $W_0 \cong \mathcal{O}$, $V_j = K^{-j}$ and $W_j = K^{-j}$ for $0 < |j| < p$ and W_{-p} is a line bundle satisfying $\deg(W_{-p}) \in (0, p(2g-2)]$.
(6) The chain is of the form (5.5) with $q = p$, and for some torsion 2 line bundle I, $V_j = IK^{-j}$ and $W_j = IK^{-j}$ for all j.

Remark 5.4. The cases (2)-(6) are special cases of the fixed points considered in Lemma 4.8. In case (2), note that if the stable invariant bundle W_0' is instead strictly polystable, then the Higgs bundle is still at a local minimum of the Hitchin function. Similarly, for case (4) replacing the stable orthogonal bundle W_0 with a strictly polystable orthogonal bundle still gives rise to a local minimum. We will prove that these are the only local minima apart from $\eta = 0$. Note also that none of the above cases have $p = 1$ and $q > 2$.

Proof. We first show that cases (1) and (2) are sufficient for the chain (5.2) to be a stable minima with $\mathbb{H}^2(C^\bullet) = 0$ by invoking Proposition 3.2. For case (1), $C_2^\bullet$ is the only isomorphism to consider. We have $\mathfrak{so}_2(V) \oplus \mathfrak{so}_2(W) = \Lambda^2 V_{-1}$ and $\mathrm{Hom}_3(W,V) \otimes K = 0$, which is an isomorphism since $\mathrm{rk}(V_{-1}) = 1$. For case (2), the holomorphic chain (5.13) is a fixed point considered in Lemma 4.8 with $W_p = 0$. By Lemma 4.9, $C_k^\bullet : \mathfrak{so}_k(V) \oplus \mathfrak{so}_k(W) \xrightarrow{\mathrm{ad}_\eta} \mathrm{Hom}_{k+1}(W,V) \otimes K$ is an isomorphism for all $k > 0$.

We now show that cases (1) and (2) are necessary for chains of the form (5.2). We have a chain

$$V_{-s} \xrightarrow{\eta_{s-1}^*} W_{1-s} \xrightarrow{\eta_{1-s}} \cdots \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\eta_0^*} W_0 \xrightarrow{\eta_0} V_1 \xrightarrow{\eta_{-2}^*} \cdots \xrightarrow{\eta_{1-s}^*} W_{s-1} \xrightarrow{\eta_{s-1}} V_s \ ,$$

with $s \geqslant 1$ odd. By Lemma 5.2 each of the bundles in the chain is line bundle except W_0. So $p = s+1$ is even and $\mathrm{rk}(W_0) = q - p + 2 \geqslant 2$. Note that $\mathcal{O} = \det(V) = \det(W) = \det(W_0)$.

If $N = \ker(\eta_0)$, then η_0^* maps V_{-1} to $N^\perp K \subset W_0 \otimes K$. By Proposition 3.7, η_0^* is nonzero, hence $\deg(N^\perp) - \deg(V_{-1}) + 2g - 2 \geqslant 0$. If N is coisotropic $N \perp$ is isotropic, and stability implies $\deg(V_{-1}) + \deg(N^\perp) < 0$, which implies $\deg(V_{-1}) < g - 1$. If N is not coisotropic, then $\eta_0 \eta_0^*$ is a nonzero section of the line bundle $V_1^2 K^2$. Thus,

$$\deg(V_{-1}) \leqslant 2g - 2 \ . \tag{5.14}$$

If $p = 2$ and $\deg(V_{-1}) < 2g - 2$ we are done. If $\deg(V_{-1}) = 2g - 2$, then $\eta_0 \eta_0^*$ is a nowhere vanishing section of the line bundle $V_1^2 K^2$, and hence the kernel of η_0 is a holomorphic orthogonal bundle $W_0' \subset W_0$ of rank $q - p + 1$. Furthermore, stability of (V, W, η) forces W_0' to be stable. Taking orthogonal complements gives a decomposition $W_0 = W_0' \oplus I$ where $KV_1 = I = \det(W_0')$ since $\mathcal{O} = \det(W_0)$. By Lemma 5.2, the holomorphic chain is given by (5.13). Thus, for $p = 2$ we are done. For $p > 2$ we will show that stability forces $\deg(V_{-1}) = 2g - 2$ and $V_{-s} = K^s I$.

For $p \geqslant 4$ and even, we have $s \geqslant 3$ and odd. Using decompositions analogous to (5.6) and (5.7) and $\mathrm{rk}(V_j) = \mathrm{rk}(W_j) = 1$ for $j \neq 0$, the isomorphism of $C_{s-1}^\bullet$ gives

$$\begin{aligned} \mathfrak{so}_{s-1}(V) \oplus \mathfrak{so}_{s-1}(W) &\cong \bigoplus_{j=0}^{\lfloor \frac{s-1}{4} \rfloor} \mathrm{Hom}(V_{2j-s}, V_{2j-1}) \oplus \bigoplus_{j=0}^{\lfloor \frac{s-3}{4} \rfloor} \mathrm{Hom}(W_{2j+1-s}, W_{2j}) \\ &\cong \ \mathrm{Hom}_s(W,V) \otimes K \ \cong \bigoplus_{j=0}^{\frac{s-1}{2}} \mathrm{Hom}(W_{2j+1-s}, V_{2j+1}) \otimes K \ . \end{aligned}$$

Since $\det(W_0) = \mathcal{O}$, the determinant of both sides of the isomorphism $C_{s-1}^\bullet$ is given by

$$V_s V_{-1} \otimes \bigotimes_{j=1}^{\lfloor \frac{s-1}{4} \rfloor} V_{s-2j} V_{2j-1} \otimes W_{s-1}^{\mathrm{rk}(W_0)} \otimes \bigotimes_{j=1}^{\lfloor \frac{s-3}{4} \rfloor} W_{s-1-2j} W_{2j} \cong \bigotimes_{j=0}^{\frac{s-3}{2}} W_{s-1-2j} V_{2j+1} K \otimes (V_s K)^{\mathrm{rk}(W_0)}. \tag{5.15}$$

From Lemma 5.2, we have $W_{s-1} \cong V_1 K^{2-s}$ and

$$V_1^2 K^{3-s} \cong \begin{cases} V_{s-2j} V_{2j-1}, & \text{for } j = 1, \ldots, \lfloor (s-1)/4 \rfloor \\ W_{s-1-2j} W_{2j}, & \text{for } j = 1, \ldots, \lfloor (s-3)/4 \rfloor \\ W_{s-1-2j} V_{2j+1} K, & \text{for } j = 0, \ldots, (s-3)/2. \end{cases}$$

This simplifies (5.15) to $(V_s V_{-1} K^{s-1})^{p-q-1} \cong (V_1 K)^2$. As in the proof of Lemma 5.2, the Higgs field gives a nonzero map $V_1 \to V_s K^{s-1}$. Therefore,

$$0 \geqslant (p - q - 1)(\deg(V_s) - \deg(V_1) + (s-1)(2g-2)) = 2(\deg(V_1) + 2g - 2),$$

and hence $\deg(V_1) \leqslant 2-2g$. By (5.14), we conclude that $\deg(V_{-1}) = 2g-2$ and $\deg(V_s) = -s(2g-2)$. As above, since $\deg(V_{-1}) = 2g-2$, the bundle W_0 decomposes as $W_0' \oplus I$, where W_0' is the kernel of η_0 and $\det(W_0)' = I = V_1K$. Moreover, we have $V_s = IK^{-s}$ since, by Lemma 5.2, $W_{s-1} = K^{1-s}I$ and $\eta_{s-1} : W_{s-1} \to V_s \otimes K$ is nonzero. This completes the proof of (2).

Cases (3) and (5) are almost identical, we will prove (3). By Lemma 5.2, the holomorphic chain (5.3) is given by

$$W_{-r} \xrightarrow{\eta_{-r}} V_{1-r} \xrightarrow{\eta^*_{r-2}} \cdots \xrightarrow{\eta^*_1} W_{-1} \xrightarrow{\eta_{-1}} V_0 \xrightarrow{\eta^*_{-1}} W_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{r-2}} V_{r-1} \xrightarrow{\eta^*_{-r}} W_r ,$$

where $\mathrm{rk}(W_j) = 1$ for all j. Thus, $r = q-1$ and either $\mathrm{rk}(V_0) = 1$ and $q = p+1$ or $\mathrm{rk}(V_0) = 2$ and $q = p$. If $q = p$, then, by switching the roles of V and W, we can assume we are in case (2). Thus we may assume $\mathrm{rk}(V_0) = 1$ and $q = p+1$. Moreover, $V_0 = \mathcal{O}$ since $\mathcal{O} \cong \det(V) \cong V_0$. Since the Higgs field defines a nonzero maps $W_{-1} \to \mathcal{O} \otimes K$ and $W_{-1} \to W_1 \otimes K^2$ we conclude that $W_{-1} \cong K$. Thus, $W_j = K^{-j}$ and $V_j = K^{-j}$ for all $|j| < r$ by Lemma 5.2. Since W_p is an invariant isotopic subbundle and the Higgs field $\eta_{-p} : W_{-p} \to V_{-p+1}K$ is nonzero, we conclude

$$0 < \deg(W_{-p} \leqslant p(2g-2).$$

Thus, the conditions in case (3) are necessary.

The holomorphic chain from case (3) is a fixed point considered in Lemma 4.8 with $W_0' = 0$ and $\mathrm{rk}(W_{-p}) = 1$. By Lemmas 4.9 and 4.10, $C^\bullet_k : \mathfrak{so}_k(V) \oplus \mathfrak{so}_k(W) \xrightarrow{\mathrm{ad}_\eta} \mathrm{Hom}_{k+1}(W,V) \otimes K$ is an isomorphism for all $k > 0$. Thus, the conditions in case (3) are also sufficient.

The holomorphic chain from case (4) is a fixed point considered in Lemma 4.8 with $W_{-p} = 0$. By Lemma 4.9, $C^\bullet_k : \mathfrak{so}_k(V) \oplus \mathfrak{so}_k(W) \xrightarrow{\mathrm{ad}_\eta} \mathrm{Hom}_{k+1}(W,V) \otimes K$ is an isomorphism for all $k > 0$. Thus, the conditions in case (4) are sufficient.

To show the conditions of (4) are necessary, note that the holomorphic chain (5.4) is given by

$$\begin{array}{c} V_{-r} \xrightarrow{\eta_{-r}} W_{1-r} \xrightarrow{\eta^*_{r-2}} \cdots \xrightarrow{\eta^*_1} W_{-1} \xrightarrow{\eta_{-1}} V_0 \xrightarrow{\eta^*_{-1}} W_1 \xrightarrow{\eta_1} \cdots \xrightarrow{\eta_{r-2}} W_{r-1} \xrightarrow{\eta^*_{-r}} V_r . \\ \oplus \\ W_0 \end{array}$$

By Lemma 5.2, $\mathrm{rk}(V_j) = 1$ for all j, thus $r = p-1$ and $\mathrm{rk}(W_0) \geqslant 1$. Also, if $V_0 = I$, then $I = \det(V) = \det(W) = \det(W_0)$, and $V_j = IK^{-j}$ for all $|j| < p-1$ and $W_j = K^{-1}I$ for all $j \neq 0$. Since $W_0 \neq 0$, $\mathfrak{so}_{p-2}(V) \oplus \mathfrak{so}_{p-2}(W) \cong \mathrm{Hom}(W_{1-r}, W_0)$ and $\mathrm{Hom}_{p-1}(W,V) \otimes K \cong \mathrm{Hom}(W_0, V_{p-1}K)$. Taking the determinant of this isomorphism and using $W_{2-p} = K^{p-2}I$ we conclude that $V_{1-p} = IK^{p-1}$, finishing the proof of case (4).

Finally, for case (6) the holomorphic chain (5.5) is given by

$$\begin{array}{c} W_{-s} \xrightarrow{\eta_{-s}} V_{1-s} \xrightarrow{\eta^*_{s-2}} \cdots \xrightarrow{\eta_{-2}} V_{-1} \xrightarrow{\eta^*_0} W_0 \xrightarrow{\eta_0} V_1 \xrightarrow{\eta^*_{-2}} \cdots \xrightarrow{\eta_{s-2}} V_{s-1} \xrightarrow{\eta^*_{-s}} W_s . \\ \oplus \\ V_0 \end{array}$$

By Lemma 5.2, $\mathrm{rk}(W_j) = 1$ for all j. Thus $s = q-1 = p-1$ and V_0 is a rank one orthogonal bundle I with $I = \det(V) = \det(W) = W_0$, $V_j = IK^{-j}$ for all j and $W_j = IK^{-j}$ for all $|j| < s$. Similar to case (4), we have $\mathfrak{so}_{p-2}(V) \oplus \mathfrak{so}_{p-2}(W) \cong \mathrm{Hom}(V_0, V_{p-2})$ and $\mathrm{Hom}_{p-1}(W,V) \otimes K \cong \mathrm{Hom}(W_{1-p}, V_0K)$. Thus, the isomorphism $C^\bullet_{p-2}$ implies $W_{1-p} \cong IK^{p-1}$. Thus, the conditions of (6) are necessary. As with the other cases, the conditions of case (6) are sufficient by Lemmas 4.8 and 4.9. □

5.2. **Stable minima with non-vanishing $\mathbb{H}^2(C^\bullet)$.** We now classify stable $\mathrm{SO}(p,q)$-Higgs bundles such that the associated $\mathrm{SO}(p+q,\mathbb{C})$-Higgs bundle is strictly polystable. By Remark 2.17, these are exactly the stable $\mathrm{SO}(p,q)$-Higgs bundles which may have $\mathbb{H}^2(C^\bullet) \neq 0$. We will prove that such $\mathrm{SO}(p,q)$-Higgs bundles define minima of the Hitchin function if and only if the Higgs field η is zero.

The $\mathrm{SO}(p+q,\mathbb{C})$-Higgs bundle associated to an $\mathrm{SO}(p,q)$-Higgs bundle (V,Q_V,W,Q_W,η) is

$$(E,Q,\Phi)=\left(V\oplus W,\begin{pmatrix} Q_V & 0\\ 0 & -Q_W\end{pmatrix},\begin{pmatrix} 0 & \eta\\ \eta^* & 0\end{pmatrix}\right). \tag{5.16}$$

Recall that a $\mathrm{GL}(p,\mathbb{R})$-Higgs bundle is defined by a triple (V,Q,η) where (V,Q) is a rank p orthogonal vector bundle and $\eta: V\to V\otimes K$ is a holomorphic map satisfying $\eta^*=Q^{-1}\eta^TQ=\eta$. Given such a $\mathrm{GL}(p,\mathbb{R})$-Higgs bundle we can construct an $\mathrm{SO}(p,p)$-Higgs bundle

$$(V,Q_V,W,Q_W,\eta)=(V,Q_V,V,Q_V,\eta)\ .$$

Using the symmetry $\eta^*=\eta$, the corresponding $\mathrm{SO}(2p,\mathbb{C})$-Higgs bundle is

$$(E,Q,\Phi)=\left(V\oplus V,\begin{pmatrix} Q_V & 0\\ 0 & -Q_V\end{pmatrix},\begin{pmatrix} 0 & \eta\\ \eta & 0\end{pmatrix}\right).$$

Even if the $\mathrm{SO}(p,p)$-Higgs bundle (V,V,η) is stable, the above $\mathrm{SO}(2p,\mathbb{C})$-Higgs bundle is strictly polystable. Indeed, the following pair of disjoint degree zero isotropic subbundles are both Φ-invariant:

$$\begin{array}{ccc} V \xrightarrow{i_1} V\oplus V & \text{and} & V\xrightarrow{i_2} V\oplus V \\ v\longmapsto (v,v) & & v\longmapsto (v,-v)\end{array}\ .$$

The following proposition shows that this example characterizes stable $\mathrm{SO}(p,q)$-Higgs bundles which are not stable as $\mathrm{SO}(p+q,\mathbb{C})$-Higgs bundles.

Proposition 5.5. *Let (V,W,η) be a stable $\mathrm{SO}(p,q)$-Higgs bundle. The associated $\mathrm{SO}(p+q,\mathbb{C})$-Higgs bundle (5.16) is strictly polystable if and only if*

$$(V,Q_V,W,Q_W,\eta)\cong\left(V_1\oplus V_2,\begin{pmatrix} Q_{V_1} & 0\\ 0 & Q_{V_2}\end{pmatrix},V_1\oplus W_2,\begin{pmatrix} Q_{V_1} & 0\\ 0 & Q_{W_2}\end{pmatrix},\begin{pmatrix} \eta_1 & 0\\ 0 & \eta_2\end{pmatrix}\right), \tag{5.17}$$

where $(V_1,Q_{V_1},V_1,Q_{V_1},\eta_1)$ is a stable $\mathrm{SO}(p_1,p_1)$-Higgs bundle with $\eta_1^*=\eta_1$ and $(V_2,Q_{V_2},W_2,Q_{W_2},\eta_2)$ is a stable $\mathrm{SO}(p_2,q_2)$-Higgs bundle.

Proof. By the above discussion, the condition (5.17) is sufficient. We now show that it is necessary. Let (V,W,η) be a stable $\mathrm{SO}(p,q)$-Higgs bundle and suppose the associated $\mathrm{SO}(p+q,\mathbb{C})$-Higgs bundle (E,Q,Φ) given by (5.16) is strictly polystable. Let $U\subset V\oplus W$ be a degree zero proper subbundle which is isotropic with respect to Q and satisfies $\Phi(U)\subset U\otimes K$. Let $V_1\subset V$ and $W_1\subset W$ be the respective image sheafs of the projection of U onto each summand of $V\oplus W$. The subsheaf $V_1\oplus W_1$ is preserved by Φ, thus $\deg(V_1)+\deg(W_1)\leqslant 0$ by polystability of the associated $\mathrm{SL}(p+q,\mathbb{C})$-Higgs bundle $(V\oplus W,\Phi)$.

Consider the sequences

$$0\longrightarrow U^w\longrightarrow U\longrightarrow V_1\longrightarrow 0 \qquad \text{and} \qquad 0\longrightarrow U^v\longrightarrow U\longrightarrow W_1\longrightarrow 0\ ,$$

where the subsheaf $U^v\subset V$ is Q_V isotropic, the subsheaf $U^w\subset W$ is Q_W isotropic, $\eta(U^w)\subset U^v\otimes K$ and $\eta^*(U^v)\subset U^w\otimes K$. Stability of the $\mathrm{SO}(p,q)$-Higgs bundle gives $\deg(U^v)+\deg(U^w)<0$, which implies $\deg(V_1)+\deg(W_1)>0$. But, since $V_1\oplus W_1$ is preserved by Φ, $\deg(V_1)+\deg(W_1)\leqslant 0$ by polystability of the Higgs bundle $(V\oplus W,\Phi)$. This contradiction implies

$$V_1\cong U\cong W_1\ .$$

We claim that V_1 and W_1 are both orthogonal subbundles. Let Q_{V_1} and Q_{W_1} be the restrictions of Q_V and Q_W to V and W respectively. Consider the following sequences

$$0\longrightarrow V_1^{\perp_{V_1}}\longrightarrow V_1\longrightarrow V_1/V_1^{\perp_{V_1}}\longrightarrow 0 \qquad \text{and} \qquad 0\longrightarrow W^{\perp_{W_1}}\longrightarrow W_1\longrightarrow W_1/W_1^{\perp_{W_1}}\longrightarrow 0\ .$$

Since $V_1^{\perp_{V_1}}$ and $W_1^{\perp_{W_1}}$ are maximally isotropic subbundles of V_1 and W_1 respectively, both $V_1/V_1^{\perp_{V_1}}$ and $W_1/W_1^{\perp_{W_1}}$ are orthogonal bundles. In particular, $V_1^{\perp_{V_1}}$ and $W_1^{\perp_{W_1}}$ are degree zero isotropic subbundles of V and W respectively. Moreover, we have

$$\eta(W_1^{\perp_{W_1}})\subset V_1^{\perp_{V_1}}\otimes K \qquad \text{and} \qquad \eta^*(V_1^{\perp_{V_1}})\subset W_1^{\perp_{W_1}}\otimes K\ .$$

Again, stability of the $\mathrm{SO}(p,q)$-Higgs bundle (V,W,η) implies both $V_1^{\perp_{V_1}}$ and $W_1^{\perp_{W_1}}$ are zero, which implies $V_1\subset V$ and $W_1\subset W$ are both orthogonal subbundles.

If $p_1=\mathrm{rk}(W_1)=\mathrm{rk}(V_1)$, then $(V_1,W_1,\eta|_{W_1})$ is an $\mathrm{SO}(p_1,p_1)$-Higgs bundle. Note that isomorphism between V_1 and W_1 is given by including V_1 into $V\oplus W$ and projecting onto W. Denote this isomorphism by $g:V_1\to W_1$, we have $\eta|_{W_1}g=(g^{-1}\otimes 1_K)\eta|_{W_1}^*$. Moreover, g is orthogonal since for any $x,y\in V_1$ we have $(x,g(x)),(y,g(y))\in U$, and

$$0=Q((x,g(x)),(y,g(y)))=Q_{V_1}(x,y)-Q_{W_1}(g(x),g(y))$$

since U is isotropic. Therefore the pair (Id_V,g^{-1}) defines an isomorphism between $(V_1,W_1,\eta|_{W_1})$ and (V_1,V_1,η_1) with $\eta_1=\eta|_{W_1}g$. In particular, $\eta_1=\eta_1^*$.

Let V_2 and W_2 be the orthogonal compliments of V_1 and W_1 respectively and let $\eta_2:W_2\to V_2\otimes K$ be the restriction of η to W_2. By the above discussion, we obtain the desired decomposition (5.17) of the $\mathrm{SO}(p,q)$-Higgs bundle (V,W,η). $\square$

Now, if a stable $\mathrm{SO}(p,q)$-Higgs bundle

$$(V,Q_V,W,Q_W,\eta)\cong\left(V_1\oplus V_2,\begin{pmatrix}Q_{V_1}&0\\0&Q_{V_2}\end{pmatrix},V_1\oplus W_2,\begin{pmatrix}Q_{V_1}&0\\0&Q_{W_2}\end{pmatrix},\begin{pmatrix}\eta_1&0\\0&\eta_2\end{pmatrix}\right),$$

with $\eta_1^*=\eta_1$ is a local minimum of the Hitchin function, then (V_1,Q_{V_1},η_1) is a local minimum local minimum of the Hitchin function on the $\mathrm{GL}(p_1,\mathbb{R})$-Higgs bundle moduli space and $(V_2,Q_{V_2},W_2,Q_{W_2},\eta_2)$ is a local minimum of the Hitchin function on the $\mathrm{SO}(p_2,q_2)$-Higgs bundle moduli space.

Recall from Example 3.4 that Hitchin proved the local minima in the $\mathrm{GL}(p,\mathbb{R})$-Higgs bundle moduli space with nonzero Higgs field are given by the minima in the Hitchin components. The holomorphic chain of such a Higgs bundle is given by

$$V_{\frac{1-p}{2}}\longrightarrow V_{\frac{3-p}{2}}\longrightarrow\cdots\longrightarrow V_{\frac{p-3}{2}}\longrightarrow V_{\frac{p-1}{2}}$$

where $V_j=IK^{-j}$ for all j and some torsion two line bundle I. The holomorphic chain of the associated $\mathrm{SO}(p,p)$-Higgs bundle is given by

$$\begin{array}{c}V_{\frac{1-p}{2}}\longrightarrow V_{\frac{3-p}{2}}\longrightarrow\cdots\longrightarrow V_{\frac{p-3}{2}}\longrightarrow V_{\frac{p-1}{2}}\\ \oplus\\ V_{\frac{1-p}{2}}\longrightarrow V_{\frac{3-p}{2}}\longrightarrow\cdots\longrightarrow V_{\frac{p-3}{2}}\longrightarrow V_{\frac{p-1}{2}}\end{array}\tag{5.18}$$

By Proposition 3.7, such an $\mathrm{SO}(p,p)$-Higgs bundle is not stable if p is even. Thus, the following proposition shows that the only stable $\mathrm{SO}(p,q)$-Higgs bundles with nonzero Higgs field are classified by Theorem 5.3.

Proposition 5.6. *For p-odd, the $\mathrm{SO}(p,p)$-Higgs bundle given by (5.18) with $V_j=IK^{-j}$ for all j and some torsion two line bundle I is not a minimum of the Hitchin function.*

Proof. By assumption $r=\frac{p-1}{2}$ is a positive integer. Set $V=\bigoplus\limits_{j=0}^{2r}V_{j-r}$ and $W=\bigoplus\limits_{j=0}^{2r}W_{j-r}$ with $V_j=IK^{-j}$ and $W_j=IK^{-j}$ for all j and some torsion two line bundle I. The holomorphic chain (5.18) is given by

$$\begin{array}{c}V_{-r}\xrightarrow{1}W_{1-r}\xrightarrow{1}\cdots\xrightarrow{1}V_{-1}\xrightarrow{1}W_0\xrightarrow{1}V_1\xrightarrow{1}\cdots\xrightarrow{1}V_{r-1}\xrightarrow{1}W_r\\ \oplus\\ W_{-r}\xrightarrow{1}V_{1-r}\xrightarrow{1}\cdots\xrightarrow{1}W_{-1}\xrightarrow{1}V_0\xrightarrow{1}W_1\xrightarrow{1}\cdots\xrightarrow{1}W_{r-1}\xrightarrow{1}V_r\end{array}$$

Consider the complex $C^\bullet_{2r-1}:\mathfrak{so}_{2r-1}(V)\oplus\mathfrak{so}_{2r-1}(W)\xrightarrow{\mathrm{ad}_\eta}\mathrm{Hom}_{2r}(W,V)\otimes K$. We have

$$\mathfrak{so}_{2r-1}(V)\cong\{(\alpha,-\alpha)\in\mathrm{Hom}(V_{-r},V_{r-1})\oplus\mathrm{Hom}(V_{1-r},V_r)\},$$
$$\mathfrak{so}_{2r-1}(W)\cong\{(\beta,-\beta)\in\mathrm{Hom}(W_{-r},W_{r-1})\oplus\mathrm{Hom}(W_{1-r},W_r)\},$$
$$\mathrm{Hom}_{2r}(W,V)\cong\mathrm{Hom}(W_{-r},V_r),$$

Using $V_j = IK^{-j}$ and $W_j \cong IK^{-j}$ for all j, the induced map on first cohomology is

$$\begin{array}{rccc} \mathrm{ad}_\eta: & H^1(K^{1-2r}) \oplus H^1(K^{1-2r}) & \longrightarrow & H^1(K^{1-2r}) \\ & ([\alpha],[\beta]) & \longmapsto & [\beta]+[\alpha] \end{array}. \tag{5.19}$$

In particular, $\mathbb{H}^2(C^\bullet_{2r-1}) = 0$ since this map is surjective and the kernel is given by $[\alpha] = -[\beta]$.

Let $\beta \in \Omega^{0,1}(K^{1-2r})$ which is nonzero in cohomology and, with respect to the above splittings of V and W, consider the deformed orthogonal holomorphic structures:

$$\bar\partial_V^\beta = \begin{pmatrix} \bar\partial_{K^r} & & & & \\ 0 & \bar\partial_{K^{r-1}} & & & \\ & & \ddots & & \\ \beta & 0 & \cdots & \bar\partial_{K^{1-r}} & \\ 0 & -\beta & \cdots & 0 & \bar\partial_{K^{-r}} \end{pmatrix} \quad \text{and} \quad \bar\partial_W^\beta = \begin{pmatrix} \bar\partial_{K^r} & & & & \\ 0 & \bar\partial_{K^{r-1}} & & & \\ & & \ddots & & \\ -\beta & 0 & \cdots & \bar\partial_{K^{1-r}} & \\ 0 & \beta & \cdots & 0 & \bar\partial_{K^{-r}} \end{pmatrix}.$$

In the above splittings of V and W, the Higgs field is given by

$$\eta = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} : W \to V \otimes K ,$$

and a calculation shows that η is still holomorphic with respect $\bar\partial_W^\beta$ and $\bar\partial_V^\beta$. Since this deformation is in the positive weight space, it decreases the Hitchin function and we conclude that such an SO(p,p)-Higgs bundle is not a local minimum of the Hitchin function. □

5.3. **Strictly polystable minima.** Recall from Proposition 2.20 that a strictly polystable SO(p,q)-Higgs bundle is isomorphic to

$$\left(E\oplus E^*\oplus V, \begin{pmatrix} 0 & \mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & Q_V \end{pmatrix}, F\oplus F^*\oplus W, \begin{pmatrix} 0 & \mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & Q_W \end{pmatrix}, \begin{pmatrix} \beta & 0 & 0 \\ 0 & \gamma^* & 0 \\ 0 & 0 & \eta \end{pmatrix}\right),$$

where (E,F,β,γ) is a polystable U(p_1,q_1)-Higgs bundle, with $\deg(E)+\deg(F)=0$, and (V,W,η) is a stable SO(p_2,q_2)-Higgs bundle.

Proposition 5.7. *Let (E,F,β,γ) be a polystable* U(p,q)*-Higgs bundle which is a fixed point of the $\mathbb{C}^*$-action. The associated* SO$(2p,2q)$*-Higgs bundle*

$$\left(E\oplus E^*, \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}, F\oplus F^*, \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \gamma^* \end{pmatrix}\right) \tag{5.20}$$

is a minimum of the Hitchin function if and only if $\beta=\gamma=0$ or $p\leqslant 1$ or $q \leqslant 1$.

Proof. If $\beta=\gamma=0$ or $p=0$ or $q=0$ the Higgs field is identically zero and we have a minimum. Now suppose $p>0$ and $q>0$ and that the SO$(2p,2q)$-Higgs bundle (5.20) is a minimum of the Hitchin function with nonzero Higgs field. This implies that the U(p,q)-Higgs bundle (E,F,β,γ) is a minimum of the Hitchin function with β and γ not both zero. Recall from Example 3.4, that this implies either $\beta=0$ or $\gamma=0$. Up to switching the roles of E, F, E^* and F^*, the relevant holomorphic chain is

$$F \xrightarrow{\begin{pmatrix} \beta \\ 0 \end{pmatrix}} \begin{matrix} E \\ \oplus \\ E^* \end{matrix} \xrightarrow{(0\ \beta^*)} F^* .$$

Since the U(p,q)-Higgs bundle $(E,F,\beta,0)$ is a polystable minimum with $\beta \neq 0$, we must have [6] $\deg(E) < 0 < \deg(F)$.

The Lie algebra bundle $\mathfrak{so}(F\oplus F^*)$ decomposes as $\mathfrak{so}_{-2}(F\oplus F^*)\oplus\mathfrak{so}_0(F\oplus F^*)\oplus\mathfrak{so}_2(F\oplus F^*)$ where

$$\mathfrak{so}_0(F\oplus F^*) = \{(d,-d^*)\in \mathrm{End}(F)\oplus\mathrm{End}(F^*)\} ,$$
$$\mathfrak{so}_2(F\oplus F^*) \cong \Lambda^2 F^* \cong \mathfrak{so}_{-2}(F\oplus F^*)^*.$$

Moreover,

$$\mathfrak{so}(E\oplus E^*) = \mathfrak{so}_0(E\oplus E^*) \cong \{(a,b,c,-c^*)\in \Lambda^2E\oplus\Lambda^2E^*\oplus\mathrm{End}(E)\oplus\mathrm{End}(E^*)\},$$

and

$$\mathrm{Hom}_1(F \oplus F^*, E \oplus E^*) \cong \mathrm{Hom}(F, E) \oplus \mathrm{Hom}(F, E^*).$$

Also, the map $\mathrm{ad}_\eta : \mathfrak{so}_2(F \oplus F^*) \to \mathrm{Hom}_3(F, E) \otimes K = 0$ is zero and

$$\begin{array}{rcl}\mathrm{ad}_\eta : \ \mathfrak{so}(E \oplus E^*) \oplus \mathfrak{so}_0(F \oplus F^*) & \longrightarrow & \mathrm{Hom}(F, E) \otimes K \oplus \mathrm{Hom}(F, E^*) \otimes K \\ (a, b, c, d) & \longmapsto & (\beta d - c\beta, -b\beta)\end{array} .$$

If $\mathrm{rk}(F) = 1$, then $\mathfrak{so}_j(F \oplus F^*) = 0$ for all $j > 0$, hence (since $\mathfrak{so}(E \oplus E^*) = \mathfrak{so}_0(E \oplus E^*)$) we have that $\mathbb{H}^1(C_j^\bullet) = 0$ for every $j > 0$. In particular, such a $\mathrm{SO}(2p, 2)$-Higgs bundle is a local minimum of the Hitchin function. Now suppose $\mathrm{rk}(F) > 1$. Since $\deg(F^*) < 0$, Riemann-Roch implies that $H^1(\Lambda^2 F^*) \neq 0$. Thus we may consider a nonzero extension $0 \to F^* \to W \to F \to 0$ given by an element of $H^1(\Lambda^2 F^*)$. Moreover, the Higgs field $\eta = \left(\begin{smallmatrix} \beta & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is still holomorphic. Such a deformation breaks the $\mathrm{U}(p, q)$ symmetry of the $\mathrm{SO}(2p, 2q)$-Higgs bundle and decreases the Hitchin function. □

By Remark 5.4, the following proposition is the final step in classifying all strictly polystable minima of the Hitchin function.

Proposition 5.8. *Let (E, F, β, γ) be a polystable $\mathrm{U}(m, n)$-Higgs bundle with $m = 1$ or $n = 1$ and $\beta = 0$ or $\gamma = 0$. Let (V', W', η') be a stable $\mathrm{SO}(p, q)$-Higgs bundle which is a minimum of the Hitchin function with $\eta' \neq 0$. The $\mathrm{SO}(p + 2m, q + 2n)$-Higgs bundle*

$$(V, Q_V, W, Q_W, \eta) = \left(E \oplus E^* \oplus V', \left(\begin{smallmatrix} 0 & \mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & Q_{V'} \end{smallmatrix}\right), F \oplus F^* \oplus W', \left(\begin{smallmatrix} 0 & \mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & Q_{W'} \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta & 0 & 0 \\ 0 & \gamma^* & 0 \\ 0 & 0 & \eta' \end{smallmatrix}\right)\right)$$

is a minimum of the Hitchin function if and only if $p = 0$ and $m = 1$ or $q = 0$ and $n = 1$.

Proof. First note that if $p = 0$ or $q = 0$, then we have a local minimum. Now suppose $p \neq 0$, $q \neq 0$ and (V, W, η') is a stable minimum from Theorem 5.3. Up to switching the roles of E, V', F, and W', it suffices holomorphic chains of one of the following six types (recall that we suppress the twisting by K from the notation):

(5.21)
$$\begin{array}{ccccc} & & F & & \\ E & \xrightarrow{\left(\begin{smallmatrix}\gamma\\0\end{smallmatrix}\right)} & \oplus & \xrightarrow{\left(\begin{smallmatrix}0 & \gamma^*\end{smallmatrix}\right)} & E^* \\ & & F^* & & \\ & & \oplus & & \\ V_{-1} & \xrightarrow{\eta_0^*} & W_0 & \xrightarrow{\eta_0} & V_1 \end{array} \quad \text{or} \quad \begin{array}{ccccc} & & E & & \\ F & \xrightarrow{\left(\begin{smallmatrix}\beta\\0\end{smallmatrix}\right)} & \oplus & \xrightarrow{\left(\begin{smallmatrix}0 & \beta^*\end{smallmatrix}\right)} & F^* \\ & & E^* & & \\ & & \oplus & & \\ V_{-1} & \xrightarrow{\eta_0^*} & W_0 & \xrightarrow{\eta_0} & V_1 \end{array}$$

where $\mathrm{rk}(V_{-1}) = 1$ and $0 < \deg(V_{-1}) \leqslant 2g - 2$.

(5.22)
$$\begin{array}{c} E \xrightarrow{\left(\begin{smallmatrix}\gamma\\0\end{smallmatrix}\right)} F \oplus F^* \xrightarrow{\left(\begin{smallmatrix}0 & \gamma^*\end{smallmatrix}\right)} E^* \\ \oplus \\ V_{1-p} \xrightarrow{1} W_{2-p} \xrightarrow{1} \cdots \xrightarrow{1} W_{p-2} \xrightarrow{1} V_{p-1} \end{array} \quad \text{or} \quad \begin{array}{c} F \xrightarrow{\left(\begin{smallmatrix}\beta\\0\end{smallmatrix}\right)} E \oplus E^* \xrightarrow{\left(\begin{smallmatrix}0 & \beta^*\end{smallmatrix}\right)} F^* \\ \oplus \\ V_{1-p} \xrightarrow{1} W_{2-p} \xrightarrow{1} \cdots \xrightarrow{1} W_{p-2} \xrightarrow{1} V_{p-1} \end{array}$$

where $V_j = IK^{-j}$ and $W_j = IK^{-j}$ for all j and some I with $I^2 \cong \mathcal{O}$.

(5.23)
$$\begin{array}{c} E \xrightarrow{\left(\begin{smallmatrix}\gamma\\0\end{smallmatrix}\right)} F \oplus F^* \xrightarrow{\left(\begin{smallmatrix}0 & \gamma^*\end{smallmatrix}\right)} E^* \\ \oplus \\ W_{-p} \xrightarrow{\eta_{-p}} V_{1-p} \xrightarrow{1} \cdots \xrightarrow{1} V_{1-p} \xrightarrow{\eta_{-p}^*} W_p \end{array} \quad \text{or} \quad \begin{array}{c} F \xrightarrow{\left(\begin{smallmatrix}\beta\\0\end{smallmatrix}\right)} E \oplus E^* \xrightarrow{\left(\begin{smallmatrix}0 & \beta^*\end{smallmatrix}\right)} F^* \\ \oplus \\ W_{-p} \xrightarrow{\eta_{-p}} V_{1-p} \xrightarrow{1} \cdots \xrightarrow{1} V_{1-p} \xrightarrow{\eta_{-p}^*} W_p \end{array}$$

where $V_j = K^{-j}$ and $W_j = K^{-j}$ for all $|j| < p$, $\mathrm{rk}(W_{-p}) = 1$, $0 < \deg(W_{-p}) \leqslant p(2g - 2)$ and $\eta_{-p} \neq 0$. Furthermore, in (5.21), (5.22) and (5.23), the first chain has $\mathrm{rk}(E) = 1$, $0 < \deg(E)$ and $\deg(F) \leqslant 0$, while the second chain has $\mathrm{rk}(F) = 1$, $0 < \deg(F)$ and $\deg(E) \leqslant 0$.

We will show that each of the above holomorphic chains is not a minimum. For the first chain of (5.21) and the first chain of (5.22), the summand of $\mathfrak{so}(V)$ given by

$$\{(\alpha, -\alpha^*) \in \mathrm{Hom}(E, V_{p-1}) \oplus \mathrm{Hom}(V_{1-p}, E^*)\}$$

lies in the kernel of $\mathrm{ad}_\eta : \mathfrak{so}(V) \oplus \mathfrak{so}(W) \to \mathrm{Hom}(W, V) \otimes K$. Since $\deg(E) > 0$ and $\deg(V_{p-1}) < 0$, we have $H^1(\mathrm{Hom}(E, V_{p-1})) \neq 0$ by Riemann-Roch. Thus, we may thus deform the holomorphic structure on V by a nonzero extension of the summand $E \oplus V_{p-1}$ of the form

$$0 \to V_{p-1} \to \widetilde{V} \to E \to 0.$$

Such a deformation decreases the Hitchin function. Similarly, for the second chain in (5.23) the summand of $\mathfrak{so}(W)$ isomorphic to $\mathrm{Hom}(F, W_p)$ is in the kernel of ad_η. As above, since $\deg(F) > 0$ and $\deg(W_p) < 0$, Riemann-Roch implies $H^1(\mathrm{Hom}(F, W_p)) \neq 0$. Hence, again, deforming the holomorphic structure by a nonzero element of $H^1(\mathrm{Hom}(F, W_p))$ decreases the Hitchin function.

For the second chain in (5.21) and the second chain in (5.22), the summand of $\mathfrak{so}(V)$ given by

$$\{(\alpha, -\alpha^*) \in \mathrm{Hom}(E^*, V_{p-1}) \oplus \mathrm{Hom}(V_{1-p}, E)\}$$

is in the kernel of ad_η. As above, $H^1(\mathrm{Hom}(E^*, V_{p-1})) \neq 0$, since $\deg(E) \leqslant 0$ and $\deg(V_{p-1}) < 0$, hence a nonzero element of $H^1(\mathrm{Hom}(E^*, V_{p-1}))$ can be used to deform the Higgs bundle and decrease the Hitchin function. Similarly, for the first chain in (5.23) the summand of $\mathfrak{so}(W)$ given by $\{(\beta, -\beta^*) \in \mathrm{Hom}(F^*, W_p) \oplus \mathrm{Hom}(W_{-p}, F)\}$ is in the kernel of ad_η. Since $\deg(F) \leqslant 0$ and $\deg(W_p) < 0$, we have $H^1(\mathrm{Hom}(F^*, W_p)) \neq 0$ by Riemann-Roch. Again, a nonzero element of $H^1(\mathrm{Hom}(F^*, W_p))$ can be used to deform the Higgs bundle and decrease the Hitchin function. □

5.4. **Summary of classification of minima of Hitchin function on $\mathcal{M}(\mathrm{SO}(p,q))$.** Putting everything together, the following theorem classifies all polystable minima of the Hitchin function in the moduli space of SO(p,q)-Higgs bundles for $p \leqslant q$.

Theorem 5.9. *For $1 \leqslant p \leqslant q$, let $f : \mathcal{M}(\mathrm{SO}(p,q)) \to \mathbb{R}$ be the Hitchin function on the moduli space of polystable $\mathrm{SO}(p,q)$-Higgs bundles given by (3.1). A polystable $\mathrm{SO}(p,q)$-Higgs bundle (V, W, η) is a local minimum of f if and only if $\eta = 0$ or (V, W, η) is isomorphic to a holomorphic chains of one of the following mutually exclusive types, where we have suppressed the twisting by K in the Higgs field from the notation:*

(1) $p = 2$ and (V, W, η) is of the form

$$V_{-1} \xrightarrow{\eta_0^*} W \xrightarrow{\eta_0} V_1 \quad ,$$

where $V = V_{-1} \oplus V_1$ with V_{-1} is a line bundle having $0 < \deg(V_{-1}) < 2g-2$, $V_1 = V_{-1}^*$ and η_0 is nonzero.

(2) $p \geqslant 2$ and (V, W, η) is of the form

$$\begin{array}{c} V_{1-p} \xrightarrow{\eta_{p-2}^*} W_{2-p} \xrightarrow{\eta_{2-p}} V_{3-p} \xrightarrow{\eta_{p-4}^*} \cdots \xrightarrow{\eta_{p-4}} V_{p-3} \xrightarrow{\eta_{2-p}^*} W_{p-2} \xrightarrow{\eta_{p-2}} V_{p-1} \ , \\ \oplus \\ W_0' \end{array}$$

where W_0' is a polystable $\mathrm{O}(q-p+1, \mathbb{C})$-bundle with $\det(W_0') = I$, $W = W_0' \oplus \bigoplus_{i=1}^{p-1} W_{-p+2i}$ with $W_j = IK^{-j}$ for all j, $V = \bigoplus_{i=0}^{p-1} V_{1-p+2i}$ with $V_j = IK^{-j}$ for all j, and each η_j is nonzero.

(3) $p = q-1$ and (V, W, η) is of the form

$$W_{-p} \xrightarrow{\eta_{-p}} V_{1-p} \xrightarrow{\eta_{p-2}^*} W_{2-p} \xrightarrow{\eta_{2-p}} V_{3-p} \xrightarrow{\eta_{p-4}^*} \cdots \xrightarrow{\eta_{p-4}} V_{p-3} \xrightarrow{\eta_{2-p}^*} W_{p-2} \xrightarrow{\eta_{p-2}} V_{p-1} \xrightarrow{\eta_{-p}^*} W_p \quad ,$$

where $V = \bigoplus_{i=0}^{p-1} V_{1-p+2i}$ *with* $V_j = K^{-j}$ *for all* j, $W = \bigoplus_{i=0}^{p-1} W_{-p+2i}$ *with* $W_j = K^{-j}$ *for all* $|j| < p$ *and* W_{-p} *is a holomorphic line bundle with* $0 < \deg(W_{-p}) \leqslant p(2g-2)$, *and each* η_j *is nonzero.*

Remark 5.10. Note that in case (2), $\det(V) = I^p = \det(W)$. Thus, if p is even, a Higgs bundle defined in case (2) of Theorem 5.9 is an $\mathrm{SO}(p,q)$-Higgs bundle which reduces to $\mathrm{SO}_0(p,q)$ if and only if $I = \mathcal{O}$, on the other hand, when p is even the Higgs bundle in case (2) always reduces to $\mathrm{SO}_0(p,q)$.

Proof. If the $\eta = 0$, then we are done, so suppose $\eta \neq 0$. By Theorem 5.3 and Proposition 5.6, the result holds if (V, W, η) is a stable $\mathrm{SO}(p,q)$-Higgs bundle. Suppose (V, W, η) is a strictly polystable $\mathrm{SO}(p,q)$-Higgs bundle with $p \leqslant q$. By Proposition 2.20,

$$(V, W, \eta) \cong \left(E \oplus E^* \oplus V', F \oplus F^* \oplus W', \begin{pmatrix} \gamma & & \\ & \beta^* & \\ & & \eta' \end{pmatrix}\right),$$

where (E, F, β, γ) is a polystable $\mathrm{U}(p_1, q_1)$-Higgs bundle and (V', W', η') is a stable $\mathrm{SO}(p_2, q_2)$-Higgs bundle which does not necessarily have $p_2 \leqslant q_2$. By Proposition 5.7 and Proposition 5.8 if such a Higgs bundle is a minimum of the Hitchin function, then one of the following hold

(a) $\beta = 0$, $\gamma = 0$ and (V', W', η') is a minimum from Theorem 5.3,
(b) $p_1 = 1$, $\beta = 0$ or $\gamma = 0$ and $\eta' = 0$,
(c) $q_1 = 1$, $\beta = 0$ or $\gamma = 0$ and $\eta' = 0$.

For case (a), note that if $p_2 = 0$ or $q_2 = 0$ then the Higgs field is zero and we are at a minimum. Consider a holomorphic chain of the form

$$\begin{array}{c} V'_{-r} \longrightarrow W'_{1-r} \longrightarrow \ \cdots \ \longrightarrow W'_{r-1} \longrightarrow V'_r \\ \oplus \\ E \oplus E^* \end{array} \qquad \text{or} \qquad \begin{array}{c} W'_{-r} \longrightarrow V'_{1-r} \longrightarrow \ \cdots \ \longrightarrow V'_{r-1} \longrightarrow W'_r \\ \oplus \\ F \oplus F^* \end{array}$$

where V'_{-r} and W'_{-r} are holomorphic line bundles of positive degree. This does not define a minimum of the Hitchin function since $\deg(E) = 0$ and $\deg(V'_{-r}) > 0$ imply $H^1(\mathrm{Hom}(V'_{-r}, E)) \neq 0$, thus we may deform such a holomorphic chain down by considering a nonzero extension in $H^1(\mathrm{Hom}(V'_{-r}, E))$. Similarly, the second chain does not define a minimum.

Since $q \geqslant p$, the only way we can have a holomorphic chain

$$\begin{array}{c} W'_{-r} \longrightarrow V'_{1-r} \longrightarrow \ \cdots \ \longrightarrow V'_{r-1} \longrightarrow W'_r \\ \oplus \\ E \oplus E^* \end{array}$$

with $\mathrm{rk}(W'_j) = \mathrm{rk}(V'_j) = 1$ for all j is if $E = 0$ and $q = p + 1$. Such a holomorphic chain is stable. By Theorem 5.3 such a holomorphic chain is a minimum of the Hitchin function if and only if it satisfies the conditions of case (3) in the statement of the Theorem. To finish case (a), consider holomorphic chains of the form

$$\begin{array}{c} V'_{-r} \longrightarrow W'_{1-r} \longrightarrow \ \cdots \ \longrightarrow W'_{r-1} \longrightarrow V'_r \\ \oplus \\ F \oplus F^* \end{array} .$$

By Theorem 5.3 and Remark 5.4, such a Higgs bundle is a polystable minimum if and only if it satisfies the conditions of case (1) or case (2) in the statement of the theorem.

For case (b), we have $\mathrm{rk}(E) = 1$ and up to switch E and E^* the holomorphic chains are given by

$$\begin{array}{ccccc} & & F & & \\ E & \xrightarrow{\left(\begin{smallmatrix}\gamma \\ 0\end{smallmatrix}\right)} & \oplus & \xrightarrow{\left(\begin{smallmatrix}0 & \gamma^*\end{smallmatrix}\right)} & E^* \\ & & F^* & & \\ & & \oplus & & \\ & & V' \oplus W' & & \end{array} , \tag{5.24}$$

where $0 < \deg(E)$. As above, (with the roles of E and V' switched) this does not define a local minimum if $V' \neq 0$. When $V' = 0$, we have a local minimum satisfying case (1) of the statement of theorem.

For case (c), we have $\mathrm{rk}(F) = 1$ and the holomorphic chain is given by (5.24) with E and F switched. As above, this is not a minimum if $W' = 0$. Since $p \leqslant q$ and $\mathrm{rk}(V) = \mathrm{rk}(V') + 2\,\mathrm{rk}(E) \leqslant 2$, we have $V' = 0$, giving a local minimum satisfying case (1) of the statement of theorem. □

6. The connected components of $\mathcal{M}(\mathrm{SO}(p, q))$

In this section we use the results from the previous sections to count the number of connected components of the moduli space $\mathcal{M}(\mathrm{SO}(p, q))$, with $1 \leqslant p \leqslant q$. If $p \neq 2$ or if $(p, q) = (2, 2)$ or $(p, q) = (2, 3)$ then we have enough information to give a precise count. In the remaining cases, namely $p = 2$, $q \geqslant 4$, we give a lower bound on the number of connected components of $\mathcal{M}(\mathrm{SO}(2, q))$ and conjecture that it this bound is sharp.

6.1. **Connected components of $\mathcal{M}(\mathrm{SO}(p, q))$ for $2 < p \leqslant q$.** Recall from (2.14) that the moduli space of $\mathrm{SO}(p, q)$-Higgs bundles decomposes as

$$\mathcal{M}(\mathrm{SO}(p,q)) = \coprod_{a,b,c} \mathcal{M}^{a,b,c}(\mathrm{SO}(p,q)), \tag{6.1}$$

where the indices (a, b, c) are classes in $H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2)$ and a polystable $\mathrm{SO}(p, q)$-Higgs bundle (V, Q_V, W, Q_W, η) is in $\mathcal{M}^{a,b,c}(\mathrm{SO}(p, q))$ if a is the first Stiefel-Whitney class of (V, Q_V) and (W, Q_W), b is the second Stiefel-Whitney class of (V, Q_V) and c is the second Stiefel-Whitney class of (W, Q_W). Notice that each $\mathcal{M}^{a,b,c}(\mathrm{SO}(p, q))$ is not necessarily connected.

When $2 < p \leqslant q$, the maximal compact subgroup $\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) \subset \mathrm{SO}(p, q)$ is semisimple. Thus by Proposition 2.23 each of the spaces $\mathcal{M}^{a,b,c}(\mathrm{SO}(p, q))$ is nonempty and has a unique connected component in which every Higgs bundle (V, Q_V, W, Q_W, η) can be deformed to one with vanishing Higgs field. Such components account for 2^{2g+2} connected components of $\mathcal{M}(\mathrm{SO}(p, q))$. These are the 'mundane' components mentioned in the Introduction. Taking into account the 'exotic' components, we obtain the following precise count of the connected components of $\mathcal{M}(\mathrm{SO}(p, q))$ for $2 < p \leqslant q$.

Theorem 6.1. *Let X be a compact Riemann surface of genus $g \geqslant 2$ and denote the moduli space of $\mathrm{SO}(p, q)$-Higgs bundles on X by $\mathcal{M}(\mathrm{SO}(p, q))$. For $2 < p \leqslant q$, we have*

$$|\pi_0(\mathcal{M}(\mathrm{SO}(p,q)))| = 2^{2g+2} + \begin{cases} 2^{2g} & \text{if } q = p \\ 2^{2g+1} - 1 + 2p(g-1) & \text{if } q = p+1 \\ 2^{2g+1} & \text{if } q > p+1\ . \end{cases}$$

Proof. By the above discussion we only need to determine the number of connected components of $\mathcal{M}(\mathrm{SO}(p, q))$ with the property that the Higgs field never vanishes. Recall that if $\mathrm{Min}(\mathcal{M}(\mathrm{SO}(p, q)))$ is the subspace of $\mathcal{M}(\mathrm{SO}(p, q))$ where the Hitchin function (3.1) attains a local minimum, then

$$|\pi_0(\mathcal{M}(\mathrm{SO}(p,q)))| \leqslant |\pi_0(\mathrm{Min}(\mathcal{M}(\mathrm{SO}(p,q))))|.$$

From Theorem 5.9, an $\mathrm{SO}(p, q)$-Higgs bundle (V, W, η), with $2 < p \leqslant q$ and $q \neq p + 1$, is a minimum of the Hitchin function with nonzero Higgs field if and only if the holomorphic chain is given by:

$$\begin{array}{c} V_{1-p} \xrightarrow{\eta^*_{p-2}} W_{2-p} \xrightarrow{\eta_{2-p}} V_{3-p} \xrightarrow{\eta^*_{p-4}} \cdots \xrightarrow{\eta_{p-4}} V_{p-3} \xrightarrow{\eta^*_{2-p}} W_{p-2} \xrightarrow{\eta_{p-2}} V_{p-1}\ , \\ \oplus \\ W'_0 \end{array} \tag{6.2}$$

where the bundle W'_0 is a polystable $\mathrm{O}(q - p + 1, \mathbb{C})$-bundle with $\det(W'_0) = I$, $V_j = IK^{-j}$ and $W_j = IK^{-j}$ for all $j \neq 0$, and each η_j is nonzero. When $q = p + 1$, so that W'_0 is a rank 2 orthogonal bundle, there are also minima of the form

$$W_{-p} \xrightarrow{\eta_{-p}} V_{1-p} \xrightarrow{\eta^*_{p-2}} W_{2-p} \xrightarrow{\eta_{2-p}} V_{3-p} \xrightarrow{\eta^*_{p-4}} \cdots \xrightarrow{\eta_{p-4}} V_{p-3} \xrightarrow{\eta^*_{2-p}} W_{p-2} \xrightarrow{\eta_{p-2}} V_{p-1} \xrightarrow{\eta^*_{-p}} W_p\ , \tag{6.3}$$

where $V_j = K^{-j}$ and $W_j = K^{-j}$ for all $|j| < p$, W_{-p} is a holomorphic line bundle with $0 < \deg(W_{-p}) \leqslant p(2g-2)$ and each η_j is nonzero.

For $2 < p = q$, we only have minima of the form (6.2) with $W_0 = I$. Each such minimum is labeled by the choice of the 2-torsion line bundle I, yielding 2^{2g} connected components. For $2 < p < q$, the connected components of the minima subvarieties of the form (6.2) are labeled by the first and second Stiefel-Whitney class of the bundle W_0' by Proposition 2.23. Thus, the number of connected components of these minima subvarieties is given by $|\mathrm{Bun}_X(\mathrm{O}(q-p+1))| = 2^{2g+1}$ for $2 < p < q-1$. For $2 < p = q-1$, when the first Stiefel-Whitney class of W_0' vanishes the second Stiefel-Whitney class also vanishes since $sw_1(W_0') = 0$ implies $W_0' = L \oplus L^{-1}$ for some degree zero line bundle L. This gives $2^{2g+1} - 1$ connected components of the minima subvarieties whose Higgs bundles are of the form (6.2). There are $p(2g-2)$ connected components of minima subvarieties of type (6.3) since its connected components are labeled by $\deg(W_{-p}) \in (0, p(2g-2)]$.

Finally, by Theorem 4.1, each of the above minima are in a different connected component of the image the map $\Psi : \mathcal{M}_{K^p}(\mathrm{SO}(1, q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(K^{2j}) \to \mathcal{M}(\mathrm{SO}(p,q))$. Thus, each such minima subvariety defines a connected component. □

The following is a direct corollary of the above proof. This formulation will be useful in Section 7. Recall notation (2.11).

Corollary 6.2. *Suppose $2 < p < q-1$. For polystable Higgs bundles $(V, W, \eta) \in \mathcal{M}(\mathrm{SO}(p,q))$ we have the following dichotomy:*

- *Either (V, W, η) can be continuously deformed to a polystable $(V', W', 0)$,*
- *or (V, W, η) can be continuously deformed to $(\mathcal{K}_{p-1} \otimes I, \widehat{W} \oplus \mathcal{K}_{p-2} \otimes I, (0 \ \ \eta_0))$, where $\widehat{W}$ is a polystable rank $q-p+1$ orthogonal bundle with $\Lambda^{q-p+1}\widehat{W} = I$ and $(\mathcal{K}_{p-1}, \mathcal{K}_{p-2}, \eta_0)$ is the unique minimum in the $\mathrm{SO}(p-1,p)$-Hitchin component.*

For minima of the form (6.2) or (6.3), the first and second Stiefel-Whitney classes of V and W are readily computed. The results are shown in the table.

Type of min.	$a = sw_1(W)$	$b = sw_2(V)$	$c = sw_2(W)$
(6.2)	0 if p is even $sw_1(W')$ if p is odd	0	$sw_2(W')$
(6.3)	0	0	$\deg(W_{-p}) \pmod 2$

The following corollaries are now immediate.

Corollary 6.3. *For $2 < p < q-1$, if $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q))$ is the union of connected components from (6.1), then*

$$|\pi_0(\mathcal{M}^{a,b,c}(\mathrm{SO}(p,q)))| = \begin{cases} 2 & \text{if } p \text{ is odd and } b = 0 \\ 2^{2g}+1 & \text{if } p \text{ is even, } a = 0 \text{ and } b = 0 \\ 1 & \text{otherwise .} \end{cases}$$

Corollary 6.4. *For $p > 2$ and $q = p$, if $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,p))$ is the union of connected components from (6.1), then*

$$|\pi_0(\mathcal{M}^{a,b,c}(\mathrm{SO}(p,p)))| = \begin{cases} 2 & \text{if } p \text{ is odd and } b = c = 0 \\ 2^{2g}+1 & \text{if } p \text{ is even and } a = b = c = 0 \\ 1 & \text{otherwise .} \end{cases}$$

Corollary 6.5. *For $p > 2$ and $q = p+1$, if $\mathcal{M}^{a,b,c}(\mathrm{SO}(p,p+1))$ is the union of connected components from (6.1), then*

$$|\pi_0(\mathcal{M}^{a,b,c}(\mathrm{SO}(p,p+1)))| = \begin{cases} 2 & \text{if } p \text{ is odd, } b=0 \text{ and } a\neq 0 \\ 2+p(g-1) & \text{if } p \text{ is odd and } a=b=c=0 \\ 1+p(g-1) & \text{if } p \text{ is odd and } a=b=0 \text{ and } c\neq 0 \\ 2+2^{2g}+p(g-1) & \text{if } p \text{ is even and } a=b=c=0 \\ 1+2^{2g}+p(g-1) & \text{if } p \text{ is even and } a=b=0 \text{ and } c\neq 0 \\ 1 & \text{otherwise .} \end{cases}$$

We observe finally that the following corollary is immediate since the map Ψ is injective.

Corollary 6.6. *For $p \geqslant 1$, the number of connected components of $\mathcal{M}_{K^p}(\mathrm{SO}(1,q))$ are given by*

$$|\pi_0(\mathcal{M}_{K^p}(\mathrm{SO}(1,q)))| = \begin{cases} 2^{2g} & q=1 \\ 2^{2g+1}-1+p(2g-2) & q=2 \\ 2^{2g+1} & q>2\ . \end{cases}$$

In particular, if $q > 2$ then every polystable K^p-twisted $\mathrm{SO}(1,q)$-Higgs bundle can be continuously deformed to one with zero Higgs field.

6.2. **Connected components of $\mathcal{M}(\mathrm{SO}(2,q))$.** In the previous section a complete component count of $\mathcal{M}(\mathrm{SO}(p,q))$ when $p \leqslant q$ and $p \neq 2$ was given. We now discuss the case $p=2$. In this special case the group $\mathrm{SO}(p,q)$ is a group of Hermitian type. Furthermore in this case the minima of type (1) from Theorem 5.9 appear. These are given by holomorphic chains of the form

$$V_{-1} \xrightarrow{\eta_0^*} W \xrightarrow{\eta_0} V_1 \quad , \tag{6.4}$$

where $0 < \deg(V_{-1}) < 2g-2$ and η_0 is nonzero.

Let (V,W,η) be an $\mathrm{SO}(2,q)$-Higgs bundle. As in the general case, the first and second Stiefel-Whitney classes of the orthogonal bundles provide primary topological invariants which help distinguish the connected components of the moduli space. However, when the first Stiefel-Whitney class vanishes, we have $(V,Q_V) \cong (L\oplus L^{-1}, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$ for some line bundle L. The natural number $|\deg(L)|$ satisfies $|\deg(L)| = sw_2(V) \pmod 2$ and provides a refinement of the second Stiefel-Whitney class invariant. This natural number is the absolute value of the so-called Toledo invariant of the $\mathrm{SO}(2,q)$-Higgs bundle. Moreover, if such an $\mathrm{SO}(2,q)$-Higgs bundle (V,W,η) is polystable then

$$|\deg(L)| \leqslant 2g-2.$$

This inequality is usually referred to as the Milnor-Wood inequality and was derived in the proof of Theorem 5.3 (see (5.14)). The special maximal case $|\deg(L)| = 2g-2$ will be discussed in Section 7.3.

Examining the minima classification of Theorem 5.9 and using Theorem 4.1, in the case $2 = p \leqslant q$ we see that the only obstruction to obtaining a full connected component count of $\mathcal{M}(\mathrm{SO}(2,q))$ is the connectedness of the fixed point set (6.4). In particular, for $2 = p < q$, we get bounds, rather than precise values, namely

$$|\pi_0(\mathcal{M}(\mathrm{SO}(2,q))| \geqslant \begin{cases} 2^{2g+2}-4+4(g-1)+2^{2g+1}+4g-5 & \text{if } q=3 \\ 2^{2g+2}-4+4(g-1)+2^{2g+1} & \text{if } q\geqslant 4 \end{cases}$$

It follows from [21], that the above inequality was shown to be an equality for $q=3$:

$$|\pi_0(\mathcal{M}(\mathrm{SO}(2,3))| = 3\times 2^{2g+1}+8g-13. \tag{6.5}$$

We conjecture that equality also holds above for $q \geqslant 4$.

For $p=q=2$ we can use Proposition 3.8 to give a complete count of the connected components of $\mathcal{M}(\mathrm{SO}(2,2))$, as we now briefly explain, leaving the details for the reader.

Proposition 6.7. $|\pi_0(\mathcal{M}(\mathrm{SO}(2,2))| = 3(2^{2g+1}-1)+2g(2g-3)$.

Proof. If (V, W, η) is a polystable $\mathrm{SO}(2,2)$-Higgs bundle with $sw_1(V) = sw_1(W) \neq 0$ then, by Proposition 3.8, it can only be a minima of the Hitchin function if $\eta = 0$. Hence the corresponding subspace with the given topological types is connected. Taking into account $sw_2(V)$ and $sw_2(W)$, this gives rise to $4(2^{2g} - 1)$ components.

If $sw_1(V) = sw_1(W) = 0$, then $V \cong L \oplus L^{-1}$ and $W \cong M \oplus M^{-1}$, with L and M isotropic line subbundles, and $\eta = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ in these decompositions. Note that $\eta^* = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$. Let $l = \deg(L)$ and $m = \deg(M)$. We can assume $l \geqslant 0$ (or $m \geqslant 0$, but not both). Indeed, this follows from the fact that $\pi_0(\mathrm{SO}(2,2)) \cong \mathbb{Z}_2$ acts nontrivially on $\pi_1(\mathrm{SO}(2,2)) \cong \mathbb{Z} \times \mathbb{Z}$ by simultaneously changing the sign of the generators, hence identifies the topological types (l, m) and $(-l, -m)$. As above, l determines $sw_2(V)$ and m determines $sw_2(W)$. Hence, supposing from now on that $l \geqslant 0$, polystability forces

$$l - 2g + 2 \leqslant m \leqslant 2g - 2 - l. \tag{6.6}$$

Indeed, suppose for instance that $m > 2g - 2 - l$. Then we must have $b = 0$ in η, so $\eta(M) \subset LK$ and $\eta^*(L) \subset MK$, but $\deg(L) + \deg(M) = l + m > 2g - 2 > 0$, contradicting polystability. If $m < l - 2g + 2$ the conclusion is similar.

Since $l \geqslant 0$, using (6.6), and recalling that the topological types $(0, m)$ and $(0, -m)$ are identified, we conclude that there are precisely $(2g-1)^2 - 2g + 2$ allowed topological types (l, m) in $\mathcal{M}(\mathrm{SO}(2,2))$ with $sw_1 = 0$.

Now, by Proposition 3.8, (V, W, η) is a minimum if and only if $\eta = 0$ or it is of the form (3.8), up to switching the roles of L, M, L^{-1} and M^{-1}. By polystability, the minima with vanishing Higgs field can only arise when $l = m = 0$, giving one connected component. If $(l, m) \notin \{(0, 2g-2), (2g-2, 0)\}$, the minima (3.8) can be described as certain connected coverings of (products of) certain symmetric products of X, depending on l, m and on the divisors of the components a, b, c, d of η. This gives $(2g-1)^2 - 2g$ connected components. Finally, the minima of type (3.8) for $(l, m) = (0, 2g-2)$ or $(l, m) = (2g-2, 0)$ are parameterized by the 2-torsion points of the Jacobian of X, thus have each 2^{2g} connected components. Summing up everything yields the count of $|\pi_0(\mathcal{M}(\mathrm{SO}(2,2))|$. □

7. Positive surface group representations and Cayley partners

In this section, we recall the Non-Abelian Hodge correspondence between the Higgs bundle moduli space and the moduli space of surface group representations. After proving some immediate consequences of Theorem 6.1, we discuss how the exotic components of Theorem 4.1 are related to recent work of Guichard and Wienhard on positive Anosov representations [25]. Finally, we show this relation with positive Anosov representations can be seen as a generalization of the phenomenon which produces the so-called Cayley partner of a G-Higgs bundle with maximal Toledo invariant for G a Hermitian group of tube type.

7.1. **Surface group representations.**

Let Γ be the fundamental group of a closed oriented surface S of genus $g \geqslant 2$ and let G be a real reductive Lie group. A representation $\rho : \Gamma \to \mathrm{G}$ is called *reductive* if the composition of ρ with the adjoint representation of G is a completely reducible representations.

Denote the set of reductive representations by $\mathrm{Hom}^{red}(\Gamma, \mathrm{G})$. The conjugation action of G on $\mathrm{Hom}(\Gamma, \mathrm{G})$ does not in general have a Hausdorff quotient. However, if we restrict to the set of reductive representations, the quotient will be Hausdorff.

Definition 7.1. *The* G-representation variety $\mathcal{R}(\Gamma, \mathrm{G})$ *of a surface group* Γ *is the space of conjugacy classes of reductive representations of* Γ *in* G*:*

$$\mathcal{R}(\Gamma, \mathrm{G}) = \mathrm{Hom}^{red}(\Gamma, \mathrm{G})/\mathrm{G}\ .$$

Example 7.2. The set of *Fuchsian representations* $\mathrm{Fuch}(\Gamma) \subset \mathcal{R}(\Gamma, \mathrm{SO}(2,1))$ is defined to be the subset of conjugacy classes of *faithful* representations with *discrete image*. The space $\mathrm{Fuch}(\Gamma)$ defines one connected components of $\mathcal{R}(\Gamma, \mathrm{SO}(2,1))$ [20] and is in one to one correspondence with the Teichmüller space of isotopy classes of marked Riemann surface structures on the surface S. Since the surface S is assumed to be orientable, every Fuchsian representation reduces to $\mathrm{SO}_0(2,1)$.

For G a split real form of a complex semisimple Lie group, there is a preferred embedding

$$\iota : \mathrm{SO}_0(2,1) \to \mathrm{G} \tag{7.1}$$

called a principal embedding. When G is an adjoint group, the principal embedding is unique. For the split real form $\mathrm{G} = \mathrm{SO}_0(p,p-1)$, the principal embedding is given by taking the $(p-1)^{st}$-symmetric product of the standard action of $\mathrm{SO}_0(2,1)$ on $\mathbb{R}^3$. The principal embedding defines a map $\iota : \mathcal{R}(\Gamma, \mathrm{SO}_0(2,1)) \to \mathcal{R}(\Gamma, \mathrm{G})$, and the Hitchin component $\mathrm{Hit}(\Gamma, \mathrm{G}) \subset \mathcal{R}(\Gamma, \mathrm{G})$ is defined to be the connected component containing $\iota(\mathrm{Fuch}(\Gamma))$.

Each representation $\rho \in \mathcal{R}(\Gamma, \mathrm{G})$ defines a flat G-bundle $E_\rho = (\widetilde{S} \times \mathrm{G})/\Gamma$. This gives a decomposition of the G representation variety:

$$\mathcal{R}(\Gamma, \mathrm{G}) = \bigsqcup_{a \in \mathrm{Bun}_S(\mathrm{G})} \mathcal{R}^a(\mathrm{G}) \ ,$$

where $a \in \mathrm{Bun}_S(\mathrm{G})$ is the topological type of the flat G-bundle of the representations in $\mathcal{R}^a(\mathrm{G})$. When G is a Hermitian Lie group $\mathrm{Bun}_S(\mathrm{G})$ is infinite. Such G-Higgs bundles and surface group representations acquire a discrete invariant called the Toledo invariant. While the Toledo invariant has several different descriptions, they all yield a finite set of allowed rational values, and hence give a notion of maximality (see for example [14, 10, 5]). In particular, $\mathcal{R}^a(\mathrm{G})$ is nonempty for only finitely many values of $a \in \mathrm{Bun}_S(\mathrm{G})$.

The following theorem links the G-representation variety and the K-twisted G-Higgs bundle moduli space. It was proven by Hitchin [26], Donaldson [15], Corlette [13] and Simpson [38] in various generalities. For the general statement below see [17].

Theorem 7.3. *Let S be a closed oriented surface of genus $g \geqslant 2$ and G be a real reductive Lie group. For each Riemann surface structure X on S there is a homeomorphism between the moduli space $\mathcal{M}_K(\mathrm{G})$ of G Higgs bundles on X and the G-representation variety $\mathcal{R}(\Gamma, \mathrm{G})$. Moreover, for each $a \in \mathrm{Bun}_S(\mathrm{G})$, this homeomorphism identifies the spaces $\mathcal{M}_K^a(\mathrm{G})$ and $\mathcal{R}^a(\mathrm{G})$.*

As in (6.1), for $(a,b,c) \in H^1(S,\mathbb{Z}_2) \times H^2(S,\mathbb{Z}_2) \times H^2(S,\mathbb{Z}_2)$, we have

$$\mathcal{R}(\mathrm{SO}(p,q)) = \coprod \mathcal{R}^{a,b,c}(\mathrm{SO}(p,q)).$$

Using Theorem 6.1 and the above correspondence we have a connected component count of $\mathcal{R}(\mathrm{SO}(p,q))$.

Theorem 7.4. *Let S be a closed surface of genus $g \geqslant 2$ and fundamental group Γ. For $2 < p \leqslant q$, the number of connected components of the representation variety $\mathcal{R}(\Gamma, \mathrm{SO}(p,q))$ is given by*

$$|\pi_0(\mathcal{R}(\Gamma, \mathrm{SO}(p,q)))| = 2^{2g+2} + \begin{cases} 2^{2g} & \text{if } q = p \\ 2^{2g+1} - 1 + 2p(g-1) & \text{if } q = p+1 \\ 2^{2g+1} & \text{if } q > p+1 \ . \end{cases}$$

Remark 7.5. The connected components of $\mathcal{R}^{a,b,c}(\mathrm{SO}(p,q))$ are given by corollaries 6.3, 6.4 and 6.5.

Corollary 6.2 can now be interpreted as a dichotomy in terms of the $\mathrm{SO}(p,q)$ representation variety.

Theorem 7.6. *Let S be a closed surface of genus $g \geqslant 2$ and fundamental group Γ. For $2 < p < q-1$, the representation variety $\mathcal{R}(\mathrm{SO}(p,q))$ of S is disjoint union of two sets*

$$\mathcal{R}(\mathrm{SO}(p,q)) = \mathcal{R}^{cpt}(\mathrm{SO}(p,q)) \sqcup \mathcal{R}^{ex}(\mathrm{SO}(p,q)) \ , \tag{7.2}$$

where

- $[\rho] \in \mathcal{R}^{cpt}(\mathrm{SO}(p,q))$ *if and only if ρ can be continuously deformed to a compact representation,*
- $[\rho] \in \mathcal{R}^{ex}(\mathrm{SO}(p,q))$ *if and only if ρ can be continuously deformed to a representation*

$$\rho' = \alpha \oplus (\iota \circ \rho_{\mathrm{Fuch}}) \otimes \det(\alpha) \ , \tag{7.3}$$

where α is a representation of Γ into the compact group $\mathrm{O}(q-p+1)$, ρ_{Fuch} is a Fuchsian representation of Γ into $\mathrm{SO}_0(2,1)$, and ι is the principal embedding from (7.1).

Proof. For the first part, note that a representation $\rho : \Gamma \to \mathrm{SO}(p,q)$ can be continuously deformed to a compact representation if and only if the corresponding Higgs bundle can be continuously deformed to one with vanishing Higgs field.

If ρ cannot be continuously deformed to a compact representation, then by Corollary 6.2, the associated $\mathrm{SO}(p,q)$-Higgs bundle (V, W, η) can be continuously deformed to (cf. (2.11))

$$(\mathcal{K}_{p-1} \otimes I, \widehat{W} \oplus \mathcal{K}_{p-2} \otimes I, (0 \ \ \eta_0)),$$

where $\widehat{W}$ is a polystable rank $q-p+1$ orthogonal bundle with $\Lambda^{q-p+1}\widehat{W} = I$ and $(\mathcal{K}_{p-1}, \mathcal{K}_{p-2}, \eta_0)$ is the unique minimum in the $\mathrm{SO}(p-1,p)$-Hitchin component. Through Theorem 7.3, the Higgs bundle description of the Hitchin component from (2.21) is identified with the representation variety from Example 7.2. In particular, if s_H is the Hitchin section from (2.12), the representation associated to $s_H(0)$ is $\iota \circ \rho_{\mathrm{Fuch}}$ for a Fuchsian representation ρ_{Fuch} [28]. In particular, the representation associated to the unique minimum in the $\mathrm{SO}_0(p,p-1)$-Hitchin component $(\mathcal{K}_{p-1}, \mathcal{K}_{p-2}, \eta_0)$ is given by $\iota \circ \rho_{\mathrm{Fuch}}$ for a Fuchsian representation ρ_{Fuch}.

If $A \in \mathrm{SO}_0(p,p-1)$ and $B \in \mathrm{O}(q-p+1)$, then $(A,B) \mapsto \left(\begin{smallmatrix} \det(B)\cdot A & 0 \\ 0 & B \end{smallmatrix}\right)$ defines an embedding

$$\mathrm{SO}_0(p,p-1) \times \mathrm{O}(q-p+1) \hookrightarrow \mathrm{SO}(p,q).$$

If $\alpha : \Gamma \to \mathrm{O}(q-p+1)$ is the representation associated to the polystable $\mathrm{O}(q-p+1,\mathbb{C})$-bundle $\widehat{W}$, then the representation associated to the $\mathrm{SO}(p,q)$-Higgs bundle $(\mathcal{K}_{p-1} \otimes I, \widehat{W} \oplus \mathcal{K}_{p-2} \otimes I, (0 \ \ \eta_0))$ is given by $\alpha \oplus (\iota \circ \rho_{\mathrm{Fuch}}) \otimes \det(\alpha)$. □

7.2. **Positive Anosov representations.** Anosov representations were introduced by Labourie [31] and have many interesting geometric and dynamic properties which generalize convex cocompact representations into rank one Lie groups. Important examples of Anosov representations include Fuchsian representations, quasi-Fuchsian representations, Hitchin representations into split real groups and maximal representations into Lie groups of Hermitian type. We will describe the necessary properties of Anosov representations and refer the reader to [31, 24, 22, 29] for more details.

Let G be a semisimple Lie group and $\mathrm{P} \subset \mathrm{G}$ be a parabolic subgroup. Let $\mathrm{L} \subset \mathrm{P}$ be the Levi factor (the maximal reductive subgroup) of P, it is given by $\mathrm{L} = \mathrm{P} \cap \mathrm{P}^{opp}$, where P^{opp} is the opposite parabolic of P. The homogeneous space G/L is the unique open G orbit in $\mathrm{G/P} \times \mathrm{G/P}$, and points $(x,y) \in \mathrm{G/P} \times \mathrm{G/P}$ in this open orbit are called *transverse*.

Definition 7.7. *Let Γ be the fundamental group of a closed surface of genus $g \geqslant 2$. Let $\partial_\infty\Gamma$ be the Gromov boundary of the group Γ. Topologically $\partial_\infty\Gamma \cong \mathbb{RP}^1$. A representation $\rho : \Gamma \to \mathrm{G}$ is* P-Anosov *if there exists a unique continuous boundary map $\xi_\rho : \partial_\infty\Gamma \to \mathrm{G/P}$ which satisfies*

- *Equivariance: $\xi(\gamma \cdot x) = \rho(\gamma) \cdot \xi(x)$ for all $\gamma \in \Gamma$ and all $x \in \partial_\infty\Gamma$.*
- *Transversality: for all distinct $x, y \in \partial_\infty\Gamma$ the generalized flags $\xi(x)$ and $\xi(y)$ are transverse.*
- *Dynamics preserving: see [31, 24, 22, 29] for the precise notion.*

The map ξ_ρ will be called the P-Anosov boundary curve*.*

One important property of Anosov representations is that they define an open subset of the representation variety $\mathcal{R}(\Gamma, \mathrm{G})$. The set of Anosov representations is however not closed. For example, for the group $\mathrm{PSL}(2,\mathbb{C})$ the set of Anosov representations corresponds to the non-closed set quasi-Fuchsian representations of $\mathcal{R}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$. The special cases of Hitchin representations and maximal representations define connected components of Anosov representations. Both Hitchin representations and maximal representations satisfy an additional "positivity" property which is a closed condition. For Hitchin representations this was proven by Labourie [31] and Fock-Goncharov [16], and for maximal representations by Burger-Iozzi-Wienhard [9]. These notions of positivity have recently been unified and generalized by Guichard and Wienhard [25].

For a parabolic subgroup $\mathrm{P} \subset \mathrm{G}$, denote the Levi factor of P by L and the unipotent subgroup by $\mathrm{U} \subset \mathrm{P}$. The Lie algebra $\mathfrak{p}$ of P admits an Ad_{L}-invariant decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ and $\mathfrak{u}$

are the Lie algebras of L and U respectively. Moreover, the unipotent Lie algebra $\mathfrak{u}$ decomposes into irreducible L-representation:

$$\mathfrak{u} = \bigoplus \mathfrak{u}_\beta \ .$$

Recall that a parabolic subgroup P is determined by fixing a simple restricted root system Δ of a maximal $\mathbb{R}$-split torus of G, and choosing a subset $\Theta \subset \Delta$ of simple roots. To each simple root $\beta_j \in \Theta$ there is a corresponding irreducible L-representation space $\mathfrak{u}_{\beta_j}$.

Definition 7.8. *([25, Definition 4.2]) A pair* $(\mathrm{G}, \mathrm{P}^\Theta)$ admits a positive structure *if for all* $\beta_j \in \Theta$, *the* L^Θ*-representation space* $\mathfrak{u}_{\beta_j}$ *has an* L_0^Θ*-invariant acute convex cone* $c_{\beta_j}^\Theta$, *where* L_0^Θ *denotes the identity component of* L^Θ.

If $(\mathrm{G}, \mathrm{P}^\Theta)$ admits a positive structure, then exponentiating certain combinations of elements in the L_0^Θ-invariant acute convex cones give rise to a semigroup $\mathrm{U}_{>0}^\Theta \subset \mathrm{U}^\Theta$ [25, Theorem 4.5]. The existence of the semigroup $U_{>0}$ gives a well defined notion of positively oriented triples of pairwise transverse points in $\mathrm{G}/\mathrm{P}^\Theta$. This notion allows one to define a *positive Anosov representation.*

Definition 7.9. *([25, Definition 5.3]) If the pair* $(\mathrm{G}, \mathrm{P}^\Theta)$ *admits a positive structure, then a* P^Θ*-Anosov representation* $\rho : \Gamma \to \mathrm{G}$ *is called* positive *if the Anosov boundary curve* $\xi : \partial_\infty\Gamma \to \mathrm{G}/\mathrm{P}^\Theta$ *sends positively ordered triples in* $\partial_\infty\Gamma$ *to positive triples in* $\mathrm{G}/\mathrm{P}^\Theta$.

Conjecture 7.10. *([23, 25]) If* $(\mathrm{G}, \mathrm{P}^\Theta)$ *admits a notion of positivity, then the set* P^Θ*-positive Anosov representations is an open and closed subset of* $\mathcal{R}(\Gamma, \mathrm{G})$.

In particular, the aim of this conjecture is to characterize the connected components of $\mathcal{R}(\Gamma, \mathrm{G})$ which are not labeled by primary topological invariants as being connected components of positive Anosov representations, such connected components are referred as higher Teichmüller components.

Remark 7.11. When G is a split real form and $\Theta = \Delta$, the corresponding parabolic is a Borel subgroup of G. In this case, the connected component of the identity of the Levi factor is $\mathrm{L}_0^\Delta \cong (\mathbb{R}^+)^{rk(\mathrm{G})}$ and each simple root space $\mathfrak{u}_{\beta_i}$ is one dimensional. The L_0^Δ-invariant acute convex cone in each simple root space $\mathfrak{u}_{\beta_i}$ is isomorphic to $\mathbb{R}^+$. The set of P^Δ-positive Anosov representations into a split group are exactly Hitchin representations. When G is a Hermitian Lie group of tube type and P is the maximal parabolic associated to the Shilov boundary of the Riemannian symmetric space of G, the pair (G, P) also admits a notion of positivity [10]. In this case, the space of maximal representations into G are exactly the P-positive Anosov representations. In particular, the above conjecture holds in these two cases.

In general, the group $\mathrm{SO}(p, q)$ is not a split group and not a group of Hermitian type. Nevertheless, if $p \neq q$, then $\mathrm{SO}(p, q)$ has a parabolic subgroup P^Θ which admits a positive structure. Here P^Θ is the stabilizer of the partial flag $V_1 \subset V_2 \subset \cdots \subset V_{p-1}$, where $V_j \subset \mathbb{R}^{p+q}$ is a j-plane which is isotropic with respect to a signature (p, q) inner product with $p < q$. Here the subgroup $\mathrm{L}_{pos}^\Theta \subset \mathrm{L}^\Theta \subset \mathrm{SO}(p, q)$ which preserves the cones $c_{\beta_j}^\Theta$ is isomorphic to $\mathrm{L}_{pos}^\Theta \cong \mathbb{R}^+ \times \mathrm{SO}(1, q - p + 1)$. We refer the reader to [25] and [12, Section 7] for more details.

To construct examples of $\mathrm{SO}(p, q)$ positive Anosov representations we have the following proposition.

Proposition 7.12. *Let* $p < q$. *Consider the signature* (p, q)*-inner product* $\langle x, x\rangle = \sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{p+q} x_j^2$. *If* $A \in \mathrm{SO}_0(p, p-1)$ *and* $B \in \mathrm{O}(q - p + 1)$, *then the set matrices* $\left(\begin{smallmatrix} \det(B)\cdot A & 0 \\ 0 & B \end{smallmatrix}\right)$ *defines an embedding*

$$\mathrm{SO}_0(p, p-1) \times \mathrm{O}(q - p + 1) \hookrightarrow \mathrm{SO}(p, q).$$

If $\rho_{Hit} : \Gamma \to \mathrm{SO}_0(p, p-1)$ *is a Hitchin representation and* $\alpha : \Gamma \to \mathrm{O}(q - p + 1)$ *is any representation, then*

$$\rho = \rho_{Hit} \otimes \det(\alpha) \oplus \alpha : \Gamma \to \mathrm{SO}(p, q)$$

is a P^Θ*-positive Anosov representation.*

This is proven for $q = p+1$ in [12, Section 7], and the proof for general q is the same. For the proof of the first part of the above proposition it suffices to show that the map $\mathrm{SO}(p,p-1) \to \mathrm{SO}(p,q)$ described above sends the positive semigroup $U^{\Delta}_{>0} \subset \mathrm{SO}(p,p-1)$ into the positive semigroup $U^{\Theta}_{>0}$. The second part follows from the fact that a representation ρ is a P-Anosov representation if and only if the restriction of ρ to any finite index subgroup is P-Anosov, and the fact that the centralizer of an Anosov representation acts trivially on the Anosov boundary curve.

Using Proposition 7.12 and Theorem 7.6, we conclude that for $q > p+1$ the connected components of $\mathcal{R}(\Gamma, \mathrm{SO}(p,q))$ from Theorem 4.1 contain P^{Θ}-positive Anosov representations.

Proposition 7.13. *Let $\mathrm{P}^{\Theta} \subset \mathrm{SO}(p,q)$ be the stabilizer of the partial flag $V_1 \subset V_2 \subset \cdots \subset V_{p-1}$, where $V_j \subset \mathbb{R}^{p+q}$ is a j-plane which is isotropic with respect to a signature (p,q) inner product with $p < q$. If $q > p+1$, then each connected component of $\mathcal{R}^{ex}(\mathrm{SO}(p,q))$ from (7.2) contains P^{Θ}-positive Anosov representations.*

Remark 7.14. When $q = p+1$, this was shown in [12] for the analogous connected components which contain minima of the form (6.2). The components which contain minima of the form (6.3) are smooth, and one cannot use Proposition 7.12 to obtain positive representations in these components. However, we note that if Conjecture 7.10 holds, then each of the these smooth connected components of $\mathcal{R}(\mathrm{SO}(p,p+1))$ consists of positive representations since each component would be contained in a component of positive representations into $\mathrm{SO}(p,p+2)$.

Proposition 7.13 gives further evidence for Conjecture 7.10, and it is thus natural to expect that all representations in the connected components from Theorem 4.1 are positive Anosov representations. Indeed, this would follow from Conjecture 7.10 and Proposition 7.13. Moreover, if Conjecture 7.10 is true, then the connected components of Theorem 4.1 correspond exactly to those connected components of $\mathcal{R}(\Gamma, \mathrm{SO}(p,q))$ which contain positive Anosov representations.

7.3. **Positivity and a generalized Cayley correspondence.** We conclude the paper by interpreting the parameterization of the 'exotic' connected components of the $\mathrm{SO}(p,q)$-Higgs bundle moduli space from Theorem 4.1 as a generalized Cayley correspondence.

Let G be a simple adjoint Hermitian Lie group of tube type and let G/P be the Shilov boundary of the symmetric space of G. In [5], it is proven that if L is the Levi factor of P, then the space of Higgs bundles with maximal Toledo invariant is isomorphic to $\mathcal{M}_{K^2}(\mathrm{L})$. More generally, an analogous statement holds when $\mathrm{G}' \to \mathrm{G}$ is a finite cover such that a G-Higgs bundle with maximal Toledo invariant lifts to a G'-Higgs bundle. This correspondence between maximal G-Higgs bundles and K^2-twisted L-Higgs bundles is called the Cayley correspondence.

Remark 7.15. In [5], the above statement is stated differently. We use the above interpretation because it relates directly with the notions of positivity discussed in the previous section.

Note that the above parabolic and Levi factor are exactly the objects which appear in the notion of positivity when G is Hermitian Lie group of tube type. When G is a split real form the Hitchin components of $\mathcal{M}(G)$ admit an analogous interpretation. Namely, if G is such a split group, then (G,P) admits a positive structure when P is a minimal parabolic subgroup. In this case, $\mathrm{L} \subset \mathrm{P}$ is $(\mathbb{R}^*)^{\mathrm{rk(G)}}$ and the identity component L_0 is given by $(\mathbb{R}^+)^{\mathrm{rk(G)}}$. Moreover, the moduli space of L-twisted $\mathbb{R}^+$-Higgs bundles is isomorphic to $H^0(L)$:

$$\mathcal{M}_L(\mathbb{R}^+) \cong H^0(L).$$

Thus, when the Hitchin base is $\bigoplus_{j=1}^{\mathrm{rk}(G)} H^0(K^{m_j+1})$, the Hitchin components are given by

$$\mathcal{M}_{K^{m_1+1}}(\mathbb{R}^+) \times \cdots \times \mathcal{M}_{K^{m_{\mathrm{rk}(G)}+1}}(\mathbb{R}^+).$$

In particular, the Higgs bundles associated surface group to representations into split real groups which are positive with respect the minimal parabolic subgroup also satisfy a 'Cayley correspondence'.

For the group $\mathrm{SO}(p,q)$, the Levi factor of the parabolic P^Θ so that $(\mathrm{SO}(p,q),\mathrm{P}^\Theta)$ has a positive structure is $\mathrm{L}^\Theta=\mathrm{SO}(1,q-p+1)\times(\mathbb{R}^*)^{p-1}$. Moreover, the subgroup L^Θ_{pos} which preserves the positive cones is

$$\mathrm{L}^\Theta_{pos}\cong\underbrace{\mathbb{R}^+\times\cdots\times\mathbb{R}^+}_{(p-1)\text{-times}}\times\mathrm{SO}(1,q-p+1).$$

Recall that the 'exotic' connected components in the image of Ψ Theorem 4.1 are given by

$$\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))\times\prod_{j=1}^{p-1}H^0(K^{2j}).$$

Using $\mathcal{M}_{K^{2j}}(\mathbb{R}^+)=H^0(K^{2j})$, this is equivalent to

$$\mathcal{M}_{K^p}(\mathrm{SO}(1,q-p+1))\times\prod_{j=1}^{p-1}\mathcal{M}_{K^{2j}}(\mathbb{R}^+).$$

When $2=p\leqslant q$, we recover the Cayley correspondence for groups of Hermitian type [7, 5]. Hence, for $2<p\leqslant q$ we have established that the Higgs bundles associated to representations into $\mathrm{SO}(p,q)$ which cannot be continuously deformed to compact representations satisfy a generalized Cayley correspondence. Moreover, when $p<q-1$ each such component of the representation variety contains positive representations by Proposition 7.13. This suggests a general theorem for positive representations which relates the connected components of the subgroup of L^Θ which preserves the cones with the product of moduli spaces of appropriately twisted L_j-Higgs bundles.

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