A Parametric Family of Subalgebras of the Weyl Algebra II. Irreducible Modules

Georgia Benkart, Samuel A. Lopes*, and Matthew Ondrus

Abstract

An Ore extension over a polynomial algebra $\mathbb{F}[x]$ is either a quantum plane, a quantum Weyl algebra, or an infinite-dimensional unital associative algebra A_h generated by elements x, y, which satisfy $yx - xy = h$, where $h \in \mathbb{F}[x]$. When $h \neq 0$, the algebras A_h are subalgebras of the Weyl algebra A_1 and can be viewed as differential operators with polynomial coefficients. In previous work, we studied the structure of A_h and determined its automorphism group $\mathsf{Aut}_{\mathbb{F}}(\mathsf{A}_h)$ and the subalgebra of invariants under $\mathsf{Aut}_{\mathbb{F}}(\mathsf{A}_h)$. Here we determine the irreducible A_h-modules. In a sequel to this paper, we completely describe the derivations of A_h over any field.

1 Introduction

In [BLO1], we investigated a family of infinite-dimensional unital associative algebras A_h parametrized by a polynomial h in one variable, whose definition is given as follows:

Definition 1.1. Let $\mathbb{F}$ be a field, and let $h \in \mathbb{F}[x]$. The algebra A_h is the unital associative algebra over $\mathbb{F}$ with generators x, y and defining relation $yx = xy + h$ (equivalently, $[y, x] = h$ where $[y, x] = yx - xy$).

These algebras arose naturally in the context of Ore extensions over a polynomial algebra $\mathbb{F}[x]$. Recall that an Ore extension $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is built from a unital associative (not necessarily commutative) algebra R over a field $\mathbb{F}$, an $\mathbb{F}$-algebra endomorphism σ of R, and a σ-derivation of R, where by a σ-derivation δ, we mean that δ is $\mathbb{F}$-linear and $\delta(rs) = \delta(r)s + \sigma(r)\delta(s)$ holds for all $r, s \in \mathsf{R}$. Then $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is the algebra generated by y over R subject to the relation

$$yr = \sigma(r)y + \delta(r) \qquad \text{for all } r \in \mathsf{R}.$$

Many algebras can be realized as iterated Ore extensions, and for that reason, Ore extensions have become a mainstay in associative theory. Ore extensions inherit

*Research funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT – Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0144/2011.

properties from the underlying algebra R. For instance, when σ is an automorphism, then A is a free left and right R-module with basis $\{y^n \mid n \geq 0\}$; if R is left (resp. right) Noetherian, then A is left (resp. right) Noetherian; and if R is a domain, then A is a domain.

The Ore extensions with $\mathsf{R} = \mathbb{F}[x]$ and σ an automorphism have the following description (compare [AVV] and [AD] for a somewhat different division into cases).

Lemma 1.2. *Assume* $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ *is an Ore extension with* $\mathsf{R} = \mathbb{F}[x]$*, a polynomial algebra over a field* $\mathbb{F}$ *of arbitrary characteristic, and* σ *an automorphism of* R*. Then* A *is isomorphic to one of the following:*

(a) *a quantum plane*

(b) *a quantum Weyl algebra*

(c) *a unital associative algebra* A_h *with generators* x, y *and defining relation* $yx = xy + h$ *for some polynomial* $h \in \mathbb{F}[x]$.

The algebra A_h is the Ore extension $\mathsf{R}[y, \mathsf{id}_\mathsf{R}, \delta]$ obtained from taking $\mathsf{R} = \mathbb{F}[x]$, $h \in \mathsf{R}$, $\sigma = \mathsf{id}_\mathsf{R}$, and $\delta : \mathsf{R} \to \mathsf{R}$ to be the $\mathbb{F}$-linear derivation with $\delta(r) = r'h$ for all $r \in \mathsf{R}$, where r' denotes the usual derivative of r with respect to x. In particular, $[y, r] = \delta(r) = r'h$ for all $r \in \mathsf{R}$. The algebra A_h is a Noetherian domain and a free left and right R-module with basis $\{y^n \mid n \geq 0\}$. Both $\{x^m y^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$ and $\{y^n x^m \mid m, n \in \mathbb{Z}_{\geq 0}\}$ are bases for A_h, and A_h has Gelfand-Kirillov dimension 2.

Several well-known algebras have the form A_h for some $h \in \mathbb{F}[x]$. For example, A_0 is the polynomial algebra $\mathbb{F}[x, y]$; A_1 is the Weyl algebra; and the algebra A_x is the universal enveloping algebra of the two-dimensional non-abelian Lie algebra (there is only one such Lie algebra up to isomorphism). The algebra A_{x^2} is often referred to as the Jordan plane. It appears in noncommutative algebraic geometry (see for example, [SZ] and [AS]) and exhibits many interesting features such as being Artin-Schelter regular of dimension 2. In a series of articles [S1]–[S3], Shirikov has undertaken an extensive study of the automorphisms, derivations, prime ideals, and modules of the algebra A_{x^2}. Recent work of Iyudu [I] has further developed the representation theory of A_{x^2}. Cibils, Lauve, and Witherspoon [CLW] have constructed new examples of finite-dimensional Hopf algebras in prime characteristic which are Nichols algebras using quotients of the algebra A_{x^2} and cyclic subgroups of their automorphisms.

Quantum planes and quantum Weyl algebras are examples of generalized Weyl algebras, and as such, have been studied extensively. There are striking similarities in the behavior of the algebras A_h as h ranges over the polynomials in $\mathbb{F}[x]$. For that reason, we believe that studying them as one family provides much insight into their structure, automorphisms, derivations, and modules. In [BLO1], we determined the center, normal elements, prime ideals, and automorphisms of A_h and their invariants in A_h. In [BLO2], we determine the derivations of an arbitrary algebra A_h over any field, derive expressions for the Lie bracket in the quotient $\mathsf{HH}^1(\mathsf{A}_h) := \mathsf{Der}_\mathbb{F}(\mathsf{A}_h)/\mathsf{Inder}_\mathbb{F}(\mathsf{A}_h)$ of $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$ modulo the ideal $\mathsf{Inder}_\mathbb{F}(\mathsf{A}_h)$ of inner derivations, and use these formulas to understand the structure of the Lie algebra $\mathsf{HH}^1(\mathsf{A}_h)$. In particular, when $\mathsf{char}(\mathbb{F}) = 0$, we construct a maximal nilpotent ideal of $\mathsf{HH}^1(\mathsf{A}_h)$ and explicitly describe the structure of the corresponding quotient in terms of the Witt algebra (centreless Virasoro algebra) of vector fields on the unit circle.

Our aim in this paper is to give a detailed investigation of the modules for the algebras A_h over arbitrary fields. In [Bl], Block undertook a comprehensive study of the irreducible modules for the Weyl algebra A_1 and for the universal enveloping algebras of $\mathfrak{sl}_2$ and of the two-dimensional solvable Lie algebra (which is the algebra A_x) over a field of characteristic zero. (Compare also [AP] for the $\mathfrak{sl}_2$ case.) Block also considered Ore extensions $\mathsf{R}[y, \mathsf{id}, \delta]$ over a Dedekind domain R of characteristic zero, with the main effort in [Bl] directed towards investigating irreducible R-torsion-free modules. Block's results were extended by Bavula in [B3] to more general Ore extensions over Dedekind domains, and by Bavula and vanOystaeyen in [BO] to develop a representation theory for generalized Weyl algebras over Dedekind domains.

The generalized weight A_h-modules over fields of arbitrary characteristic will form the main focus of the present paper. Included also will be results on indecomposable A_h-modules, on primitive ideals of A_h (that is, the annihilators of irreducible A_h-modules), and on some combinatorial connections as well.

Since the representation theory of polynomial algebras is well developed, we will assume that $h \neq 0$ throughout the paper.

It is an easy consequence of the relation $[y, r] = \delta(r)$ for $r \in \mathsf{R}$ and induction that the following identity holds in any Ore extension $\mathsf{R}[y, \mathsf{id}_\mathsf{R}, \delta]$ for all $n \geq 0$:

$$r y^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} y^{n-j} \delta^j(r). \tag{1.3}$$

Using that identity, we obtained the following description of the center of A_h:

Theorem 1.4. [BLO1, Thm. 5.5] *Let $\mathsf{Z}(\mathsf{A}_h)$ denote the center of A_h.*

(1) *If $\mathsf{char}(\mathbb{F}) = 0$, then $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}1$.*

(2) *If $\mathsf{char}(\mathbb{F}) = p > 0$, then $\mathsf{Z}(\mathsf{A}_h)$ is isomorphic to the polynomial algebra $\mathbb{F}[x^p, z_p]$, where*

$$z_p := y(y + h')(y + 2h') \cdots (y + (p-1)h') = y^p - y\frac{\delta^p(x)}{h(x)}, \tag{1.5}$$

and $'$ denotes the usual derivative. Moreover $\frac{\delta^p(x)}{h(x)} \in \mathsf{Z}(\mathsf{A}_h) \cap \mathbb{F}[x] = \mathbb{F}[x^p]$.

Remark 1.6. The proof of Theorem 5.5 in [BLO1] shows that y commutes with $\frac{\delta^p(x)}{h(x)}$, but since $\frac{\delta^p(x)}{h(x)}$ is a polynomial in x, it commutes with x as well, hence is central in A_h.

When $\mathsf{char}(\mathbb{F}) = p > 0$, it follows from Theorem 1.4 that A_h is free of rank p^2 as a module over its center (see [BLO1, Prop. 5.9]). This implies that A_h is a polynomial identity ring (e.g. [McR, Cor. 13.1.13 (iii)]). Applying [McR, Thm. 13.10.3 (i)], we can conclude the following:

Proposition 1.7. *Assume $\mathsf{char}(\mathbb{F}) = p > 0$. Then all irreducible A_h-modules are finite dimensional.*

In Section 2, we review basic facts about modules for Ore extensions over Dedekind domains. Our approach here follows [Bl] (see also [B3] for results for more general Ore extensions). For such Ore extensions, the irreducible modules are either generalized weight modules relative to R (equivalently, have R-torsion), or are R-torsion-free. We show in Section 3 that for any field $\mathbb{F}$, when $h \notin \mathbb{F}^*$, the algebra A_h has a family of indecomposable modules of arbitrarily large dimension. Section 4 is devoted to generalized weight modules for A_h. In particular, we consider induced generalized weight modules for A_h, which play a role analogous to Verma modules in the representation theory of semisimple Lie algebras, and also finite-dimensional irreducible modules for A_h. In Section 5, we determine the primitive ideals of A_h. Corollary 5.5 gives an A_h-version of Duflo's well-known result [Du, Cor. 1] on the primitive ideals of enveloping algebras of complex semisimple Lie algebras.

Section 6 is dedicated to the $\mathsf{char}(\mathbb{F}) = 0$ case. Corollary 6.1 of that section shows that the irreducible generalized weight modules for A_h are either induced modules or finite-dimensional quotients of them (compare [Bl, Prop. 4.1]). The classification of the irreducible generalized weight modules for A_h when $\mathbb{F}$ is algebraically closed of characteristic zero is given in Corollary 6.5. Part (i) of that corollary may be regarded as the analogue of Lie's theorem for the algebras A_h, and in fact, it is Lie's theorem for A_x. In Section 6.3, we investigate irreducible R-torsion-free A_h-modules when $\mathsf{char}(\mathbb{F}) = 0$ and determine a criterion for when an irreducible R-torsion-free module for the Weyl algebra A_1 restricts to one for A_h. When $\mathsf{char}(\mathbb{F}) = p > 0$, all irreducible modules are finite dimensional, so R-torsion-free irreducible modules only exist when $\mathsf{char}(\mathbb{F}) = 0$. When $\mathbb{F}$ is an algebraically closed field of characteristic $p > 0$, we show in Section 7 that the irreducible A_h-modules have dimension 1 or p and give an explicit description of them in Corollary 7.8. The expressions for the A_h-action on irreducible modules often entail terms of the form $\delta^k(x)$. Section 8 presents some interesting combinatorics for these terms phrased in the language of partitions.

2 Modules for Ore Extensions

Assume $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is an Ore extension with R a Dedekind domain. Let E denote the field of fractions of R. Thus $\mathsf{E} = \mathsf{S}^{-1}\mathsf{R}$ where $\mathsf{S} = \mathsf{R} \setminus \{0\}$. The localization $\mathsf{B} = \mathsf{S}^{-1}\mathsf{A}$ is the Ore extension $\mathsf{B} = \mathsf{E}[y, \sigma, \delta]$, where σ and δ have natural extensions to E.

Given an A-module M, $\operatorname{tor}_{\mathsf{R}}(\mathsf{M}) := \{v \in \mathsf{M} \mid rv = 0 \text{ for some } 0 \neq r \in \mathsf{R}\}$ is an A-submodule called the R-*torsion submodule* of M. We say that M is an R-*torsion* (resp. R-*torsion-free*) module if $\operatorname{tor}_{\mathsf{R}}(\mathsf{M}) = \mathsf{M}$ (resp. $\operatorname{tor}_{\mathsf{R}}(\mathsf{M}) = 0$). If M is irreducible, then $\mathsf{S}^{-1}\mathsf{M} = \mathsf{B} \otimes_{\mathsf{A}} \mathsf{M}$ is either 0 or a nonzero irreducible B-module. In the former case, M has R-torsion, and in the latter, M is R-torsion-free. Thus, the set $\widehat{\mathsf{A}}$ of isomorphism classes of irreducible A-modules decomposes into two disjoint subsets,

$$\widehat{\mathsf{A}} = \widehat{\mathsf{A}}(\mathsf{R}\text{-torsion}) \cup \widehat{\mathsf{A}}(\mathsf{R}\text{-torsion-free}).$$

Assume M is an A-module. For any ideal $\mathfrak{n}$ of R, let

$$\mathsf{M}_{\mathfrak{n}} = \{v \in \mathsf{M} \mid \mathfrak{n}v = 0\} \quad \text{and} \quad \mathsf{M}^{\mathfrak{n}} = \{v \in \mathsf{M} \mid \mathfrak{n}^k v = 0 \text{ for some } k = k(v)\}. \tag{2.1}$$

Let $\mathfrak{max}(\mathsf{R})$ denote the set of maximal ideals of R. An A-module M is said to be an R-*weight module* (resp. R-*generalized weight module*) if $\mathsf{M} = \bigoplus_{\mathfrak{n} \in \mathfrak{max}(\mathsf{R})} \mathsf{M}_{\mathfrak{n}}$ (resp. if $\mathsf{M} = \bigoplus_{\mathfrak{n} \in \mathfrak{max}(\mathsf{R})} \mathsf{M}^{\mathfrak{n}}$). When R is a Dedekind domain, the irreducible R-torsion modules are precisely the irreducible R-generalized weight modules. We present a proof of this fact next (compare the arguments in [Bl, Proof of Prop. 4.1] and also [B3, Sec. 4]).

Proposition 2.2. *Suppose that* $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_{\mathsf{R}}, \delta]$ *is an Ore extension with* R *a Dedekind domain.*

(i) *If* V *is an* A*-module such that* $\mathsf{V} = \mathsf{A}u$ *for* $u \in \mathsf{V}^{\mathfrak{m}}$ *and some ideal* $\mathfrak{m}$ *of* R*, then* $\mathsf{V} = \mathsf{V}^{\mathfrak{m}}$*. Moreover, if* $\mathfrak{m}$ *is* δ*-invariant and* $u \in \mathsf{V}_{\mathfrak{m}}$*, then* $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$*.*

(ii) $\widehat{\mathsf{A}}(\mathsf{R}\text{-torsion}) = \widehat{\mathsf{A}}(\mathsf{R}\text{-generalized weight})$.

(iii) *If* V *is an irreducible* R*-torsion* A*-module, then* $\mathsf{V} = \mathsf{V}^{\mathfrak{m}}$ *for some* $\mathfrak{m} \in \mathfrak{max}(\mathsf{R})$*, and when* $\mathfrak{m}$ *is* δ*-invariant,* $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$*.*

Proof. Let $\mathfrak{m}$ be an ideal of R and suppose $\ell \geq 1$. Then by Leibniz's rule, $\delta(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^{\ell-1}$ (where $\mathfrak{m}^0 = \mathsf{R}$). In case $\mathfrak{m}$ is δ-invariant, then $\delta(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^\ell$.

(i) Assume $\mathsf{V} = \mathsf{A}u$ and $\mathfrak{m}^k u = 0$ for some $k \geq 1$. Then by (1.3), we have

$$\mathfrak{m}^{k+n} y^n \mathsf{R}u \subseteq \sum_{j=0}^{n} y^{n-j} \delta^j(\mathfrak{m}^{k+n}) \mathsf{R}u \subseteq \sum_{j=0}^{n} y^{n-j} \mathfrak{m}^{k+n-j} u = 0.$$

Thus, $y^n \mathsf{R}u \subseteq \mathsf{V}^{\mathfrak{m}}$ for all $n \geq 0$, which proves that $\mathsf{V} = \mathsf{A}u = \mathsf{V}^{\mathfrak{m}}$ (and hence that V is an R-generalized weight module if $\mathfrak{m} \in \mathfrak{max}(\mathsf{R})$). If $\mathfrak{m}$ is δ-invariant, then $\mathfrak{m}^k y^n \mathsf{R}u \subseteq \sum_{j=0}^{n} y^{n-j} \mathfrak{m}^k u = 0$. Therefore, if $u \in \mathsf{V}_{\mathfrak{m}}$, we can take $k = 1$ and obtain $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$, (so that V is an R-weight module if $\mathfrak{m} \in \mathfrak{max}(\mathsf{R})$).

It remains to prove (ii), and then (iii) will be a consequence of that and (i). The inclusion

$$\widehat{\mathsf{A}}(\mathsf{R}\text{-generalized weight}) \subseteq \widehat{\mathsf{A}}(\mathsf{R}\text{-torsion})$$

is clear, so we show that if V is an irreducible R-torsion A-module, then V is an R-generalized weight module. Since R is Noetherian, the set $\{\mathsf{Ann}_{\mathsf{R}}(v) \mid 0 \neq v \in \mathsf{V}\}$ has a maximal element $\mathfrak{p} = \mathsf{Ann}_{\mathsf{R}}(u)$, which is nonzero, as V has R-torsion. The maximality condition implies that $\mathfrak{p}$ is a prime ideal of R. Indeed, if $ab \in \mathfrak{p}$ and $b \notin \mathfrak{p}$, then $\mathfrak{p} = \mathsf{Ann}_{\mathsf{R}}(u) \subseteq \mathsf{Ann}_{\mathsf{R}}(bu)$, so $a \in \mathsf{Ann}_{\mathsf{R}}(bu) = \mathfrak{p}$. As $\mathfrak{p} \neq 0$, $\mathfrak{p}$ is a maximal ideal of the Dedekind domain R. Thus, $u \in \mathsf{V}_{\mathfrak{p}}$ and $\mathsf{V} = \mathsf{A}u = \mathsf{V}^{\mathfrak{p}}$, by irreducibility and the first part of the proof. □

Remark 2.3. In the remainder of the paper, we will simply say weight module and generalized weight module with the understanding that always they are with respect to R.

Lemma 2.4. *Assume $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_{\mathsf{R}}, \delta]$ is an Ore extension with R a Dedekind domain. Let $\mathfrak{m}$ be any δ-invariant ideal of R, and let q be a fixed element of R. Then the following hold.*

(i) *The space $\mathsf{N}(\mathfrak{m}, q) := \mathsf{R}/\mathfrak{m}$ with the action*

$$s.(r + \mathfrak{m}) = sr + \mathfrak{m}, \qquad y.(r + \mathfrak{m}) = (qr + \delta(r)) + \mathfrak{m},$$

for $r, s \in \mathsf{R}$, is an A-module. The A-submodules of $\mathsf{N}(\mathfrak{m}, q)$ are the submodules of the form $\mathfrak{p}/\mathfrak{m}$ where $\mathfrak{p}$ is a δ-invariant ideal of R containing $\mathfrak{m}$.

(ii) *If $\mathfrak{m}$ is a maximal ideal of R, then $\mathsf{N}(\mathfrak{m}, q) = \mathsf{N}(\mathfrak{m}, q)_{\mathfrak{m}}$ is an irreducible weight module.*

(iii) *If $\mathfrak{m}$ is a maximal ideal of R and $n \geq 1$, then $\mathsf{N}(\mathfrak{m}^n, q) = \mathsf{N}(\mathfrak{m}^n, q)^{\mathfrak{m}}$ is a generalized weight A-module and it is uniserial (its submodules are linearly ordered by inclusion), hence it is indecomposable.*

(iv) *Assume $\mathsf{char}(\mathbb{F}) = 0$. If $\mathfrak{m}$ is a maximal ideal of R, then*

$$\bigcap_{n \geq 1} \mathsf{Ann}_{\mathsf{A}}\big(\mathsf{N}(\mathfrak{m}^n, q)\big) = (0).$$

In particular, if R is a finitely generated $\mathbb{F}$-algebra (e.g. if $\mathsf{R} = \mathbb{F}[x]$), then A is residually finite dimensional (that is to say, there is a family of ideals of A of finite co-dimension having trivial intersection).

Proof. We leave the verification that $\mathsf{N}(\mathfrak{m}, q)$ is an A-module as an exercise for the reader. It is clear for any δ-invariant ideal $\mathfrak{p}$ of R containing $\mathfrak{m}$ that $\mathfrak{p}/\mathfrak{m}$ is an A-submodule of $\mathsf{N}(\mathfrak{m}, q)$. Conversely, any A-submodule of $\mathsf{N}(\mathfrak{m}, q)$ is necessarily an R-submodule of $\mathsf{R}/\mathfrak{m}$, and thus has the form $\mathfrak{p}/\mathfrak{m}$ for some ideal $\mathfrak{p} \supseteq \mathfrak{m}$ of R. Given $r \in \mathfrak{p}$, we have $y.(r + \mathfrak{m}) = (qr + \delta(r)) + \mathfrak{m}$, so $qr + \delta(r) \in \mathfrak{p}$. As $qr \in \mathfrak{p}$ also, it follows that $\delta(r) \in \mathfrak{p}$, which proves that $\mathfrak{p}$ is δ-invariant. Part (ii) follows immediately.

For part (iii), observe first that whenever $\mathfrak{m}$ is δ-invariant, then $\delta(\mathfrak{m}^k) \subseteq \mathfrak{m}^k$ for all $k \geq 1$, so that $\mathfrak{m}^k$ is δ-invariant. Thus, $\mathsf{N}(\mathfrak{m}^n, q)$ is an A-module by (i).

Moreover, $\mathsf{N}(\mathfrak{m}^n, q)$ is generated by $1 + \mathfrak{m}^n \in \mathsf{N}(\mathfrak{m}^n, q)^{\mathfrak{m}}$, so $\mathsf{N}(\mathfrak{m}^n, q) = \mathsf{N}(\mathfrak{m}^n, q)^{\mathfrak{m}}$ by Proposition 2.2. As R is Dedekind, the ideals of R which contain $\mathfrak{m}^n$ are the ideals of the form $\mathfrak{m}^k$, with $0 \le k \le n$, and these are all δ-invariant. Thus by (i), the A-submodules of $\mathsf{N}(\mathfrak{m}^n, q)$ are $\mathfrak{m}^k/\mathfrak{m}^n$ for $k = 0, 1, \ldots, n$, where $\mathfrak{m}^0 = \mathsf{R}$, which are obviously linearly ordered by inclusion. This shows that $\mathsf{N}(\mathfrak{m}^n, q)$ is uniserial; in particular, it is indecomposable.

For (iv), note first that $\mathsf{Ann}_{\mathsf{R}}\big(\mathsf{N}(\mathfrak{m}^n, q)\big) = \mathfrak{m}^n$, so

$$\bigcap_{n\ge 1} \mathsf{Ann}_{\mathsf{R}}\big(\mathsf{N}(\mathfrak{m}^n, q)\big) = \bigcap_{n\ge 1} \mathfrak{m}^n = (0),$$

because R is Dedekind. Now observe for any nonzero ideal J of A that $\mathsf{J} \cap \mathsf{R} \neq (0)$. To see this, assume $a = \sum_{i=0}^{k} y^i s_i$ ($s_i \in \mathsf{R}$ for all i) is a nonzero element of minimal y-degree in J. Since $h \neq 0$, we may take $r \in \mathsf{R}$ so that $\delta(r) \neq 0$. Then by (1.3),

$$\mathsf{J} \ni [r, a] = \sum_{i=0}^{k} [r, y^i] s_i = -k y^{k-1} \delta(r) s_k + \text{lower order terms in } y.$$

Since $\mathsf{char}(\mathbb{F}) = 0$, the minimality of k forces $k = 0$ to hold, and $a \in \mathsf{J} \cap \mathsf{R}$.

If $\bigcap_{n\ge1} \mathsf{Ann}_{\mathsf{A}}(\mathsf{N}(\mathfrak{m}^n, q)) \neq (0)$, then it contains a nonzero $r \in \mathsf{R}$. But then $r \in \bigcap_{n\ge1} \mathsf{Ann}_{\mathsf{R}}(\mathsf{N}(\mathfrak{m}^n, q)) = (0)$. Hence the ideal $\bigcap_{n\ge1} \mathsf{Ann}_{\mathsf{A}}(\mathsf{N}(\mathfrak{m}^n, q))$ of A must be trivial, as claimed.

Suppose R is a finitely generated $\mathbb{F}$-algebra. Then the Nullstellensatz implies that $\mathsf{R}/\mathfrak{m}$ is finite dimensional over $\mathbb{F}$. Since $\mathsf{N}(\mathfrak{m}^n, q)$ has finite length, with composition factors isomorphic to $\mathsf{R}/\mathfrak{m}$ as R-modules, it follows that $\mathsf{N}(\mathfrak{m}^n, q)$ is finite dimensional over $\mathbb{F}$, and so is $\mathsf{A}/\mathsf{Ann}_{\mathsf{A}}(\mathsf{N}(\mathfrak{m}^n, q))$ for $n \ge 1$. Since, $\bigcap_{n\ge1} \mathsf{Ann}_{\mathsf{A}}(\mathsf{N}(\mathfrak{m}^n, q)) = (0)$ we have that A is residually finite dimensional. $\square$

Remark 2.5. The *Jacobson radical* $\mathcal{J}(\mathsf{A}_h)$ is the intersection of all the primitive ideals of A_h. If $a \in \mathcal{J}(\mathsf{A}_h)$, then $1 - a$ is invertible. But the invertible elements of A_h belong to $\mathbb{F}$ according to [BLO1, Thm. 2.1], so it follows that $a \in \mathbb{F}$. Since $\mathcal{J}(\mathsf{A}_h) \neq \mathsf{A}_h$, it must be that $a = 0$ and $\mathcal{J}(\mathsf{A}_h) = (0)$. Now if $\mathsf{char}(\mathbb{F}) = p > 0$, then all irreducible modules are finite dimensional by Proposition 1.7, so the ideal (0) is the intersection of ideals of A_h having finite co-dimension, and A_h is residually finite dimensional.

The above results show that special behavior occurs when an ideal of R is invariant under the derivation δ. Such ideals are related to normal elements of A as the next result shows. Recall that an element $b \in \mathsf{A}$ is *normal* if $\mathsf{A}b = b\mathsf{A}$.

Lemma 2.6. *Assume* $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_{\mathsf{R}}, \delta]$ *is any Ore extension, and let* $\mathfrak{m}$ *be an ideal of* R*. Then* $\mathfrak{m}$ *is* δ*-invariant if and only if* $\mathfrak{m}\mathsf{A} = \mathsf{A}\mathfrak{m}$*. If* $\mathfrak{m} = \mathsf{R}f$ *and* R *is commutative, then* $\mathfrak{m}$ *is* δ*-invariant if and only if* f *is a normal element of* A*.*

Proof. Suppose that $\mathfrak{m}$ is a δ-invariant ideal of R. Since $y\mathfrak{m} \subseteq \mathfrak{m}y + \delta(\mathfrak{m}) \subseteq \mathfrak{m}\mathsf{A}$ and A is generated by R and y, it follows that $\mathsf{A}\mathfrak{m} \subseteq \mathfrak{m}\mathsf{A}$. A similar argument shows $\mathfrak{m}\mathsf{A} \subseteq \mathsf{A}\mathfrak{m}$, so indeed $\mathfrak{m}\mathsf{A} = \mathsf{A}\mathfrak{m}$. If $\mathfrak{m}$ is an ideal of R with $\mathfrak{m}\mathsf{A} = \mathsf{A}\mathfrak{m}$, then $y\mathfrak{m} \subseteq \mathfrak{m}\mathsf{A}$. Thus for any $r \in \mathfrak{m}$, $ry + \delta(r) = yr \in \mathsf{A}\mathfrak{m} = \mathfrak{m}\mathsf{A}$, and so $\delta(r) \in \mathfrak{m}\mathsf{A} - ry \subseteq \mathfrak{m}\mathsf{A}$. Since $\mathfrak{m}\mathsf{A} = \bigoplus_{i \geq 0} \mathfrak{m}y^i$ and $\delta(r) \in \mathfrak{m}\mathsf{A} \cap \mathsf{R}$, it follows that $\delta(r) \in \mathfrak{m}$, and thus $\mathfrak{m}$ is δ-invariant. Now if $\mathfrak{m} = \mathsf{R}f$ and R is commutative, then $\mathfrak{m}$ is δ-invariant if and only if $\mathsf{A}\mathsf{R}f = f\mathsf{R}\mathsf{A}$ if and only if $f\mathsf{A} = \mathsf{A}f$ (i.e. f is normal in A). $\square$

Now assume as before that $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_\mathsf{R}, \delta]$ with R a Dedekind domain, and fix $\mathfrak{m}$ an ideal of R. We can induce the R-module $\mathsf{R}/\mathfrak{m}$ to an A-module

$$\mathsf{U}(\mathfrak{m}) := \mathsf{A} \otimes_\mathsf{R} \mathsf{R}/\mathfrak{m}. \tag{2.7}$$

Set $u_\mathfrak{m} := 1 \otimes (1 + \mathfrak{m}) \in \mathsf{U}(\mathfrak{m})$. Since $\mathsf{U}(\mathfrak{m}) = \mathsf{A}u_\mathfrak{m}$ and $\mathfrak{m}u_\mathfrak{m} = 0$, Proposition 2.2 (i) implies that $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})^\mathfrak{m}$ (and hence that $\mathsf{U}(\mathfrak{m})$ is a generalized weight A-module if $\mathfrak{m}$ is maximal). Furthermore, if $\mathfrak{m}$ is δ-invariant then $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})_\mathfrak{m}$ (which is a weight module when $\mathfrak{m} \in \mathfrak{max}(\mathsf{R})$).

As A is a free right R-module with basis $\{y^k \mid k \in \mathbb{Z}_{\geq 0}\}$, it follows (with a slight abuse of notation) that any element of $\mathsf{U}(\mathfrak{m})$ can be written uniquely as a finite sum $\sum_{k \geq 0} y^k \bar{r}_k u_\mathfrak{m}$, with $\bar{r}_k \in \mathsf{R}/\mathfrak{m}$.

By the tensor product construction, the A-module $\mathsf{U}(\mathfrak{m})$ has the following universal property:

Proposition 2.8. *Let V be an A-module for $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_\mathsf{R}, \delta]$, where R is a Dedekind domain, and suppose for some ideal $\mathfrak{m}$ of R that $v \in \mathsf{V}_\mathfrak{m}$. Then there is a unique A-module homomorphism $\mathsf{U}(\mathfrak{m}) \to \mathsf{V}$ with $u_\mathfrak{m} \mapsto v$, where $u_\mathfrak{m} = 1 \otimes (1 + \mathfrak{m})$. If $\mathsf{V} = \mathsf{A}v$, then V is a homomorphic image of $\mathsf{U}(\mathfrak{m})$.*

Proof. The map $\zeta : \mathsf{A} \times \mathsf{R}/\mathfrak{m} \to \mathsf{V}$ given by $\zeta(a, (r+\mathfrak{m})) = arv$ is well defined because $\mathfrak{m}v = 0$, and it is clearly R-balanced (see [P, Chap. 9]), so it induces an abelian group homomorphism $\mathsf{A} \otimes_\mathsf{R} \mathsf{R}/\mathfrak{m} \to \mathsf{V}$, satisfying $a \otimes (r + \mathfrak{m}) \mapsto arv$. This is an A-module homomorphism and $u_\mathfrak{m} = 1 \otimes (1+\mathfrak{m}) \mapsto v$. The uniqueness is trivial as $\mathsf{U}(\mathfrak{m}) = \mathsf{A}u_\mathfrak{m}$, and the remaining statements follow. $\square$

Proposition 2.9. *Assume $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_\mathsf{R}, \delta]$ is an Ore extension with R a Dedekind domain, and let $\mathfrak{m}$ be a δ-invariant ideal of R. Assume $\mathsf{N}(\mathfrak{m}, q) = \mathsf{R}/\mathfrak{m}$ is as in Lemma 2.4 for some fixed element $q \in \mathsf{R}$. Then*

$$\mathsf{N}(\mathfrak{m}, q) \cong \mathsf{U}(\mathfrak{m})/\mathsf{A}(y-q)u_\mathfrak{m} \cong \mathsf{A}/\big(\mathsf{A}(y-q) + \mathfrak{m}\big).$$

Proof. By Proposition 2.8, there is an A-module map $\zeta : \mathsf{U}(\mathfrak{m}) \to \mathsf{N}(\mathfrak{m}, q)$ such that $\zeta(au_\mathfrak{m}) = a(1+\mathfrak{m})$ for all $a \in \mathsf{A}$. We claim that kernel of ζ is the space $\mathsf{K} = \mathsf{A}(y-q)u_\mathfrak{m}$. It is easy to check that $\mathsf{K} \subseteq \mathsf{ker}(\zeta)$. Note that $\{(y-q)^j \mid j \in \mathbb{Z}_{\geq 0}\}$ is a basis for A viewed as a left R-module, and $\sum_{j \geq 0} r_j (y-q)^j u_\mathfrak{m} \in \mathsf{ker}(\zeta)$ (where $r_j \in \mathsf{R}$ for all j) if and only if $r_0 u_\mathfrak{m} \in \mathsf{ker}(\zeta)$ if and only if $r_0 + \mathfrak{m} = \bar{0}$. But since $r_0 u_\mathfrak{m} = 0$ when $r_0 \in \mathfrak{m}$, we have $\mathsf{ker}(\zeta) = \mathsf{K}$, and $\mathsf{N}(\mathfrak{m}, q) \cong \mathsf{U}(\mathfrak{m})/\ker(\zeta) = \mathsf{U}(\mathfrak{m})/\mathsf{A}(y-q)u_\mathfrak{m}$, as asserted.

Suppose $\mathsf{J} := \mathsf{A}(y-q)+\mathfrak{m}$. (This sum is actually a vector space direct sum, which can be seen from the fact that $\{(y-q)^j, j \in \mathbb{Z}_{\geq 0}\}$ is an R-basis of A.) Since $\mathfrak{m}$ is δ-invariant, $(y-q)^k r = \sum_{j=0}^{k} \binom{k}{j} \delta^j(r)(y-q)^{k-j} \in \mathsf{J}$ for all $r \in \mathfrak{m}$. Hence, J is a left ideal of A and $\mathsf{A}/\mathsf{J} = \mathsf{A}v$ is an A-module generated by $v = 1+\mathsf{J}$. Since $\mathfrak{m}v = 0$, there is a homomorphism $\vartheta : \mathsf{U}(\mathfrak{m}) \to \mathsf{A}/\mathsf{J}$ with $\vartheta(au_{\mathfrak{m}}) = a+\mathsf{J}$ for all $a \in \mathsf{A}$. Clearly, $\mathsf{A}(y-q)u_{\mathfrak{m}}$ is in the kernel, and $ru_{\mathfrak{m}}$ is in the kernel for $r \in \mathsf{R}$ if and only if $r \in \mathfrak{m}$. Thus, $\mathsf{U}(\mathfrak{m})/\mathsf{A}(y-q)u_{\mathfrak{m}} \cong \mathsf{A}/\mathsf{J}$. □

Remark 2.10. Part (i) of Proposition 2.2 and parts (i) and (ii) of Lemma 2.4 are valid when R is an arbitrary ring. Thus, the same induced module $\mathsf{U}(\mathfrak{m})$ can be constructed, and the results in Proposition 2.8 and Proposition 2.9 hold in the more general setting of an Ore extension $\mathsf{A} = \mathsf{R}[y, \mathsf{id}_{\mathsf{R}}, \delta]$ over any ring R.

3 Indecomposable A_h-modules

For the remainder of this paper, we specialize to the case that the Ore extension is the algebra $\mathsf{A}_h = \mathsf{R}[y, \mathsf{id}_{\mathsf{R}}, \delta]$, where $\mathsf{R} = \mathbb{F}[x]$ and $\delta(r) = r'h$ for all $r \in \mathsf{R}$.

In this section, we use the modules $\mathsf{N}(\mathfrak{m}^{n+1}, q)$ for $n \geq 0$ from Section 2 to show that for any field $\mathbb{F}$, if $h \notin \mathbb{F}^*$, then A_h can have an indecomposable module of dimension $n+1$ for any $n \geq 0$. To provide an explicit description of the action of A_h, we will use a modified version of the usual kth derivative $f^{(k)}$ of $f \in \mathbb{F}[x]$ when $\mathsf{char}(\mathbb{F}) = p > 0$, which we introduce next.

For any $k \in \mathbb{Z}_{\geq 0}$, we write its p-adic expansion as $k = \sum_{i \geq 0} k_i p^i$, where $0 \leq k_i < p$ for all i. It is well known that in characteristic $p > 0$, if $k, \ell \in \mathbb{Z}_{\geq 0}$, then

$$\binom{\ell}{k} = \prod_{i \geq 0} \binom{\ell_i}{k_i}.$$

Set

$$(x^\ell)^{[k]} = \left(\prod_{i \geq 0} \ell_i(\ell_i - 1) \cdots (\ell_i - k_i + 1) \right) x^{\ell - k}. \tag{3.1}$$

When $k_i = 0$, we interpret the product $\ell_i(\ell_i - 1) \cdots (\ell_i - k_i + 1)$ as being 1. This "p-adic" derivative can be extended linearly to arbitrary polynomials $f \in \mathbb{F}[x]$. We write $f^{[k]}$ for the result and note that $f^{[0]} = f$.

Proposition 3.2. *Assume $h \notin \mathbb{F}^*$ and $\mathfrak{m} = \mathsf{R}(x - \lambda)$, where $h(\lambda) = 0$. Let q be a fixed element of R. Then for all $n \geq 0$, the module $\mathsf{N}(\mathfrak{m}^{n+1}, q)$ is an indecomposable A_h-module of dimension $n+1$ with basis $\{v_j \mid j = 0, 1, \ldots, n\}$ such that for each j,*

(i) *$x.v_j = \lambda v_j + v_{j-1}$, where $v_{-1} = 0$;*

(ii) $y.v_j = q.v_j + (n-j)h.v_{j+1} = q.v_j + (n-j)\sum_{\ell=0}^{j} \eta_{j+1-\ell} v_\ell$, *where*

$$\eta_k = \begin{cases} \dfrac{h^{(k)}(x)\mid_{x=\lambda}}{k!} & \text{if } \mathsf{char}(\mathbb{F}) = 0, \\ \dfrac{h^{[k]}(x)\mid_{x=\lambda}}{\prod_{i\geq 0} k_i!} & \text{if } \mathsf{char}(\mathbb{F}) = p > 0, \end{cases} \tag{3.3}$$

for all $k \geq 0$, *and* $q.v_j$ *is computed using* (3.4) *below.*

Proof. Since $\mathfrak{m}$ is a maximal, δ-invariant ideal of $\mathsf{R} = \mathbb{F}[x]$, we know by Lemma 2.4 (iii) that $\mathsf{N}(\mathfrak{m}^{n+1}, q)$ is an indecomposable generalized weight A_h-module for all $n \geq 0$.

To simplify the notation in the remainder of the proof, set $\mathfrak{p} = \mathfrak{m}^{n+1}$. Let $v_j := (x-\lambda)^{n-j} + \mathfrak{p}$ for $j = 0, 1, \ldots, n$, and set $v_j = 0$ if $j < 0$. Then

$$(x-\lambda).v_j = (x-\lambda)^{n-(j-1)} + \mathfrak{p} = v_{j-1},$$

so that $x.v_j = \lambda v_j + v_{j-1}$ holds for all j as in (i). Arguing by induction, we have

$$x^\ell.v_j = \sum_{k=0}^{\ell} \binom{\ell}{k} \lambda^{\ell-k} v_{j-k}$$

for all $\ell \geq 0$. Hence, it follows that for any polynomial $f = f(x) \in \mathsf{R}$,

$$f.v_j = \begin{cases} \displaystyle\sum_{k\geq 0} \frac{f^{(k)}(x)\mid_{x=\lambda}}{k!} v_{j-k} & \text{if } \mathsf{char}(\mathbb{F}) = 0, \\ \displaystyle\sum_{k\geq 0} \frac{f^{[k]}(x)\mid_{x=\lambda}}{\prod_{i\geq 0} k_i!} v_{j-k} & \text{if } \mathsf{char}(\mathbb{F}) = p > 0, \end{cases} \tag{3.4}$$

where $f^{(0)} = f = f^{[0]}$. In particular,

$$h.v_j = \sum_{k=1}^{j} \eta_k v_{j-k} = \sum_{k=0}^{j-1} \eta_{j-k} v_k,$$

where η_k is as in (3.3), and $\eta_0 = 0$ since $h(\lambda) = 0$.

Now

$$\begin{aligned} y.v_j &= y.\Big((x-\lambda)^{n-j} + \mathfrak{p}\Big) = q(x-\lambda)^{n-j} + \delta\big((x-\lambda)^{n-j}\big) + \mathfrak{p} \\ &= q.v_j + (n-j)h.v_{j+1} \\ &= q.v_j + (n-j)\sum_{k=0}^{j} \eta_{j+1-k} v_k, \end{aligned}$$

where $q.v_j$ can be computed using (3.4), to give (ii). □

Remark 3.5. In the preceding result, the space $\mathsf{N}_j := \mathsf{span}_{\mathbb{F}}\{v_0, v_1, \ldots, v_j\}$ is an A_h-submodule of $\mathsf{N}(\mathfrak{m}^{n+1}, q)$ for each $j = 0, 1, \ldots, n$. Set $\mathsf{N}_{-1} = \mathfrak{m}^{n+1}$. If $\overline{v}_j = v_j + \mathsf{N}_{j-1}$, then for $j = 0, 1, \ldots, n$, we have $x.\overline{v}_j = \lambda \overline{v}_j$ and $y.\overline{v}_j = \mu_j \overline{v}_j$, where $\mu_j = q(\lambda) + (n-j)\eta_1$. Therefore, $\mathsf{N}_j/\mathsf{N}_{j-1} = \mathbb{F}\overline{v}_j \cong \mathsf{V}_{\lambda,\mu_j}$ in the notation used in Theorem 6.3 and Corollary 7.8 below.

4 Generalized Weight Modules for A_h

For the algebra $\mathsf{A}_h = \mathsf{R}[y, \mathsf{id}_\mathsf{R}, \delta]$, a maximal ideal $\mathfrak{m} = \mathsf{R}f$ of $\mathsf{R} = \mathbb{F}[x]$ is δ-invariant if and only if f divides $\delta(f) = f'h$. Since f is a prime polynomial, the only way that can happen when $\mathsf{char}(\mathbb{F}) = 0$ is if f is a prime factor of h. Therefore, the δ-invariant maximal ideals are exactly the ideals generated by the prime factors of h when $\mathsf{char}(\mathbb{F}) = 0$. When $\mathsf{char}(\mathbb{F}) = p > 0$, then f divides $\delta(f) = f'h$ exactly when f is a prime factor of h or when $f' = 0$. In the latter case, $f \in \mathbb{F}[x^p]$. (This could also be deduced using Lemma 2.6 above and Theorem 7.3 of [BLO1], which gives a complete description of all the normal elements of A_h.) We record these facts for later use.

Lemma 4.1. *Assume $\mathfrak{m} = \mathsf{R}f$ is a δ-invariant maximal ideal of $\mathsf{R} = \mathbb{F}[x]$, where $\delta(r) = r'h$ for all $r \in \mathsf{R}$. If $\mathsf{char}(\mathbb{F}) = 0$, then f is a prime factor of h; if $\mathsf{char}(\mathbb{F}) = p > 0$, then either f is a prime factor of h or $f \in \mathbb{F}[x^p]$.*

4.1 Induced A_h-modules

Assume $\mathfrak{m}$ is an ideal of R, not necessarily δ-invariant. The induced A_h-module, $\mathsf{U}(\mathfrak{m}) := \mathsf{A}_h \otimes_\mathsf{R} \mathsf{R}/\mathfrak{m} = \mathsf{A}_h u_m$, where $u_\mathfrak{m} := 1 \otimes (1 + \mathfrak{m})$, has a basis

$$\{y^k x^\ell u_\mathfrak{m} = y^k \otimes (x^\ell + \mathfrak{m}) \mid 0 \le \ell < \mathsf{dim}(\mathsf{R}/\mathfrak{m}),\ k \in \mathbb{Z}_{\ge 0}\}$$

with A_h-action given by

$$\begin{aligned} x.y^n x^\ell u_\mathfrak{m} &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} y^{n-j} \delta^j(x) x^\ell u_\mathfrak{m}, \\ y.y^n x^\ell u_\mathfrak{m} &= y^{n+1} x^\ell u_\mathfrak{m}. \end{aligned} \tag{4.2}$$

Then for $r \in \mathbb{F}[x]$,

$$y^k r u_\mathfrak{m} = 0 \ \text{ if and only if } \ r \in \mathfrak{m}. \tag{4.3}$$

Since $\mathsf{U}(\mathfrak{m}) = \mathsf{A}_h u_\mathfrak{m}$ and $\mathfrak{m} u_\mathfrak{m} = 0$, by Proposition 2.2 (i) we have $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})^\mathfrak{m}$, and when $\mathfrak{m}$ is δ-invariant, $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})_\mathfrak{m}$.

We assume now that $\mathfrak{m} \in \mathfrak{max}(\mathsf{R})$ so that $\mathfrak{m} = \mathsf{R}f$ for some prime polynomial $f \in \mathsf{R}$, and consider first the following case:

***f* is a factor of *h*:** Since $\mathfrak{m}$ is δ-invariant when f is a factor of h, $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})_{\mathfrak{m}}$. Lemmas 6.1 and 7.1 of [BLO1] show that $[\mathsf{A}_h, \mathsf{A}_h] \subseteq h\mathsf{A}_h = \mathsf{A}_h h$. Thus, for any $a, b \in \mathsf{A}_h$ and $w \in \mathsf{U}(\mathfrak{m})$, we have

$$baw = abw + [b, a]w = abw, \tag{4.4}$$

and $a\mathsf{U}(\mathfrak{m})$ is an A_h-submodule of $\mathsf{U}(\mathfrak{m})$ for any $a \in \mathsf{A}_h$.

If $a = \sum_{i\geq 0} y^i r_i$ and $b = \sum_{i\geq 0} y^i s_i$, where $r_i, s_i \in \mathsf{R}$, then $a\mathsf{U}(\mathfrak{m}) = b\mathsf{U}(\mathfrak{m})$ if and only if $r_i - s_i \in \mathfrak{m}$ for all $i \geq 0$. In particular, $a\mathsf{U}(\mathfrak{m}) = 0$ if and only if $r_i \in \mathfrak{m}$ for all i. Hence $\mathfrak{m}[y]$ annihilates $\mathsf{U}(\mathfrak{m})$, and the action of A_h on $\mathsf{U}(\mathfrak{m})$ is the same as the action of the commutative polynomial algebra $\mathsf{Q}_{\mathfrak{m}} = (\mathsf{R}/\mathfrak{m})[y] \cong \mathsf{R}[y]/\mathfrak{m}[y]$.

Let W be a submodule of $\mathsf{U}(\mathfrak{m})$, and set

$$\mathsf{J}_{\mathsf{W}} = \left\{ \bar{a} = \sum_{i\geq 0} y^i \bar{r}_i \in \mathsf{Q}_{\mathfrak{m}} \;\middle|\; \bar{a}\mathsf{U}(\mathfrak{m}) \subseteq \mathsf{W} \right\} \tag{4.5}$$

Then J_{W} is an ideal of the PID $\mathsf{Q}_{\mathfrak{m}}$, and we may assume $\mathsf{J}_{\mathsf{W}} = \mathsf{Q}_{\mathfrak{m}}\bar{g}$ for some monic polynomial $\bar{g} = \sum_{i\geq 0} y^i \bar{g}_i \in \mathsf{Q}_{\mathfrak{m}}$. The map $\mathsf{Q}_{\mathfrak{m}} \to \mathsf{U}(\mathfrak{m})/\mathsf{W}$ given by $\bar{a} \mapsto \bar{a}(u_{\mathfrak{m}} + \mathsf{W})$ is onto and has kernel $\mathsf{Q}_{\mathfrak{m}}\bar{g}$. Thus, $\mathsf{U}(\mathfrak{m})/\mathsf{W} \cong \mathsf{Q}_{\mathfrak{m}}/\mathsf{Q}_{\mathfrak{m}}\bar{g}$, which has dimension $\mathsf{deg}\,(f)\mathsf{deg}\,(\bar{g})$ over $\mathbb{F}$, and $\mathsf{U}(\mathfrak{m})/\mathsf{W}$ is irreducible when $\bar{g}$ is a prime polynomial in $\mathsf{Q}_{\mathfrak{m}}$.

Conversely, if $\bar{g} \in \mathsf{Q}_{\mathfrak{m}}$, then $\bar{g}\mathsf{U}(\mathfrak{m})$ is a submodule of $\mathsf{U}(\mathfrak{m})$ and $\mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m}) \cong \mathsf{Q}_{\mathfrak{m}}/\mathsf{Q}_{\mathfrak{m}}\bar{g}$. When $\bar{g}$ is a monic prime polynomial in $\mathsf{Q}_{\mathfrak{m}}$, the quotient

$$\mathsf{L}(\mathfrak{m}, \bar{g}) := \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m}) \cong \mathsf{Q}_{\mathfrak{m}}/\mathsf{Q}_{\mathfrak{m}}\bar{g} \tag{4.6}$$

is irreducible, and by the preceding paragraph, every irreducible quotient of $\mathsf{U}(\mathfrak{m})$ has this form. Any irreducible generalized weight A_h-module $\mathsf{V} = \mathsf{V}^{\mathfrak{m}}$ must be a weight module, $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$ by Proposition 2.2 (iii), since $\mathfrak{m}$ is δ-invariant. Moreover, since V is a homomorphic image of $\mathsf{U}(\mathfrak{m})$, it is isomorphic to some irreducible quotient of $\mathsf{U}(\mathfrak{m})$. Hence, $\mathsf{V} \cong \mathsf{L}(\mathfrak{m}, \bar{g})$ for some monic prime polynomial $\bar{g}$ of $\mathsf{Q}_{\mathfrak{m}}$.

***f* is not a factor of *h*:** Assume now that $\overline{\mathsf{char}(\mathbb{F}) = 0}$ and f is not a factor of h. Let W be a nonzero submodule of $\mathsf{U}(\mathfrak{m})$. Let $0 \neq w = \sum_{k=0}^{n} y^k r_k u_{\mathfrak{m}}$ be an element of minimal degree in y lying in W, where $r_k \in \mathsf{R}$ for all k and $\mathsf{deg}\, r_k < \mathsf{deg}\, f$. Then f does not divide r_n by the minimality assumption. Applying f we have

$$fw = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} \delta^j(f) r_k u_{\mathfrak{m}} \in \mathsf{W}.$$

Since $\delta^0(f) = f$, and $f r_k u_{\mathfrak{m}} = 0$ for all $k = 0, 1, \ldots, n$, the element fw has smaller degree in y, and so must be 0. Now if $n \geq 1$, this implies that $n y^{n-1} \delta(f) r_n u_{\mathfrak{m}} = 0$. Since $\delta(f) r_n$ is not divisible by f and $\mathsf{char}(\mathbb{F}) = 0$, we have arrived at a contradiction. Hence, any nonzero element of minimal y-degree in W must have the form $w = r_0 u_{\mathfrak{m}}$.

But since $\mathsf{R}/\mathfrak{m}$ is a field, there exists an $s \in \mathsf{R}$ so that $sr_0 \equiv 1 \bmod \mathfrak{m}$. Thus, $u_\mathfrak{m} = sr_0 u_\mathfrak{m} = sw \in \mathsf{W}$. Consequently, $\mathsf{U}(\mathfrak{m}) = \mathsf{A}_h u_\mathfrak{m} \subseteq \mathsf{W}$, and $\mathsf{U}(\mathfrak{m})$ is an irreducible generalized weight module for A_h. We summarize what we have just shown.

Theorem 4.7. *Let $\mathfrak{m} = \mathsf{R}f$ be the maximal ideal of $\mathsf{R} = \mathbb{F}[x]$ generated by the prime polynomial f, and let $\mathsf{U}(\mathfrak{m}) := \mathsf{A}_h \otimes_\mathsf{R} \mathsf{R}/\mathfrak{m}$ be the A_h-module induced from the irreducible R-module $\mathsf{R}/\mathfrak{m}$. Then the following hold:*

(i) $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})^\mathfrak{m}$ *is a generalized weight module for A_h. If $\mathfrak{m}$ is δ-invariant, then* $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})_\mathfrak{m}$ *is a weight module for A_h.*

(ii) *If f is a factor of h, then $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})_\mathfrak{m}$. For any monic prime polynomial $\bar{g} \in \mathsf{Q}_\mathfrak{m} = (\mathsf{R}/\mathfrak{m})[y]$, the quotient $\mathsf{L}(\mathfrak{m}, \bar{g}) = \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m})$ is an irreducible weight A_h-module of dimension $\deg(f)\deg(\bar{g})$ over $\mathbb{F}$, and any irreducible generalized weight module $\mathsf{V} = \mathsf{V}^\mathfrak{m}$ for A_h is isomorphic to $\mathsf{L}(\mathfrak{m}, \bar{g})$ for some monic prime polynomial $\bar{g} \in \mathsf{Q}_\mathfrak{m}$.*

(iii) *If $\mathsf{char}(\mathbb{F}) = 0$, and f is not a factor of h, then $\mathsf{U}(\mathfrak{m}) = \mathsf{U}(\mathfrak{m})^\mathfrak{m}$ is an irreducible generalized weight module for A_h.*

4.2 Finite-dimensional A_h-modules

Let V be an irreducible weight module for A_h such that $\mathsf{V} = \mathsf{V}_\mathfrak{m}$ for some δ-invariant maximal ideal $\mathfrak{m} = \mathsf{R}f$ of R. Recall that the ideal $\mathfrak{m}$ is δ-invariant if and only if f divides $\delta(f) = f'h$, which says that either f is a prime factor of h or else $\mathsf{char}(\mathbb{F}) = p > 0$ and $f \in \mathbb{F}[x^p]$ (as in Lemma 4.1). Since V is a homomorphic image of $\mathsf{U}(\mathfrak{m})$ by Proposition 2.8, Theorem 4.7(ii) shows that V is finite dimensional whenever f is a prime factor of h. Since by Proposition 1.7, *any* irreducible module is finite dimensional when $\mathsf{char}(\mathbb{F}) = p > 0$, an irreducible A_h-module V such that $\mathsf{V} = \mathsf{V}_\mathfrak{m}$ and $\mathfrak{m}$ is δ-invariant is always finite dimensional. Next we explore the converse.

Lemma 4.8. *Assume M is any finite-dimensional irreducible A_h-module. Then there exists a monic prime polynomial $f \in \mathsf{R}$ so that $\mathsf{M} = \mathsf{M}^\mathfrak{m}$ for $\mathfrak{m} = \mathsf{R}f$. Either $\mathfrak{m}$ is δ-invariant and $\mathsf{M} = \mathsf{M}_\mathfrak{m}$, or $\mathsf{char}(\mathbb{F}) = p > 0$ and $\mathsf{Ann}_\mathsf{R}(\mathsf{M}) = \mathfrak{m}^p = \mathsf{R}f^p$, where $f \notin \mathbb{F}[x^p]$.*

Proof. Since M has R-torsion and is irreducible, $\mathsf{M} = \mathsf{M}^\mathfrak{m}$ for some maximal ideal $\mathfrak{m} = \mathsf{R}f$ generated by a monic prime polynomial $f \in \mathsf{R}$ by Proposition 2.2. As M is finite dimensional, there is a least integer $k \geq 1$ so that $\mathfrak{m}^k\mathsf{M} = 0$. Hence $\mathfrak{m}^k = \mathsf{Ann}_\mathsf{R}(\mathsf{M})$. Since for any $v \in \mathsf{M}$, we have $0 = yf^kv - f^kyv = \delta(f^k)v$, it must be that $\delta(f^k) \in \mathfrak{m}^k = \mathsf{R}f^k$. But this says, f^k divides $kf^{k-1}f'h$, and hence that f divides $kf'h = k\delta(f)$. If f divides $\delta(f)$, then $\mathfrak{m} = \mathsf{R}f$ is δ-invariant and $\mathsf{M} = \mathsf{M}_\mathfrak{m}$. If that is not the case, then $\mathsf{char}(\mathbb{F}) = p > 0$, $f' \neq 0$, and $k \equiv 0 \bmod p$ must hold.

Assume now that $\mathsf{char}(\mathbb{F}) = p > 0$. Since $\mathfrak{m}^p = \mathsf{R}f^p$ and $f^p \in \mathbb{F}[x^p] \subseteq \mathsf{Z}(\mathsf{A}_h)$, it follows that $\mathfrak{m}^p\mathsf{M}$ is an A_h-submodule of M. Because M is irreducible, either $\mathfrak{m}^p\mathsf{M} = 0$

or $\mathfrak{m}^p\mathsf{M} = \mathsf{M}$. If $\mathfrak{m}^p\mathsf{M} = \mathsf{M}$, then $\mathfrak{m}^{2p}\mathsf{M} = \mathfrak{m}^p(\mathfrak{m}^p\mathsf{M}) = \mathfrak{m}^p\mathsf{M} = \mathsf{M}$, and (proceeding inductively) $\mathfrak{m}^{(n+1)p}\mathsf{M} = \mathfrak{m}^{np}(\mathfrak{m}^p\mathsf{M}) = \mathfrak{m}^{np}\mathsf{M} = \mathsf{M}$. Since some power of $\mathfrak{m}$ must annihilate M, it is necessarily the case that $\mathfrak{m}^p\mathsf{M} = 0$. □

Remark 4.9. When $\mathsf{char}(\mathbb{F}) = p > 0$ and λ is not a root of $h(x)$, the irreducible A_h-modules $\mathsf{M} = \mathsf{L}(\mathfrak{m}, z_\beta)$ appearing in Lemma 7.6 below have the property that $\mathsf{Ann}_\mathsf{R}(\mathsf{M}) = \mathfrak{m}^p$ where $\mathfrak{m} = \mathsf{R}(x - \lambda)$. As we show in Corollary 7.8, they, along with the one-dimensional modules, are the only irreducible A_h-modules when $\mathbb{F}$ is algebraically closed of characteristic p.

5 Primitive ideals of A_h

Recall that a *primitive ideal* is the annihilator of an irreducible module; in other words, it is the kernel of an irreducible representation. A ring is primitive if it has a faithful irreducible module. In any ring, primitive ideals are prime, and maximal ideals are primitive, but the converses of these statements generally fail to be true. For the universal enveloping algebra of a finite-dimensional nilpotent Lie algebra over a field of characteristic 0, [D, Prop. 4.7.4] shows that all primitive ideals are maximal. We will see below that this does not hold for A_h. In fact, if $\mathsf{char}(\mathbb{F}) = 0$, then A_h has faithful irreducible modules.

In [BLO1, Thm. 7.7] we determined the height-one prime ideals of A_h and noted in [BLO1, Remark 7.9] that the maximal ideals of A_h are the prime ideals of height two. (The *height* of a prime ideal is the largest length of a chain of prime ideals contained in it, or is said to be ∞ if no bound exists.) In Proposition 5.3 below, we determine the primitive ideals of A_h. Our argument uses the following result, which holds quite generally.

Lemma 5.1. *Let A be an associative $\mathbb{F}$-algebra. Suppose M is a finite-dimensional irreducible A-module, and let $\mathsf{P} = \mathsf{Ann}_\mathsf{A}(\mathsf{M})$. Then P is a maximal ideal of A, and $\mathsf{A}/\mathsf{P} \cong \mathsf{End}_\mathsf{D}(\mathsf{M})$, where $\mathsf{D} = \mathsf{End}_\mathsf{A}(\mathsf{M})$.*

Proof. The representation $\mathsf{A} \to \mathsf{End}_\mathbb{F}(\mathsf{M})$ induces an injective homomorphism

$$\mathsf{A}/\mathsf{P} \hookrightarrow \mathsf{End}_\mathbb{F}(\mathsf{M}). \tag{5.2}$$

Let $\mathsf{D} = \mathsf{End}_\mathsf{A}(\mathsf{M})$. By Schur's Lemma, D is a division ring containing $\mathbb{F}\mathsf{id}_\mathsf{M}$, and M is finite dimensional over D. The image of (5.2) is contained in $\mathsf{End}_\mathsf{D}(\mathsf{M})$, and the Jacobson Density Theorem implies that $\mathsf{A}/\mathsf{P} \cong \mathsf{End}_\mathsf{D}(\mathsf{M})$. Hence A/P is simple, and P is maximal. □

Proposition 5.3. *An ideal P of A_h is primitive if and only if P is maximal, or $\mathsf{char}(\mathbb{F}) = 0$ and $\mathsf{P} = (0)$. In particular, if $\mathsf{char}(\mathbb{F}) = 0$, then A_h is a primitive algebra, and all infinite-dimensional irreducible A_h-modules are faithful.*

Proof. As mentioned earlier, any maximal ideal is primitive. Let P be a primitive ideal of A_h, and let M be an irreducible A_h-module with annihilator P.

If $\mathsf{char}(\mathbb{F}) = p > 0$, then by Proposition 1.7, M is finite dimensional, and Lemma 5.1 implies that P is maximal.

Now assume $\mathsf{char}(\mathbb{F}) = 0$. If $\mathsf{P} \neq (0)$, then P contains a height-one prime ideal. By [BLO1, Thm. 7.7], we deduce that P contains a prime factor of h. But then $h \in \mathsf{P}$, and A_h/P is commutative, as $[y, x] \in \mathsf{P}$. Hence $\mathsf{M} \cong \mathsf{A}_h/\mathsf{P}$, and P must be a maximal ideal. In particular, in this case A_h/P is finite dimensional (it is a finitely generated field extension of $\mathbb{F}$), and thus M is also finite dimensional. This shows that if M is an infinite-dimensional irreducible A_h-module, then $\mathsf{P} = \mathsf{Ann}_{\mathsf{A}_h}(\mathsf{M}) = (0)$ and M is faithful. It remains to show that (0) is a primitive ideal when $\mathsf{char}(\mathbb{F}) = 0$. But that follows from the existence of infinite-dimensional irreducible A_h-modules. Indeed, by Theorem 4.7 (iii), if $\mathsf{char}(\mathbb{F}) = 0$ and $f \in \mathsf{R}$ is a prime polynomial which is not a factor of h (e.g. if $f = x - \lambda$ with $h(\lambda) \neq 0$), then the induced generalized weight module $\mathsf{U}(\mathfrak{m})$ for $\mathfrak{m} = \mathsf{R}f$ is irreducible and infinite dimensional, thus faithful. □

Remark 5.4. Iyudu [I] has shown that this result holds for the algebra A_{x^2} over algebraically closed fields of characteristic 0. It should be noted that the roles of x and y in [I] are reversed, and the ideal (0) needs to be added to statement of Corollary 5.4 in [I].

In the proof of Proposition 5.3, we have seen that when $\mathsf{char}(\mathbb{F}) = 0$ and P is a nonzero primitive ideal, then P is a maximal ideal containing a prime factor f of h. Let $\mathfrak{m} = \mathsf{R}f$. Since $\mathsf{A}_h/\mathsf{P} = (\mathsf{A}_h/\mathsf{P})_{\mathfrak{m}}$ is an irreducible weight module, by Theorem 4.7 (ii) there exists a monic prime polynomial $\bar{g}$ in $\mathsf{Q}_{\mathfrak{m}} = (\mathsf{R}/\mathfrak{m})[y]$ such that $\mathsf{A}_h/\mathsf{P} \cong \mathsf{L}(\mathfrak{m}, \bar{g})$. Hence, P is the annihilator of one of the finite-dimensional irreducible modules $\mathsf{L}(\mathfrak{m}, \bar{g})$. We have the following analogue of Duflo's result on the primitive ideals of the universal enveloping algebra of a finite-dimensional complex semisimple Lie algebra (see [Du]).

Corollary 5.5. (a) *Assume* $\mathsf{char}(\mathbb{F}) = 0$ *and* $h \notin \mathbb{F}^*$. *A primitive ideal of* A_h *is* (0) *or is the annihilator of an irreducible module* $\mathsf{L}(\mathfrak{m}, \bar{g})$ *for* $\mathfrak{m} = \mathsf{R}f$, *where* f *is a prime factor of* h, *and* $\bar{g}$ *is a monic prime polynomial of* $\mathsf{Q}_{\mathfrak{m}} = (\mathsf{R}/\mathfrak{m})[y]$. *The primitive ideal* (0) *is the annihilator of* $\mathsf{U}(\mathfrak{m})$ *for any maximal ideal* $\mathfrak{m}$ *of* R *which is not* δ*-invariant.*

(b) *Over any field* $\mathbb{F}$, *if* $\mathfrak{m} = \mathsf{R}f$, *where* f *is a prime factor of* h, *and if* $g = \sum_{j \geq 0} y^j g_j \in \mathsf{A}_h$ *(where* $g_j \in \mathsf{R}$ *for all* j*) has the property that* $\bar{g} = \sum_{j \geq 0} y^j \bar{g}_j$ *is a monic prime polynomial in* $\mathsf{Q}_{\mathfrak{m}}$, *then* $\mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}, \bar{g})) = \mathsf{A}_h g + \mathsf{A}_h \mathfrak{m}$.

Proof. Only part (b) remains to be shown. Clearly, $\mathsf{A}_h g + \mathsf{A}_h \mathfrak{m} \subseteq \mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}, \bar{g}))$. For the other direction, assume $a = \sum_{j \geq 0} y^j r_j \in \mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}, \bar{g}))$. Since the action of a on $\mathsf{L}(\mathfrak{m}, \bar{g}) = \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m})$ is the same as the action of $\bar{a} = \sum_{j \geq 0} y^j \bar{r}_j$ on $\mathsf{Q}_{\mathfrak{m}}/\bar{g}\mathsf{Q}_{\mathfrak{m}}$, it must be that $\bar{a}$ is divisible by $\bar{g}$. Thus, there exists a $b = \sum_{j \geq 0} y^j b_j \in \mathsf{A}_h$ (with $b_j \in \mathsf{R}$

for all j) so that $\bar{a} = \bar{b}\bar{g}$ in $\mathsf{Q}_{\mathfrak{m}}$, where $\bar{b} = \sum_{j\geq 0} y^j \bar{b}_j$. Hence $a - bg \in \mathfrak{m}[y] = \mathsf{A}_h\mathfrak{m}$, and $a \in \mathsf{A}_h g + \mathsf{A}_h\mathfrak{m}$. $\square$

Corollary 5.6. *Assume $\mathsf{L}(\mathfrak{m}_i, \bar{g}_i)$ for $i = 1, 2$ are two irreducible A_h-modules as in* Corollary 5.5 (b). *Then the following are equivalent:*

(a) $\mathsf{L}(\mathfrak{m}_1, \bar{g}_1) \cong \mathsf{L}(\mathfrak{m}_2, \bar{g}_2)$.

(b) $\mathfrak{m}_1 = \mathfrak{m}_2$, *and* $\bar{g}_1 = \bar{g}_2$ *as polynomials in* $\mathsf{Q}_{\mathfrak{m}_1} = \mathsf{Q}_{\mathfrak{m}_2}$.

(c) $\mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}_1, \bar{g}_1)) = \mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}_2, \bar{g}_2))$.

Proof. For $i = 1, 2$, the maximal ideal $\mathfrak{m}_i$ is determined by $\mathfrak{m}_i = \mathsf{Ann}_{\mathsf{R}}\big(\mathsf{L}(\mathfrak{m}_i, \bar{g}_i)\big) = \mathsf{R} \cap \mathsf{Ann}_{\mathsf{A}_h}\big(\mathsf{L}(\mathfrak{m}_i, \bar{g}_i)\big)$. In particular, if the generator f_i of $\mathfrak{m}_i$ is assumed to be monic, it is uniquely determined. Then $\bar{g}_i$ is the monic prime polynomial in $(\mathsf{R}/\mathfrak{m}_i)[y]$ which annihilates $\mathsf{L}(\mathfrak{m}_i, \bar{g}_i)$. Equivalently, it is the generator of $\mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}_i, \bar{g}_i))/\mathsf{A}_h\mathfrak{m}_i$ as an ideal of $\mathsf{Q}_{\mathfrak{m}_i}$. Since $\mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}_i, \bar{g}_i))$ is determined by the isomorphism class of $\mathsf{L}(\mathfrak{m}_i, \bar{g}_i)$ we have $(a) \implies (c)$, and $(c) \implies (b)$ by the above. Finally, since $\mathfrak{m}_i$ and $\bar{g}_i$ determine $\mathsf{L}(\mathfrak{m}_i, \bar{g}_i)$, we have $(b) \implies (a)$. $\square$

The equivalence of (a) and (c) in the previous corollary is a general phenomenon. We include a proof of this equivalence in a very general context next for the convenience of the reader, and also because the following proposition can be used to deduce information about the primitive ideals in Corollary 7.8 below.

Proposition 5.7. *Let A be an associative $\mathbb{F}$-algebra, and let V, W be finite-dimensional irreducible A-modules. Then $\mathsf{V} \cong \mathsf{W}$ if and only if $\mathsf{Ann}_{\mathsf{A}}(\mathsf{V}) = \mathsf{Ann}_{\mathsf{A}}(\mathsf{W})$. Thus, the isomorphism classes of finite-dimensional irreducible A-modules are in bijection with the maximal ideals of A of finite co-dimension.*

Proof. Assume $\phi : \mathsf{V} \to \mathsf{W}$ is a surjective A-homomorphism. Then

$$\mathsf{Ann}_{\mathsf{A}}(\mathsf{W}) = \mathsf{Ann}_{\mathsf{A}}(\phi(\mathsf{V})) \supseteq \mathsf{Ann}_{\mathsf{A}}(\mathsf{V}),$$

so $\mathsf{Ann}_{\mathsf{A}}(\mathsf{W}) \supseteq \mathsf{Ann}_{\mathsf{A}}(\mathsf{V})$, and equality holds if ϕ is an isomorphism.

Conversely, suppose V is a finite-dimensional irreducible A-module, and let $\mathsf{P} = \mathsf{Ann}_{\mathsf{A}}(\mathsf{V})$. Lemma 5.1 implies that P is maximal and of finite co-dimension in A. Furthermore, if W is another irreducible A-module with $\mathsf{Ann}_{\mathsf{A}}(\mathsf{W}) = \mathsf{Ann}_{\mathsf{A}}(\mathsf{V}) = \mathsf{P}$, then V and W are two irreducible modules over the simple Artinian ring $\mathsf{End}_{\mathsf{D}}(\mathsf{V}) \cong \mathsf{A}/\mathsf{P}$, where $\mathsf{D} = \mathsf{End}_{\mathsf{A}}(\mathsf{V})$. But this ring has only one irreducible module up to isomorphism. Thus $\mathsf{V} \cong \mathsf{W}$ as A/P-modules, hence also as A-modules. $\square$

6 Irreducible A_h-modules when $\mathsf{char}(\mathbb{F}) = 0$

6.1 Irreducible generalized weight modules for A_h

It is an immediate consequence of Theorem 4.7 and the fact that a maximal ideal $\mathfrak{m} = \mathsf{R}f$ is δ-invariant if and only if f divides h when $\mathsf{char}(\mathbb{F}) = 0$ that the following holds.

Corollary 6.1. *Assume $\mathsf{char}(\mathbb{F}) = 0$. Let V be an irreducible generalized weight A_h-module. Then $\mathsf{V} = \mathsf{V}^{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m} = \mathsf{R}f$ of R generated by a prime polynomial f.*

(i) *If f is a factor of h, then $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$ and $\mathsf{V} \cong \mathsf{L}(\mathfrak{m}, \bar{g}) = \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m})$ for some monic prime polynomial $\bar{g} \in (\mathsf{R}/\mathfrak{m})[y]$.*

(ii) *If f is not a factor of h, then V is isomorphic to the induced module $\mathsf{U}(\mathfrak{m}) = \mathsf{A}_h \otimes_{\mathsf{R}} \mathsf{R}/\mathfrak{m}$.*

Remark 6.2. When $h \in \mathbb{F}^*$, the algebra A_h is isomorphic to the Weyl algebra A_1. There are no prime polynomial factors of h in this case, Thus, when $\mathsf{char}(\mathbb{F}) = 0$, all the irreducible generalized weight modules for A_1 are induced modules $\mathsf{U}(\mathfrak{m}) = \mathsf{A}_1 \otimes_{\mathsf{R}} \mathsf{R}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R by Corollary 6.1. Modules for the Weyl algebra A_1, and more generally for the Weyl algebras in arbitrarily many variables, and for generalized Weyl algebras over fields of arbitrary characteristic, have been studied extensively by many authors (see for example, [B1], [B2], [Bl], [DGO], [C], [BBF]).

6.2 Finite-dimensional irreducible A_h-modules when $\mathsf{char}(\mathbb{F}) = 0$

When $\mathsf{char}(\mathbb{F}) = 0$, Lemma 4.8 shows that for any finite-dimensional irreducible A_h-module V, there is a δ-invariant maximal ideal $\mathfrak{m} = \mathsf{R}f$ such that $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$, and f is a prime factor of h. Here we determine more information about these finite-dimensional modules first in the algebraically closed case, then for arbitrary $\mathbb{F}$.

6.2.1 $\mathbb{F}$ algebraically closed of characteristic 0

Let M be a finite-dimensional irreducible A_h-module. As noted above, we may assume $\mathsf{M} = \mathsf{M}_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal generated by a prime factor f of h, and x and y are commuting transformations on M (compare (4.4)). When $\mathbb{F}$ is algebraically closed, this implies that x and y have a common eigenvector, which then is a basis for M by irreducibility. Since f must be a linear factor of h in this case, we have the following.

Theorem 6.3. *Assume $\mathbb{F}$ is an algebraically closed field of characteristic 0 and $h \notin \mathbb{F}$. Then every finite-dimensional irreducible A_h-module M is one dimensional. In particular, there exist $\lambda, \mu \in \mathbb{F}$, with λ a root of h, so that $\mathsf{M} \cong \mathsf{V}_{\lambda,\mu} := \mathbb{F}v_{\lambda,\mu}$, where the A_h-module action is given by $x.v_{\lambda,\mu} = \lambda v_{\lambda,\mu}$ and $y.v_{\lambda,\mu} = \mu v_{\lambda,\mu}$. Thus, in* *the notation of Theorem 4.7,* $\mathsf{M} \cong \mathsf{V}_{\lambda,\mu} \cong \mathsf{L}(\mathfrak{m}, \bar{g})$*, where* $f = x - \lambda$*,* $\mathfrak{m} = \mathsf{R}f$*, and* $g = y - \mu$*.*

Remark 6.4. For the algebra A_x, which is the universal enveloping algebra of the 2-dimensional solvable, non-abelian Lie algebra, Theorem 6.3 is Lie's theorem. For the algebra A_{x^2}, this result appears in [I]. In both these cases (and more generally when $h = x^n$ for any $n \geq 1$) $\lambda = 0$ in Theorem 6.3.

Corollary 6.5. *Assume* $\mathbb{F}$ *is an algebraically closed field of characteristic 0, and let* $\mathsf{V} = \mathsf{V}^{\mathfrak{m}}$ *be an irreducible generalized weight module for* A_h *with* $\mathfrak{m} = \mathsf{R}f$*. Either*

(i) $f = x - \lambda$*, where* λ *is a root of* h*, and* $\mathsf{V} = \mathsf{V}_{\lambda,\mu}$ *for some* $\mu \in \mathbb{F}$*, where* $\mathsf{V}_{\lambda,\mu}$ *is the one-dimensional* A_h*-module determined by* λ, μ *in Theorem 6.3; or*

(ii) f *is not a factor of* h *and* V *is isomorphic to the induced module* $\mathsf{U}(\mathfrak{m}) = \mathsf{A}_h \otimes_{\mathsf{R}} \mathsf{R}/\mathfrak{m}$*.*

6.2.2 $\mathbb{F}$ an arbitrary field of characteristic 0

Assume $\mathbb{F}$ is an arbitrary field of characteristic 0, and M is as above, a finite-dimensional irreducible A_h-module. We may suppose that $\mathsf{M} = \mathsf{M}_{\mathfrak{m}}$, where $\mathfrak{m}$ is a maximal ideal generated by a prime factor f of h of degree d. By Corollary 6.1, we know that $\mathsf{M} \cong \mathsf{L}(\mathfrak{m}, \bar{g}) = \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m})$ for some monic prime polynomial $\bar{g} = y^n - \sum_{j=0}^{n-1} y^j \bar{g}_j \in \mathsf{Q}_{\mathfrak{m}} = (\mathsf{R}/\mathfrak{m})[y]$. Taking v any nonzero element of M, we have that $\{y^k x^\ell v \mid 0 \leq k < n, 0 \leq \ell < d\}$ is a basis for M.

Assuming $f = x^d - \sum_{i=0}^{d-1} \zeta_i x^i$ and $g = y^n - \sum_{j=0}^{n-1} y^j g_j$, where $\zeta_i \in \mathbb{F}$ for all i and the polynomial $g_j \in \mathsf{R}$ is of degree less than d for all j, we have

$$
\begin{aligned}
x.y^k x^\ell v &= \begin{cases} y^k x^{\ell+1} v & \text{if } \ 0 \leq \ell < d-1, \\ \displaystyle\sum_{i=0}^{d-1} \zeta_i \, y^k x^i v & \text{if } \ \ell = d-1, \end{cases} \\
y.y^k x^\ell v &= \begin{cases} y^{k+1} x^j v & \text{if } \ 0 \leq k < n-1, \\ \displaystyle\sum_{j=0}^{n-1} y^j g_j x^\ell v = \sum_{j=0}^{n-1} y^j s_{j,\ell} v & \text{if } \ k = n-1, \end{cases}
\end{aligned}
$$

where $s_{j,\ell}$ is the remainder when $g_j x^\ell$ is divided by f.

Example 6.6. Assume $h = (x - \lambda)^\ell$ for some $\lambda \in \mathbb{F}$ and some $\ell \geq 1$; $f = x - \lambda$; and $\mathfrak{m} = \mathsf{R}f$. Let $g = y^n - \sum_{j=0}^{n-1} y^j g_j \in \mathsf{A}_h$ be such that $g_j \in \mathsf{R}$ for all j and $\bar{g} = y^n - \sum_{j=0}^{n-1} y^j \bar{g}_j$ is prime in $(\mathsf{R}/\mathfrak{m})[y]$, i.e. $y^n - \sum_{j=0}^{n-1} g_j(\lambda) y^j$ is a prime polynomial in $\mathbb{F}[y]$. Then the irreducible module $\mathsf{L}(\mathfrak{m},\bar{g}) = \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m})$ has a basis $\{y^k v \mid 0 \leq k < n\}$, where $v := u_\mathfrak{m} + \bar{g}\mathsf{U}(\mathfrak{m})$, and the A_h-action is given by

$$x.y^k v = \lambda y^k v, \quad y.y^k v = y^{k+1} v \ \ (0 \leq k < n-1), \quad y.y^{n-1} v = \sum_{j=0}^{n-1} g_j(\lambda) y^j v.$$

6.3 Irreducible R-torsion-free A_h-modules when $\mathsf{char}(\mathbb{F}) = 0$

In order to discuss the R-torsion-free irreducible A_h-modules when $\mathsf{char}(\mathbb{F}) = 0$, we assume $\mathsf{S} = \mathsf{R}\setminus\{0\}$ and $\mathsf{E} = \mathsf{S}^{-1}\mathsf{R}$ is the field of fractions of $\mathsf{R} = \mathbb{F}[x]$ as in Section 2. The localization $\mathsf{B} = \mathsf{S}^{-1}\mathsf{A}_h$ is the Ore extension $\mathsf{B} = \mathsf{E}[y, \mathsf{id}_\mathsf{E}, \delta]$, where $\delta(e) = e'h$ for all $e \in \mathsf{E}$. (Note that B does not depend on h, up to isomorphism.) First we briefly review Block's correspondence between $\widehat{\mathsf{A}}_h(\mathsf{R}\text{-torsion-free})$ and $\widehat{\mathsf{B}}(\mathsf{A}_h\text{-socle})$, where the latter denotes the set of isomorphism classes of irreducible B-modules V such that $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V}) \neq 0$. Recall that the *socle* of an A_h-module V is the submodule $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V})$ generated by the irreducible A_h-submodules of V. Block's correspondence [Bl, Lem. 2.2.1] gives the following (see also [B3, Sec. 5] for the same correspondence in a more general setting).

Proposition 6.7. *Let* M *be an irreducible* R*-torsion-free* A_h*-module. Then* $\mathsf{S}^{-1}\mathsf{M} = \mathsf{B} \otimes_{\mathsf{A}_h} \mathsf{M}$ *is an irreducible* B*-module, and the map*

$$\widehat{\mathsf{A}}_h(\mathsf{R}\text{-torsion-free}) \longrightarrow \widehat{\mathsf{B}}(\mathsf{A}_h\text{-socle}), \qquad [\mathsf{M}] \mapsto [\mathsf{S}^{-1}\mathsf{M}] \tag{6.8}$$

is a bijection.

Proof. Let M be an irreducible R-torsion-free A_h-module. Then $\mathsf{S}^{-1}\mathsf{M} = \mathsf{B} \otimes_{\mathsf{A}_h} \mathsf{M}$ is an irreducible B-module. Thus, there is a map $\Psi : \widehat{\mathsf{A}}_h(\mathsf{R}\text{-torsion-free}) \longrightarrow \widehat{\mathsf{B}}$ given by $[\mathsf{M}] \mapsto [\mathsf{S}^{-1}\mathsf{M}]$. Since M embeds in $\mathsf{S}^{-1}\mathsf{M}$ as an A_h-module, we have $\mathsf{M} \subseteq \mathsf{Soc}_{\mathsf{A}_h}(\mathsf{S}^{-1}\mathsf{M})$.

Recall that a submodule of a module V is said to be *essential* if its intersection with any nonzero submodule of V is nonzero. It is easy to see that M is an essential A_h-submodule of $\mathsf{S}^{-1}\mathsf{M}$, thus $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{S}^{-1}\mathsf{M}) = \mathsf{M}$. This shows that the map Ψ is injective, and its image is contained in $\widehat{\mathsf{B}}(\mathsf{A}_h\text{-socle})$. Conversely, if V is an irreducible B-module such that $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V}) \neq 0$, then we claim that $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V})$ is an irreducible A_h-module and $\mathsf{S}^{-1}\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V}) = \mathsf{V}$. Indeed, if $\mathsf{L} \subseteq \mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V})$ is an irreducible A_h-submodule, then $\mathsf{S}^{-1}\mathsf{L} = \mathsf{V}$ by the irreducibility of V and the fact that $\mathsf{L} \subseteq \mathsf{V}$ is R-torsion-free. Thus, L is an essential A_h-submodule of V which implies that $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V}) = \mathsf{L}$ is irreducible. Hence, Ψ gives a bijection onto $\widehat{\mathsf{B}}(\mathsf{A}_h\text{-socle})$, with inverse

$$\widehat{\mathsf{B}}(\mathsf{A}_h\text{-socle}) \longrightarrow \widehat{\mathsf{A}}_h(\mathsf{R}\text{-torsion-free}), \quad [\mathsf{V}] \mapsto [\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{V})]. \tag{6.9}$$

$\square$

Since B is an Ore extension over the field $\mathsf{S}^{-1}\mathsf{R}$, B is a principal left ideal domain so that the irreducible B-modules are the B-modules of the form $\mathsf{B}/\mathsf{B}b$, where $b \in \mathsf{B}$ is an irreducible element. In particular, any R-torsion-free irreducible A_h-module has the form $\mathsf{A}_h/(\mathsf{A}_h \cap \mathsf{B}b)$, for $b \in \mathsf{B}$ irreducible, but not all such A_h-modules are irreducible (compare [Bl, Thm. 4.3]). In [Bl, Cor. 2.2, Cor. 4.4.1], Block showed that for the Weyl algebra A_1, the map $\Psi : \widehat{\mathsf{A}}_1(\text{R-torsion-free}) \longrightarrow \widehat{\mathsf{B}}$ is in fact surjective (i.e., $\widehat{\mathsf{B}} = \widehat{\mathsf{B}}(\mathsf{A}_1\text{-socle})$), so the irreducible R-torsion-free A_1-modules correspond to B-modules of the form $\mathsf{B}/\mathsf{B}b$ and are classified by the similarity classes of irreducible elements of B. This does not hold for A_h if $h \notin \mathbb{F}$, by [Bl, Cor. 4.4.1]. We illustrate this phenomenon with a specific example.

Example 6.10. Suppose $\mathsf{char}(\mathbb{F}) = 0$. Let $\mathsf{B} = \mathsf{S}^{-1}\mathsf{A}_h$, and consider the B-module $\mathsf{B}/\mathsf{B}y$. Then as an $\mathsf{S}^{-1}\mathsf{R}$-module, $\mathsf{B}/\mathsf{B}y \cong \mathbb{F}(x)$, the field of fractions of R, with $y.\frac{q}{r} = h\left(\frac{q}{r}\right)' = h\frac{q'r - qr'}{r^2}$ for all $q, r \in \mathsf{R}$, $r \neq 0$. It is clear that $\mathsf{B}/\mathsf{B}y$ is an irreducible B-module, as $h^{-1}y$ acts as $\frac{d}{dx}$. Now consider the A_h-submodule $\mathsf{A}_h/(\mathsf{A}_h \cap \mathsf{B}y) = \mathsf{A}_h/\mathsf{A}_h y$. As an R-module, $\mathsf{A}_h/\mathsf{A}_h y \cong \mathbb{F}[x] = \mathsf{R}$, with y acting as $h\frac{d}{dx}$. For any $k \geq 0$, $h^k\mathsf{R}$ is an A_h-submodule of $\mathsf{A}_h/\mathsf{A}_h y$ and $\left\{h^k\mathsf{R}\right\}_{k\geq 0}$ is a strictly descending chain of submodules of $\mathsf{A}_h/\mathsf{A}_h y$ if $h \notin \mathbb{F}$. In particular, $\mathsf{A}_h/\mathsf{A}_h y$ is irreducible if and only if $h \in \mathbb{F}^*$.

Similarly, suppose $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{B}/\mathsf{B}y) \neq 0$, and assume $\mathsf{A}_h.\frac{q}{r} \subseteq \mathsf{Soc}_{\mathsf{A}_h}(\mathsf{B}/\mathsf{B}y)$ is an irreducible A_h-submodule of $\mathsf{B}/\mathsf{B}y \cong \mathbb{F}(x)$. As $0 \neq q = r\frac{q}{r} \in \mathsf{A}_h.\frac{q}{r}$, which is an irreducible submodule, we have $\mathsf{A}_h.q = \mathsf{A}_h.\frac{q}{r}$, so we can assume $r = 1$; in particular, $\mathsf{A}_h.q \subseteq \mathsf{R}$. The irreducibility argument also shows that $\mathsf{A}_h.(hq) = \mathsf{A}_h.q$, so $q \in \mathsf{A}_h.(hq)$. Assume further that $h \notin \mathbb{F}$ and take $k \geq 0$ maximal such that h^k divides q. Then every nonzero element in $\mathsf{A}_h.(hq)$ is divisible by h^{k+1}, which contradicts the maximality of k. Thus, $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{B}/\mathsf{B}y) = 0$ if $h \notin \mathbb{F}$. If $h \in \mathbb{F}^*$, then clearly $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{B}/\mathsf{B}y) = \mathsf{A}_h.1 = \mathsf{R}$.

Next we will characterize the isomorphism classes of irreducible R-torsion-free A_h-modules in terms of the irreducible R-torsion-free A_1-modules, without involving localization. For this, we will view A_h as a subalgebra of the Weyl algebra A_1 via the embedding $\mathsf{A}_h \hookrightarrow \mathsf{A}_1$, $x \mapsto x$, $\hat{y} \mapsto yh$, where $x, \hat{y}$ are the generators of A_h with $[\hat{y}, x] = h$ and x, y are the generators of the Weyl algebra, satisfying $[y, x] = 1$.

Let M be an irreducible R-torsion-free A_h-module. Since h is normal in A_h, (that is $h\mathsf{A}_h = \mathsf{A}_h h$, as shown in [BLO1, Lem. 7.1]), $h\mathsf{M}$ is a submodule. But then $h\mathsf{M} = \mathsf{M}$, as M is R-torsion-free. Given $m \in \mathsf{M}$, there exists an $\widetilde{m} \in \mathsf{M}$ with $m = h\widetilde{m}$, and $\widetilde{m}$ is unique since M is R-torsion-free. Define

$$y.m := \hat{y}.\widetilde{m}.$$

It is apparent that this extends the action of A_h on M to an action of A_1 on M, so that M is an irreducible R-torsion-free A_1-module. Thus, we have an injective map

$$\widehat{\mathsf{A}}_h(\text{R-torsion-free}) \longrightarrow \widehat{\mathsf{A}}_1(\text{R-torsion-free}), \quad [\mathsf{M}] \mapsto [\mathsf{M}]. \tag{6.11}$$

The next result describes the image of this map.

Proposition 6.12. *Suppose* M *is an irreducible* R*-torsion-free* A_1*-module. The following conditions are equivalent:*

(i) *The restriction of* M *to* A_h *is an irreducible* A_h*-module.*

(ii) $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{M}) \neq 0$*.*

(iii) $h\mathsf{M} = \mathsf{M}$ *and* M *is a Noetherian* A_h*-module.*

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious, and (i) $\Longrightarrow$ (iii) follows from the preceding considerations. Suppose $\mathsf{Soc}_{\mathsf{A}_h}(\mathsf{M}) \neq 0$, and let L be an irreducible A_h-submodule of M. Then as before, $h\mathsf{L} = \mathsf{L}$, and L is an A_1-submodule of M. Thus $\mathsf{L} = \mathsf{M}$ which shows that M is an irreducible A_h-module, so that (ii) $\Longrightarrow$ (i) holds.

Finally, assume that $h\mathsf{M} = \mathsf{M}$ and M is a Noetherian A_h-module. Let N be a maximal A_h-submodule of M. Thus, since h is normal, $\{m \in \mathsf{M} \mid hm \in \mathsf{N}\}$ is an A_h-submodule of M containing N. As N is maximal and $h\mathsf{M} = \mathsf{M} \not\subseteq \mathsf{N}$, it follows that $\{m \in \mathsf{M} \mid hm \in \mathsf{N}\} = \mathsf{N}$. Given $v \in \mathsf{N} \subseteq \mathsf{M} = h\mathsf{M}$, there exists $m \in \mathsf{M}$ so that $v = hm$; hence $m \in \mathsf{N}$ and $h\mathsf{N} = \mathsf{N}$. Now we can conclude that N is a proper A_1-submodule of M. Therefore, $\mathsf{N} = 0$, proving that M is an irreducible A_h-module. This shows that (iii) $\Longrightarrow$ (i). $\square$

7 Irreducible A_h-modules when $\mathsf{char}(\mathbb{F}) = p > 0$

In this section, we investigate the irreducible A_h-modules when $\mathsf{char}(\mathbb{F}) = p > 0$ and completely determine them when $\mathbb{F}$ is algebraically closed. When $\mathsf{char}(\mathbb{F}) = p > 0$, all irreducible A_h-modules are finite dimensional by Proposition 1.7 and therefore have R-torsion. We have seen in Theorem 1.4 that the center of A_h is the polynomial algebra $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}[x^p, z_p]$, where $z_p = y(y+h')\cdots(y+(p-1)h') = y^p - y\frac{\delta^p(x)}{h(x)}$, and $\frac{\delta^p(x)}{h(x)} \in \mathbb{F}[x^p]$. Quillen's extension of Schur's Lemma tells us that $\mathsf{Z}(\mathsf{A}_h)$ must act as scalars on any irreducible A_h-module V when $\mathbb{F}$ is algebraically closed.

Since our ultimate goal is a description of the irreducibles when $\mathbb{F}$ is algebraically closed, we make the following assumptions throughout the section:

Assumptions 7.1. V *is an irreducible* A_h*-module, and there exist scalars* $\beta \in \mathbb{F}$ *and* λ, α *in the algebraic closure* $\overline{\mathbb{F}}$ *of* $\mathbb{F}$ *such that* $\lambda^p, \alpha^{p-1} \in \mathbb{F}$*, and as transformations on* V*,*

- $x^p = \lambda^p \,\mathsf{id}_{\mathsf{V}}$ *(equivalently,* $(x - \lambda\,\mathsf{id}_{\mathsf{V}})^p = 0$*),*
- $\frac{\delta^p(x)}{h(x)} = \alpha^{p-1}\,\mathsf{id}_{\mathsf{V}}$*,*
- $y^p - \alpha^{p-1} y = \beta\,\mathsf{id}_{\mathsf{V}}$*.*

Suppose $\mu \in \overline{\mathbb{F}}$ is a root of the polynomial $g(t) := t^p - \alpha^{p-1}t - \beta$. Then

$$\Theta = \{\mu + j\alpha \mid j = 0, 1, \ldots, p-1\} \tag{7.2}$$

is the complete set of roots of $g(t)$ in $\overline{\mathbb{F}}$. Now if $g(t)$ has a monic factor in $\mathbb{F}[t]$, say of degree m where $1 \le m < p$, then the coefficient of t^{m-1} in that factor has the form $-(m\mu + n\alpha)$ for some n. This implies $\mu + m^{-1}n\alpha \in \mathbb{F}$, hence $g(t)$ has a root in $\mathbb{F}$. From this we see that either $t^p - \alpha^{p-1}t - \beta$ has a root in $\mathbb{F}$ or is a prime polynomial in $\mathbb{F}[t]$.

Lemma 7.3. *Suppose* V *is an* A_h*-module and* $\lambda \in \mathbb{F}$ *is such that* $h(\lambda) = 0$ *and* $x = \lambda\,\mathsf{id}_{\mathsf{V}}$ *as a transformation on* V*. Then* $\frac{\delta^p(x)}{h(x)} = h'(\lambda)^{p-1}\,\mathsf{id}_{\mathsf{V}}$.

Proof. Note that $\delta^1(x) = h$ and $\delta^2(x) = h'h$. It is evident by induction that for all $k \ge 1$, $\delta^k(x) = (h')^{k-1}h + f_k h^2$ for some $f_k \in \mathsf{R}$ (compare Lemma 8.1 and Corollary 8.5 below). Therefore $\frac{\delta^p(x)}{h(x)} = h'(x)^{p-1} + f_p(x)h(x)$ and $\frac{\delta^p(x)}{h(x)}\Big|_{x=\lambda} = h'(\lambda)^{p-1} + 0$. $\square$

Theorem 7.4. *Suppose* $\mathsf{char}(\mathbb{F}) = p > 0$*, and let* V *be an irreducible* A_h*-module satisfying the assumptions in 7.1. Suppose further that* $\lambda \in \mathbb{F}$*. Then one of the following holds:*

(i) $h(\lambda) = 0$ *and there exists* $\theta \in \Theta \cap \mathbb{F}$ *so that* $\mathsf{V} = \mathbb{F}v$ *where* $x.v = \lambda v$, $y.v = \theta v$.

(ii) $h(\lambda) = 0$, $\Theta \cap \mathbb{F} = \emptyset$, *and* V *has a basis* $\{v_n \mid n = 0, 1, \ldots, p-1\}$ *such that* $x.v_n = \lambda v_n$ *for all* n, $y.v_n = v_{n+1}$ *for* $n < p-1$ *and* $y.v_{p-1} = h'(\lambda)^{p-1}v_1 + \beta v_0$.

(iii) $h(\lambda) \ne 0$ *and* V *has a basis* $\{v_n \mid n = 0, 1, \ldots, p-1\}$ *such that*

$$\begin{aligned} y.v_n &= \begin{cases} v_{n+1} & \text{if} \quad 0 \le n < p-1, \\ \alpha^{p-1}v_1 + \beta v_0 & \text{if} \quad n = p-1; \end{cases} \\ x.v_n &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \delta^j(x)\,|_{x=\lambda}\, v_{n-j}. \end{aligned}$$

Proof. Assume first that $h(\lambda) = 0$. Then $x - \lambda$ is a factor of h and $\mathfrak{m} = \mathsf{R}(x-\lambda)$ is a maximal δ-invariant ideal. Since $\mathsf{V}_{\mathfrak{m}} \ne 0$, Proposition 2.2 (iii) implies that $\mathsf{V} = \mathsf{V}_{\mathfrak{m}}$. In particular, $x = \lambda\,\mathsf{id}_{\mathsf{V}}$, and x and y commute as transformations on V. Since y satisfies the polynomial $t^p - \alpha^{p-1}t - \beta$ on V, V is a homomorphic image of the module $\mathsf{L}(\mathfrak{m}, \bar{g}) = \mathsf{U}(\mathfrak{m})/\bar{g}\mathsf{U}(\mathfrak{m})$, where $\bar{g}(y) = y^p - \alpha^{p-1}y - \beta$, under the identification $\mathsf{Q}_{\mathfrak{m}} = (\mathsf{R}/\mathfrak{m})[y] \cong \mathbb{F}[y]$. By Lemma 7.3, we may write $\bar{g}(y) = y^p - h'(\lambda)^{p-1}y - \beta$, where $h'(\lambda) \in \mathbb{F}$ since $\lambda \in \mathbb{F}$. We have seen that either $\bar{g}$ has a root in $\mathbb{F}$ or is a prime polynomial. If $\mu \in \mathbb{F}$ is a root of $\bar{g}$, then $\Theta = \{\mu + jh'(\lambda) \mid j = 0, 1, \ldots, p-1\} \subseteq \mathbb{F}$ is the complete set of roots of $\bar{g}$, and it follows that y has an eigenvalue $\theta \in \mathbb{F}$ on V, so case (i) holds. If $\bar{g}$ is prime in $\mathsf{Q}_{\mathfrak{m}}$, then $\mathsf{L}(\mathfrak{m}, \bar{g})$ is irreducible, so $\mathsf{V} = \mathsf{L}(\mathfrak{m}, \bar{g})$ and $\dim_{\mathbb{F}} \mathsf{V} = p$, by Theorem 4.7 (ii). Taking a nonzero vector $v_0 \in \mathsf{V}$ and setting

$v_n = y^n.v_0$ for $n = 0, 1, \ldots, p-1$, we see that the v_n are linearly independent, and hence are a basis of V. Moreover, $y.v_n = v_{n+1}$ for $n < p-1$ and $y.v_{p-1} = y^p.v_0 = \alpha^{p-1}y.v_0 + \beta v_0 = \alpha^{p-1}v_1 + \beta v_0$, so we have case (ii).

Now suppose that $h(\lambda) \neq 0$, and take $0 \neq v_0 \in \mathsf{V}$ such that $x.v_0 = \lambda v_0$. Assume $v_m = y^m.v_0$ for $m = 0, 1, \ldots$. Let n be minimal such that there is a dependence relation $v_n = \sum_{k=0}^{n-1} \xi_k v_k$. Observe that $n \leq p$, as the minimum polynomial in $\mathbb{F}[t]$ of y on V divides $t^p - \alpha^{p-1}t - \beta$. Applying x to this relation and using (1.3), we obtain

$$\begin{aligned} x.v_n &= xy^n.v_0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \delta^j(x)\,\big|_{x=\lambda} v_{n-j} \qquad (7.5) \\ &= \sum_{k=0}^{n-1} \xi_k \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \delta^\ell(x)\,\big|_{x=\lambda} v_{k-\ell}. \end{aligned}$$

The $j = 0$ term cancels with the sum of the $\ell = 0$ terms on the right by the minimal dependence relation. The term v_{n-1} occurs in the resulting expression only when $j = 1$, and in this case, we have $(-1)n\delta(x)\big|_{x=\lambda} v_{n-1}$. Since $\delta(x)\big|_{x=\lambda} = h(\lambda) \neq 0$, we will achieve a dependence relation involving v_{n-1}, except when $n = p$. Thus, we have case (iii). □

Lemma 7.6. *Let* $\mathsf{char}(\mathbb{F}) = p > 0$ *and* $\beta, \lambda \in \mathbb{F}$, *and assume* $h(\lambda) \neq 0$. *Let* $\mathfrak{m} = \mathsf{R}(x - \lambda)$ *and set* $z_\beta = y^p - y\frac{\delta^p(x)}{h(x)} - \beta$. *Then the quotient* $\mathsf{L}(\mathfrak{m}, z_\beta) := \mathsf{U}(\mathfrak{m})/z_\beta \mathsf{U}(\mathfrak{m})$ *is a* p*-dimensional irreducible* A_h*-module with basis* $v_n = y^n.\overline{u}_\mathfrak{m}$, $0 \leq n < p$, *where* $\overline{u}_\mathfrak{m}$ *is the image of* $u_\mathfrak{m} = 1 \otimes (1 + \mathfrak{m})$ *in* $\mathsf{L}(\mathfrak{m}, z_\beta)$. *The* A_h*-action is given by*

$$\begin{aligned} y.v_n &= \begin{cases} v_{n+1} & \text{if} \quad 0 \leq n < p-1, \\ \frac{\delta^p(x)}{h(x)}\Big|_{x=\lambda} v_1 + \beta v_0 & \text{if} \quad n = p-1; \end{cases} \qquad (7.7) \\ x.v_n &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \delta^j(x)\,|_{x=\lambda}\, v_{n-j}. \end{aligned}$$

Proof. Since $y^p - y\frac{\delta^p(x)}{h(x)}$ is central in A_h, it is apparent that $z_\beta \mathsf{U}(\mathfrak{m})$ is a submodule of $\mathsf{U}(\mathfrak{m})$, and hence that the corresponding quotient $\mathsf{L}(\mathfrak{m}, z_\beta)$ is an A_h-module. As $\{y^n.u_\mathfrak{m} \mid n = 0, 1, \ldots\}$ is a basis for $\mathsf{U}(\mathfrak{m})$, the module $\mathsf{L}(\mathfrak{m}, z_\beta)$ is spanned by the vectors $y^n.\overline{u}_\mathfrak{m}$, $n = 0, 1, \ldots$, where $\overline{u}_\mathfrak{m}$ is the image of $u_\mathfrak{m}$ in $\mathsf{L}(\mathfrak{m}, z_\beta)$. However, since $y^p.\overline{u}_\mathfrak{m} = \frac{\delta^p(x)}{h(x)}\Big|_{x=\lambda} y.\overline{u}_\mathfrak{m} + \beta\overline{u}_\mathfrak{m}$, we see that the dimension of $\mathsf{L}(\mathfrak{m}, z_\beta)$ is at most p. The argument that the vectors $v_n := y^n.\overline{u}_\mathfrak{m}$ are linearly independent for $n = 0, 1, \ldots, p-1$ is the same as that given in (7.5).

Now if W is a nonzero submodule of $\mathsf{L}(\mathfrak{m}, z_\beta)$, and $0 \neq w = \sum_{k=0}^{n} \gamma_k v_k \in \mathsf{W}$ with n minimal, then

$$(x - \lambda).w = \gamma_n \sum_{j=1}^{n} (-1)^j \binom{n}{j} \delta^j(x)\,|_{x=\lambda}\, v_{n-j} - \sum_{k=1}^{n-1} \gamma_k \sum_{\ell=1}^{k} (-1)^\ell \binom{k}{\ell} \delta^\ell(x)\,|_{x=\lambda}\, v_{k-\ell},$$

will give a smaller length element in W if $(-1)\binom{n}{1}\gamma_n\delta(x)\mid_{x=\lambda}= -n\gamma_n h(\lambda) \neq 0$. As $h(\lambda) \neq 0$, it must be that $n = 0$, and $w = \gamma_0 v_0$. But then applying y^n to w shows that $v_n \in \mathsf{W}$ for all $n = 0, 1, \ldots, p-1$. Hence, $\mathsf{W} = \mathsf{L}(\mathfrak{m}, z_\beta)$, which is irreducible. $\square$

Corollary 7.8. *Assume $\mathbb{F}$ is an algebraically closed field of characteristic $p > 0$, and let V be an irreducible A_h-module. Then either*

(i) *for some $\lambda, \theta \in \mathbb{F}$ with $h(\lambda) = 0$, $\mathsf{V} \cong \mathsf{V}_{\lambda,\theta} = \mathbb{F}v_{\lambda,\theta}$, where*

$$x.v_{\lambda,\theta} = \lambda v_{\lambda,\theta}, \quad y.v_{\lambda,\theta} = \theta v_{\lambda,\theta}, \quad \text{or}$$

(ii) *for some $\lambda, \beta \in \mathbb{F}$ with $h(\lambda) \neq 0$, $\mathsf{V} \cong \mathsf{L}(\mathfrak{m}, z_\beta) = \bigoplus_{n=0}^{p-1} \mathbb{F}v_n$, where $\mathfrak{m} = \mathsf{R}(x-\lambda)$ and the action of A_h is given in* (7.7).

Hence, if P is a primitive ideal of A_h, then P is isomorphic to one of the following:

(i) $\mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}, \bar{g}))$ *for some $\mathfrak{m} = \mathsf{R}(x-\lambda)$, where $h(\lambda) = 0$, and some $g = y - \theta$, where $\theta \in \mathbb{F}$, or*

(ii) $\mathsf{Ann}_{\mathsf{A}_h}(\mathsf{L}(\mathfrak{m}, z_\beta))$ *for some $\mathfrak{m} = \mathsf{R}(x-\lambda)$, where $h(\lambda) \neq 0$, and some $z_\beta = y^p - y\frac{\delta^p(x)}{h(x)} - \beta \in \mathsf{Z}(\mathsf{A}_h)$, where $\beta \in \mathbb{F}$.*

Proof. This is a direct consequence of Theorem 7.4 and Lemma 7.6, since only cases (i) and (iii) of that theorem occur when $\mathbb{F}$ is algebraically closed. In case (iii), V must be a homomorphic image of the irreducible A_h-module $\mathsf{L}(\mathfrak{m}, z_\beta)$ for some λ and β in $\mathbb{F}$ by Lemma 7.6, so V must be isomorphic to $\mathsf{L}(\mathfrak{m}, z_\beta)$. $\square$

8 The Combinatorics of $\delta^k(x)$

We have seen that many of the expressions for the action of A_h on an irreducible module involve terms $\delta^k(x)$ for some $k \geq 1$, where δ is the derivation of R given by $\delta(f) = f'h$, and $'$ denotes the usual derivative. Here, we first determine an expression for $\delta^k(f)$ for arbitrary f and then specialize to the case $f = x$.

Suppose ν is a partition of some integer n, and let $\ell(\nu)$ denote the number of nonzero parts of ν. We write $\nu = (n^{\nu_n}, \ldots, 2^{\nu_2}, 1^{\nu_1})$ to indicate that ν has ν_1 parts equal to 1, ν_2 parts equal to 2, and so forth. Thus, $\sum_{k=1}^n k\nu_k = n$ and $\sum_{k=1}^n \nu_k = \ell(\nu)$. For example, $\nu = (4, 2^2, 1^3)$ is a partition of 11, which we write $\nu \vdash 11$, with $\nu_1 = 3$, $\nu_2 = 2$, $\nu_3 = 0$, $\nu_4 = 1$, and $\ell(\nu) = 6$.

Let $\emptyset$ denote the unique partition of 0 and set $h^{(\emptyset)} = 1$. For $j \geq 1$, let $h^{(j)} = (\frac{d}{dx})^j(h)$. Then for $\nu = (n^{\nu_n}, \ldots, 2^{\nu_2}, 1^{\nu_1}) \vdash n$, we define

$$h^{(\nu)} := (h^{(1)})^{\nu_1}(h^{(2)})^{\nu_2} \cdots (h^{(n)})^{\nu_n}.$$

Lemma 8.1. *For $k \geq 1$,*

$$\delta^k(f) = \sum_{n=0}^{k-1} \sum_{\nu \vdash n} b_\nu^k f^{(k-n)} h^{(\nu)} h^{k-\ell(\nu)},$$

where the b_ν^k are nonnegative integer coefficients.

Before beginning the proof, and as the initial inductive steps, we present some examples:

$$\begin{aligned}
\delta^1(f) &= f'h = f^{(1)}h, \\
\delta^2(f) &= f''h^2 + f'h'h = f^{(2)}h^2 + f^{(1)}h^{(1)}h, \\
\delta^3(f) &= f'''h^3 + 3f''h'h^2 + f'h''h^2 + f'(h')^2h \\
&= f^{(3)}h^3 + 3f^{(2)}h^{(1)}h^2 + f^{(1)}h^{(2)}h^2 + f^{(1)}(h^{(1)})^2h.
\end{aligned}$$

Proof. We can assume the lemma is true for k and prove it for $k+1$. Suppose there are nonnegative integers b_ν^k so that

$$\delta^k(f) = \sum_{n=0}^{k-1} \sum_{\nu \vdash n} b_\nu^k f^{(k-n)} h^{(\nu)} h^{k-\ell(\nu)}.$$

Then

$$\begin{aligned}
\delta^{k+1}(f) &= \sum_{n=0}^{k-1} \sum_{\nu \vdash n} b_\nu^k \Big(f^{(k-n)} h^{(\nu)} h^{k-\ell(\nu)} \Big)' h \qquad (8.2) \\
&= \sum_{n=0}^{k-1} \sum_{\nu \vdash n} b_\nu^k f^{(k+1-n)} h^{(\nu)} h^{k+1-\ell(\nu)} + \sum_{n=0}^{k-1} \sum_{\nu \vdash n} b_\nu^k f^{(k-n)} \Big(h^{(\nu)} \Big)' h^{k+1-\ell(\nu)} \\
&\quad + \sum_{n=0}^{k-1} \sum_{\nu \vdash n} \big(k - \ell(\nu)\big) b_\nu^k f^{(k-n)} h^{(\nu)} h' h^{k-\ell(\nu)}.
\end{aligned}$$

Observe for $\nu \vdash n$ that

$$\Big(h^{(\nu)} \Big)' = \sum_{j=1}^{n} \nu_j (h^{(1)})^{\nu_1} \cdots (h^{(j)})^{\nu_j - 1} (h^{(j+1)})^{\nu_{j+1}+1} \cdots (h^{(n)})^{\nu_n}.$$

In the jth summand on the right, a part of size j has been converted to a part of size $j+1$. Now if $\nu_j \neq 0$ for some j such that $1 \leq j \leq n$, we set

$$\nu[j] = \begin{cases} (n^{\nu_n}, \ldots, (j+1)^{\nu_{j+1}+1}, j^{\nu_j - 1}, \ldots, 2^{\nu_2}, 1^{\nu_1}) & \text{if } 1 \leq j < n, \\ ((n+1)^1) \quad \text{if } j = n. \end{cases} \qquad (8.3)$$

Then $\nu[j] \vdash n+1$, and $\ell(\nu[j]) = \sum_{i=1}^{n} \nu_i = \ell(\nu)$. Hence,

$$b_\nu^k f^{(k-n)} \left(h^{(\nu)}\right)' h^{k+1-\ell(\nu)} = \sum_{j=1}^{n} b_\nu^k \nu_j f^{(k+1-(n+1))} h^{(\nu[j])} h^{k+1-\ell(\nu[j])},$$

where $h^{(\nu[j])}$ should be interpreted as 1 if $\nu_j = 0$.

Now let's consider a term $h^{(\nu)} h' h^{k-\ell(\nu)}$ in the last sum of (8.2), where $\nu \vdash n$. Then $h^{(\nu)} h'$ corresponds to the partition

$$\nu^+ = (n^{\nu_n}, \ldots, 2^{\nu_2}, 1^{\nu_1+1}) \vdash n+1, \tag{8.4}$$

which has one more part equal to 1 than does ν. Hence $k+1-\ell(\nu^+) = k-\ell(\nu)$, and the corresponding term is

$$\big(k-\ell(\nu)\big) b_\nu^k f^{(k-n)} h^{(\nu)} h' h^{k-\ell(\nu)} = \big(k+1-\ell(\nu^+)\big) b_\nu^k f^{(k+1-(n+1))} h^{(\nu^+)} h^{k+1-\ell(\nu^+)}.$$

For $\mu \vdash m$, where $0 \le m < k+1$, if $f^{(k+1-m)} h^{(\mu)} h^{k+1-\ell(\mu)} \neq 0$, it appears in (8.2) with a nonnegative integer coefficient b_μ^{k+1}, which is obtained from summing the following:

(i) b_μ^k if $m < k$,

(ii) $\nu_j b_\nu^k$ if $\nu \vdash m-1$ is a partition such that $\nu[j] = \mu$,

(iii) $(k-\ell(\nu)) b_\nu^k$ if $\nu \vdash m-1$ is a partition such that $\nu^+ = \mu$.

Hence b_μ^{k+1} is a nonnegative integer and

$$\delta^{k+1}(f) = \sum_{m=0}^{k} \sum_{\mu \vdash m} b_\mu^{k+1} f^{(k+1-m)} h^{(\mu)} h^{k+1-\ell(\mu)}.$$

□

Since $f^{(j)} = 0$ for all $j \ge 2$ when $f = x$, Lemma 8.1 reduces in this special case to

Corollary 8.5. *For $k \ge 1$,*

$$\delta^k(x) = \sum_{\mu \vdash k-1} c_\mu^k h^{(\mu)} h^{k-\ell(\mu)},$$

where the coefficients c_μ^k are nonnegative integer coefficients, which are obtained from the coefficients c_ν^{k-1} appearing in $\delta^{k-1}(x)$ by summing all the following terms:

(a) $\nu_j c_\nu^{k-1}$ *if $\nu \vdash k-2$ is a partition such that $\nu[j] = \mu$, where $\nu[j]$ is as in* (8.3)*;*

(b) $(k-1-\ell(\nu))c_\nu^{k-1}$ *if* $\nu \vdash k-2$ *is a partition such that* $\nu^+ = \mu$, *where* ν^+ *is as in* (8.4).

In the table below, for $k = 1, \ldots, 7$ and for each partitition $\mu \vdash k-1$, we display the coefficient c_μ^k as a subscript on μ.

k	c_μ^k						
1	$(0)_1$						
2	$(1)_1$						
3	$(2)_1$	$(1^2)_1$					
4	$(3)_1$	$(2,1)_4$	$(1^3)_1$				
5	$(4)_1$	$(3,1)_7$	$(2^2)_4$	$(2,1^2)_{11}$	$(1^4)_1$		
6	$(5)_1$	$(4,1)_{11}$	$(3,2)_{15}$	$(3,1^2)_{32}$	$(2^2,1)_{34}$	$(2,1^3)_{26}$	$(1^5)_1$
7	$(6)_1$	$(5,1)_{16}$	$(4,2)_{26}$	$(4,1^2)_{76}$	$(3^2)_{15}$	$(3,2,1)_{192}$	$(3,1^3)_{122}$
7 cont.				$(2^3)_{34}$	$(2^2,1^2)_{180}$	$(2,1^4)_{57}$	$(1^6)_1$

Examples 8.6. (1) Consider the partition $\mu = (3,2) \vdash 5$, so here $k = 6$. Since $\mu = \nu[2]$, for $\nu = (2^2)$, and $\mu = \pi[1]$ for $\pi = (3,1)$, we have $c_\mu^6 = 2c_\nu^5 + c_\pi^5 = 2 \cdot 4 + 7 = 15$, as displayed in the table.

(2) As another example, consider the partition $\mu = (2^2,1) \vdash 5$. Now $\mu = \nu^+$ for $\nu = (2^2) \vdash 4$, and $\mu = \pi[1]$ for $\pi = (2,1^2) \vdash 4$. Thus, $c_\mu^6 = (5-\ell(\nu))c_\nu^5 + 2c_\pi^5 = 3 \cdot 4 + 2 \cdot 11 = 34$, as shown.

The coefficients c_μ^k satisfy some intriguing properties. We illustrate this with one particular example in the next proposition.

Proposition 8.7. *Assume* c_μ^k *are the coefficients appearing in Corollary 8.5. Then*

$$\sum_{\mu \vdash k-1} c_\mu^k = (k-1)\,!.$$

Proof. We proceed by induction on k. Verification for small values of k can be done by adding the subscripts in the kth row of the table. We assume the result for k and show it for $k+1$. To accomplish this, we define a new sort of "multiplication" that will help to reveal the proof.

- **Step 1**. *List the parts of a partition ν of $k-1$ with multiplicity in descending order, and add sufficiently many 0's to get a k-tuple $\tilde{\nu}$ with weakly descending components. Multiply the k-tuple $\tilde{\nu}$ by c_ν^k, then sum over $\nu \vdash k-1$.*

  To illustrate this, consider the line corresponding to $k=4$ in the table, which is $\mathbf{(3)_1 \quad (2,1)_4 \quad (1^3)_1}$. In this step we rewrite it as

$$\mathbf{(3,0,0,0) + 4(2,1,0,0) + (1,1,1,0)}.$$

- **Step 2**. *"Multiply" by (1); i.e. add 1 to each component in all possible ways and sum the result.*

$$\begin{aligned}(\mathbf{1}) * \Big((\mathbf{3},\mathbf{0},\mathbf{0},\mathbf{0}) + \mathbf{4}(\mathbf{2},\mathbf{1},\mathbf{0},\mathbf{0}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0})\Big) = \\ (\mathbf{4},\mathbf{0},\mathbf{0},\mathbf{0}) + (\mathbf{3},\mathbf{1},\mathbf{0},\mathbf{0}) + (\mathbf{3},\mathbf{0},\mathbf{1},\mathbf{0}) + (\mathbf{3},\mathbf{0},\mathbf{0},\mathbf{1}) \\ +\mathbf{4}(\mathbf{3},\mathbf{1},\mathbf{0},\mathbf{0}) + \mathbf{4}(\mathbf{2},\mathbf{2},\mathbf{0},\mathbf{0}) + \mathbf{4}(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{0}) + \mathbf{4}(\mathbf{2},\mathbf{1},\mathbf{0},\mathbf{1}) \\ +(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{0}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{0}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{0}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}).\end{aligned}$$

- **Step 3**. *Collect terms that are the same after permutation of the components.*

$$\mathbf{(4,0,0,0) + 7(3,1,0,0) + 4(2,2,0,0) + 11(2,1,1,0) + (1,1,1,1).}$$

  We can read off the line $k=5$ in the table from this.

This process is just a different way of doing what is described in Corollary 8.5 to determine the coefficient c_μ^{k+1}. Indeed, adding 1 to the nonzero parts of a k-tuple takes into account the multiplicities in (a) of that corollary, and adding 1 to the $k-\ell(\mu)$ components that are 0 accounts for (b). Thus, the resulting coefficient of each $\mu \vdash k$ is c_μ^{k+1}. Now suppose that $\sum_{\nu\vdash k-1} c_\nu^k = (k-1)!$. The sum of the coefficients in $(1) * \sum_{\nu\vdash k-1} c_\nu^k \tilde{\nu}$ is just $\sum_{\mu\vdash k} c_\mu^{k+1}$. But each c_ν^k appears k times in $(1) * \sum_{\nu\vdash k-1} c_\nu^k \tilde{\nu}$. Thus,

$$\sum_{\mu\vdash k} c_\mu^{k+1} = k \sum_{\nu\vdash k-1} c_\nu^k = k\cdot(k-1)! = k!.$$

□

References

[AD] J. Alev and F. Dumas, *Invariants du corps de Weyl sous l'action de groupes finis*, Comm. Algebra 25 (1997), no. 5, 1655–1672.

[AP] D. Arnal and G. Pinczon, *On algebraically irreducible representations of the Lie algebra* sl_2, J. Math. Phys. **15** (1974), 350–359.

[AS] M. Artin and J.T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **122** (1995), no. 2, 231–276.

[AVV] M. Awami, M. Van den Bergh, and F. Van Oystaeyen, *Note on derivations of graded rings and classification of differential polynomial rings*, Bull. Soc. Math. Belg. Sr. A **40** (1988), no. 2, 175–183.

[B1] V.V. Bavula, *Generalized Weyl algebras and their representations*, Algebra i Analiz, **4** (1) (1992), 75–97; English transl.: St. Petersburg Math. J. **4** (1) (1993), 71–92.

[B2] V.V. Bavula, *The simple $D[X, Y; \sigma, a]$-modules*, Ukrainian Math. J. **44** (1992), 1628–1644.

[B3] V.V. Bavula, *The irreducible modules of the Ore extensions with coefficients from a Dedekind ring*, Comm. Algebra **27**(6) (1999), 2665-2699.

[BO] V.V. Bavula and F. van Oystaeyen, *The simple modules of certain generalized crossed products*, J. Algebra **194** (1997), no. 2, 521–566.

[BBF] V. Bekkert, G. Benkart, and V. Futorny, *Weight modules for Weyl algebras*, Kac-Moody Lie Algebras and Related Topics, "Ramanujan International Symposium on Kac-Moody Algebras and its Applications," Contemp. Math. Amer. Math. Soc. **343** (2004), 17-42.

[BLO1] G. Benkart, S.A. Lopes, and M. Ondrus, *A parametric family of subalgebras of the Weyl algebra: I. Structure and automorphisms*, Trans. Amer. Math. Soc. to appear, (arXiv #1210-4631).

[BLO2] G. Benkart, S. Lopes, and M. Ondrus, *A parametric family of subalgebras of the Weyl algebra III. Derivations, in preparation.*

[Bl] R. Block, *The irreducible representations of the Lie algebra $\mathfrak{sl}(2)$ and of the Weyl algebra*, Adv. Math. **39** (1981), 69-110.

[C] M. Chakrabarti, *Representations of Modular Weyl Algebras and Quantum Weyl Algebras*, Ph.D. thesis, University of Wisconsin-Madison 2002.

[CLW] C. Cibils, A. Lauve, and S. Witherspoon, *Hopf quivers and Nichols algebras in positive characteristic*, Proc. Amer. Math. Soc. **137** (2009), no. 12, 4029–4041.

[D] J. Dixmier, *Enveloping Algebras*, Revised reprint of the 1977 translation, Graduate Studies in Mathematics **11**, American Mathematical Society, Providence, RI, 1996.

[DGO] Yu. A. Drozd, B.L. Guzner, S.A. Ovsienko, *Weight modules over generalized Weyl algebras*, J. of Algebra **184** (1996), 491–504.

[Du] M. Duflo, *Sur la Classification des ideaux primitifs dans làlgebre enveloppante d'une algebre de Lie semi-simple*, Ann. Math. **105** No. 1, (1977), 107-120.

[GW] K.R. Goodearl and R.B. Warfield, *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society Student Texts **16**, Cambridge University Press, Cambridge, 1989.

[I] N.K. Iyudu, *Representation spaces of the Jordan plane*, arXiv:1209.0746v1.

[McR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, With the cooperation of L.W. Small, Revised edition, Graduate Studies in Mathematics **30**, American Mathematical Society, Providence, RI, 2001.

[P] D.S. Passman, *A Course in Ring Theory*, AMS Chelsea Publ. vol. **348** (2004).

[S1] E. N. Shirikov, *Two-generated graded algebras*, Algebra Discrete Math. (2005), no. 3, 60–84.

[S2] E. N. Shirikov, *The Jordan plane over a field of positive characteristic*, Mat. Zametki **82** (2007), no. 2, 272–292; transl. Math. Notes **82** (2007), no. 1-2, 238–256.

[S3] E. N. Shirikov, *The Jordanian plane*, Fundam. Prikl. Mat. **13** (2007), no. 2, 217-230, transl. J. Math. Sci. (N.Y.) **154** (2008), 270–278.

[SZ] J.T. Stafford and J.J. Zhang, *Examples in noncommutative projective geometry*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 3, 415–433.

Georgia Benkart
Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA
benkart@math.wisc.edu

Samuel A. Lopes
CMUP, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687
4169-007 Porto, Portugal slopes@fc.up.pt

Matthew Ondrus
Mathematics Department, Weber State University, Ogden, Utah, 84408 USA
mattondrus@weber.edu