Automorphisms of Generalized Down-Up Algebras

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Abstract

A generalization of down-up algebras was introduced by Cassidy and Shelton in [11], the so-called generalized down-up algebras. We describe the automorphism group of conformal Noetherian generalized down-up algebras $L(f, r, s, \gamma)$ such that r is not a root of unity, listing explicitly the elements of the group. In the last section we apply these results to Noetherian down-up algebras, thus obtaining a characterization of the automorphism group of Noetherian down-up algebras $A(\alpha, \beta, \gamma)$ for which the roots of the polynomial $X^2 - \alpha X - \beta$ are not both roots of unity.

Keywords: down-up algebra; generalized down-up algebra; automorphisms; enveloping algebra; generalized Weyl algebra.

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Introduction

Generalized down-up algebras were introduced by Cassidy and Shelton in [11] as a generalization of the down-up algebras $A(\alpha, \beta, \gamma)$ of Benkart and Roby [7]. Generalized down-up algebras include all down-up algebras, the algebras *similar to the enveloping algebra of* $\mathfrak{sl}_2$ defined by Smith [25], Le Bruyn's *conformal* $\mathfrak{sl}_2$ *enveloping algebras* [18] and Rueda's algebras *similar to the enveloping algebra of* $\mathfrak{sl}_2$ [24]. The reader is encouraged to consult [11] for further details and references.

Two of the most remarkable examples of down-up algebras are $U(\mathfrak{sl}_2)$ and $U(\mathfrak{h})$, the enveloping algebras of the 3-dimensional complex simple Lie algebra $\mathfrak{sl}_2$ and of the 3-dimensional nilpotent, non-abelian Heisenberg Lie algebra $\mathfrak{h}$, respectively. These algebras have a very rich structure and representation theory which has been extensively studied, having an unquestionable impact on the theory of semisimple and nilpotent Lie algebras. Nevertheless, a precise description of their symmetries, as given by the understanding of their automorphism group, is yet to be obtained (see [12, 13] and [16, 1]). The problem of describing the automorphism group seems to be considerably simpler when a deformation is introduced. Indeed, the automorphism group of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ was computed in [2], and in [9, 3] the authors independently described the group of automorphisms of the quantum Heisenberg algebra; in all cases it was assumed that the deformation parameter is not a root of unity. Despite these and other successful results on the description of automorphism groups of quantum algebras, e.g. [2, 3, 9, 14, 19, 20, 21], there is yet much to be done. For example, regarding the quantized enveloping algebras $U_q(\mathfrak{g}^+)$, where $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra and $\mathfrak{g}^+$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, there is a conjecture of Andruskiewitsch and Dumas [4] describing the automorphism group of $U_q(\mathfrak{g}^+)$ as a semidirect product of a torus of rank equal to the rank of $\mathfrak{g}$ by a finite group corresponding to the automorphisms of the Dynkin diagram of $\mathfrak{g}$. So far, only particular cases of this conjecture have been verified, for $\mathfrak{g}$ of rank at most 3 [9, 3, 19, 21]. Another difficulty that arises is when the deformation parameter is a root of unity. Very few results are known in this case, e.g. [2, Prop. 1.4.4, Thé. 1.4.5] and [3, Sec. I].

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It is reasonable to think of a Noetherian generalized down-up algebra as a deformation of an enveloping algebra of a 3-dimensional Lie algebra. Working over an algebraically closed field of characteristic 0, we use elementary methods to compute the automorphism groups of Noetherian generalized down-up algebras, under certain assumptions. This is the content of Theorem 2.19. Specializing, in Section 3, our results to down-up algebras, we obtain in Theorem 3.1 a complete description of the automorphism groups of all Noetherian down-up algebras $A(\alpha, \beta, \gamma)$, under the restriction that at least one of the roots of the polynomial $X^2 - \alpha X - \beta$ is not a root of unity.

1 Generalized down-up algebras

Throughout this paper, $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{K}$ denotes an algebraically closed field of characteristic 0 and $\mathbb{K}^*$ is the multiplicative group of units of $\mathbb{K}$. If A is a subset of the ring R then the two-sided ideal of R generated by A is denoted by $\langle A \rangle$; we also write $\langle x_1, \ldots, x_n \rangle$ in place of $\langle \{x_1, \ldots, x_n\} \rangle$.

Given a polynomial $f = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{K}[X]$, with all $a_i \in \mathbb{K}$, we define the support of f to be the set $\operatorname{supp}(f) = \{i \mid a_i \neq 0\}$ and the degree of f, denoted $\deg(f)$, as the supremum of $\operatorname{supp}(f)$. In particular, the zero polynomial has degree $-\infty$, the supremum of the empty set.

1.1 Preliminaries

Let $f \in \mathbb{K}[X]$ be a polynomial and fix scalars $r, s, \gamma \in \mathbb{K}$. The generalized down-up algebra $L = L(f, r, s, \gamma)$ was defined in [11] as the unital associative $\mathbb{K}$-algebra generated by d, u and h, subject to the relations:

$$dh - rhd + \gamma d = 0, \tag{1}$$
$$hu - ruh + \gamma u = 0, \tag{2}$$
$$du - sud + f(h) = 0. \tag{3}$$

When f has degree one, we retrieve all down-up algebras $A(\alpha, \beta, \gamma)$, α, β, $\gamma \in \mathbb{K}$, for suitable choices of the parameters of L. This is argued in [11, Ex. 1.2]. To correct some typos, we will explicitly construct isomorphisms between the algebras $A(\alpha, \beta, \gamma)$ and the algebras $L = L(f, r, s, \gamma)$, in case f has degree one. The reader is referred to [7] for the definition of $A(\alpha, \beta, \gamma)$.

Lemma 1.1 ([11, Ex. 1.2]). (a) *Given α, β, $\gamma \in \mathbb{K}$, let r and s be the roots of $X^2 - \alpha X - \beta$. Then,*

$$L(X, r, s, \gamma) \simeq A(\alpha, \beta, \gamma).$$

(b) *Let λ, μ, r, s, $\gamma \in \mathbb{K}$ with $\lambda \neq 0$. Then,*

$$L(\lambda X + \mu, r, s, \gamma) \simeq A(r + s, -rs, \lambda\gamma + (r-1)\mu).$$

In both cases, there is an isomorphism taking the canonical generators d and u of L to the canonical generators d and u of A, respectively. Under that isomorphism, h is sent to $sud - du$ in case (a) and to $\lambda^{-1}(sud - du - \mu)$ in case (b).

Other natural isomorphisms between generalized down-up algebras are the following, for $\lambda \in \mathbb{K}^*$:

- $L(f, r, s, \gamma) \xrightarrow{\simeq} L(f(\lambda^{-1}X), r, s, \lambda\gamma)$, where $u \mapsto u$, $d \mapsto d$, $h \mapsto \lambda^{-1}h$;
- $L(f, r, s, \gamma) \xrightarrow{\simeq} L(\lambda f, r, s, \gamma)$, where $u \mapsto \lambda^{-1}u$, $d \mapsto d$, $h \mapsto h$.

Therefore, if convenient, it can be assumed that either $f = 0$ or f is monic, and that either $\gamma = 0$ or $\gamma = 1$. An additional symmetry comes from an antiautomorphism of $L(f, r, s, \gamma)$ interchanging u and d and fixing h. Because of this antiautomorphism, one can carry over properties of the generator u to properties of d, and vice-versa.

1.2 Noetherian generalized down-up algebras

Several ring-theoretical and homological properties of L were derived by Cassidy and Shelton [11, Secs. 2, 3], and in [11, Sec. 4] they classified all simple weight modules of L under the assumption that $rs \neq 0$, which is precisely when L is a Noetherian domain. This classification was later extended by Praton [22] to the non-Noetherian case.

Let us briefly recall some of the results from [11] which we will use often. We begin with [11, Props. 2.5–2.6], which extend to L results from [17]:

Proposition 1.2. *The following conditions are equivalent:*

(a) *L is Noetherian;*

(b) *L is a domain;*

(c) *$rs \neq 0$.*

Given a ring D, an automorphism σ of D and a central element $a \in D$, the generalized Weyl algebra $D(\sigma, a)$ is the ring extension of D generated by x and y, subject to the relations:

$$xb = \sigma(b)x, \qquad by = y\sigma(b), \qquad \text{for all } b \in D; \tag{4}$$

$$yx = a, \qquad xy = \sigma(a). \tag{5}$$

Generalized Weyl algebras were introduced and studied by Bavula [5], and their properties and representation theory have been subsequently studied by himself and several other authors. If D is a Noetherian $\mathbb{K}$-algebra which is a domain, the automorphism σ is $\mathbb{K}$-linear and $a \neq 0$ then $D(\sigma, a)$ is a Noetherian domain (see [5] for example).

As occurs with down-up algebras [17], the Noetherian generalized down-up algebras can be presented as generalized Weyl algebras. In fact, set $a = ud$, let D be the commutative polynomial algebra $\mathbb{K}[h, a]$ and define the automorphism σ of D by the rules $\sigma(h) = rh - \gamma$ and $\sigma(a) = sa - f(h)$.

Lemma 1.3 ([11, Lem. 2.7])**.** *With the notation introduced above, L is isomorphic to the generalized Weyl algebra $D(\sigma, a)$, under an isomorphism taking $d \in L$ (resp. u, resp. h) to $x \in D(\sigma, a)$ (resp. y, resp. h).*

Let R be a ring and let τ be an endomorphism of R. Recall that a (left) τ-derivation of R is an additive map $\delta : R \to R$ which satisfies the relation $\delta(ab) = \tau(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Given R, τ and δ as above, we can form the skew polynomial ring $R[\theta; \tau, \delta]$. As a left R-module, $R[\theta; \tau, \delta]$ is free with basis $\{\theta^i \mid i \geq 0\}$ and the multiplication in $R[\theta; \tau, \delta]$ is determined by that of R and the rule:

$$\theta a = \tau(a)\theta + \delta(a),$$

for $a \in R$. Naturally, if τ' is an endomorphism of $R[\theta; \tau, \delta]$ and δ' is a τ'-derivation of $R[\theta; \tau, \delta]$, this construction can be repeated to obtain an iterated skew polynomial ring $R[\theta; \tau, \delta][\Phi; \tau', \delta']$, and so on.

The next remark will be useful when comparing normal elements of L generating the same ideal.

Lemma 1.4. *If $rs \neq 0$ then L is an iterated skew polynomial ring over $\mathbb{K}$ and the group of units of L is $\mathbb{K}^*$.*

Proof. We can realize L as the iterated skew polynomial ring

$$\mathbb{K}[h][d; \sigma][u; \sigma^{-1}, \delta],$$

where the automorphism σ of $\mathbb{K}[h]$ given by $\sigma(h) = rh - \gamma$ is extended to an automorphism of $\mathbb{K}[h][d; \sigma]$ by defining $\sigma(d) = sd$ and the σ-derivation δ of $\mathbb{K}[h][d; \sigma]$ is determined by the rules $\delta(h) = 0$ and $\delta(d) = s^{-1}f(h)$. Now it follows from well-known results on skew polynomial rings that the units of L are just the non-zero scalars. □

Remark 1.5. The hypothesis $rs \neq 0$ in the previous lemma is not unnecessary. For example, if $r = 0$ then the calculation

$$(1 + \gamma u + uh)(1 - \gamma u - uh) = 1 = (1 - \gamma u - uh)(1 + \gamma u + uh)$$

shows that $1 + \gamma u + uh$ is a non-scalar unit of $L = L(f, 0, s, \gamma)$.

1.3 Conformal generalized down-up algebras

Generalized down-up algebras can also be viewed as ambiskew polynomial rings (see [11, Sec. 2] and [15]). In this context, L is said to be conformal if there exists a polynomial $g \in \mathbb{K}[X]$ such that $f(X) = sg(X) - g(rX - \gamma)$. One of the advantages of L being conformal is that in this case the element $z = du - g(rh - \gamma) = s(ud - g(h))$ is normal and satisfies the relations $zh = hz$, $dz = szd$ and $zu = suz$; furthermore, z is nonzero provided $s \neq 0$.

If $f = 0$, then clearly L is conformal. Otherwise, write

$$f(X) = a_0 + a_1 X + \cdots + a_n X^n \tag{6}$$

with $a_i \in \mathbb{K}$, $n \geq 0$ and $a_n \neq 0$. Hence $\deg(f) = n$. Cassidy and Shelton [11, Lem. 2.8] give a sufficient condition for L to be conformal, namely that $s \neq r^i$ for all $0 \leq i \leq n$. As is pointed out, this condition is not necessary (take for example $f(X) = X$, $r = s = \gamma = 1$ and $g(X) = \frac{1}{2}(X^2 + X)$).

If $\gamma = 0$ it is easy to give a necessary and sufficient condition for L to be conformal. We will see shortly that, up to isomorphism, the condition $\gamma = 0$ is not very restrictive.

Lemma 1.6. *Let f be as in (6). Then $L(f, r, s, 0)$ is conformal if and only if $s \neq r^i$ for all i such that $a_i \neq 0$. In that case, a polynomial g satisfying $f(X) = sg(X) - g(rX)$ exists and is unique if we impose the additional condition that* $\operatorname{supp}(f) = \operatorname{supp}(g)$*; in particular, g can be chosen so that* $\deg(g) = \deg(f)$.

Proof. Write $g(X) = b_0 + b_1 X + \cdots + b_m X^m$, so

$$sg(X) - g(rX) = \sum_{i=0}^{m} (s - r^i) b_i X^i.$$

Thus, if $f(X) = sg(X) - g(rX)$ then $m \geq n$, $a_i = (s - r^i) b_i$ for all $0 \leq i \leq n$ and $(s - r^i) b_i = 0$ for $i > n$. In particular, $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$ and the condition that $s \neq r^i$ for all i such that $a_i \neq 0$ is necessary for L to be conformal. Moreover, if this condition is satisfied and we take

$$g(X) = \sum_{a_i \neq 0} \frac{a_i}{s - r^i} X^i,$$

then we see that L is indeed conformal with $f(X) = sg(X) - g(rX)$ and $\operatorname{supp}(f) = \operatorname{supp}(g)$. The uniqueness is clear from the construction. □

Proposition 1.7. *If $r \neq 1$ then $L(f, r, s, \gamma) \simeq L(\tilde{f}, r, s, 0)$ for some $\tilde{f} \in \mathbb{K}[X]$ of the same degree as f. Furthermore, $L(f, r, s, \gamma)$ is conformal if and only if $L(\tilde{f}, r, s, 0)$ is conformal.*

Proof. Define $\tilde{f}$ by the formula $\tilde{f}(X) = f\left(\frac{1}{r-1}(X + \gamma)\right)$. Now consider the algebra epimomorphism $\phi : \mathbb{K}\langle d, u, h\rangle \to L(\tilde{f}, r, s, 0)$ defined on the free $\mathbb{K}$-algebra on free generators d, u, h by:

$$\phi(d) = d, \qquad \phi(u) = u \qquad \text{and} \qquad \phi(h) = \frac{1}{r-1}(h + \gamma).$$

Using the relations in $L(\tilde{f}, r, s, 0)$ and the definition of $\tilde{f}$ we find that:

$$\begin{aligned}
\phi(dh - rhd + \gamma d) &= \frac{1}{r-1}d(h+\gamma) - \frac{r}{r-1}(h+\gamma)d + \gamma d \\
&= \frac{1}{r-1}\left((dh - rhd) + (1-r)\gamma d\right) + \gamma d \\
&= \frac{1-r}{r-1}\gamma d + \gamma d = 0;
\end{aligned}$$

similarly, $\phi(hu - ruh + \gamma u) = 0$; and finally

$$\begin{aligned}
\phi(du - sud + f(h)) &= du - sud + f\left(\frac{1}{r-1}(h+\gamma)\right) \\
&= du - sud + \tilde{f}(h) = 0.
\end{aligned}$$

Therefore, by (1)–(3), ϕ induces an algebra epimorphism, still denoted ϕ, $L(f, r, s, \gamma) \to L(\tilde{f}, r, s, 0)$. To conclude that this map is an isomorphism it is enough to proceed similarly and define an algebra map $\psi : L(\tilde{f}, r, s, 0) \to L(f, r, s, \gamma)$ satisfying:

$$\psi(d) = d, \qquad \psi(u) = u \qquad \text{and} \qquad \psi(h) = (r-1)h - \gamma.$$

The maps ϕ and ψ are mutual inverses.

For $g \in \mathbb{K}[X]$ define $\tilde{g}$ by $\tilde{g}(X) = g\left(\frac{1}{r-1}(X+\gamma)\right)$. The last statement follows because the equations $f(X) = sg(X) - g(rX - \gamma)$ and $\tilde{f}(X) = s\tilde{g}(X) - \tilde{g}(rX)$ are equivalent in $\mathbb{K}[X]$. □

In view of Lemma 1.6 and Proposition 1.7, it remains to determine when $L(f, 1, s, \gamma)$ is conformal, which is what we do next.

Proposition 1.8. *$L(f, 1, s, \gamma)$ is conformal in all cases except the case $L(f, 1, 1, 0)$ with $f \neq 0$.*

Proof. Suppose $s \neq 1$. Then by [11, Lem. 2.8] $L(f, 1, s, \gamma)$ is conformal ($s \neq r^i$ for all i). Also, $L(f, 1, s, \gamma)$ is conformal if $f = 0$. Hence, if $L(f, 1, s, \gamma)$ is not conformal then $s = 1$ and $f \neq 0$.

Suppose first that $s = 1$, $f \neq 0$ and $\gamma = 0$. Then the conformality condition becomes $f(X) = g(X) - g(X) = 0$, so indeed $L(f, 1, 1, 0)$ is not conformal for $f \neq 0$. It remains to show that $L(f, 1, 1, \gamma)$ is conformal if $f \neq 0$ and $\gamma \neq 0$. This amounts to showing that the linear map $\mathbb{K}[X] \to \mathbb{K}[X]$ defined by $g(X) \mapsto g(X) - g(X - \gamma)$ is onto. A routine induction on n shows that X^n is in the image of this map for all $n \geq 0$, so the map is indeed onto . □

Remark 1.9. The notion of conformality is not invariant under isomorphism. For example, there is an isomorphism $L(X, 1, 2, 1) \to L(X + 1, 2, 1, 0)$, taking d to d, u to u and h to $ud + h + 1$. Nevertheless, $L(X, 1, 2, 1)$ is conformal, by Proposition 1.8, whereas by Lemma 1.6, $L(X+1, 2, 1, 0)$ is not conformal.

1.4 The $\mathbb{Z}$-grading

Given the defining relations (1)–(3), there is a $\mathbb{Z}$-grading of L obtained by assigning to the generators d, u and h the degrees -1, 1 and 0, respectively [11, Sec. 4]. We thus get a decomposition $L = \bigoplus_{i \in \mathbb{Z}} L_i$ of L into homogeneous subspaces. Whenever $rs \neq 0$ these are easy to describe, either by using the isomorphism $L \simeq D(\sigma, a)$ of Lemma 1.3, or by invoking [11, Prop. 4.1]:

Proposition 1.10. *Assume $rs \neq 0$. Then $L_0 = D = \mathbb{K}[h, a]$ is the commutative polynomial algebra generated by h and $a = ud$, $L_{-i} = Dd^i = d^i D$ and $L_i = Du^i = u^i D$, for $i > 0$.*

This result has some interesting consequences, as the next Corollary shows. We recall the reader that an element t of a ring R is said to be normal if $tR = Rt$.

Corollary 1.11. *Assume $rs \neq 0$ and let $t \in L_i$, for some $i \in \mathbb{Z}$. Recall the automorphism σ of D defined just before Lemma 1.3 by $\sigma(h) = rh - \gamma$ and $\sigma(a) = sa - f(h)$. Then:*

(a) $pt = t\sigma^i(p)$ *for all* $p \in D$;

(b) *If* t *is also assumed to be normal then there exist* $\lambda, \mu \in \mathbb{K}^*$ *such that* $td = \lambda dt$ *and* $tu = \mu ut$.

Proof. Let $t \in L_i$ and suppose $i \geq 0$. By Proposition 1.10, there exists $b \in D$ such that $t = bu^i$. Then, for $p \in D$, $pt = pbu^i = bpu^i = bu^i\sigma^i(p) = t\sigma^i(p)$.

Assume further that t is normal and nonzero. There exist $\xi, \zeta \in L$ satisfying $td = \xi t$ and $dt = t\zeta$. Since td and dt are homogeneous of degree $i-1$, the elements ξ and ζ must also be homogeneous of degree -1, as L is a domain. Thus, there are $p, q \in D$ so that $\xi = pd$ and $\zeta = qd$. The computation

$$td = \xi t = pdt = pt\zeta = ptqd = t\sigma^i(p)qd$$

implies $\sigma^i(p)q = 1$. So $\sigma^i(p)$ and q are units of D. In particular, $p = \lambda \in \mathbb{K}^*$ as σ is an automorphism of D. Then, $td = \lambda dt$. Similarly, $tu = \mu ut$.

The proof of the case $i < 0$ is symmetric. □

2 Automorphisms of generalized down-up algebras

In this section we will describe the group of automorphisms of the Noetherian, conformal generalized down-up algebras $L(f, r, s, \gamma)$, under the additional assumption that the parameter r is not a root of unity. As $r \neq 1$, it can be assumed by Proposition 1.7 that $\gamma = 0$ and that there is $g \in \mathbb{K}[X]$ satisfying $f(X) = sg(X) - g(rX)$. Recalling Lemma 1.6, it can be further assumed that $\text{supp}\,(f) = \text{supp}\,(g)$, so that g is uniquely determined by f; in particular, $\deg(f) = \deg(g)$. Hence, for the remainder of Section 2 we assume $\gamma = 0$.

It will be more convenient for us to use the generalized Weyl algebra approach. Let $a = ud$ and $k = a - g(h)$. Then h and k are generators of the polynomial algebra $D = \mathbb{K}[h, a]$ and the automorphism σ acts on k by

$$\sigma(k) = \sigma(a - g(h)) = sa - f(h) - g(rh) = sa - sg(h) = sk.$$

Therefore, L is presented as the generalized Weyl algebra $D(\sigma, k + g(h))$, where $D = \mathbb{K}[h, k]$ and σ is the automorphism of D defined by $\sigma(h) = rh$, $\sigma(k) = sk$. The relations are thus:

$$xp(h, k) = p(rh, sk)x, \qquad p(h, k)y = yp(rh, sk), \qquad \text{for all } p \in D, \tag{7}$$

$$yx = k + g(h), \qquad xy = sk + g(rh). \tag{8}$$

The parameters $r, s \in \mathbb{K}$ satisfy $rs \neq 0$ and $r^i = 1 \iff i = 0$. The connection between $D(\sigma, k + g(h))$ and $L(f, r, s, 0)$ is given by the isomorphism $h \mapsto h$, $k \mapsto ud - g(h)$, $x \mapsto d$, $y \mapsto u$.

2.1 The center of L

Define $\epsilon \in \mathbb{Z}$ and $\tau \in \mathbb{N}$ by

$$\begin{aligned} \tau =& \min\{i > 0 \mid s^i = r^j \quad \text{for some } j \in \mathbb{Z}\} \qquad \text{and} \qquad r^\epsilon = s^\tau \\ & \text{if} \quad \{i > 0 \mid s^i = r^j \quad \text{for some } j \in \mathbb{Z}\} \neq \emptyset, \\ \tau =& 0 = \epsilon \quad \text{otherwise.} \end{aligned}$$

Since r is not a root of unity, ϵ is uniquely defined.

The next lemma is a routine exercise.

Lemma 2.1. *Let* $\delta, \eta \in \mathbb{Z}$. *Then*

$$r^\delta s^\eta = 1 \iff (\delta, \eta) = \lambda(-\epsilon, \tau) \quad \textit{for some } \lambda \in \mathbb{Z}.$$

Proposition 2.2. *The center* $\mathbb{Z}(L)$ *of* L *is* $\mathbb{K}$ *if either* $\tau = 0$ *or* $\epsilon > 0$, *and it is the polynomial algebra* $\mathbb{K}[h^{-\epsilon}k^\tau]$ *if* $\tau > 0$ *and* $\epsilon \leq 0$.

Proof. As the canonical generators of L are homogeneous with respect to the $\mathbb{Z}$-grading defined in 1.4, it follows that $\mathbb{Z}(L)$ is graded. This means that if z is central and $z = z_{i_1} + \cdots + z_{i_m}$ is the decomposition of z into homogeneous components, then each of the z_{i_j} is itself central. So we just need to determine $\mathbb{Z}(L) \cap L_i$ for all $i \in \mathbb{Z}$.

Let $i \leq 0$ and take $px^i \in \mathbb{Z}(L) \cap L_i$, with p a nonzero element of D. Then, since L is a domain and r is not a root of 1, the computation

$$0 = hpx^i - px^i h = hpx^i - r^i phx^i = (1 - r^i)hpx^i$$

implies that $i = 0$. The situation is identical if we take $i \geq 0$; hence $\mathbb{Z}(L) \subseteq D$.

Now take $p = p(h, k) \in D$. Again we compute:

$$xp(h, k) - p(h, k)x = (p(rh, sk) - p(h, k))x,$$

and likewise for y. Therefore, as p commutes with h and k, p is central if and only if $p(rh, sk) = p(h, k)$. Write $p = \sum a_{ij} h^i k^j$. Thus $p \in \mathbb{Z}(L)$ precisely when $r^i s^j = 1$ whenever $a_{ij} \neq 0$. In view of Lemma 2.1, this condition means that $(i, j) = \lambda(-\epsilon, \tau)$ for some $\lambda \in \mathbb{Z}$.

If $\tau > 0$ and $\epsilon \leq 0$ then $\lambda \geq 0$ as $\lambda\tau = j \geq 0$ and thus $h^i k^j = (h^{-\epsilon} k^{\tau})^{\lambda}$. In this case $p \in \mathbb{K}[h^{-\epsilon} k^{\tau}]$ and $\mathbb{Z}[L] = \mathbb{K}[h^{-\epsilon} k^{\tau}]$. Otherwise either $\tau, \epsilon > 0$ or $\tau = 0 = \epsilon$. In the first of these cases λ must be zero and $(i, j) = (0, 0)$; in the second case $(i, j) = (0, 0)$ as well. Thus $\mathbb{Z}(L) = \mathbb{K}$. □

2.2 The normal elements of L

We start out by classifying the normal elements of L of degree zero.

Lemma 2.3. *Let $p \in D$ be a nonzero normal element of L. Write $p(h, k) = h^{\alpha} k^{\beta} q(h, k)$, with $\alpha, \beta \in \mathbb{N}$ and $q \in D$ not a multiple of h or k. Then:*

(a) *$q \in \mathbb{K}^*$, if $\tau = 0$;*

(b) *$q \in \mathbb{Z}(L)$ and q has a nonzero constant term as a polynomial in $(h^{-\epsilon} k^{\tau})$, if $\tau > 0$ and $\epsilon \leq 0$;*

(c) *$q = \sum_{i=0}^{l} d_i (h^{\epsilon})^{l-i} (k^{\tau})^i$ with $l \geq 0$, $d_i \in \mathbb{K}$ and $d_0, d_l \neq 0$, if $\tau, \epsilon > 0$.*

Proof. Since h and k are themselves normal and L is a domain, it follows that q is normal, and nonzero. Write $q(h, k) = \sum_{i=0}^{l} q_i(h) k^i$ with $l \geq 0$, $q_i(h) \in \mathbb{K}[h]$ and $q_l(h) \neq 0$.

By Corollary 1.11 there exists $\lambda \in \mathbb{K}^*$ such that $xq = \lambda qx$. Thus

$$\lambda \left(\sum_{i=0}^{l} q_i(h) k^i \right) x = \lambda qx = xq = \sigma(q)x = \left(\sum_{i=0}^{l} s^i q_i(rh) k^i \right) x$$

and we conclude that

$$\lambda q_i(h) = s^i q_i(rh), \qquad \text{for all } 0 \leq i \leq l. \tag{9}$$

Now fix i and write $q_i(h) = \sum_j \alpha_j h^j$. If $\alpha_j \neq 0$ then (9) implies that $r^j = \lambda s^{-i}$. As r is not a root of 1, $j = n_i$ is determined by i and $q_i(h) = a_i h^{n_i}$, for some $a_i \in \mathbb{K}$.

So far we have

$$q(h, k) = \sum_{i=0}^{l} a_i h^{n_i} k^i.$$

As q is not a multiple of k, it must be that $a_0 \neq 0$ and consequently $\lambda = r^{n_0}$. Therefore, for every i such that $a_i \neq 0$, we have $r^{n_i - n_0} s^i = 1$. By Lemma 2.1, there is $T_i \in \mathbb{Z}$ such that

$$n_i - n_0 = -\epsilon T_i \qquad \text{and} \qquad i = \tau T_i. \tag{10}$$

To finish our argument, we just need to distinguish between the three possibilities for the pair (ϵ, τ) and use (10).

If $\tau = 0$ then necessarily $i = 0$ and $q = a_0 h^{n_0}$. Also, n_0 must be zero so that q is not a multiple of h. This establishes (a).

If $\tau > 0$ and $\epsilon \leq 0$ then $T_i \geq 0$ and $n_i = n_0 - \epsilon T_i \geq n_0$. Hence $n_0 = 0$, to ensure that q is not a multiple of h, and $h^{n_i} k^i = (h^{-\epsilon} k^{\tau})^{T_i} \in \mathbb{Z}(L)$. So q is indeed central and the constant term of q when written as a polynomial in $(h^{-\epsilon} k^{\tau})$ must be nonzero, or otherwise q would be a multiple of k.

Let us analyze the final case with $\tau > 0$ and $\epsilon > 0$. Again, $T_i \geq 0$ by (10). Moreover, there is $0 \leq i \leq l$ such that $a_i \neq 0$ and $n_i = 0$, by the condition that q is not a multiple of h. It follows from (10) that ϵ divides n_0, say $n_0 = \epsilon m$. Hence, for all i such that $a_i \neq 0$, $h^{n_i} k^i = (h^{\epsilon})^{m - T_i} (k^{\tau})^{T_i}$. In particular, $0 \leq T_i \leq m$ and q can be written as

$$q(h,k) = \sum_{i=0}^{m} d_i \, (h^{\epsilon})^{m-i} \, (k^{\tau})^i$$

with $d_i \in \mathbb{K}$ and $d_0, d_m \neq 0$, ensuring q is neither a multiple of k nor of h.

□

Our next step in describing the monoid of normal elements of L is to determine when x^n and y^n are normal.

Lemma 2.4. *The following conditions are equivalent, for $n \geq 1$:*

- (a) *x^n is normal;*
- (b) *Either $f = 0$ or $f(X) = \mu(s - r^m)X^m$ for some $\mu \in \mathbb{K}^*$ and some $m \geq 0$ so that $\epsilon = \tau m$ and τ divides n;*
- (c) *$yx^n = s^{-n} x^n y$;*
- (d) *$xy^n = s^n y^n x$;*
- (e) *y^n is normal.*

In particular, if x^n is normal then either $f = 0$ or $\tau, n > 1$.

Proof. The algebra antiautomorphism interchanging x and y referred to at the end of Section 1.1 proves the equivalence of statements (a) and (e) and of statements (c) and (d). It remains to show the series of implications: $(a) \implies (b) \implies (c) \implies (a)$.

So assume x^n is normal, for some $n \geq 1$. By Corollary 1.11 there exists $\lambda \in \mathbb{K}^*$ such that $yx^n = \lambda x^n y$. Hence,

$$(k + g(h))x^{n-1} = yx^n = \lambda x^n y = \lambda x^{n-1}(sk + g(rh)) = \lambda (s^n k + g(r^n h)) x^{n-1},$$

from which the following equality in D is deduced: $k + g(h) = s^n \lambda k + \lambda g(r^n h)$. Comparing coefficients of k in this last equation yields $s^n \lambda = 1$ and $\lambda g(r^n h) = g(h)$. Thus,

$$g(r^n h) = s^n g(h). \tag{11}$$

Since r is not a root of 1, there exist $\mu \in \mathbb{K}$ and $m \in \mathbb{N}$ so that $g(X) = \mu X^m$ and $f(X) = \mu(s - r^m)X^m$.

If $\mu = 0$ then $0 = g = f$. Otherwise, assume $\mu \neq 0$. Then condition (11) translates to $r^{nm} = s^n$. By Lemma 2.1 we have

$$nm = \epsilon T \quad \text{and} \quad n = \tau T, \qquad \text{for some } T \in \mathbb{Z}.$$

In particular, τ divides n and $\epsilon = \tau m$. Notice that, in this case, we cannot have $n = 1$, as this would imply $\tau = 1$, $\epsilon = m$ and $f(X) = 0$, contrary to our supposition. Hence both integers n and τ must be greater than 1, if $f \neq 0$.

Now assume (b) holds. If $f = 0$ then $g = 0$ and by (8), $xy = sk = syx$. It follows that $x^n y = s^n y x^n$. Instead, suppose $g(X) = \mu X^m$ for $\mu \in \mathbb{K}^*$ and $m \in \mathbb{N}$. As τ divides n, it is enough to show that $x^\tau y = s^\tau y x^\tau$. This is indeed a true statement as, by hypothesis, $r^{\tau m} = r^\epsilon = s^\tau$:

$$\begin{aligned} x^\tau y &= x^{\tau-1}(sk + g(rh)) = (s^\tau k + g(r^\tau h))x^{\tau-1} \\ &= (s^\tau k + r^{\tau m} g(h))x^{\tau-1} = s^\tau (k + g(h))x^{\tau-1} = s^\tau y x^\tau. \end{aligned}$$

In either case, (c) holds.

Finally, if (c) holds then clearly x^n is normal, by (7). □

We are finally ready to describe all normal elements of L.

Proposition 2.5. *The normal elements of L are the elements of the form $p(h, k)x^n$ and $p(h, k)y^n$, with $n \geq 0$ and $p(h, k) \in D$ such that $p(h, k)$, x^n and y^n are normal.*

Proof. Since the product of normal elements is normal, it is clear that all of the indicated elements are normal. Conversely, let $0 \neq t \in L$ be normal. Write $t = \sum_{j \in J} t_j$ with J a finite nonempty subset of $\mathbb{Z}$ and $0 \neq t_j \in L_j$.

Claim: t is homogeneous, i.e., $|J| = 1$.

Proof of claim: By the normality of t, $ht = tt'$ for some $t' \in L$. The $\mathbb{Z}$-grading of L, together with the fact that L is a domain, imply that $t' \in D$. Note that by Corollary 1.11(a), $ht_j = r^j t_j h$ for all $j \in J$. Thus,

$$\sum_{j \in J} r^j t_j h = \sum_{j \in J} ht_j = ht = tt' = \sum_{j \in J} t_j t'.$$

Using again the $\mathbb{Z}$-grading and the fact that L is a domain, we infer that $r^j h = t'$, for all $j \in J$. So, as claimed, $|J| = 1$ because r is not a root of unity.

We can assume, without loss of generality, that $t = p(h, k)x^n \in L_{-n}$, $n \in \mathbb{N}$, the case $t \in L_n$ being symmetric. As t is homogeneous, Corollary 1.11(b) can be invoked to guarantee the existence of $\lambda \in \mathbb{K}^*$ satisfying $xt = \lambda tx$. Working out this equation in L leads to the equivalent equation $p(rh, sk) = \lambda p(h, k)$ in D. Hence $xp(h, k) = \lambda p(h, k)x$ and $yp(h, k) = \lambda^{-1} p(h, k)y$, showing that $p(h, k)$ is normal in L.

Finally, to prove that x^n is normal we use Corollary 1.11(b) once more: there is $\mu \in \mathbb{K}^*$ such that $ty = \mu yt$. So

$$p(h, k)x^n y = ty = \mu yt = \mu y p(h, k)x^n = \mu \lambda^{-1} p(h, k) y x^n$$

and $x^n y = \mu \lambda^{-1} y x^n$. Thus x^n is normal as well, by (7). □

Combining Proposition 2.5 with Lemma 2.3 and Lemma 2.4, we obtain a complete description of all normal elements of L. Before we end this section, we record a straightforward, yet useful, result, which holds in any domain if we replace $\mathbb{K}^*$ by its group of units.

Lemma 2.6. *Assume $t, v \in L$ are nonzero normal elements whose product generates a prime ideal of L. Then either $t \in \mathbb{K}^*$ or $v \in \mathbb{K}^*$.*

2.3 Some properties of the automorphisms of L

In this section we gather some general information about the automorphisms of L. We denote the group of algebra automorphisms of L by $\mathrm{Aut}_{\mathbb{K}}(L)$.

Recall that a proper ideal P of a ring R is said to be completely prime if the factor ring R/P is a domain. In particular, completely prime ideals are prime.

Lemma 2.7. *Let $\phi \in \mathrm{Aut}_{\mathbb{K}}(L)$. Then $\phi(h)$ is a normal element of L which generates a completely prime ideal of L.*

Proof. It needs to be shown that h is normal and generates a completely prime ideal of L, as these two properties are invariant by automorphisms. The first one is clear, as h is central in D and $\sigma(h) = rh$ (see (7)). To prove that $\langle h \rangle$ is completely prime we need to argue that the factor algebra $L/\langle h \rangle$ is a domain.

By relations (1)–(3), with $\gamma = 0$, $L/\langle h \rangle$ is the algebra generated by $\bar{x}$ and $\bar{y}$, subject only to the relation $\bar{x}\bar{y} - s\bar{y}\bar{x} + f_0 = 0$, f_0 being the constant term of the polynomial f. There are four possibilities, depending on the scalars s and f_0. If $s = 1$ we are in the classical setting and the factor algebra is either a commutative polynomial algebra in two variables ($f_0 = 0$) or the first Weyl algebra over $\mathbb{K}$ ($f_0 \neq 0$). If $s \neq 1$ we are in the quantum setting. Recalling that we are assuming also $s \neq 0$, the factor algebra is either a quantum plane ($f_0 = 0$) or the first quantum Weyl algebra ($f_0 \neq 0$). Any of these four algebras is a domain, so the ideal $\langle h \rangle$ is completely prime. □

Lemma 2.8. *Assume $n \geq 1$. Then x^n (resp. y^n) is normal and generates a completely prime ideal of L if and only if $n = 1$ and $f = 0$.*

Proof. If $f = 0$ then x is normal, by Lemma 2.4. In this case, the ideal $\langle x \rangle$ is completely prime since the factor algebra $L/\langle x \rangle$ is easily seen to be a quantum plane, generated by $\bar{h}$ and $\bar{y}$, satisfying the relation $\bar{h}\bar{y} = r\bar{y}\bar{h}$.

Conversely, assume that x^n is normal and that the ideal $\langle x^n \rangle = x^n L = Lx^n$ is completely prime. Suppose, by way of contradiction, that $n > 1$. Then, since $xx^{n-1} \in Lx^n$, it must be that either $x \in Lx^n$ or $x^{n-1} \in Lx^n$. In any case, $x^{n-1} \in Lx^n$, as $n - 1 \geq 1$, and there is $v \in L$ so that $1 = vx$ because L is a domain. Similarly, there is $v' \in L$ so that $1 = xv'$. This is a contradiction because x is not a unit in L, by Lemma 1.4. Therefore $n = 1$. By Lemma 2.4, $f = 0$. □

Lemma 2.9. *Let $t, v \in D \setminus \{0\}$. Suppose $xt = \lambda tx$ and $xv = \mu vx$, for some $\lambda, \mu \in \mathbb{K}^*$. If there is $\phi \in \mathrm{Aut}_{\mathbb{K}}(L)$ such that $\phi(t) = v$ then $\langle\langle \lambda \rangle\rangle = \langle\langle \mu \rangle\rangle$, where $\langle\langle \xi \rangle\rangle$ denotes the subgroup of $\mathbb{K}^*$ generated by ξ.*

Proof. Replacing ϕ by ϕ^{-1}, it is enough to show that $\langle\langle \lambda \rangle\rangle \subseteq \langle\langle \mu \rangle\rangle$.

If we multiply both sides of equation $xv = \mu vx$ on the left by y, observe that $yx \in D$ commutes with v and use the fact that L is a domain, we obtain $vy = \mu yv$. Therefore,

$$vl = \mu^i lv, \qquad \text{for all } l \in L_i. \tag{12}$$

Let us write $\phi(x) = \sum_j x_j$, with $x_j \in L_j$. Applying ϕ to equation $xt = \lambda tx$ and using (12), we deduce the following: $\sum_j x_j v = \sum_j \lambda\mu^j x_j v$. By the $\mathbb{Z}$-grading, $\lambda\mu^j = 1$ for all $j \in \mathbb{Z}$ such that $x_j \neq 0$. Since $\phi(x) \neq 0$, there is $i \in \mathbb{Z}$ with $\lambda = \mu^i$. Thus $\lambda \in \langle\langle \mu \rangle\rangle$ and $\langle\langle \lambda \rangle\rangle \subseteq \langle\langle \mu \rangle\rangle$, as desired. □

2.4 The automorphisms of L

Now we describe the algebra automorphisms of L in detail. The results of this section will be combined in the next section to determine the group $\mathrm{Aut}_{\mathbb{K}}(L)$.

Recall that $f(X) = sg(X) - g(rX)$. In case $f \neq 0$, we define a nonnegative integer ρ by

$$\rho = \gcd\{\deg(f) - i \mid i \in \mathrm{supp}\,(f)\}$$

if $\{\deg(f) - i \mid i \in \mathrm{supp}\,(f)\} \neq \{0\}$, and $\rho = 0$ otherwise. We do not define ρ if $f = 0$.

Lemma 2.10. *Let $\lambda, \mu \in \mathbb{K}$ with $\lambda \neq 0$. Then $f(\lambda X) = \mu f(X) \iff$ either $f = 0$, or $\lambda^\rho = 1$ and $\lambda^{\deg(f)} = \mu$.*

Proof. Notice that $f(\lambda X) = \mu f(X) \iff \lambda^i = \mu$ for all $i \in \mathrm{supp}\,(f)$. If $f = 0$ this condition is clearly satisfied, so assume $\lambda^\rho = 1$ and $\lambda^{\deg(f)} = \mu$. Given $i \in \mathrm{supp}\,(f)$ we have, by the definition of ρ, $\lambda^{\deg(f)-i} = 1$. Hence, $\mu = \lambda^{\deg(f)} = \lambda^i$.

Conversely, assume that $f(\lambda X) = \mu f(X)$ and $f \neq 0$. Then, in particular, $\lambda^{\deg(f)} = \mu$. If $\rho = 0$ there is nothing else to prove. Assume $\rho \neq 0$. By hypothesis, $\lambda^i = \mu = \lambda^{\deg f}$ for all $i \in \mathrm{supp}\,(f)$. As $\lambda \neq 0$ we have $\lambda^{\deg(f)-i} = 1$ for all $i \in \mathrm{supp}\,(f)$. Hence $\lambda^\rho = 1$. □

Consider the following subgroup of $\mathrm{Aut}_{\mathbb{K}}(L)$:

$$\mathcal{H} = \{\phi \in \mathrm{Aut}_{\mathbb{K}}(L) \mid \phi(h) = \lambda h, \text{ for some } \lambda \in \mathbb{K}^*\}.$$

Lemma 2.11. *The following define elements of* $\mathcal{H}$*:*

(a) *If* $f = 0$ *there is a unique* $\phi_{(\alpha,\beta,\gamma)} \in \mathcal{H}$ *defined on the generators by* $\phi_{(\alpha,\beta,\gamma)}(h) = \alpha h$, $\phi_{(\alpha,\beta,\gamma)}(x) = \beta x$, $\phi_{(\alpha,\beta,\gamma)}(y) = \gamma y$, *for any* $(\alpha, \beta, \gamma) \in (\mathbb{K}^*)^3$*;*

(b) *If* $f \neq 0$ *there is a unique* $\phi_{(\alpha,\beta)} \in \mathcal{H}$ *defined on the generators by* $\phi_{(\alpha,\beta)}(h) = \alpha h$, $\phi_{(\alpha,\beta)}(x) = \beta x$, $\phi_{(\alpha,\beta)}(y) = \beta^{-1}\alpha^{\deg(f)}y$, *for any* $(\alpha, \beta) \in (\mathbb{K}^*)^2$ *such that* $\alpha^\rho = 1$.

Proof. The uniqueness is clear, as L is generated by h, x and y. To prove the existence, we need to check that relations (1)–(3), with $\gamma = 0$ and d (resp. u) replaced by x (resp. y), are preserved when we define the homomorphism on the free algebra on generators h, x and y, and to argue the existence of an inverse.

If $f = 0$ then the relations are homogeneous in the generators and hence $\phi_{(\alpha,\beta,\gamma)}$ is indeed an automorphism, with inverse $\phi_{(\alpha^{-1},\beta^{-1},\gamma^{-1})}$.

Now consider the case $f \neq 0$. Relations $xh = rhx$ and $hy = ryh$ are trivial to check. When we apply $\phi_{(\alpha,\beta)}$ to $xy - syx + f(h)$ we obtain $\alpha^{\deg(f)}(xy - syx) + f(\alpha h)$. Thus, we must have $f(\alpha h) = \alpha^{\deg(f)}f(h)$ for $\phi_{(\alpha,\beta)}$ to be a homomorphism of L. By Lemma 2.10, this is indeed the case, as we have the additional restriction that $\alpha^\rho = 1$. Furthermore, $\phi^{-1}_{(\alpha,\beta)} = \phi_{(\alpha^{-1},\beta^{-1})}$. □

Our next result describes the group $\mathcal{H}$.

Proposition 2.12. (a) *If* $f = 0$ *then* $\mathcal{H} = \{\phi_{(\alpha,\beta,\gamma)} \mid (\alpha, \beta, \gamma) \in (\mathbb{K}^*)^3\} \simeq (\mathbb{K}^*)^3$*, with* $\phi_{(\alpha,\beta,\gamma)}$ *as given in Lemma 2.11.*

(b) *If* $f \neq 0$ *then* $\mathcal{H} = \{\phi_{(\alpha,\beta)} \mid (\alpha, \beta) \in (\mathbb{K}^*)^2$ *and* $\alpha^\rho = 1\}$*, with* $\phi_{(\alpha,\beta)}$ *as given in Lemma 2.11. Consequently,* $\mathcal{H} \simeq (\mathbb{K}^*)^2$ *if* $\rho = 0$ *and* $\mathcal{H} \simeq \mathbb{Z}/\rho\mathbb{Z} \times \mathbb{K}^*$ *if* $\rho > 0$.

Proof. Let $\phi \in \mathrm{Aut}_{\mathbb{K}}(L)$ with $\phi(h) = \alpha h$, for some $\alpha \in \mathbb{K}^*$. For $i \in \mathbb{Z}$ and $l \in L_i$, we have the relation $hl = r^i lh$. Upon applying ϕ to this relation and dividing by α we obtain the relation $h\phi(l) = r^i\phi(l)h$. Given the $\mathbb{Z}$-grading and since L is a domain and r is not a root of unity, it is routine to conclude that $\phi(L_i) \subseteq L_i$. Moreover, as ϕ is onto it follows that $\phi(L_i) = L_i$, for all $i \in \mathbb{Z}$.

Take t, $v \in D$ so that $\phi(x) = tx$ and $\phi(vx) = x$. Then, $x = \phi(vx) = \phi(v)\phi(x) = \phi(v)tx$ and thus $\phi(v)t = 1$. Since both $\phi(v)$ and t are elements of D, the latter implies that t is a unit. So $\phi(x) = \beta x$, for some $\beta \in \mathbb{K}^*$; similarly, $\phi(y) = \gamma y$, for some $\gamma \in \mathbb{K}^*$.

If $f = 0$ then $\phi = \phi_{(\alpha,\beta,\gamma)}$, as described in Lemma 2.11(a), and the map $(\mathbb{K}^*)^3 \to \mathcal{H}$, $(\alpha, \beta, \gamma) \mapsto \phi_{(\alpha,\beta,\gamma)}$ is a group isomorphism.

Now suppose $f \neq 0$. Applying ϕ to both sides of the relation $xy - syx + f(h) = 0$ yields $\beta\gamma(xy - syx) + f(\alpha h) = 0$, which is equivalent in L to $\beta\gamma f(h) = f(\alpha h)$. Since the elements $\{h^j\}_{j\geq 0}$ are linearly independent over $\mathbb{K}$, we have $\beta\gamma f(X) = f(\alpha X)$. Thus, by Lemma 2.10, $\alpha^\rho = 1$ and $\beta\gamma = \alpha^{\deg(f)}$. So $\phi = \phi_{(\alpha,\beta)}$.

If $\rho = 0$ then α and $\beta \in \mathbb{K}^*$ are arbitrary and $(\mathbb{K}^*)^2 \to \mathcal{H}$, $(\alpha, \beta) \mapsto \phi_{(\alpha,\beta)}$ is a group isomorphism. Otherwise, if $\rho \geq 1$, let $\xi \in \mathbb{K}$ be a primitive ρ-th root of unity. Then the multiplicative group $\{\alpha \in \mathbb{K}^* \mid \alpha^\rho = 1\} = \{\xi^i \mid 0 \leq i \leq \rho - 1\}$ is isomorphic to the additive group $\mathbb{Z}/\rho\mathbb{Z}$ of integers modulo ρ and $\mathbb{Z}/\rho\mathbb{Z} \times \mathbb{K}^* \to \mathcal{H}$, $(i + \rho\mathbb{Z}, \beta) \mapsto \phi_{(\xi^i,\beta)}$ is a group isomorphism. □

Now we turn our attention to automorphisms of L not necessarily fixing the ideal $\langle h \rangle$.

Lemma 2.13. *Assume* $\tau > 0$ *and* $f(X) = \alpha X + \beta$ *for some* $\alpha, \beta \in \mathbb{K}$*. The following define automorphisms of* L*:*

(a) *If* $\epsilon = 1$ *and* $\alpha = 0$ *then there is a unique* $\psi^+_{(\mu,\mu',\nu,\eta)} \in \mathrm{Aut}_{\mathbb{K}}(L)$ *so that* $\psi^+_{(\mu,\mu',\nu,\eta)}(x) = \mu x$, $\psi^+_{(\mu,\mu',\nu,\eta)}(y) = \mu' y$ *and* $\psi^+_{(\mu,\mu',\nu,\eta)}(h) = \nu h + \eta k^\tau$, *for all* $(\mu, \mu', \nu, \eta) \in (\mathbb{K}^*)^3 \times \mathbb{K}$ *with* $\beta(\mu\mu' - 1) = 0$.

(b) *If* $\tau = 1$, $\epsilon = -1$ *and* $\alpha \neq 0$ *then there is a unique* $\psi^-_{(\mu,\nu)} \in \mathrm{Aut}_{\mathbb{K}}(L)$ *so that* $\psi^-_{(\mu,\nu)}(x) = \mu y$, $\psi^-_{(\mu,\nu)}(y) = \frac{r\alpha\nu}{\mu(s-r)} x$ *and* $\psi^-_{(\mu,\nu)}(h) = \nu k$, *for all* $(\mu, \nu) \in (\mathbb{K}^*)^2$ *with* $\beta(\frac{r\alpha\nu}{s-r} - 1) = 0$.

Proof. As in the proof of Lemma 2.11, the uniqueness is clear, and the existence follows from checking that relations (1)–(3) are preserved and from the construction of an inverse homomorphism.

Suppose first that $\epsilon = 1$, $\alpha = 0$ and the scalars μ, μ', ν, $\eta \in \mathbb{K}$ satisfy $\mu\mu'\nu \neq 0$ and $\beta(\mu\mu' - 1) = 0$. In this case, $f(X) =$ is a constant polynomial and

$$\begin{aligned}\psi^+_{(\mu,\mu',\nu,\eta)}(xh - rhx) &= \mu\left(x(\nu h + \eta k^\tau) - r(\nu h + \eta k^\tau)x\right)\\ &= 0, \qquad \text{as } r = s^\tau;\end{aligned}$$

$$\begin{aligned}\psi^+_{(\mu,\mu',\nu,\eta)}(hy - ryh) &= \mu'\left((\nu h + \eta k^\tau)y - ry(\nu h + \eta k^\tau)\right)\\ &= 0, \qquad \text{as } r = s^\tau;\end{aligned}$$

$$\begin{aligned}\psi^+_{(\mu,\mu',\nu,\eta)}(xy - syx + f(h)) &= \mu\mu'(xy - syx) + \beta\\ &= \mu\mu'\left(xy - syx + \beta\right), \qquad \text{as } \beta = \beta\mu\mu',\\ &= 0.\end{aligned}$$

Thus $\psi^+_{(\mu,\mu',\nu,\eta)}$ does indeed define an algebra endomorphism of L. Note also that $g(X) = \frac{\beta}{s-1}$ and thus $\psi^+_{(\mu,\mu',\nu,\eta)}(k) = \psi^+_{(\mu,\mu',\nu,\eta)}(yx - g(h)) = \mu\mu'(yx - g(h)) = \mu\mu' k$. Finally, we can check that $\psi^+_{(\mu^{-1},\mu'^{-1},\nu^{-1},\frac{-\eta}{(\mu\mu')^\tau\nu})}$ is the inverse of $\psi^+_{(\mu,\mu',\nu,\eta)}$.

Now assume $\tau = 1$, $\epsilon = -1$, $\alpha \neq 0$, $(\mu, \nu) \in (\mathbb{K}^*)^2$ and $\beta(\frac{r\alpha\nu}{s-r} - 1) = 0$. Since $s = r^{-1}$ and $\beta = \frac{\alpha\beta\nu r}{s-r}$, we obtain

$$\psi^-_{(\mu,\nu)}(xh - rhx) = \mu\nu(yk - rky) = 0;$$

$$\psi^-_{(\mu,\nu)}(hy - ryh) = \frac{r\alpha\nu^2}{\mu(s-r)}(kx - rxk) = 0;$$

$$\begin{aligned}\psi^-_{(\mu,\nu)}(xy - syx + f(h)) &= \frac{r\alpha\nu}{s-r}(yx - sxy) + f(\nu k)\\ &= \frac{r\alpha\nu}{s-r}((1 - r^{-2})k + g(h) - r^{-1}g(rh)) + f(\nu k)\\ &= \frac{r\alpha\nu}{s-r}((1 - r^{-2})k + \frac{1 - r^{-1}}{r^{-1} - 1}\beta) + f(\nu k)\\ &= -(\alpha\nu k + \frac{\alpha\beta\nu r}{s-r}) + f(\nu k)\\ &= 0.\end{aligned}$$

So indeed $\psi^-_{(\mu,\nu)}$ defines an algebra endomorphism of L. It can be checked that $\psi^-_{(\mu,\nu)}(k) = \left(\frac{\alpha r}{s-r}\right)^2 \nu h$ and that the inverse of $\psi^-_{(\mu,\nu)}$ is $\psi^-_{(\frac{\mu(s-r)}{r\alpha\nu},(\frac{s-r}{\alpha r})^2\nu^{-1})}$. ☐

Proposition 2.14. *Suppose* $\phi(h) = \lambda k$ *for some* $\phi \in \mathrm{Aut}_{\mathbb{K}}(L)$ *with* $\lambda \in \mathbb{K}^*$. *Then* $s = r^{-1}$, *there exists* $(\alpha, \beta) \in \mathbb{K}^* \times \mathbb{K}$ *so that* $f(X) = \alpha X + \beta$, $\beta(\frac{r\alpha\lambda}{s-r} - 1) = 0$ *and* $\phi = \psi^-_{(\mu,\lambda)}$ *for some* $\mu \in \mathbb{K}^*$.

Proof. By Lemma 2.9, r and s generate the same multiplicative subgroup of $\mathbb{K}^*$ and so $s = r^{\pm 1}$, as $\langle\langle r\rangle\rangle$ is the infinite cyclic group. If we argue as in the proof of Proposition 2.12, we can deduce that, for any $i \in \mathbb{Z}$, $\phi(L_i) = L_i$ if $s = r$ or $\phi(L_i) = L_{-i}$ if $s = r^{-1}$. In either case, $\phi(D) = D$.

Suppose, by way of contradiction, that $s = r$. Then, again following the proof of Proposition 2.12, there exist $\mu, \mu' \in \mathbb{K}^*$ so that $\phi(x) = \mu x$ and $\phi(y) = \mu' y$. Upon applying ϕ to the relation $xy - syx + f(h) = 0$ we obtain the equality $\mu\mu'(xy - syx) + f(\lambda k) = 0$, which is equivalent to

$$\mu\mu' f(h) = f(\lambda k). \tag{13}$$

The left-hand-side of (13) being a polynomial in h whereas the right-hand-side is one in k implies that f is a constant polynomial, say $f = \beta \in \mathbb{K}$. Thus $g = \frac{\beta}{s-1}$ and equation (13) yields $\beta = \beta\mu\mu'$. Hence, $\phi(k) = \phi(yx - \frac{\beta}{s-1}) = \mu\mu' yx - \frac{\beta}{s-1} = \mu\mu' k$. This contradicts the injectivity of ϕ, as we would have $\phi(\lambda k) = \mu\mu'\lambda k = \phi(\mu\mu' h)$, with $\lambda\mu\mu' \neq 0$. So indeed $s = r^{-1}$.

As before, given that $\phi(L_{\pm 1}) = L_{\mp 1}$, there exist $\mu, \mu' \in \mathbb{K}^*$ so that $\phi(x) = \mu y$ and $\phi(y) = \mu' x$. This time, if we apply ϕ to relation $xy - r^{-1}yx + f(h) = 0$ and work it out in L using (8), we arrive at the equivalent equation

$$\mu\mu'(1 - r^{-2})k + f(\lambda k) = \mu\mu'(r^{-1}g(rh) - g(h)). \tag{14}$$

So each one of the two sides of (14) must be a scalar. In particular, the condition $\mu\mu'(1 - r^{-2})k + f(\lambda k) \in \mathbb{K}$ implies $\deg(f) = \deg(g) = 1$, as r is not a root of 1. Thus $f(X) = \alpha X + \beta$, for $\alpha, \beta \in \mathbb{K}$ with $\alpha \neq 0$ and $g(X) = \frac{\alpha}{r^{-1}-r}X + \frac{\beta}{r^{-1}-1}$. Equation (14) becomes $\left(\mu\mu'(1 - r^{-2}) + \lambda\alpha\right)k + \beta = \mu\mu'\beta$, which is equivalent to $\mu' = \frac{r\alpha\lambda}{\mu(s-r)}$ and $\beta(\frac{r\alpha\lambda}{s-r} - 1) = 0$. So $\phi = \psi^-_{(\mu,\lambda)}$, as we wished to conclude. □

We will continue our study of those automorphisms of L which are not in the subgroup $\mathcal{H}$. The following proposition will be useful in the proof of Proposition 2.17. Lacking a precise reference for part (a), we provide a simple sketch of the proof. Note, however, that parts (b) and (c) can be deduced from [2, Prop. 1.4.4].

In what follows, given $q \in \mathbb{K} \setminus \{0, 1\}$, $\mathbb{K}_q[z, w]$ denotes the quantum plane, generated over $\mathbb{K}$ by indeterminates z, w satisfying the q-commutation relation $zw = qwz$.

Proposition 2.15. *Let $q, q' \in \mathbb{K} \setminus \{0, 1\}$ and assume $\phi : \mathbb{K}_q[z, w] \to \mathbb{K}_{q'}[z', w']$ is an algebra isomorphism. Then:*

(a) *$q' = q^{\pm 1}$;*

(b) *if $q \neq -1$ then either $q' = q$, $\phi(z) = \lambda_1 z'$, $\phi(w) = \lambda_2 w'$ or $q' = q^{-1}$, $\phi(z) = \lambda_1 w'$, $\phi(w) = \lambda_2 z'$, for $\lambda_1, \lambda_2 \in \mathbb{K}^*$;*

(c) *if $q = -1$ then either $\phi(z) = \lambda_1 z'$, $\phi(w) = \lambda_2 w'$ or $\phi(z) = \lambda_1 w'$, $\phi(w) = \lambda_2 z'$, for $\lambda_1, \lambda_2 \in \mathbb{K}^*$.*

Proof. It can easily be shown that the set of normal elements of $\mathbb{K}_{q'}[z', w']$ is $\{z'^a w'^b v \mid a, b \in \mathbb{N}, v \in \mathbb{Z}(\mathbb{K}_{q'}[z', w'])\}$. Therefore, $\phi(z) = z'^a w'^b v$ for some $a, b \in \mathbb{N}$ and $v \in \mathbb{Z}(\mathbb{K}_{q'}[z', w'])$. If $a = 0 = b$ then $\phi(z)$ would be central, implying that also z is central and $q = 1$, which is a contradiction. Since z generates a completely prime ideal of $\mathbb{K}_q[z, w]$, it must be that $v = \lambda_1 \in \mathbb{K}^*$ and either $a = 1$, $b = 0$ or $a = 0$, $b = 1$. A similar statement holds for $\phi(w)$. If $\phi(z) = \lambda_1 z'$ then, by the surjectivity of ϕ, $\phi(w) = \lambda_2 w'$ for some $\lambda_2 \in \mathbb{K}^*$. By relation $zw = qwz$, it follows that $q' = q$. Similarly, if $\phi(z) = \lambda_1 w'$ then $q' = q^{-1}$. □

The proof of the next lemma is omitted, as it is obvious.

Lemma 2.16. *Assume $f = 0$ and $s = r^{-1}$. There are $\phi, \psi \in \mathrm{Aut}_{\mathbb{K}}(L)$, uniquely determined by the rules:*

$$\phi(x) = \lambda_1 y, \qquad \phi(y) = \lambda_2 h \quad \textit{and} \quad \phi(h) = \lambda_3 x,$$
$$\psi(x) = \lambda_1 h, \qquad \psi(y) = \lambda_2 x \quad \textit{and} \quad \psi(h) = \lambda_3 y,$$

for any $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{K}^*)^3$.

Proposition 2.17. *Suppose there exists* $\phi \in \operatorname{Aut}_{\mathbb{K}}(L)$ *satisfying* $\phi(h) = \mu x^n$ *(resp.* $\phi(h) = \mu y^n$*), for some* $n \geq 1$ *and* $\mu \in \mathbb{K}^*$*. Then:*

(a) $n = 1$*,* $f = 0$*,* $s = r^{-1}$*;*

(b) $\phi(x) = \lambda_1 y$ *and* $\phi(y) = \lambda_2 h$ *(resp.* $\phi(x) = \lambda_1 h$ *and* $\phi(y) = \lambda_2 x$*), for some* $\lambda_1, \lambda_2 \in \mathbb{K}^*$*.*

Proof. Let $\phi \in \operatorname{Aut}_{\mathbb{K}}(L)$ with $\phi(h) = \mu x^n$, $n \geq 1$ and $\mu \in \mathbb{K}^*$. By Lemmas 2.7 and 2.8, $n = 1$ and $f = 0$. Then ϕ induces an isomorphism $\bar{\phi} : L/\langle h \rangle \to L/\langle x \rangle$. Clearly, $L/\langle h \rangle \simeq \mathbb{K}_s[\bar{x}, \bar{y}]$ and $L/\langle x \rangle \simeq \mathbb{K}_r[\bar{h}, \bar{y}]$, so Proposition 2.15 implies that $s = r^{\pm 1}$.

Suppose first that $s = r$. Then, again by Proposition 2.15 ($r \neq -1$), there are $\lambda_1, \lambda_2 \in \mathbb{K}^*$ so that $\bar{\phi}(\bar{x}) = \lambda_1 \bar{h}$ and $\bar{\phi}(\bar{y}) = \lambda_2 \bar{y}$. Hence there exist also $v_1, v_2 \in L$ such that $\phi(x) = \lambda_1 h + v_1 x$ and $\phi(y) = \lambda_2 y + v_2 x$. If we apply ϕ to the relation $xh = rhx$ and compute, we arrive at the relation $\lambda_1(1 - r^2)h = rxv_1 - v_1 x$. This implies the contradiction $h \in \langle x \rangle$. Therefore it must be that $s = r^{-1}$.

Proceeding in the same manner as in the last paragraph, we conclude that $\phi(x) = \lambda_1 y + v_1 x$ and $\phi(y) = \lambda_2 h + v_2 x$. Furthermore, relation $xh = rhx$ implies $v_1 x = rxv_1$. Writing v_1 in the basis $\left\{x^\alpha y^\beta h^\gamma\right\}_{\alpha,\beta,\gamma \geq 0}$, we easily deduce that v_1 can be written as $v_1 = y\xi$, for some $\xi \in L$. Thus $\phi(x) = y(\lambda_1 + \xi x)$. Since $\phi(x)$ must be normal and generate a completely prime ideal of L, as x does, it follows that $\lambda_1 + \xi x$ is normal (because both $y(\lambda_1 + \xi x)$ and y are normal, and L is a domain). We can then invoke Lemma 2.6 and infer that $\lambda_1 + \xi x \in \mathbb{K}^*$. In such a case, necessarily $\xi = 0$ and $\phi(x) = \lambda_1 y$. Similarly, $\phi(y) = \lambda_2 h$.

The case $\phi(h) = \mu y^n$ is symmetric. □

Proposition 2.18. *Suppose* $\tau, \epsilon > 0$ *and* $\phi(h) = \sum_{i=0}^{l} d_i \left(h^\epsilon\right)^{l-i} \left(k^\tau\right)^i$*, for some* $\phi \in \operatorname{Aut}_{\mathbb{K}}(L)$ *with* $l > 0$*,* $d_i \in \mathbb{K}$ *and* $d_0, d_l \neq 0$*. Then* $\epsilon = l = 1$*,* $f(X) = \beta \in \mathbb{K}$ *and* $\phi = \psi^+_{(\mu,\mu',d_0,d_1)}$ *for some* $\mu, \mu' \in \mathbb{K}^*$ *so that* $\beta(\mu\mu' - 1) = 0$*.*

Proof. Let $\phi \in \operatorname{Aut}_{\mathbb{K}}(L)$ and assume the hypotheses of the proposition are satisfied. For simplicity of notation write $N := \sum_{i=0}^{l} d_i \left(h^\epsilon\right)^{l-i} \left(k^\tau\right)^i$. Then $xN = r^{\epsilon l} Nx$ and $yN = r^{-\epsilon l} Ny$. By Lemma 2.9, r and $r^{\epsilon l}$ generate the same multiplicative subgroup of $\mathbb{K}^*$ and hence, r not being a root of unity, $\epsilon l = \pm 1$. As both integers ϵ and l are positive, it must be that $\epsilon = l = 1$. In particular, $\phi(h) = N = d_0 h + d_1 k^\tau$, $xN = rNx$ and $yN = r^{-1} Ny$. As we have argued in the proof of Proposition 2.12, this implies that $\phi(L_i) = L_i$ for all $i \in \mathbb{Z}$, and that $\phi(x) = \mu x$, $\phi(y) = \mu' y$ for some $\mu, \mu' \in \mathbb{K}^*$.

If we apply ϕ to the relation $xy - syx + f(h) = 0$ and simplify, we obtain $\mu\mu' f(h) = f(N)$. Since the left-hand side of the latter equation is a polynomial in h and $N = d_0 h + d_1 k^\tau$ with $\tau > 0$ and $d_1 \neq 0$, by hypothesis, the given relation forces f to be a constant polynomial, say $f(X) = \beta \in \mathbb{K}$. In that case, equation $\mu\mu' f(h) = f(N)$ reduces to $\beta(\mu\mu' - 1) = 0$ and ϕ must be the automorphism $\phi = \psi^+_{(\mu,\mu',d_0,d_1)}$ of Lemma 2.13(a). □

2.5 The group $\operatorname{Aut}_{\mathbb{K}}(L)$

Using the information obtained this far, especially in Section 2.4, the group $\operatorname{Aut}_{\mathbb{K}}(L)$ is explicitly determined in the following theorem. Recall the definition of τ and ϵ given in Section 2.1.

Theorem 2.19. *Let* $L = L(f, r, s, \gamma)$ *be a generalized down-up algebra. Assume* r*,* $s \in \mathbb{K}^*$*,* r *is not a root of unity and* $f(X) = sg(X) - g(rX - \gamma)$ *for some* $g \in \mathbb{K}[X]$*. Then the group* $\operatorname{Aut}_{\mathbb{K}}(L)$ *of algebra automorphisms of* L *is isomorphic to:*

(a) $(\mathbb{K}^*)^3 \rtimes \mathbb{Z}/3\mathbb{Z}$ *if* $f = 0$ *and* $s = r^{-1}$*, where the generator* $1 + 3\mathbb{Z}$ *of* $\mathbb{Z}/3\mathbb{Z}$ *acts on the torus* $(\mathbb{K}^*)^3$ *via the automorphism* $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_3, \lambda_1, \lambda_2)$*;*

(b) $\mathbb{K} \rtimes (\mathbb{K}^*)^3$ *if* $f = 0$ *and* $s^\tau = r$*, where* $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{K}^*)^3$ *acts on the additive group* $\mathbb{K}$ *via the automorphism* $t \mapsto \lambda_1^{-1} \left(\lambda_2 \lambda_3\right)^\tau t$*;*

(c) $\mathbb{K} \rtimes (\mathbb{K}^*)^2$ *if* $\deg(f) = 0$ *and* $s^\tau = r$, *where* $(\lambda_1, \lambda_2) \in (\mathbb{K}^*)^2$ *acts on the additive group* $\mathbb{K}$ *via the automorphism* $t \mapsto \lambda_1^{-1} t$;

(d) $(\mathbb{K}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ *if* $\deg(f) = 1$, $s = r^{-1}$ *and* $f(\frac{\gamma}{r-1}) = 0$, *where the generator* $1 + 2\mathbb{Z}$ *of* $\mathbb{Z}/2\mathbb{Z}$ *acts on the torus* $(\mathbb{K}^*)^2$ *via the automorphism* $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2^{-1}\lambda_1)$;

(e) $\mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ *if* $\deg(f) = 1$, $s = r^{-1}$ *and* $f(\frac{\gamma}{r-1}) \neq 0$, *where the generator* $1 + 2\mathbb{Z}$ *of* $\mathbb{Z}/2\mathbb{Z}$ *acts on the torus* $\mathbb{K}^*$ *via the automorphism* $\lambda \mapsto \lambda^{-1}$;

(f) $\mathcal{H}$ *otherwise, where* $\mathcal{H}$ *should be taken as described in Proposition 2.12, with* ρ *determined with respect to the polynomial* $\tilde{f}(X) = f\left(\frac{1}{r-1}(X + \gamma)\right)$.

In view of Proposition 1.7 and the hypothesis that r is not a root of unity, we can assume that $\gamma = 0$, by replacing $f \in \mathbb{K}[X]$ with $\tilde{f}(X) = f\left(\frac{1}{r-1}(X + \gamma)\right)$. Notice that $f = 0$ *iff* $\tilde{f} = 0$, $\deg(f) = \deg(\tilde{f})$ and $f(\frac{\gamma}{r-1}) = 0$ *iff* $\tilde{f}(0) = 0$. Hence, for the proof of Theorem 2.19, we assume that $\gamma = 0$ and $\mathcal{H}$ is the subgroup of $\mathrm{Aut}_{\mathbb{K}}(L)$ defined in Section 2.4 and computed in Proposition 2.12, relative to $\tilde{f}$. For a group G, $\langle\langle a \rangle\rangle$ denotes the cyclic subgroup of G generated by $a \in G$.

For the sake of clarity, we split the proof of this theorem into three propositions, dealing separately with the cases $f = 0$; $f \neq 0$, $\epsilon > 0$; and $f \neq 0$, $\epsilon \leq 0$. Each of these propositions gives additional insight into the group $\mathrm{Aut}_{\mathbb{K}}(L)$, as it explicitly lists the elements of this group, rather than just describing the group up to isomorphism.

Proposition 2.20. *Let* $L = L(f, r, s, 0)$ *be as before and suppose* $f = 0$. *Then:*

(a) $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H} \rtimes \langle\langle \phi \rangle\rangle$ *if* $s = r^{-1}$, *where* ϕ *is defined by* $\phi(x) = y$, $\phi(y) = h$, $\phi(h) = x$, *as given in Lemma 2.16; for all* $\phi_{(\lambda_1,\lambda_2,\lambda_3)} \in \mathcal{H}$, $\phi \circ \phi_{(\lambda_1,\lambda_2,\lambda_3)} \circ \phi^{-1} = \phi_{(\lambda_3,\lambda_1,\lambda_2)}$.

(b) $\mathrm{Aut}_{\mathbb{K}}(L) = \{\psi^+_{(1,1,1,t)} \mid t \in \mathbb{K}\} \rtimes \mathcal{H}$ *if* $s^\tau = r$, *where* $\psi^+_{(1,1,1,t)}$ *is given in Lemma 2.13(a); for all* $\phi_{(\lambda_1,\lambda_2,\lambda_3)} \in \mathcal{H}$, $\phi_{(\lambda_1,\lambda_2,\lambda_3)} \circ \psi^+_{(1,1,1,t)} \circ \phi^{-1}_{(\lambda_1,\lambda_2,\lambda_3)} = \psi^+_{(1,1,1,\lambda_1^{-1}(\lambda_2\lambda_3)^\tau t)}$.

(c) $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H}$ *otherwise.*

Proof. Let $\phi \in \mathrm{Aut}_{\mathbb{K}}(L)$. By Lemma 2.7, $\phi(h)$ is normal and generates a (completely) prime ideal of L. Hence, by Proposition 2.5, $\phi(h) = h^a k^b q x^n$ or $\phi(h) = h^a k^b q y^n$, for $a, b, n \geq 0$ and $q \in D$ as described in Lemma 2.3. Observing Lemma 2.6, we see that one of the following must occur:

$$\phi(h) = \lambda h, \quad \phi(h) = \lambda k, \quad \phi(h) = q, \quad \phi(h) = \lambda x \quad \text{or} \quad \phi(h) = \lambda y,$$

for some $\lambda \in \mathbb{K}^*$.

The case $\phi(h) = \lambda k$ cannot occur, by Proposition 2.14, as $f = 0$. Notice that, since h is not central ($r \neq 1$), if $\phi(h) = q$ then necessarily $\tau, \epsilon > 0$ and $q = \sum_{i=0}^{l} d_i \left(h^\epsilon\right)^{l-i} \left(k^\tau\right)^i$, with $l > 0$ and $d_0, d_l \neq 0$. Furthermore, in this case, Proposition 2.18 can be applied and we deduce that $\epsilon = 1 = l$ and $\phi = \psi^+_{(\mu,\mu',d_0,d_1)}$ for $\mu, \mu' \in \mathbb{K}^*$. Similarly, if either $\phi(h) = \lambda x$ or $\phi(h) = \lambda y$ then Proposition 2.17 implies that $(\tau, \epsilon) = (1, -1)$.

Suppose first that neither $\epsilon = 1$ nor $(\tau, \epsilon) = (1, -1)$. Then the only possibility is $\phi(h) = \lambda h$ and $\phi \in \mathcal{H}$, as given in Proposition 2.12(a).

Now consider the case $\epsilon = 1$. Then either $\phi(h) = \lambda h$ and $\phi \in \mathcal{H}$, or $\phi = \psi^+_{(\mu,\mu',d_0,d_1)}$ for $\mu, \mu' \in \mathbb{K}^*$. As $\psi^+_{(\mu,\mu',d_0,d_1)} = \phi_{(d_0,\mu,\mu')} \circ \psi^+_{(1,1,1,t)}$, with $\phi_{(d_0,\mu,\mu')} \in \mathcal{H}$ and $t = d_1(\mu\mu')^{-\tau}$, we can assume without loss of generality that $\phi = \psi^+_{(1,1,1,t)}$, for some $t \in \mathbb{K}$. Now note that $\psi^+_{(1,1,1,0)} = \mathrm{id}_L$ and $\psi^+_{(1,1,1,t)} \circ \psi^+_{(1,1,1,t')} = \psi^+_{(1,1,1,t+t')}$, so $\{\psi^+_{(1,1,1,t)} \mid t \in \mathbb{K}\}$ is isomorphic to the additive group of $\mathbb{K}$. Furthermore, $\mathcal{H} \cap \{\psi^+_{(1,1,1,t)} \mid t \in \mathbb{K}\} = \{\mathrm{id}_L\}$ and it is routine to verify that

$$\phi_{(\lambda_1,\lambda_2,\lambda_3)} \circ \psi^+_{(1,1,1,t)} \circ \phi^{-1}_{(\lambda_1,\lambda_2,\lambda_3)} = \psi^+_{(1,1,1,\lambda_1^{-1}(\lambda_2\lambda_3)^\tau t)}.$$

So indeed $\mathrm{Aut}_{\mathbb{K}}(L) = \{\psi^{+}_{(1,1,1,t)} \mid t \in \mathbb{K}\} \rtimes \mathcal{H} \simeq \mathbb{K} \rtimes (\mathbb{K}^*)^3$, in this case.

Finally, let us consider the case $(\tau, \epsilon) = (1, -1)$, i.e., $s = r^{-1}$. As we have seen, either $\phi(h) = \lambda h$ and $\phi \in \mathcal{H}$ or $\phi(h) = \lambda x$ or $\phi(h) = \lambda y$. Assume that $\phi(h) = \lambda x$ (the case $\phi(h) = \lambda y$ is symmetric). Then by Proposition 2.17, $\phi(x) = \lambda_1 y$ and $\phi(y) = \lambda_2 h$. Composing ϕ with an appropriate element of $\mathcal{H}$, we can assume that $\phi(h) = x$, $\phi(x) = y$ and $\phi(y) = h$. Then $\phi^2(h) = y$ and $\phi^3 = \mathrm{id}_L$. Hence $\langle\langle \phi \rangle\rangle$ is the cyclic group of order 3, $\mathcal{H} \cap \langle\langle \phi \rangle\rangle = \{\mathrm{id}_L\}$ and

$$\phi \circ \phi_{(\lambda_1,\lambda_2,\lambda_3)} \circ \phi^{-1} = \phi_{(\lambda_3,\lambda_1,\lambda_2)}.$$

This proves that, in this case, $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H} \rtimes \langle\langle \phi \rangle\rangle$. □

Proposition 2.21. *Let $L = L(f, r, s, 0)$ be as before and suppose $f \neq 0$ and $\epsilon > 0$. Then:*

(a) $\mathrm{Aut}_{\mathbb{K}}(L) = \{\psi^{+}_{(1,1,1,t)} \mid t \in \mathbb{K}\} \rtimes \mathcal{H}$ *if $s^\tau = r$ and $\deg(f) = 0$, where $\psi^{+}_{(1,1,1,t)}$ is given in Lemma 2.13(a); for all $\phi_{(\lambda_1,\lambda_2)} \in \mathcal{H}$, $\phi_{(\lambda_1,\lambda_2)} \circ \psi^{+}_{(1,1,1,t)} \circ \phi^{-1}_{(\lambda_1,\lambda_2)} = \psi^{+}_{(1,1,1,\lambda_1^{-1}t)}$.*

(b) $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H}$ *otherwise.*

Proof. Let $\phi \in \mathrm{Aut}_{\mathbb{K}}(L)$. As in the proof of Proposition 2.20, only two possibilities can occur: $\phi(h) = \lambda h$ or $\phi(h) = q$, with $q \in D$ as described in Lemma 2.3(c) (see Proposition 2.14 and Proposition 2.17).

If $\phi(h) = \lambda h$ then $\phi \in \mathcal{H}$. Otherwise, Proposition 2.18 implies that $\epsilon = 1$, $\deg(f) = 0$ and $\phi = \psi^{+}_{(\mu,\mu^{-1},d_0,d_1)}$ for $\mu, d_0, d_1 \in \mathbb{K}^*$. Therefore, if either $\epsilon \neq 1$ or $\deg(f) \neq 0$ then $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H}$. If $\epsilon = 1$ and $\deg(f) = 0$ then $\rho = 0$ and $\mathcal{H} = \{\phi_{(\lambda_1,\lambda_2)} \mid (\lambda_1, \lambda_2) \in (\mathbb{K}^*)^2\}$, with $\phi_{(\lambda_1,\lambda_2)}(h) = \lambda_1 h$, $\phi_{(\lambda_1,\lambda_2)}(x) = \lambda_2 x$, $\phi_{(\lambda_1,\lambda_2)}(y) = \lambda_2^{-1} y$. In case $\phi \notin \mathcal{H}$ we can assume $\phi = \psi^{+}_{(1,1,1,t)}$ and proceed as in the proof of Proposition 2.20 to conclude that $\mathrm{Aut}_{\mathbb{K}}(L) = \{\psi^{+}_{(1,1,1,t)} \mid t \in \mathbb{K}\} \rtimes \mathcal{H}$. □

Proposition 2.22. *Let $L = L(f, r, s, 0)$ be as before and suppose $f \neq 0$ and $\epsilon \leq 0$. Then:*

(a) $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H} \rtimes \langle\langle \psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})} \rangle\rangle$ *if $s = r^{-1}$ and $f(X) = \alpha X$ for $\alpha \in \mathbb{K}^*$, where $\psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})}$ is given in Lemma 2.13(b); for all $\phi_{(\lambda_1,\lambda_2)} \in \mathcal{H}$, $\psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})} \circ \phi_{(\lambda_1,\lambda_2)} \circ (\psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})})^{-1} = \phi_{(\lambda_1,\lambda_2^{-1}\lambda_1)}$.*

(b) $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H} \rtimes \langle\langle \psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})} \rangle\rangle$ *if $s = r^{-1}$ and $f(X) = \alpha X + \beta$ for $\alpha, \beta \in \mathbb{K}^*$, where $\psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})}$ is given in Lemma 2.13(b); for all $\phi_{(1,\lambda)} \in \mathcal{H}$, $\psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})} \circ \phi_{(1,\lambda)} \circ (\psi^{-}_{(1,\frac{r^{-1}-r}{r\alpha})})^{-1} = \phi_{(1,\lambda^{-1})}$.*

(c) $\mathrm{Aut}_{\mathbb{K}}(L) = \mathcal{H}$ *otherwise.*

Proof. We only sketch the proof, as it is similar to the proof of the two previous results.

Assume $\phi \notin \mathcal{H}$. Hence, as before, the only other possibility is $\phi(h) = \lambda k$, for some $\lambda \in \mathbb{K}^*$. Then, by Proposition 2.14, $s = r^{-1}$ and $\deg(f) = 1$. Write $f(X) = \alpha X + \beta$, with $\alpha \neq 0$. Thus $\phi = \psi^{-}_{(\mu,\lambda)}$ for $\mu \in \mathbb{K}^*$ and $\beta(\frac{r\alpha\lambda}{s-r} - 1) = 0$.

Suppose $\beta = 0$. Then $\rho = 0$, $\mathcal{H} = \{\phi_{(\lambda_1,\lambda_2)} \mid (\lambda_1, \lambda_2) \in (\mathbb{K}^*)^2\}$ and it can be assumed that $\phi = \psi^{-}_{(1,\frac{s-r}{r\alpha})}$, as $\psi^{-}_{(\mu,\lambda)} = \phi_{(\frac{r\alpha\lambda}{s-r},\frac{r\alpha\lambda}{\mu(s-r)})} \circ \psi^{-}_{(1,\frac{s-r}{r\alpha})}$. The result follows in this case because $(\psi^{-}_{(1,\frac{s-r}{r\alpha})})^2 = \mathrm{id}_L$, $\mathcal{H} \cap \langle\langle \psi^{-}_{(1,\frac{s-r}{r\alpha})} \rangle\rangle = \{\mathrm{id}_L\}$ and

$$\psi^{-}_{(1,\frac{s-r}{r\alpha})} \circ \phi_{(\lambda_1,\lambda_2)} \circ (\psi^{-}_{(1,\frac{s-r}{r\alpha})})^{-1} = \phi_{(\lambda_1,\lambda_2^{-1}\lambda_1)}.$$

The case $\beta \neq 0$ is analogous, with $\rho = 1$ and $\frac{r\alpha\lambda}{s-r} = 1$. □

3 Automorphisms of down-up algebras

Having computed in Section 2 the automorphism group of the generalized down-up algebras $L(f, r, s, \gamma)$ which are conformal, Noetherian and for which r is not a root of unity, we specialize in this section our results to the case of down-up algebras. We remark that the isomorphism problem for Noetherian down-up algebras has already been solved in [10].

Other classes of algebras to which our study applies are Le Bruyn's *conformal* $\mathfrak{sl}_2$ *enveloping algebras* [18], occuring as $L(bx^2 + x, r, s, \gamma)$, for $b \in \mathbb{K}$ and $rs \neq 0$, and some of Witten's seven parameter deformations of the enveloping algebra of $\mathfrak{sl}_2$ [26] (see also [8, Thm. 2.6] and [11, Ex. 1.4]). We leave it to the reader to apply Theorem 2.19 to these and perhaps to other classes of generalized down-up algebras.

3.1 Some well-known examples

We start by computing some examples, which have appeared elsewhere in the literature, namely [2], [3] and [9].

The quantum Heisenberg algebra $\mathbb{H}_q$ is a deformation of the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. It can be viewed as the positive part in the triangular decomposition of the quantized enveloping algebra corresponding to the simple complex Lie algebra $\mathfrak{sl}_3$ of traceless 3×3 matrices. It is presented as the unital associative $\mathbb{K}$-algebra generated by X, Y, Z, with relations:

$$XZ = qZX, \qquad ZY = qYZ, \qquad XY - q^{-1}YX = Z, \tag{15}$$

where $q \in \mathbb{K}^*$.

The automorphism group of the quantum Heisenberg algebra was computed by Caldero [9] in case q is transcendental over $\mathbb{Q}$ and, independently, by Alev and Dumas [3] just assuming q is not a root of 1. It is the semidirect product of the 2-torus $(\mathbb{K}^*)^2$, acting diagonally on the generators X and Y, and the finite group $\mathbb{Z}/2\mathbb{Z}$, acting as the symmetric group on X and Y. From relations (15), we see that $\mathbb{H}_q$ is the algebra $L(-X, q, q^{-1}, 0)$, isomorphic to the down-up algebra $A(q + q^{-1}, -1, 0)$. Thus, $L(-X, q, q^{-1}, 0)$ is Noetherian for all choices of $q \in \mathbb{K}^*$, and conformal provided $q \neq 1, -1$. If we assume, as in [3], that q is not a root of 1, then $\tau = 1$, $\epsilon = -1$ and $\rho = 0$. Hence, we retrieve [3, Prop. 2.3] in Theorem 2.19(d).

Another example, which is not that of a down-up algebra, is the algebra of regular functions on quantum affine 3-space. This is the unital associative $\mathbb{K}$-algebra with generators x_1, x_2, x_3, satisfying the relations:

$$x_1x_2 = q_{12}x_2x_1, \qquad x_1x_3 = q_{13}x_3x_1, \qquad x_2x_3 = q_{23}x_3x_2, \tag{16}$$

where $q_{12}, q_{13}, q_{23} \in \mathbb{K}^*$. In case $q_{12}q_{13} = 1$, this algebra coincides with the generalized down-up algebra $L(0, r, s, 0)$, with $r = q_{13} = q_{12}^{-1}$ and $s = q_{23}$. Therefore, Proposition 2.20 can be used to compute $\mathrm{Aut}_{\mathbb{K}}(L)$ whenever r is not a root of 1. In particular, if $r = s$ is not a root of unity, it was seen in [2, Thé. 1.4.6i)] that $\mathrm{Aut}_{\mathbb{K}}(L)$ is isomorphic to the semidirect product $\mathbb{K} \rtimes (\mathbb{K}^*)^3$. Since, in this case, $\tau = \epsilon = 1$, we also obtain this description of $\mathrm{Aut}_{\mathbb{K}}(L)$ in Theorem 2.19(b).

3.2 Down-up algebras

Using Proposition 1.7, Theorem 2.19 and also some results of Carvalho and Musson [10] and Jordan [15], we will now compute the automorphism group of all down-up algebras $A(\alpha, \beta, \gamma)$, except in the cases where either $\beta = 0$ or where both roots r and s of the polynomial $X^2 - \alpha X - \beta$ are roots of 1.

Our exceptions include two remarkable examples, corresponding to $r = s = 1$. One is $U(\mathfrak{sl}_2)$, the enveloping algebra of the complex simple Lie algebra $\mathfrak{sl}_2$, which occurs in the family of down-up algebras as $A(2, -1, 1)$. The other one is $U(\mathfrak{h})$, the enveloping algebra of the 3-dimensional Heisenberg Lie algebra, occurring as $A(2, -1, 0)$. Neither for $U(\mathfrak{sl}_2)$ nor for $U(\mathfrak{h})$ is the full group of automorphisms known, and in both cases *wild* automorphisms have been shown to exist, by work of Joseph [16] and Alev [1], respectively. Regarding $U(\mathfrak{sl}_2)$, Dixmier computed the automorphism group of the minimal primitive quotients of this algebra in [13]. As for the primitive quotients of $U(\mathfrak{h})$ which are not one-dimensional, these are isomorphic to the first Weyl algebra $A_1(\mathbb{K})$, whose group of automorphisms was also computed by Dixmier in [12].

In [6], Bavula and Jordan solved the isomorphism problem and found generators for the automorphism group of generalized Weyl algebras of the form $\mathbb{K}[X](X \overset{\sigma}{\mapsto} X-1, a)$, a class which includes the infinite-dimensional primitive quotients of both $U(\mathfrak{sl}_2)$ and $U(\mathfrak{h})$. They also solved the isomorphism problem for Smith's algebras $L(f,1,1,1)$ similar to $U(\mathfrak{sl}_2)$. We note that our results do not overlap with those of [6]. Other generalized Weyl algebras of the form $\mathbb{K}[X](X \overset{\sigma}{\mapsto} qX, a)$, with q not a root of unity, were studied in [23], and their automorphism group was determined. With minor changes, [23, Cor. 2.2.7] can be adapted to describe the automorphism group of the down-up algebras of the form $A(r+1,-r,0)$, with $r \in \mathbb{K}^*$ not a root of unity. This may be achieved by replacing in [23, Cor. 2.2.7] the base field $\mathbb{K}$ by the domain $\mathbb{K}[X]$, and observing that the arguments used are still valid. As a result, we would retrieve a subcase of Theorem 3.1(b) below.

Fix α, β, $\gamma \in \mathbb{K}$ with $\beta \neq 0$, and let r and s be the roots of the polynomial $X^2 - \alpha X - \beta$ in $\mathbb{K}$. Thus $\alpha = r+s$ and $\beta = -rs$. The down-up algebra $A = A(\alpha,\beta,\gamma)$, as defined in [7], coincides with $L(X,r,s,\gamma)$, upon identifying the canonical generators d and u of A with the generalized Weyl algebra generators x and y of L, respectively. Since r and s have symmetric roles in A, it should be no surprise that $L(X,r,s,\gamma) \simeq L(X,s,r,\gamma)$, under an isomorphism taking x to x, y to y and $h \in L(X,r,s,\gamma)$ to $h + (s-r)yx \in L(X,s,r,\gamma)$. Hence, when dealing with down-up algebras, we can interchange the roles of r and s in L. Also, the generator h of L is redundant when $\deg(f) = 1$, so in this case it will suffice to give the action of an automorphism of L on the generators x and y.

Our results in this section will apply to all down-up algebras A under the restrictions that $rs \neq 0$ and that one of r or s is not a root of 1. In view of the symmetric roles of r and s, we always assume that r is not a root of 1. In particular, $A = L(X,r,s,\gamma) \simeq L(\frac{1}{r-1}(X+\gamma), r, s, 0)$, by Proposition 1.7. We distinguish three cases:

Case 1: $r \neq s$, r is not a root of 1 and $s \neq 1$. In this case, $A \simeq L(\frac{1}{r-1}(X+\gamma), r, s, 0)$ is conformal and $\mathrm{Aut}_{\mathbb{K}}(A)$ is given in Theorem 2.19(d)–(f). If $\gamma = 0$ then $\rho = 0$; if $\gamma \neq 0$ then $\rho = 1$.

Assume first that $\gamma = 0$. Then $\mathcal{H} \simeq (\mathbb{K}^*)^2$ acts diagonally on the generators x and y. If, in addition, $s = r^{-1}$ then there is an automorphism of A of order 2 which interchanges x and y.

Now assume $\gamma \neq 0$. Then $\mathcal{H} \simeq \mathbb{K}^*$ and $\lambda \in \mathbb{K}^*$ acts on x by multiplication by λ and on y by multiplication by λ^{-1}. If, in addition, $s = r^{-1}$ then there is an automorphism of A of order 2 which interchanges x and y.

Case 2: r is not a root of 1 and $s = 1$. Assume $\gamma = 0$. Then $A = L(X,r,s,0)$ is conformal, as $r \neq 1$. Also, $\tau = 1$, $\epsilon = 0$ and $\rho = 0$. Thus, by Theorem 2.19, $\mathrm{Aut}_{\mathbb{K}}(A) = \mathcal{H} \simeq (\mathbb{K}^*)^2$, acting diagonally on x and y.

Now assume $\gamma \neq 0$. Then $A = L(X,r,s,\gamma)$ is not conformal, by Proposition 1.7 and Lemma 1.6, so Theorem 2.19 cannot be applied. Let $\omega = yx - xy + \frac{\gamma}{1-r}$. By [10, Cor. 4.10], any automorphism ϕ of A must fix the ideal ωA. By Lemma 1.4, there exists $\lambda \in \mathbb{K}^*$ so that $\phi(\omega) = \lambda\omega$. The proof of Proposition 2.12 can be readily adapted to show that $\phi(x) = \mu x$ and $\phi(y) = \mu^{-1} y$, for some $\mu \in \mathbb{K}^*$. Thus, $\mathrm{Aut}_{\mathbb{K}}(A) \simeq \mathbb{K}^*$.

Case 3: $r = s$ is not a root of 1. In this case, $A = L(X,r,r,\gamma)$ is not conformal, by Proposition 1.7 and Lemma 1.6. We can use the description of the height one prime ideals of A that appears in [15, Prop. 6.13] precisely for the case that r is not a root of one. Indeed, let $\omega = ryx - xy + \frac{\gamma}{1-r}$. Then $\omega \in D$, $x\omega = r\omega x$ and $\omega y = ry\omega$, so ω is normal. If $\gamma = 0$ then ωA is the unique height one prime ideal of A. If $\gamma \neq 0$ then the height one primes of A are ωA and the annihilators of certain simple finite-dimensional A-modules. In either case, ωA is the unique height one prime ideal of A not having finite codimension, as $A/\omega A$ is either a quantum plane ($\gamma = 0$) or a quantum Weyl algebra ($\gamma \neq 0$). Thus, all automorphisms of A fix the ideal generated by ω. As above, we deduce that, given $\phi \in \mathrm{Aut}_{\mathbb{K}}(A)$, there exist nonzero scalars λ, μ so that $\phi(x) = \lambda x$ and $\phi(y) = \mu y$. If $\gamma = 0$, no further restrictions arise on the parameters λ, μ and $\mathrm{Aut}_{\mathbb{K}}(A) \simeq (\mathbb{K}^*)^2$. In case $\gamma \neq 0$, there is only the additional restriction that $\lambda = \mu^{-1}$, so $\mathrm{Aut}_{\mathbb{K}}(A) \simeq \mathbb{K}^*$.

We summarize out results on down-up algebras in the following theorem. For the convenience of those readers who are mostly interested in down-up algebras, we replace our usual generators x and y of L with the canonical generators d and u of A, respectively.

Theorem 3.1. *Let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra, with $\alpha = r+s$ and $\beta = -rs$. Assume that $\beta \neq 0$ and that one of r or s is not a root of unity. The group $\mathrm{Aut}_{\mathbb{K}}(A)$ of algebra automorphisms of A is described bellow.*

(a) *If $\gamma = 0$ and $\beta = -1$ then $\mathrm{Aut}_{\mathbb{K}}(A) \simeq (\mathbb{K}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$;*

(b) *If $\gamma = 0$ and $\beta \neq -1$ then $\mathrm{Aut}_{\mathbb{K}}(A) \simeq (\mathbb{K}^*)^2$;*

(c) *If $\gamma \neq 0$ and $\beta = -1$ then $\mathrm{Aut}_{\mathbb{K}}(A) \simeq \mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}$;*

(d) *If $\gamma \neq 0$ and $\beta \neq -1$ then $\mathrm{Aut}_{\mathbb{K}}(A) \simeq \mathbb{K}^*$.*

In all cases, the 2-torus $(\mathbb{K}^*)^2$ acts diagonally on the generators d and u, $\mu \in \mathbb{K}^*$ acts as multiplication by μ on d and as multiplication by μ^{-1} on u, and the generator of the finite group $\mathbb{Z}/2\mathbb{Z}$ interchanges d and u.

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