

Dynamics for exclusion processes with random environments of particle jump rates

Mikhail Menshikov*

Serguei Popov[†]

Andrew Wade*

5th October 2026

Abstract

We consider a continuous-time particle system with exclusion interaction on the integer lattice. Each particle is assigned intrinsic left and right jump rates by an independent draw from a common random environment. We identify the decomposition of the system into maximal stable subsystems, which we call clouds. We show different qualitative behaviour for the cloud decomposition corresponding to different regimes for the random environment law that parallel the classical Sinai and Kesten–Kozlov–Spitzer regimes from one-dimensional random walk in random environment. Our main result in the Sinai regime is that the system is decomposed into an infinite number of finite clouds which all go to minus infinity with rapidly decreasing speed magnitudes, and we obtain explicit associated distributional limits. We analyse the model through the potential function associated with the random environment. In the Sinai regime, deep potential wells trap particles, producing stable clouds whose speeds are exponentially small in the well depth. Since successive wells are deeper, the resulting clouds are successively slower. Thus the localization mechanism of Sinai’s random walk manifests itself here as a jamming phenomenon, in which a potential well traps not one walker but an entire block of mutually excluding particles.

Key words: Exclusion process, interacting particle systems, random environment, Sinai’s regime, stable clouds, infinite Jackson network, asymptotic speeds, potential function.

AMS Subject Classification: 60K35 (Primary), 60K37, 60K25, 60J25 (Secondary).

This paper is dedicated to the memory of Mikhail Menshikov, who passed away on May 1, 2026. We (S.P. and A.W.) wish to acknowledge that Mikhail was the driving force behind the research programme to which this work belongs (see [27–30]), and that the original idea for this line of investigation was his. We are deeply grateful for his vision, insight, and guidance throughout the project. Beyond his scientific contributions, we will profoundly miss him as a dear friend and a lifelong mentor.

1 Introduction and results

1.1 Formulation of the model

We study a semi-infinite exclusion process in which each particle has its own randomly chosen left and right jump rates. The interaction causes particles with different intrinsic drifts to organize into maximal stable groups, or clouds, whose internal separations remain tight while

*Department of Mathematical Sciences, Durham University, Durham, UK.
mikhail.menshikov@durham.ac.uk, andrew.wade@durham.ac.uk.

[†]Centro de Matemática, University of Porto, Porto, Portugal. serguei.popov@fc.up.pt.

distinct clouds separate. Our aim is to describe the typical cloud structure generated by a random environment of particle rates.

Our main results describe the cloud partitions for natural classes of random environments, which parallel the phase diagram for environments in *random walk in random environment* (RWRE) on $\mathbb{Z}$ [19, 33, 36, 37]. For example, in the analogue of Sinai’s regime [35], we show that, for almost all environments, the system decomposes into an infinite collection of finite subsystems whose size and intrinsic speed can be described asymptotically via a *potential* function, indexed by particle labels, associated to the random environment. We show that the localization phenomenon for random walk in random environment has an analogous *jamming* phenomenon in the particle system context.

The system is described as follows. Particles are enumerated by $\mathbb{N} := \{1, 2, 3, \dots\}$ from the left to the right (particle 1 is the leftmost particle, and there is no rightmost particle). The k th particle has intrinsic jump rates $a_k, b_k \in (0, \infty)$, given by the realization of the random environment, where a_k is the rate at which the particle attempts to jump one site to the left, and b_k the rate at which it attempts to jump one site to the right. The exclusion interaction means that jumps that target a site occupied by another particle are suppressed; consequently, the order of the particles is preserved. See Figure 1 for a picture.

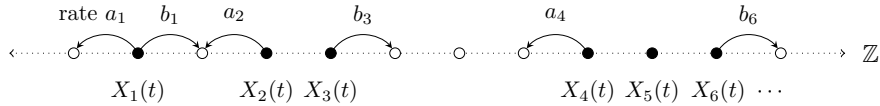


Figure 1: Schematic of the model showing the leftmost 6 particles, illustrating some of the main notation. Filled circles represent particles, empty circles represent unoccupied lattice sites, and directed arcs represent admissible transitions, with exponential rates indicated.

The configuration space of the system is $\mathbb{X}$, given by

$$\mathbb{X} := \{(x_1, x_2, \dots) \in \mathbb{Z}^{\mathbb{N}} : x_1 < x_2 < \dots\}. \quad (1.1)$$

Let $X(t) = (X_1(t), X_2(t), \dots) \in \mathbb{X}$ be the configuration of the process of particle positions at time $t \in \mathbb{R}_+ := [0, \infty)$, started from a fixed initial configuration $X(0) \in \mathbb{X}$.

For deterministic jump rates, cloud decompositions were studied for finite systems in [27], while [28, 29] investigated stability and dynamics of semi-infinite systems forming a single cloud. The goals and methods of the present paper are different. Here we study the cloud structure generated by an i.i.d. environment of particle rates, combining potential methods from one-dimensional RWRE with coupling arguments and the finite-system theory of [27].

We use the notation $\omega := ((a_k, b_k), k \in \mathbb{N})$ for the random environment, which we assume lives on a probability space with probability measure $\mathbb{P}$ and expectation $\mathbb{E}$. We suppose that, under $\mathbb{P}$, $(a_i, b_i)_{i \geq 1}$ are independently sampled as random copies of a vector $(\Theta^-, \Theta^+) \in (0, \infty)^2$. The particle system in random environment is then, for fixed ω , defined by taking (a_k, b_k) to be Markov transition rates of the k th particle, as described above. When the initial configuration is $x \in \mathbb{X}$, we use the notation $\mathbb{P}_\omega^x$ for the *quenched* (i.e., with fixed environment ω) law of $(X(t), t \geq 0)$ started from $X(0) = x$, and we denote by $\mathbb{E}_\omega^x$ the corresponding expectation.

The central object in our analysis is the random *potential* obtained by summing the log-ratios of the particle jump rates, which we define formally in Section 3 below. Roughly speaking, stable clouds correspond to *traps* or *wells* of this potential: a sufficiently deep well produces a group of particles which moves together, with a speed exponentially small in the depth of the well. Thus a major part of the narrative of this paper is explaining how the large-scale cloud structure is governed by the successive record wells of a one-dimensional random walk (in the limit, Brownian motion).

We need to assume some regularity conditions on the random environment. The first “nestling” condition (the terminology comes from the RWRE literature; see [37, 38]) ensures that

the particles may have drifts in both directions; in the absence of such an assumption we would not get the possibility of deep potential wells.

(N) *Nestling random environment.* It holds that $\mathbb{P}[\Theta^- < \Theta^+] > 0$ and $\mathbb{P}[\Theta^- > \Theta^+] > 0$.

We assume a *uniform ellipticity* condition, i.e., that the rates are uniformly bounded away from zero and infinity (this is standard in RWRE, for example).

(UE) *Uniform ellipticity.* For some $\varepsilon_0 \in (0, 1)$ we have $\mathbb{P}[\varepsilon_0 \leq \Theta^\pm \leq \varepsilon_0^{-1}] = 1$.

Observe that Condition **(UE)** implies that

$$\text{for } \kappa := 2|\log \varepsilon_0| < \infty \text{ we have } \mathbb{P}\text{-a.s. that } -\kappa \leq \log(\Theta^-/\Theta^+) \leq \kappa. \quad (1.2)$$

For technical reasons we also need a mild regularity assumption on ascending ladder heights of the random walk with independent increments distributed as $\log(\Theta^-/\Theta^+)$. If we denote such random walk by $V = (V_n, n \geq 0)$ and assume $V_0 = 0$, then we can denote $\tau_+ := \min\{k : V_k > 0\}$ and let $\eta := V_{\tau_+} > 0$ be the first ascending ladder height. The next condition assures that the first ladder height is unlikely to be exceptionally small.

(LH) *Ladder height.* Suppose there exist positive constants γ and δ_0 such that for all small enough $y > 0$ it holds that $\mathbb{P}[\eta \geq y] \geq 1 - \gamma y^{\delta_0}$.

Condition **(LH)** holds, in particular, for lattice environments and under natural bounded-density hypotheses; see Lemma A.1. It need not follow from **(UE)** and **(N)** alone. Here is an example for which all of the above conditions are satisfied.

Example 1.1. Let $\varepsilon \in (0, 1)$, and take Θ^-, Θ^+ to be independent $U[\varepsilon, \varepsilon^{-1}]$ random variables. Then hypotheses **(N)**, **(UE)**, and **(LH)** are satisfied (for the latter, apply Lemma A.1). $\triangle$

The sign of $\mathbb{E} \log(\Theta^-/\Theta^+)$, which is finite under hypothesis **(UE)**, informs a phase diagram resembling that for one-dimensional RWRE, and our results cover all three regimes. We give our results for the case $\mathbb{E} \log(\Theta^-/\Theta^+) = 0$ in §1.3 and for the cases $\mathbb{E} \log(\Theta^-/\Theta^+) \neq 0$ in §1.4. In the zero-mean case, we show that the system consists of infinitely many finite clouds with doubly exponentially decreasing speed magnitudes. When the mean is negative, all but finitely many particles form a single semi-infinite stable cloud. When the mean is positive, there are again infinitely many finite clouds, but magnitudes of particle speeds decrease polynomially. Before formally stating our results, we formally define the concept of *stable clouds* that we have so far discussed informally.

1.2 Stable clouds

Informally, a stable cloud is a maximal consecutive group of particles in which pairwise particle separations remain tight while its distance from particles outside the cloud diverges. Note that, typically, the decomposition into stable clouds will depend on the environment ω , and, in principle, can depend also on the initial configuration, as explained in Section 2.3 below. We use notation $[i, j] := \{\ell \in \mathbb{Z}_+ : i \leq \ell \leq j\}$ ($i \leq j$) for a discrete interval, and write $[n, \infty) := \{\ell \in \mathbb{Z}_+ : \ell \geq n\}$ (throughout $\mathbb{Z}_+ := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$).

The next definition says that a finite group of consecutive particles is a stable cloud if the distance between the outermost particles in the group stays tight, and the distances of each outermost particle with its neighbour outside the group tends to ∞ in probability.

Definition 1.2 (Stable cloud, finite case). For $i \leq j$ the group $[i, j]$ is an (ω, x) -stable cloud, if
(i) for every $\varepsilon > 0$ there exists $B \in \mathbb{R}_+$ such that

$$\mathbb{P}_\omega^x(X_j(t) - X_i(t) \geq B) \leq \varepsilon, \text{ for all } t \in \mathbb{R}_+,$$

and (ii) for every $B \in \mathbb{R}_+$ and every $\varepsilon > 0$ there exists $t_0 \in \mathbb{R}_+$ such that $\mathbb{P}_\omega^x(X_{j+1}(t) - X_j(t) \geq B) \geq 1 - \varepsilon$ for all $t \geq t_0$, and (provided $i > 1$) $\mathbb{P}_\omega^x(X_i(t) - X_{i-1}(t) \geq B) \geq 1 - \varepsilon$ for all $t \geq t_0$.

The semi-infinite case is similar.

Definition 1.3 (Stable cloud, semi-infinite case). The group $[i, \infty)$ is an (ω, x) -stable cloud, if
(i) for every $j, k \in [i, \infty)$, for every $\varepsilon > 0$ there exists $B \in \mathbb{R}_+$ such that

$$\mathbb{P}_\omega^x(|X_j(t) - X_k(t)| \geq B) \leq \varepsilon, \text{ for all } t \in \mathbb{R}_+,$$

and (ii) (provided $i > 1$) for every $B \in \mathbb{R}_+$ and every $\varepsilon > 0$ there exists $t_0 \in \mathbb{R}_+$ such that $\mathbb{P}_\omega^x(X_i(t) - X_{i-1}(t) \geq B) \geq 1 - \varepsilon$ for all $t \geq t_0$.

1.3 Results: Sinai regime

Suppose that

$$\mathbb{E} \log \frac{\Theta^-}{\Theta^+} = 0. \quad (1.3)$$

By analogy with the RWRE case [35], we say this is *Sinai's regime* for the random environment. Note that (1.3) holds whenever Θ^- and Θ^+ have the same distribution, as they do in Example 1.1 above. Let $\sigma^2 := \text{Var} \log(\Theta^-/\Theta^+)$; note $\sigma^2 \in (0, \infty)$ under **(N)** and **(UE)**.

Our main result for Sinai's regime is the following. Informally, in the Sinai regime the particles organize themselves into successively larger finite clouds, which, up to a finite initial section, we can identify explicitly with potential wells, and whose speeds decrease doubly exponentially with the cloud index. The ratios of successive logarithmic speeds have an asymptotic Pareto law, while cloud sizes and squared logarithmic speeds satisfy a distributional limit result. We write “a.a.” for “almost all”.

Theorem 1.4. Assume Conditions **(N)**, **(UE)**, and **(LH)**, and suppose that (1.3) holds. Let $X(0) = x \in \mathbb{X}$. Then for $\mathbb{P}$ -almost all environments the system is decomposed into infinitely many finite stable clouds, i.e., there exist pairwise disjoint, non-empty $\mathcal{C}_1, \mathcal{C}_2, \dots$ such that (i) for every $k \in \mathbb{N}$ and all $i \in \mathcal{C}_k$ and $j \in \mathcal{C}_{k+1}$, $i < j$, and (ii) $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots = \mathbb{N}$, such that each $\mathcal{C}_k$ is stable in the sense of Definition 1.2. Moreover, the cloud decomposition is independent of the initial configuration, and, for a.a. ω there exist k_1 and k_2 such that sequences $(\mathcal{C}_k, k \geq k_1)$ and $([h_k + 1, h_{k+1}], k \geq k_2)$ coincide, where h_k are defined in terms of the potential in §4.1 below.

To every cloud $k \in \mathbb{N}$ there is an associated speed $s_k = s_k(\omega)$ for which the following hold.

(i) For every $k \in \mathbb{N}$ and every $j \in \mathcal{C}_k$, for $\mathbb{P}$ -a.a. ω we have the strong law

$$\lim_{t \rightarrow \infty} t^{-1} X_j(t) = s_k, \text{ } \mathbb{P}_\omega^x\text{-a.s.} \quad (1.4)$$

(ii) For $\mathbb{P}$ -a.a. ω , it holds that, as $k \rightarrow \infty$,

$$s_k = -\exp\left(-e^{k(1+o(1))}\right). \quad (1.5)$$

(iii) We have, for all $y \geq 1$,

$$\lim_{k \rightarrow \infty} \mathbb{P}[\xi_k > y] = \frac{1}{y}, \text{ where } \xi_k := \frac{\log |s_{k+1}|}{\log |s_k|}. \quad (1.6)$$

Also, the random variables ξ_k are asymptotically independent in the sense that, for each $m \in \mathbb{Z}_+$, $(\xi_n, \xi_{n+1}, \dots, \xi_{n+m})$ converges in distribution, as $n \rightarrow \infty$, to $(\theta_0, \theta_1, \dots, \theta_m)$ a vector of i.i.d. variables with $\mathbb{P}[\theta_0 > y] = 1/y$ for $y \geq 1$.

(iv) Let $S_k = S_k(\omega)$ denote the size of cloud $k \in \mathbb{N}$ (i.e., the number of particles it contains). There is a non-negative random variable Y , the first hitting time of 2 by a 3-dimensional Bessel process started at 0, with

$$\mathbb{P}(Y \leq y) = 1 - 2 \sum_{m=1}^{\infty} (-1)^{m+1} \exp \left\{ -\frac{\pi^2 m^2 y}{8} \right\}, \quad (1.7)$$

such that (where $\Rightarrow$ denotes convergence in distribution)

$$\frac{S_k}{\log^2 |s_k|^{-1}} \Rightarrow \frac{Y}{\sigma^2}, \text{ as } k \rightarrow \infty. \quad (1.8)$$

While Theorem 1.4 describes the speeds and sizes of stable clouds, the next result describes the speed of a given particle. The two results are related, since determining the particle's speed essentially amounts to determining which cloud it belongs to.

Theorem 1.5. *Assume Conditions (N), (UE), and (LH), suppose that (1.3) holds. Let $X(0) = x \in \mathbb{X}$. To every particle $n \in \mathbb{N}$ there is an associated speed $v_n = v_n(\omega) < 0$ such that, for $\mathbb{P}$ -a.a. ω ,*

$$\lim_{t \rightarrow \infty} t^{-1} X_n(t) = v_n, \quad \mathbb{P}_\omega^x\text{-a.s.} \quad (1.9)$$

Moreover, there is a non-negative random variable Z , given in terms of standard Brownian motion at (4.30) below, with

$$\mathbb{P}(Z \leq z) = \frac{8}{\pi^2} \sum_{m=0}^{\infty} (2m+1)^{-2} \exp \left\{ -\frac{\pi^2 (2m+1)^2}{8z^2} \right\}, \quad (1.10)$$

such that the speed of the n th particle satisfies

$$n^{-1/2} \log |v_n|^{-1} \Rightarrow \sigma Z, \text{ as } n \rightarrow \infty. \quad (1.11)$$

Remark 1.6. The explicit distributions for the limit random variables Y and Z from (1.7) and (1.10) are reminiscent of certain limit distributions in the theory of RWRE, such as identified by Golosov [16] and Kesten [20]. These distributions likewise arise from potential-well-type functionals of Brownian motion. We have, however, been unable to find either of the distributions of Y or Z explicitly in the RWRE literature.

1.4 Results: Kesten–Kozlov–Spitzer regime

The case when $\mathbb{E} \log(\Theta^-/\Theta^+) \neq 0$ is sometimes referred to in the RWRE literature as the *Kesten–Kozlov–Spitzer* or KKS regime, after [19]. For some of the results in this section, we restrict our initial configuration to the *finite configurations* (meaning configurations with finitely many holes), $\mathbb{X}_F \subset \mathbb{X}$, given by

$$\mathbb{X}_F := \{x \in \mathbb{X} : x_{i+1} - x_i = 1 \text{ for all but finitely many } i\}. \quad (1.12)$$

First, we will show that the case

$$\mathbb{E} \log \frac{\Theta^-}{\Theta^+} < 0 \quad (1.13)$$

is relatively easy to treat: some finite (ω -dependent) number of leftmost particles will escape to $-\infty$ (at least in probability), and the rest forms a stable system. Indeed:

Theorem 1.7. *Assume Condition (UE) and (1.13). Let $X(0) = x \in \mathbb{X}_F$. Then there exists an a.s. finite random variable $N = N(\omega)$, whose precise definition is given in (5.1) below, such that, for some $c_1, c_2 > 0$, $\mathbb{P}[N > k] \leq c_1 e^{-c_2 k}$ for all $k \in \mathbb{Z}_+$, and, $\mathbb{P}$ -a.s., $\{N, N+1, \dots\}$ is a semi-infinite (ω, x) -stable cloud, as defined in Definition 1.3.*

Note that Theorem 1.7 does not need Condition **(N)** or **(LH)**. We expect that it should be possible to extend the range of initial conditions for which Theorem 1.7 holds: see Remark 5.1 below. A more interesting case is

$$\mathbb{E} \log \frac{\Theta^-}{\Theta^+} > 0. \quad (1.14)$$

In this case, we show that there are infinitely many stable clouds, and all particles will go to $-\infty$ at speeds whose magnitudes decrease polynomially with the particle label.

Theorem 1.8. *Assume Conditions **(N)**, **(UE)**, and (1.14). Let $\beta > 0$ be the unique positive number such that*

$$\mathbb{E} \left[(\Theta^+ / \Theta^-)^{1/\beta} \right] = 1.$$

Let $X(0) = x \in \mathbb{X}$. Then for almost all environments the system is decomposed into an infinite number of finite stable clouds. To every particle $n \in \mathbb{N}$ there is an associated speed $v_n = v_n(\omega)$ such that, for $\mathbb{P}$ -a.a. ω we have the strong law

$$\lim_{t \rightarrow \infty} t^{-1} X_n(t) = v_n, \quad \mathbb{P}_\omega^x\text{-a.s.}, \quad (1.15)$$

and we have, for $\mathbb{P}$ -a.a. ω , $v_n = -n^{-\beta(1+o(1))}$ as $n \rightarrow \infty$.

Remarks 1.9. (i) Existence and uniqueness of $\beta > 0$ in the hypothesis of Theorem 1.8 follows from the fact that, using **(N)** and (1.14), the function $s \mapsto \log \mathbb{E}[(\Theta^+ / \Theta^-)^s]$ is strictly convex, has value 0 and negative derivative at $s = 0$, and tends to $+\infty$ as $s \rightarrow \infty$, so crosses 0 exactly once for $s > 0$.

(ii) In the KKS regime it is harder to “separate” clouds. In Sinai’s regime, the successive record-well depths grow exponentially with the well index, yielding strong separation between the associated speeds. In the KKS regime, the deepest relevant wells up to particle n have depth only of order $\log n$, and the resulting speeds are separated only on polynomial scales. For this reason we do not provide an analogue of the detailed cloud identification in Theorem 1.4.

1.5 Some comments on the literature

The process of inter-particle separations associated to the exclusion process is a particular case of the *zero-range* process. Since the 1990s, there has been extensive study of zero-range and exclusion processes with disordered environments, including heterogeneous but deterministic, or randomly chosen, rates; in the exclusion process, rates attached to particles (as in our case) correspond to rates attached to *sites* in the zero-range process. However, to the best of the authors’ knowledge, the present setting is not considered in any previous work. We refer to [2–5, 11, 15] and references therein for extensive results on phenomena concerning convergence to equilibrium and hydrodynamic limits. Our emphasis is different: rather than equilibrium or hydrodynamic behaviour, we study the almost-sure large-scale decomposition of a fixed random environment into dynamically coherent particle clouds, and relate this decomposition to the record-well structure of its random potential.

The papers closest to the present setting include work of Benjamini *et al.* [6], Krug and Ferrari [23], and Evans [8, 9], and a more recent programme of Bahadoran *et al.* [3–5]. In several of these papers the (non-random or random) environment is chosen so that there is a global drift of particles in one direction (for example, in [9, 23] the process is totally asymmetric with a random environment of one-directional rates attached to particles and taking values in a bounded interval), and various aspects of the dynamics are dominated by the slowest particles.

Another set of literature looks at exclusion processes in which the random environment is attached to sites, rather than particles: we refer to, e.g. [10, 12, 18, 24, 34]. Because of the connection between exclusion processes and Jackson queueing networks (see Section 2.1 below) there is also to be found some adjacent literature on queueing networks in random environment: we refer to, e.g. [14, 22, 39].

1.6 Outline of the paper and proof strategy

Section 2 gives some preparatory results, partially known from prior work, including the representation of the particle system as a Jackson network (§2.1), some results on finite systems (§2.2), and some results on stable clouds of infinite systems (§2.3). Section 3 introduces the potential function that is our main tool in the proofs in both Sinai and KKS regimes. Section 4 gives the proofs in the Sinai regime, constructing potential wells and analysing their depths (§4.1), identifying the candidate clouds and controlling their speeds (§4.2), and then presenting in turn the proofs of Theorems 1.4 and 1.5 (§4.3), up to the exact identifications of the distributions of Y and Z , which are given in §4.4. The proofs for the KKS regimes are given in Section 5. The Appendix contains Lemma A.1 on the condition **(LH)**.

Before getting into technical details, let us give some heuristics about the role of the potential function that lies at the heart of this paper. First, it turns out (due to explicit formulas seen e.g. in [27]) that a putative cloud of particles is stable if its shape corresponds to a well in the potential, and, moreover, the depth of that well determines, roughly, the speed of that cloud. The main difficulty is to pass from potential wells to clouds in the interacting infinite system. A block that is stable in isolation need not remain a separate cloud when other particles are present. In Sinai's regime, the potential function is approximately a trajectory of Brownian motion, and, roughly, scale invariance shows that the statistics of successive wells are related one to the next by a random scale factor. Understanding this picture gives rise to the main results on Sinai's regime. In the KKS regime the same potential picture persists, but the potential has non-zero drift, leading to a different hierarchy of wells and hence of cloud speeds.

2 Preparatory results

2.1 Jackson network representation

In this section we recall from [28] a representation of the particle system in terms of a Jackson network of infinitely many queues; the link between exclusion processes and Jackson networks is well known, going back at least to [21] (see [27, 28] for further literature).

First, in some generality, consider a countably infinite system of queues, enumerated by $\mathbb{N}$. The parameters of the system are arrival rates $\lambda = (\lambda_i)_{i \in \mathbb{N}}$, service rates $\mu = (\mu_i)_{i \in \mathbb{N}}$ and $P = (P_{ij})_{i,j \in \mathbb{N}}$, a sub-stochastic routing matrix. Exogenous customers entering the system arrive at queue $i \in \mathbb{N}$ via an independent Poisson process of rate $\lambda_i \in \mathbb{R}_+$. Queue $i \in \mathbb{N}$ serves customers at exponential rate $\mu_i \in \mathbb{R}_+$ (i.e., in Kendall's notation, we have an $M/M/1$ queue at each server). Once a customer at queue i is served, it is routed to a queue j with probability P_{ij} , while with probability $\delta_i := 1 - \sum_{j \in \mathbb{N}} P_{ij}$ the customer departs from the system.

Provided $\sum_{i \in \mathbb{N}} \lambda_i > 0$ and $\sum_{i \in \mathbb{N}} \delta_i > 0$, customers both enter and leave the system, and it is called an *open Jackson network*. We assume that every queue can be *filled*, meaning that, for every $i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ for which $\lambda_j > 0$ and $(P^k)_{ji} > 0$, and that every queue can be *drained*, meaning that, for every $i \in \mathbb{N}$, there is a $j \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ for which $\delta_j > 0$ and $(P^k)_{ij} > 0$.

We now explain the Jackson network connection to the particle system model as described in Section 1.1; we use the notation introduced there. Define $\eta_k(t) := X_{k+1}(t) - X_k(t) - 1$, to be the number of holes between consecutive particles at time $t \in \mathbb{R}_+$. Define

$$\mu_i := b_i + a_{i+1}, \text{ for } i \in \mathbb{N}, \quad (2.1)$$

and

$$\lambda_1 := a_1, \text{ and } \lambda_i := 0 \text{ for } i \geq 2. \quad (2.2)$$

Also define the matrix $P := (P_{i,j})_{i,j \in \mathbb{N}}$ by

$$\begin{aligned} P_{i,i-1} &:= \frac{b_i}{\mu_i} = \frac{b_i}{b_i + a_{i+1}}, \text{ for } i \geq 2; \\ P_{i,i+1} &:= \frac{a_{i+1}}{\mu_i} = \frac{a_{i+1}}{b_i + a_{i+1}}, \text{ for } i \geq 1; \end{aligned} \quad (2.3)$$

and $P_{i,j} := 0$ for all i, j with $|i - j| \neq 1$.

The process $\eta := (\eta_i)_{i \in \mathbb{N}}$ can, under appropriate conditions on the rates and initial states, be realised as the queue-length process for a corresponding Jackson network, namely, the Jackson network with parameters λ, μ and P given as functions of $(a_i, b_i)_{i \in \mathbb{N}}$ through formulas (2.1), (2.2) and (2.3).

In the queueing network, the customers waiting at queue k are equal in number to the unoccupied sites between particles k and $k+1$ in the particle system. Exogenous customers enter the queueing network only when the leftmost particle jumps to the left (rate a_1). Customers at queue k are served if either particle k jumps right (rate b_k) or if particle $k+1$ jumps left (rate a_{k+1}). If particle k jumps right, the customer is routed to queue $k-1$ (if $k \geq 2$, at rate $b_k = \mu_k P_{k,k-1}$) or leaves the system (if $k = 1$). If particle $k+1$ jumps left, the customer is routed to queue $k+1$ (at rate $a_{k+1} = \mu_k P_{k,k+1}$). Customers leave the system only when the leftmost particle jumps to the right (rate $\mu_1 \delta_1 = b_1$). Then a particle conservation argument leads to a construction of X_1 , the position of the left-most particle, via

$$X_1(t) - X_1(0) = - \sum_{k \in \mathbb{N}} (\eta_k(t) - \eta_k(0)), \text{ whenever } \sum_{k \in \mathbb{N}} \eta_k(0) < \infty. \quad (2.4)$$

Under condition **(UE)**, existence and construction for both the queueing network and the associated particle system is given for fixed environment in [28].

2.2 Some results on finite systems

Later we will use repeatedly the concept of *intrinsic stability* of a group $[i, j]$ of particles in the semi-infinite system, which means that if all particles outside the group were removed, then the whole system would be a single stable cloud (in the sense of Definition 1.2); we give a definition in terms of formulas at Definition 3.1 below. In our later argument we often also use coupling and comparison results between the infinite system and various finite systems. To study these finite systems we rely on some relevant results from [27], which we summarize in this section.

The finite system of n particles (hence $n-1$ queues) has parameters $a_1, b_1, \dots, a_n, b_n$, having the same meaning as in Section 1.1, so that particle $k \in [1, n]$ performs a nearest neighbour random walk, rate a_k to the left and b_k to the right, subject to the exclusion rule. The Jackson network correspondence is similar to that described in Section 2.1 (see [27, §3]), but now customers may also enter or depart the system from queue $n-1$ (at rates b_n and a_n , respectively). We denote by $X_{n,k}(t)$ the location of particle k at time t .

Let us recall the following notations from [28]:

$$\alpha_0 := 1, \text{ and } \alpha_k := \frac{a_1 \cdots a_k}{b_1 \cdots b_k} \text{ for } k \in \mathbb{N}; \quad (2.5)$$

$$\beta_0 := 0, \text{ and } \beta_k := \frac{1}{b_k} + \frac{a_k}{b_k b_{k-1}} + \cdots + \frac{a_k \cdots a_2}{b_k \cdots b_1} \text{ for } k \in \mathbb{N}. \quad (2.6)$$

Also define

$$v_n := \frac{1 - \alpha_n}{\beta_n} = \frac{b_1 \cdots b_n - a_1 \cdots a_n}{b_1 \cdots b_{n-1} + b_1 \cdots b_{n-2} a_n + \cdots + a_2 \cdots a_n}. \quad (2.7)$$

and set $\rho_k := \alpha_k + v_n \beta_k$ for $k \in [1, n-1]$.

The following result gives the intrinsic speed of a finite stable system.

Proposition 2.1 (Finite stable systems; Corollary 2.6 in [27]). *Let $n \in \mathbb{N}$ and suppose that $a_k, b_k \in (0, \infty)$ for $1 \leq k \leq n$. Then if $\rho_k < 1$ for all $k \in [1, n-1]$ the n -particle system is a single stable cloud and, for every $k \in [1, n]$, we have*

$$\lim_{t \rightarrow \infty} \frac{X_{n,k}(t)}{t} = v_n, \text{ a.s.}$$

We will also consider modifications of the dynamics of a group $[i, j]$ in the semi-infinite system which we call *infinite pressure* from the right, which corresponds to setting $b_j := 0$, that is, the rightmost particle of the group is forbidden to jump to the right (i.e., one may imagine that there is *always* another particle just next to it, hence “infinite pressure”). We discuss some applications of that later; for now, let us note that it is intuitively clear that this operation preserves the intrinsic stability of a group. Putting the infinite pressure from the right on the n -particle system described above amounts to setting $b_n := 0$.

Remark 2.2. The results of [27] permit $a_i = 0$ and so, by swapping the roles of the a_i and b_i apply to the finite systems with infinite pressure considered here.

We need the following result about infinite-pressure systems.

Proposition 2.3 (Finite stable systems with infinite pressure). *Assume that the n -particle system is stable, and then change the value of b_n to zero. Then the resulting system is stable and its speed is given by*

$$\tilde{v}_n := -\frac{\alpha_n}{\beta_n} = -\frac{a_1 \cdots a_n}{b_1 \cdots b_{n-1} + b_1 \cdots b_{n-2}a_n + \cdots + a_2 \cdots a_n}. \quad (2.8)$$

(Observe that, even though α_n and β_n depend on b_n , the expression in (2.8) does not.)

Proof. Since the original system is stable, $\rho_k = \alpha_k + v_n \beta_k$ satisfies $\rho_k < 1$ for all $k \in [1, n-1]$. Setting $b_n = 0$ does not change the α_k, β_k for $k \in [1, n-1]$, but replaces v_n by $\tilde{v}_n \leq v_n$, obtained from removing the first term in the numerator from the rightmost expression in (2.7) (Corollary 2.6 in [27] applies, see Remark 2.2). Hence the ρ_k corresponding to the modified system also satisfy $\rho_k < 1$ for all $k \in [1, n-1]$. So the modified system is stable and its speed is $\tilde{v}_n$. $\square$

We finish this section with some stochastic monotonicity results.

Proposition 2.4. *Consider a finite system of particles labelled by $[1, n]$.*

(i) *For any $k \in [1, n-1]$, there is a coupling of the system $[1, n]$ and the system $[1, k]$ with infinite pressure from the right at k , such that particles $[1, k]$ start in the same relative positions and every particle in the smaller system is no farther to the right than its counterpart in the original system.*

(ii) *For any $k \in [1, n-1]$, there is a coupling of the system $[1, n]$ and the system $[k, n]$, such that particles $[k, n]$ start in the same relative positions and every particle in the smaller system is no farther to the right than its counterpart in the original system.*

Proof. We prove part (i). The coupling construction uses the *basic coupling* (or Harris graphical coupling, see [25, Ch. II]) for the exclusion dynamics. We describe the coupling between system $[1, n]$ (with particle positions $X_1(t), \dots, X_n(t)$) and system $[1, k]$ with infinite pressure at k , whose particle positions we denote $Y_1(t), \dots, Y_k(t)$. We start the coupling with $X_i(0) = Y_i(0)$ for $1 \leq i \leq k$ and use the same Harris graphical construction to execute jumps. We claim the ordering $Y_i(t) \leq X_i(t)$ for all $i \in [1, k]$ and all $t \geq 0$ is preserved. To verify the ordering is preserved, suppose it holds immediately before a clock rings. For the left-jump of particle i , we must show that X_i cannot jump left when Y_i is blocked at a time when $X_i = Y_i$ (we suppress

the time index for ease of reading). Indeed, if Y_i is blocked then $Y_{i-1} = Y_i - 1$ and since $Y_{i-1} \leq X_{i-1} < X_i = Y_i$ we must have $X_{i-1} = X_i - 1$, so the jump of X_i is also blocked. A similar argument shows that for $i < k$, we cannot have that Y_i jumps right when X_i is blocked, from $X_i = Y_i$. It follows that the ordering is preserved at every graphical event, demonstrating the existence of the coupling claimed in part (i). A similar coupling applies for the removal of the particles to the left and proves part (ii). $\square$

2.3 On stable clouds of infinite systems

Here we briefly discuss a general question about how to identify *finite* stable clouds of a semi-infinite system. This question in its full generality was intentionally left out of our paper [28]. To indicate the difficulty, note that the notion of a stable cloud in the semi-infinite case *may* depend on the initial configuration, as the following example shows. Consider first a stable semi-infinite system that has more than one admissible solution, with the corresponding speeds of these admissible solutions corresponding (say) to the interval $[0, 2]$ (to construct such an example see Example 1.17 of [28]). To the left of this semi-infinite system, add one particle with intrinsic speed equal to 1. Then, when starting from a finite initial configuration, the whole system will be stable (that is, we would have only one stable cloud), with zero speed. However, if we start at a sufficiently sparse configuration where the semi-infinite cloud is able to achieve speed 2, then it will separate from the unit speed leftmost particle, meaning that there will be two stable clouds in the system.

In the context of the present paper, the phenomenon described above does not occur. Instead, we will be typically in the situation where there are infinitely many stable clouds with strictly ordered speeds. In this case we are able to make use of couplings between the infinite system and appropriate finite systems as described in the following lemma.

Lemma 2.5. *Suppose that, for some $k_0 > 0$, after putting an infinite pressure (from the right) to k_0 th particle (i.e., setting $b_{k_0} := 0$, thus “detaching” the first k_0 particles from the rest of the system), the resulting finite system can be decomposed into strictly more than one stable clouds, $\mathcal{C}_1, \dots, \mathcal{C}_m, \mathcal{C}_{m+1}$, where $k_0 \in \mathcal{C}_{m+1}$. Let $s_1 \leq s_2 \leq \dots \leq s_{m+1} < 0$ be the speeds of these clouds; assume now that there exists $k \leq m$ such that $s_k < s_{k+1}$ (i.e., there is a transient queue separating $\mathcal{C}_k$ from $\mathcal{C}_{k+1}$).*

In such situation, there exists a coupling of the original queue process η with the infinite pressure queue process $\tilde{\eta}$, such that there is a random time T with the property that for all $j = 1, \dots, k$ it holds that $\eta(t)|_{\mathcal{C}_j} = \tilde{\eta}(t)|_{\mathcal{C}_j}$ for all $t \geq T$.

Proof. The stated coupling can be described in the following way: the customers in $\tilde{\eta}$ are *first-class customers*, and additional customers that may come to the original system in comparison with the infinite pressure one are *second-class customers*¹ which have lesser priority in service (i.e., they can only be served when there are no first-class customers in the queue). For the queue process $\tilde{\eta}$, let T_0 be a random time (which is not Markovian) after which the transient queues which separate the stable clouds never become empty. Then the crucial observation is that the second-class customers never leave such queues after T_0 , and therefore they are not transferred to $\mathcal{C}_1, \dots, \mathcal{C}_k$. Then, those second-class customers which were present in $\eta|_{\mathcal{C}_j}$, $j = 1, \dots, k$ at T_0 will eventually either leave the system or get stuck in a transient queue forever.

Indeed, consider any fixed second-class customer which is to the left of the transient queue separating $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$. For each stable cloud $\mathcal{C}_j$, the first-class Jackson network on its finitely many internal queues is irreducible positive recurrent, so it returns to the all-zero state infinitely often a.s. At each such return there is a strictly positive probability that a given second-class customer present in that cloud is served through a prescribed finite sequence of transitions and exits the cloud before the next first-class arrival. Since there are only finitely many second-class

¹That is, the second-class customers are those that come to the rightmost queue from the outside.

customers to the left of the transient queue at time T_0 , and no new second-class customers can enter this region after T_0 , the preceding probabilities may be bounded below uniformly by a positive constant. Consider now any separating queue lying to the left of the transient queue between $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$. Pathwise, either this queue is empty infinitely often, or there is a finite random time after which it is never empty. In the latter case, any second-class customer which reaches this queue is trapped there forever. In the former case, at each time at which the first-class queue is empty, a second-class customer present there has a probability bounded below by a strictly positive constant of being served before the next first-class arrival. Hence, on the event that the first-class queue is empty infinitely often, every second-class customer present there is eventually served from the queue almost surely.

Thus a tagged second-class customer cannot remain forever in any stable cloud, and at each separating queue it is either eventually passed on or becomes trapped there forever. Since there are only finitely many stable clouds and separating queues to the left of the transient queue between $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$, every second-class customer present to the left of this separator at time T_0 is eventually either removed from the system to the left or trapped forever in a separating queue. Since there are only finitely many such second-class customers, there is a finite random time T after which no second-class customer is present in any of the internal queues of $\mathcal{C}_1, \dots, \mathcal{C}_k$. $\square$

Corollary 2.6. *Assume that there are two sequences of positive integers $k_1 < k_2 < k_3 < \dots$ and $k'_1 < k'_2 < k'_3 < \dots$ also with the property that $k'_j < k_j$ for all j , such that, when putting infinite pressure at k_j , the k'_j th queue is transient. Then the system is decomposed into infinitely many finite clouds with non-decreasing speeds, and there is an infinite number of transient queues.*

Proof. Couple two infinite-pressure truncations at $k < \hat{k}$ by regarding the additional customers generated by the particles in $(k, \hat{k}]$ as second-class customers. Starting from the same queues on $[1, k-1]$, this gives $\eta_i^{(k)}(t) \leq \eta_i^{(\hat{k})}(t)$ (superscript indicating the truncation), for all $i < k$ and all $t \geq 0$. Hence any queue which is transient in the truncation at k remains transient after moving the truncation point to the right.

Fix r , and choose J sufficiently large. Then the truncation at k_J has the transient queues $k'_1, \dots, k'_r$, and hence at least $r+1$ stable clouds separated by these queues. Apply Lemma 2.5 using the transient separator k'_r . The internal queue processes of all clouds lying to its left eventually coincide with those in the infinite system. Moreover, under the same coupling the separating queues in the infinite system dominate their finite-system counterparts, so the separating gaps still diverge in probability. Thus these are stable clouds of the infinite system, with the same, non-decreasing speeds as in the finite truncation. Since $k'_r \rightarrow \infty$, every particle belongs to such a finite cloud. Finally, the domination also shows that every k'_r remains transient in the infinite system. Hence there are infinitely many finite clouds and infinitely many transient queues. $\square$

3 Finding stable clouds with the potential

In this section we define the potential and argue that it is a convenient tool for identifying stable clouds and their speeds. The *potential function* (or simply *potential*) $V : \mathbb{Z}_+ \rightarrow \mathbb{R}$ is defined by

$$V(0) := 0, \text{ and } V(n) := \sum_{j=1}^n \log \frac{a_j}{b_j} \text{ for } n \in \mathbb{N}. \quad (3.1)$$

The potential is used extensively in the context of one-dimensional RWRE: see e.g. the definition of $w^{(n)}(t)$ on p. 257 of [35], or the first display in Section 2 of [13]. In the context of RWRE the idea is that one frequently represents various quantities associated to the behaviour of the random walk in terms of the potential, so that an analysis of the behaviour of the potential

leads to conclusions about the behaviour of the random walk (in a particular environment). In our context, something similar is true, as we shall explore in this section.

We point out that the choice of the initial condition $V(0) = 0$ is rather arbitrary (although, of course, natural). In the following, when discussing intrinsic stability of some finite groups of particles, we will frequently consider finite pieces of the potential that do not necessarily start at zero; therefore, we typically present formulas in terms of the potential *differences* so that they can be applied whether or not the leftmost particle we are considering at the time is the leftmost particle overall.

We will frequently need to consider finite groups of consecutive particles, and answer questions like if that group would be intrinsically stable (meaning it would be stable if all the other particles were removed from the system). First, let us note that the formulas (2.5)–(2.6) can be rewritten in the following way in terms of the potential:

$$\alpha_k := e^{V(k)-V(0)} \text{ for } k \in \mathbb{Z}_+, \text{ and } \beta_k := \sum_{\ell=1}^k a_\ell^{-1} e^{V(k)-V(\ell-1)} \text{ for } k \in \mathbb{N}. \quad (3.2)$$

Then, consider a group $[i_0+1, \dots, i_0+n]$ together with the potential V on the interval $[i_0, \dots, i_0+n]$ and observe that the formulas (2.16) and (2.17) of [27] can be rewritten as

$$\rho_k := e^{V(k)-V(i_0)} + \hat{v} \sum_{\ell=i_0+1}^k a_\ell^{-1} e^{V(k)-V(\ell-1)}, \text{ for } k \in [i_0+1, i_0+n], \text{ where} \quad (3.3)$$

$$\hat{v} := \frac{1 - e^{V(i_0+n)-V(i_0)}}{\sum_{\ell=i_0+1}^{i_0+n} a_\ell^{-1} e^{V(i_0+n)-V(\ell-1)}}. \quad (3.4)$$

Definition 3.1 (Intrinsic stability). We say the group $[i_0+1, \dots, i_0+n]$ of n particles is *intrinsically stable* if $\rho_k < 1$ for all $k \in [i_0+1, \dots, i_0+n-1]$, where ρ_k is given by (3.3). If the group is intrinsically stable, we say its *intrinsic speed* is $\hat{v}$ given by (3.4).

Remarks 3.2. (i) If the group $[i_0+1, \dots, i_0+n]$ is intrinsically stable, then when considered as an isolated n -particle system it forms a single stable cloud (in the sense of Definition 1.2) and has speed $\hat{v}$ given by (3.4) (cf. Proposition 2.1).

(ii) If the group $[i_0+1, \dots, i_0+n]$ is intrinsically stable, then so is the same group with infinite pressure from the right, i.e., when we set $b_{i_0+n} = 0$. This follows in the same way as for isolated systems as in Proposition 2.3.

(iii) If the group $[i_0+1, \dots, i_0+n]$ is intrinsically stable, then, due to an elementary coupling argument, it is contained in a single stable cloud in the system $[1, m]$ (with or without infinite pressure at m), where $m \geq i_0+n$.

An immediate observation is that, in the case when the group $[i_0+1, \dots, i_0+n]$ is intrinsically stable, the speed $\hat{v}$ is positive (respectively, negative, zero) if $V(i_0+n) < V(i_0)$ (respectively, $V(i_0+n) > V(i_0)$, $V(i_0+n) = V(i_0)$). So, we will frequently study finite pieces of the potential in order to draw conclusions about the intrinsic stability and speed of the corresponding group of particles. Also, in this section we will typically assume $i_0 = 0$, since this obviously does not restrict generality.

Having motivated the relevance of the potential, the main goal of this section is to understand how the potential V of stable (and unstable) clouds should behave. As an initial easy observation, note that if V looks like in Figure 2, then we have a stable cloud. More precisely, the following fact holds:

Lemma 3.3. *Assume that $V(0) = V(n) = 0$; then, the group $[1, n]$ is intrinsically stable (and has zero speed) if and only if $V(k) < 0$ for all $k \in [1, n-1]$.*

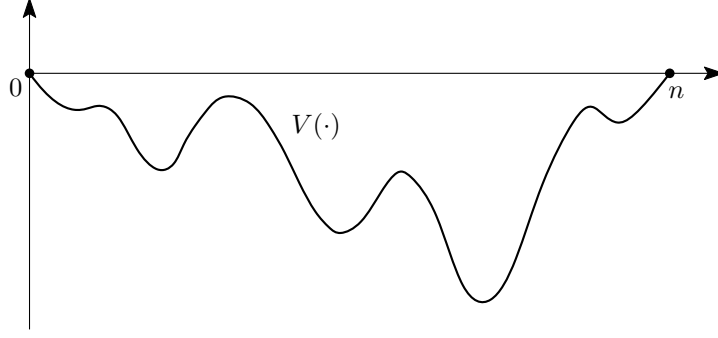


Figure 2: An example of a potential profile of an intrinsically stable, in the sense of Definition 3.1, system $[1, n]$ (see Lemma 3.3): $V(0) = V(n) = 0$, and $V(k) < 0$ for $k \in [1, n - 1]$. In this and subsequent illustrations of the potential, we often take the licence to represent the potential by a smooth curve.

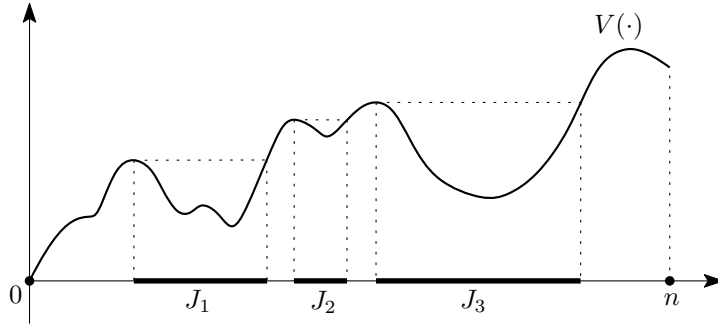


Figure 3: Finding stable clouds: for each $j = 1, 2, 3$, the particles from J_j form an intrinsically stable group. We do *not* claim that, in the infinite system, e.g., J_1 and J_2 should be in *different* stable clouds in the sense of Definition 1.2.

Proof. Indeed, (3.4) implies that $\hat{v} = 0$ so, by Definition 3.1 and (3.3), we have that the cloud is stable iff $\rho_k = e^{V(k)} < 1$ for all $k \in [1, n - 1]$, which means $V(k) < 0$ for all $k \in [1, n - 1]$. $\square$

The above is a very basic and also somewhat “rare” example (it is not easy to be so lucky to have $V(n)$ exactly equal to 0); the next situation is more general:

Lemma 3.4. *Assume that $V(k) \leq 0 \leq V(n)$ and $V(k) < V(n)$ for all $k \in [1, n - 1]$. Then $[1, n]$ is intrinsically stable.*

Proof. Suppose that $[1, n]$ is *not* intrinsically stable; consider its decomposition into stable clouds. Let v be the speed of the leftmost stable cloud and v^* be the speed of the rightmost one. Then, (3.4) would imply that $v \geq 0$ and $v^* < 0$; this is a contradiction, since cloud speeds are non-decreasing from left to right (see [27, Theorem 2.1]). $\square$

It is important to note that we can use Lemma 3.4 to identify groups that should be parts of stable clouds; see Figure 3. Also note that this procedure finds clouds that have non-positive intrinsic speeds (one can also formulate an analogous result for finding clouds with non-negative intrinsic speeds, but the present form of the lemma is better tailored for the needs of this paper).

The following result goes in the converse direction: it shows that for stable clouds the global maximum for the potential must be attained at (one or both of) the extremities.

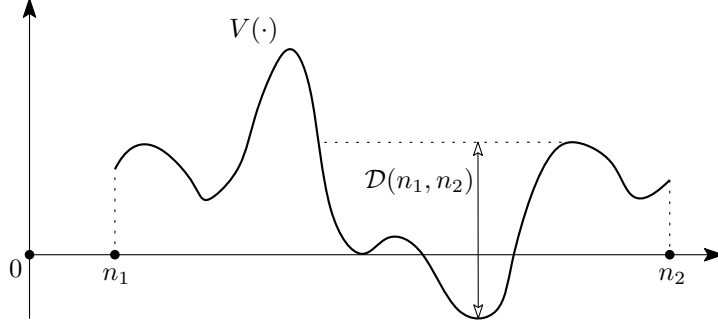


Figure 4: Illustration of the definition of the depth at (3.6).

Lemma 3.5. *Assume that group $[1, n]$ is intrinsically stable, and let*

$$M = \{k \in \{0, \dots, n\} : V(k) = \max_{0 \leq j \leq n} V(j)\}$$

be the set of the global maxima of the potential on the interval $[0, n]$. Then $M \subseteq \{0, n\}$.

Proof. Note that we can assume without restricting generality² that $V(0) \geq V(n)$. Then $\hat{v} \geq 0$ by (3.4), so (3.3) gives $\rho_k \geq e^{V(k)-V(0)}$, and the stability condition $\rho_k < 1$ implies $V(k) < V(0)$ for all $k \in [1, n-1]$. Since also $V(n) \leq V(0)$, we get $M \subseteq \{0, n\}$. $\square$

As discussed in Section 2.2, it is a simple observation that, if we have a stable finite group $[1, n]$, then setting b_n to zero preserves the stability (and makes the speed smaller); recall that we refer to this operation as putting *infinite pressure* to the right extreme of the group. As before, we will typically use the notation $\tilde{v}$ for denoting the speed with infinite pressure from the right. This is also useful for the original semi-infinite system: if we have a good estimate on $\tilde{v}$, then we know that, whatever the particles on the right do, they cannot increase the magnitude of the speed of our cloud beyond $|\tilde{v}|$. Let us rewrite the formula for the speed under infinite pressure (see Proposition 2.3) in terms of the potential, with the help of (3.2):

$$\tilde{v} = -\frac{e^{V(n)-V(0)}}{\frac{1}{a_n}e^{V(n)-V(n-1)} + \dots + \frac{1}{a_1}e^{V(n)-V(0)}} \quad (3.5)$$

(note that $e^{V(n)}$ will cancel, so, in fact, the above expression does not depend on b_n).

Now, we define one of the central concepts of this paper. For $[n_1, n_2] \subset \mathbb{Z}_+$, let the *depth* of the potential on this subinterval be defined by

$$\mathcal{D}(n_1, n_2) = \max_{n_1 \leq k \leq n_2} \max_{\substack{n_1 \leq \ell_1 \leq k \\ k \leq \ell_2 \leq n_2}} ((V(\ell_1) - V(k)) \wedge (V(\ell_2) - V(k))). \quad (3.6)$$

In words, the depth is the “maximal height to overcome” when going out of the interval, maximized over all starting points: see Figure 4. Also note that (3.6) takes the convenient form

$$\mathcal{D}(n_1, n_2) = \max_{n_1 \leq k \leq n_2} \left(\max_{n_1 \leq \ell \leq k} V(\ell) - V(k) \right) \text{ whenever } V(n_2) = \max_{n_1 \leq j \leq n_2} V(j), \quad (3.7)$$

i.e., the depth is equal to the maximal descent from the running maximum when the maximum potential is attained at the end of the interval.

A very important observation (which will be made rigorous in Lemma 3.6 below), is that regions of significant depth, or *potential wells* (such as the one below the horizontal dotted line

²This is because one can “mirror” the system by setting the transition rates to $(b_n, a_n), \dots, (b_1, a_1)$; then the potential will also be “mirrored”.

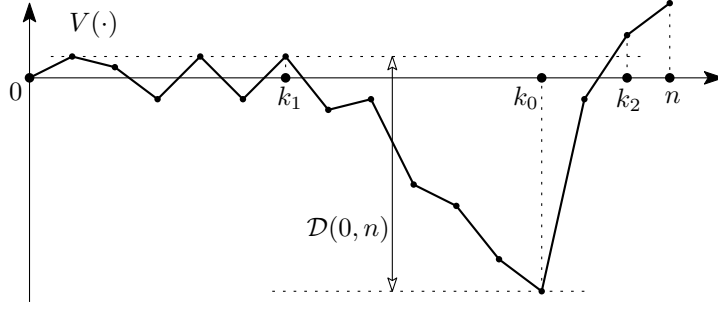


Figure 5: Illustration for the proof of Lemma 3.6; this time, for more precision, the illustration retains the discrete nature of the potential.

in Figure 4) tend to slow down the corresponding particles due to what we call *jamming*: the particles in the left part of the well tend to want to go to the right, while the particles in the right part of the well tend to want to go to the left, which results in a “traffic jam”. For this reason, we also refer to such regions as *traps*; the role of traps is analogous to their role in localization for RWRE.

The next result says that the speed of a stable cloud is roughly inversely proportional to the exponential of the depth of the associated potential well.

Lemma 3.6. *Assume that $\varepsilon_0 \leq a_k \leq \varepsilon_0^{-1}$ and $\varepsilon_0 \leq b_k \leq \varepsilon_0^{-1}$ for some $\varepsilon_0 > 0$, and all $k \in [1, n]$ (i.e., the same assumption as in (UE)). Suppose that group $[1, n]$ is intrinsically stable, and assume also that $V(n) = \max_{0 \leq j \leq n} V(j)$, which implies that the intrinsic speed satisfies $\hat{v} \leq 0$. Denote by $\tilde{v} < 0$ the speed of group $[1, n]$ with the infinite pressure from the right³ (note that $|\tilde{v}| > |\hat{v}|$ by assumption that all $b_k > 0$). Then with $\mathcal{D}$ defined at (3.6) we have*

$$|\tilde{v}| \leq \varepsilon_0^{-1} e^{-\mathcal{D}(0, n)}, \quad (3.8)$$

and

$$|\hat{v}| \geq \varepsilon_0 n^{-1} (1 - e^{-(V(n) - V(0))}) e^{-\mathcal{D}(0, n)}. \quad (3.9)$$

Proof. We refer to Figure 5 for a visualization of several of the quantities defined in this proof. First observe from (3.7) and the fact that $V(n) = \max_{0 \leq j \leq n} V(j)$ that

$$\mathcal{D}(0, n) = \max_{0 \leq k \leq n} \left(\max_{0 \leq \ell \leq k} V(\ell) - V(k) \right). \quad (3.10)$$

To prove (3.9), let $k' \in [0, n]$ be the smallest k such that $V(k) = \min_{0 \leq j \leq n} V(j)$. In the formula (3.4) for $\hat{v}$, since $V(n) \geq V(0)$, the absolute value of the numerator equals $e^{V(n) - V(0)} (1 - e^{-(V(n) - V(0))})$, while the denominator is bounded above by $\varepsilon_0^{-1} n e^{V(n) - V(k')}$. That is,

$$|\hat{v}| \geq \varepsilon_0 n^{-1} (1 - e^{-(V(n) - V(0))}) e^{-(V(0) - V(k'))},$$

where, by (3.10), $\mathcal{D}(0, n) \geq \max_{0 \leq k \leq n} (V(0) - V(k)) = V(0) - V(k')$, which gives (3.9).

Next, we prove (3.8), so we consider the system with infinite pressure from the right. Let k_0 be the largest $k \in [0, n]$ for which we have

$$\max_{0 \leq \ell \leq k} V(\ell) - V(k) = \mathcal{D}(0, n),$$

i.e., the largest site where the maximal descent from the running maximum is achieved. Define k_1 to be the largest $k \in [0, k_0]$ such that $V(k) = \max_{0 \leq j \leq k_0} V(j)$. Note that $V(k_1) \leq V(n)$, by hypothesis. Observe that, by construction and (3.10), $V(k_1) - V(k_0) = \mathcal{D}(0, n)$.

³Recall from Remark 3.2(ii) that the system with infinite pressure is also stable.

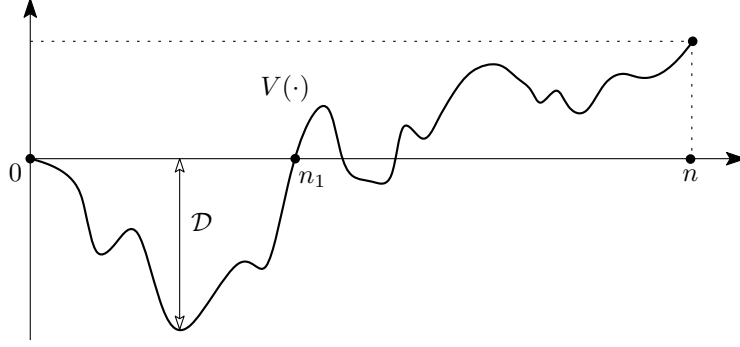


Figure 6: Another situation with one stable cloud.

Next note that $k_0 = n$ if and only if $\mathcal{D}(0, n) = 0$, which by (3.10), occurs if and only if $V(0) \leq V(1) \leq \dots \leq V(n)$. In this case (3.8) follows directly from the fact that the denominator in (3.5) is bounded below by $\varepsilon_0 e^{V(n)-V(0)}$. Suppose instead that $k_0 < n$. Then define

$$k_2 := \min \{k > k_0 : V(k) \geq V(k_1)\},$$

and observe that the group $[k_1 + 1, k_2]$ is intrinsically stable, by Lemma 3.4. Then, observe that the magnitude of the speed of this cloud with infinite pressure from the right will be at least $|\tilde{v}|$, because (i) the infinite pressure at k_2 is at least the “pressure” from the presence of any particles in $[k_2 + 1, n]$ and infinite pressure at n , and (ii) any particles in $[0, k_1]$ serve to slow down the cloud’s movement to the left. More formally, start the system $[1, n]$ with infinite pressure at n , and the subsystem $[k_1 + 1, k_2]$ from corresponding positions, with particles $[1, k_1]$ removed and infinite pressure imposed at k_2 . Then applying both parts of Proposition 2.4 shows that there is a coupling under which each particle of the subsystem remains no farther to the right than its counterpart in the original system. Hence an upper bound for $|\tilde{v}|$ is the magnitude of the speed of the system $[k_1 + 1, k_2]$ with infinite pressure from the right. For the latter, we use the appropriate version of the formula (3.5), the denominator of which is bounded below by $\varepsilon_0 e^{V(k_2)-V(k_0)}$ (recall that $k_0 < k_2$). Hence we get

$$|\tilde{v}| \leq \varepsilon_0^{-1} e^{V(k_2)-V(k_1)-(V(k_2)-V(k_0))} = \varepsilon_0^{-1} e^{-(V(k_1)-V(k_0))},$$

which yields (3.8). $\square$

To give a flavour of what we are going to do in the next section, let us discuss the situation shown in Figure 6: we want to argue that the whole cloud has to be intrinsically stable. The idea is that, first, by Lemma 3.4, the particles in $[1, n_1]$ should belong to the same stable cloud, and, by (3.8), the absolute value of the speed of that cloud has to be quite small, around $e^{-\mathcal{D}}$ at most. Now, let us think of the stable cloud that contains the rightmost particle (n). If we assume that this cloud does not contain those particles with indices in $[1, n_1]$, then, by (3.9), the absolute value of its speed would be much larger than $e^{-\mathcal{D}}$ due to the fact that $\mathcal{D}(n_1, n) < \mathcal{D}$ (note also that the speed has to be strictly negative since the maximum of the potential is at n). The associated strong law for finite systems (see Theorem 2.1 of [27]) then gives a contradiction, as particles to the right would want to go to the left faster than particles to the left. It follows that there can be only one stable cloud, travelling to the left with speed of roughly $e^{-\mathcal{D}}$.

4 Proofs in Sinai’s regime

4.1 The construction and growth of record wells

In Sinai’s regime, conditions (UE) and (1.3) ensure that the potential defined by (3.1) is a trajectory of a random walk whose increments are bounded and have mean zero. As already

discussed in Section 3, the behaviour of the potential can be linked to the problem of determining which groups of the particles in the particle system form stable clouds, and what the intrinsic speeds of these clouds are. In this section we give an explicit scheme for determining the stable clouds from the potential, and use the fact that the potential is well-approximated by a Brownian motion trajectory to draw conclusions about statistics of those stable clouds.

We describe a construction that lets us identify *potential wells* through successive local maxima and minima of the potential, and to define associated well depths: see Figure 7 for a diagram. First define the descent depth from the current maximum as

$$\mathcal{L}(x) := \max_{0 \leq y \leq x} V(y) - V(x). \quad (4.1)$$

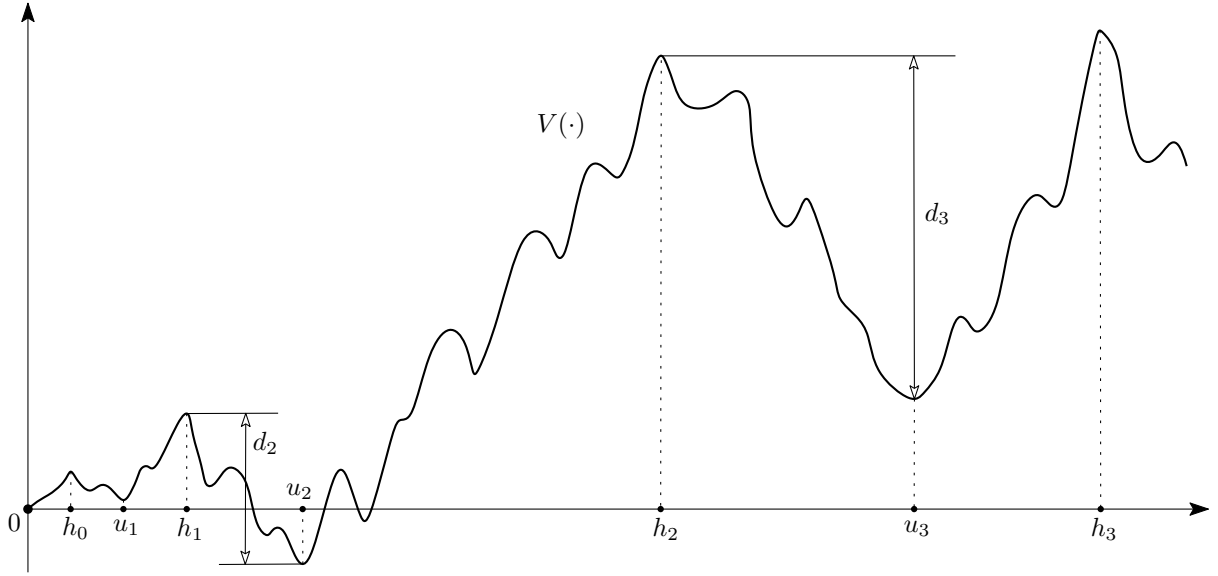


Figure 7: Illustration of the definition of potential wells. Roughly, the h_k and u_k are interlacing local maxima and minima, and the d_k are the corresponding depths of the wells.

Then, we set (here and in the sequel $\arg \min$ and $\arg \max$ always choose the leftmost candidate in case there are ties)

$$\begin{aligned} h_0 &:= \min\{n \geq 0 : V(n+1) < V(n)\}, \\ \ell_1 &:= \min\{x > h_0 : \mathcal{L}(x) > 0\}, \\ m_1 &:= \min\{x > \ell_1 : V(x) > V(h_0)\}, \\ u_1 &:= \arg \min_{x \in [h_0, m_1]} V(x), \\ d_1 &:= \mathcal{L}(u_1), \end{aligned}$$

and then, inductively, for $k \in \mathbb{N}$,

$$\begin{aligned} \ell_{k+1} &:= \min\{x > m_k : \mathcal{L}(x) > d_k\}, \\ h_k &:= \arg \max_{x \in [m_k, \ell_{k+1}]} V(x), \\ m_{k+1} &:= \min\{x > \ell_{k+1} : V(x) > V(h_k)\}, \\ u_{k+1} &:= \arg \min_{x \in [h_k, m_{k+1}]} V(x), \\ d_{k+1} &:= \mathcal{L}(u_{k+1}). \end{aligned}$$

Also, denote for $k \in \mathbb{N}$,

$$\Delta_k := V(h_k) - \max_{0 \leq j < h_k} V(j) > 0 \quad (4.2)$$

for the ladder height at h_k . Since h_k is a strict record, $V(h_k) - V(j) \geq \Delta_k$ for all $j < h_k$. We note that u_k and h_k are *not* stopping times with respect to the sequence $V(0), V(1), \dots$, but m_k and ℓ_k are stopping times, which will be useful below. We will need the following properties of this construction (again, see Figure 7 for an illustration).

Lemma 4.1. *The above construction defines two integer-valued sequences $(h_k)_{k \geq 0}$ and $(u_k)_{k \geq 1}$ such that $0 \leq h_0 < u_1 < h_1 < u_2 < h_2 < \dots$ and the following properties hold.*

- (i) For all $k \in \mathbb{Z}_+$, $V(h_{k+1}) > V(h_k)$.
- (ii) For all $k \in \mathbb{Z}_+$, $V(h_k) = \max_{0 \leq x \leq u_{k+1}} V(x) = \max_{u_k \leq x \leq u_{k+1}} V(x)$ (set $u_0 := 0$).
- (iii) For all $k \in \mathbb{N}$, $d_k = V(h_{k-1}) - V(u_k) = \mathcal{D}(0, h_k) = \mathcal{D}(h_{k-1}, h_k)$.
- (iv) For all $k \in \mathbb{N}$, $\mathcal{D}(h_k, h_{k+1}) > \mathcal{D}(h_{k-1}, h_k)$.

Proof. First note that by the definition of m_k , for all $k \in \mathbb{N}$,

$$V(h_k) = \max_{m_k \leq x \leq \ell_{k+1}} V(x) \geq V(m_k) > V(h_{k-1}),$$

which proves part (i).

Next we prove (ii). By definition of h_0 , $V(h_0 + 1) < V(h_0)$ and $V(h_0) = \max_{0 \leq y \leq h_0+1} V(y)$, so that $\mathcal{L}(h_0 + 1) = V(h_0) - V(h_0 + 1) > 0$. Thus $\ell_1 = h_0 + 1$. Then by definition of m_1 we get $V(h_0) = \max_{0 \leq y < m_1} V(y)$, and since $u_1 < m_1$ this means $V(h_0) = \max_{0 \leq x \leq u_1} V(x)$, which verifies the $k = 0$ case of (ii). Suppose $k \in \mathbb{N}$. By definitions of h_k and m_{k+1} we have $V(x) \leq V(h_k)$ for all $m_k \leq x < m_{k+1}$. Hence also $V(x) \leq V(h_{k-1}) \leq V(h_k)$ for all $m_{k-1} \leq x < m_k$. Since $m_{k-1} \leq u_k \leq m_k \leq u_{k+1} < m_{k+1}$, it follows that $V(h_k) = \max_{u_k \leq x \leq u_{k+1}} V(x)$. This completes the proof of (ii).

For (iii), first observe that $d_k = \mathcal{L}(u_k) = V(h_{k-1}) - V(u_k)$, for $k \in \mathbb{N}$, by part (ii). Then by definition of ℓ_{k+1} we have $\mathcal{L}(x) \leq \mathcal{L}(u_k)$ for $m_k \leq x < \ell_{k+1}$. Also, for $h_{k-1} \leq x < m_k$ we have $V(h_{k-1}) \geq V(x) \geq V(u_k)$ so $\mathcal{L}(x) = V(h_{k-1}) - V(x) \leq \mathcal{L}(u_k)$. That is, $\mathcal{L}(x) \leq \mathcal{L}(u_k)$ for $h_{k-1} \leq x < \ell_{k+1}$ and in particular $\mathcal{L}(u_k) = \max_{h_{k-1} \leq x \leq h_k} \mathcal{L}(x)$ for all $k \in \mathbb{N}$. Also, $\mathcal{L}(\ell_{k+1}) > \mathcal{L}(u_k)$ where $h_k \leq \ell_{k+1} < h_{k+1}$, meaning that $\mathcal{L}(u_k) < \mathcal{L}(u_{k+1})$ for all $k \in \mathbb{N}$. Since $\max_{0 \leq x \leq h_0} \mathcal{L}(x) = 0$, we conclude that

$$d_k = \mathcal{L}(u_k) = \max_{0 \leq x \leq h_k} \mathcal{L}(x) = \max_{h_{k-1} \leq x \leq h_k} \mathcal{L}(x). \quad (4.3)$$

Now using part (ii) and the formula (3.7) with (4.3), we get $\mathcal{D}(0, h_k) = \max_{0 \leq x \leq h_k} \mathcal{L}(x) = d_k$. Moreover, by (3.7),

$$\mathcal{D}(0, h_k) \geq \mathcal{D}(h_{k-1}, h_k) \geq \max_{h_{k-1} \leq \ell \leq u_k} V(\ell) - V(u_k) = V(h_{k-1}) - V(u_k) = d_k.$$

This completes the proof of (iii), and (iv) also follows from the observation made above that $\mathcal{L}(u_k) < \mathcal{L}(u_{k+1})$ for all $k \in \mathbb{N}$. $\square$

In principle, one can understand how the above quantities behave by comparing the potential V to the Brownian motion, via Donsker's theorem, for example. Roughly speaking, we will have $h_{k+1} = Y_k h_k$ and $d_{k+1} = Z_k d_k$, where Y s and Z s are approximately i.i.d.; the details will be provided below.

Now, our programme is to show that

- For all large enough k , $[h_k + 1, h_{k+1}]$ are distinct stable clouds.

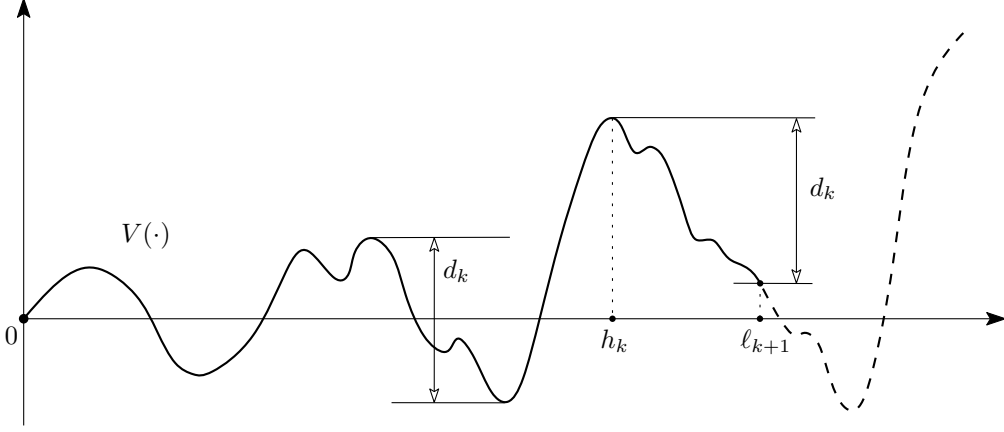


Figure 8: Constructing the next well in the proof of Lemma 4.2.

- All stable clouds go to the left (i.e., have negative speeds), and the speed of the cloud of $[h_k + 1, h_{k+1}]$ is roughly $-e^{-d_{k+1}}$.
- With v_n being the speed of n th particle and d being the depth of the well containing n , we have $v_n \approx -\exp(-d) \approx -e^{-\sigma Z \sqrt{n}}$, where Z is a function of the environment with distribution “almost independent” of n .

To make the above rigorous, we need some preparations. We intend to use the techniques of Section 3 (in particular, Lemma 3.4 and the discussion after it, together with the ideas described in the last paragraph of that section) to identify the stable clouds. Then, we will try to use Lemma 3.6 to estimate their speeds, and, in particular, show that these speeds strictly decrease (so that our candidate stable clouds are really stable clouds). In Lemma 3.6, the estimates for the speed are off by essentially a factor proportional to the cloud’s size, which, in our case (for the would-be cloud $[h_k + 1, h_{k+1}]$), can be bounded above by h_{k+1} , so, in principle, to “separate” the clouds, we need that $\frac{e^{-d_k}}{h_k} \gg e^{-d_{k+1}}$. Also, we need to take care of the height difference between the extremes, with Condition **(LH)**; as explained later, this would mean that instead of h_k we would have h_k to some power, which changes essentially nothing; that is, we need the following relation to hold, for a (large) fixed M

$$\frac{e^{-d_k}}{h_k^M} \gg e^{-d_{k+1}}. \quad (4.4)$$

The following result addresses this, showing that with high probability, each d_{k+1} has to be appreciably larger than d_k . Recall the definition of κ from (1.2), and denote $\mathcal{F}_m := \sigma(V(j) : 0 \leq j \leq m)$.

Lemma 4.2. *Assume Conditions **(N)**, **(UE)**, and (1.3). Then, for any $\delta > 0$, we have*

$$1 - \frac{\delta d_k + \kappa}{(1 + \delta)d_k + \kappa} \leq \mathbb{P}[d_{k+1} > (1 + \delta)d_k \mid \mathcal{F}_{\ell_{k+1}}] \leq 1 - \frac{\delta d_k - \kappa}{(1 + \delta)d_k + \kappa}. \quad (4.5)$$

Proof. We have to be a bit careful with the preceding construction since h_k ’s are not stopping times with respect to the random walk V . However, ℓ_k ’s are stopping times, so, when ℓ_{k+1} happens, the process V is “free to evolve” (see Figure 8). Define

$$\tau := \min \{n \geq \ell_{k+1} : V(n) \notin [V(h_k) - (1 + \delta)d_k, V(h_k)]\},$$

and

$$A := \{V(\tau) < V(h_k) - (1 + \delta)d_k\} = \{d_{k+1} > (1 + \delta)d_k\}.$$

Note that, using (1.2), we have

$$V(h_k) - d_k - \kappa \leq V(\ell_{k+1}) \leq V(h_k) - d_k. \quad (4.6)$$

We now use the optional stopping theorem for the martingale V with the stopping time τ : abbreviating $p := \mathbb{P}(A \mid \mathcal{F}_{\ell_{k+1}})$ and using (1.2) again, we get

$$\begin{aligned} V(\ell_{k+1}) &= \mathbb{E}(V(\tau)\mathbb{1}_A \mid \mathcal{F}_{\ell_{k+1}}) + \mathbb{E}(V(\tau)\mathbb{1}_{A^c} \mid \mathcal{F}_{\ell_{k+1}}) \\ &\geq p(V(h_k) - (1+\delta)d_k - \kappa) + (1-p)V(h_k) \\ &= V(h_k) - p((1+\delta)d_k + \kappa), \end{aligned}$$

so (4.6) implies that $p \geq \frac{d_k}{(1+\delta)d_k + \kappa} = 1 - \frac{\delta d_k + \kappa}{(1+\delta)d_k + \kappa}$. Analogously, we obtain that

$$\begin{aligned} V(\ell_{k+1}) &\leq p(V(h_k) - (1+\delta)d_k) + (1-p)(V(h_k) + \kappa) \\ &= V(h_k) + \kappa - p((1+\delta)d_k + \kappa); \end{aligned}$$

again using (4.6) we obtain that $p \leq 1 - \frac{\delta d_k - \kappa}{(1+\delta)d_k + \kappa}$ and conclude the proof of the lemma. $\square$

Next, we need a result that permits us to control the “second deepest trap” in $[h_{k-1} + 1, h_k]$.

Lemma 4.3. *Assume Conditions (N), (UE), and (1.3). For any $\delta \in (0, 1)$, for all $k \in \mathbb{N}$,*

$$\mathbb{P}[\mathcal{D}(m_k, h_k) \leq (1-\delta)d_k \mid \mathcal{F}_{m_k}] \geq \frac{1-\delta}{1+\kappa d_k^{-1}}. \quad (4.7)$$

Proof. Let $\delta \in (0, 1)$. For convenience write $\psi_k := (1-\delta)d_k$. Define

$$\sigma := \min\{n > m_k : \max_{m_k \leq s \leq n} V(s) - V(n) > \psi_k\},$$

and set $M := \max_{m_k \leq s \leq \sigma} V(s)$. Then define

$$\gamma := \min\{n \geq \sigma : M - V(n) > d_k\}, \text{ and } \tau := \min\{n \geq \sigma : V(n) > M\}.$$

We claim that $\{\gamma < \tau\} \subseteq \{\mathcal{D}(m_k, h_k) \leq \psi_k\}$. Indeed, if $\gamma < \tau$, then no new record above M is attained between σ and γ . Hence at γ the running maximum is still M , so $\mathcal{L}(\gamma) > d_k$. By definition of σ , this means that all descents before σ are at most $\psi_k < d_k$, so $\ell_{k+1} \in [\sigma, \gamma]$. Thus $\max_{m_k \leq s \leq \ell_{k+1}} V(s) = M$ and its leftmost attainment h_k occurs before σ . Therefore, by definition of σ , $\mathcal{D}(m_k, h_k) \leq \psi_k$.

Now consider the martingale $Y_n = M - V(n)$, $n \geq \sigma$. Applying the optional stopping theorem, and using the fact that $\psi_k < Y_\sigma \leq \psi_k + \kappa$, by (1.2), we get

$$\psi_k < Y_\sigma = \mathbb{E}(Y_{\gamma \wedge \tau} \mid \mathcal{F}_\sigma) \leq \mathbb{P}(\gamma < \tau \mid \mathcal{F}_\sigma)(d_k + \kappa),$$

since $Y_\tau < 0$. Hence

$$\mathbb{P}(\gamma < \tau \mid \mathcal{F}_\sigma) \geq \frac{\psi_k}{d_k + \kappa} = \frac{1-\delta}{1+\kappa d_k^{-1}}.$$

Taking conditional expectations given $\mathcal{F}_{m_k}$ yields the claimed bound. $\square$

The next result gives exponential growth bounds on the d_k .

Lemma 4.4. *Assume Conditions (N), (UE), and (1.3). Then for any $\delta > 0$ there exist $C_0 < \infty$ and $c_0 > 0$ (depending on δ and on d_1) such that, for all $k \geq 2$,*

$$\mathbb{P}[e^{(1-\delta)k} \leq d_k \leq e^{(1+\delta)k} \mid \mathcal{F}_{\ell_2}] \geq 1 - C_0 e^{-c_0 k}. \quad (4.8)$$

Remark 4.5. Consider

$$d_k = d_1 \cdot \frac{d_2}{d_1} \cdot \frac{d_3}{d_2} \cdots \frac{d_k}{d_{k-1}},$$

and note that Lemma 4.2, says, for d_k large, for $\theta \geq 1$,

$$\mathbb{P}\left[\frac{d_{k+1}}{d_k} > \theta \mid \mathcal{F}_{\ell_{k+1}}\right] \approx \frac{1}{\theta}.$$

Roughly speaking, this says that d_k is approximated by $Y_1 \cdots Y_k = \exp(\log Y_1 + \cdots + \log Y_k)$ where $\mathbb{P}[Y_j > \theta] = 1/\theta$ for $\theta \geq 1$. This means that $\log Y_j$ is a unit mean exponential random variable, and then the intuition for the result is by comparison with concentration for exponential sums. The following proof makes this idea precise.

Proof of Lemma 4.4. For the duration of this proof, write $X_k := \log(d_{k+1}/d_k)$ and $\mathcal{G}_k := \mathcal{F}_{\ell_{k+1}}$; then d_k is $\mathcal{G}_k$ -measurable. Several times in this proof we use the fact that $d_{k+1} > d_k$ for all $k \in \mathbb{N}$ (see Lemma 4.1). First note that, for $\lambda \in (0, 1)$, using the upper bound in Lemma 4.2,

$$\begin{aligned} \mathbb{E}[e^{\lambda X_k} \mid \mathcal{G}_k] &= \mathbb{E}[(d_{k+1}/d_k)^\lambda \mid \mathcal{G}_k] \\ &= 1 + \lambda \int_1^\infty \theta^{\lambda-1} \mathbb{P}[d_{k+1} > \theta d_k \mid \mathcal{G}_k] d\theta \\ &\leq 1 + \lambda \left(1 + \frac{2\kappa}{d_k}\right) \int_1^\infty \theta^{\lambda-2} d\theta. \end{aligned}$$

It follows that, for all $\lambda \in (0, 1)$,

$$\mathbb{E}[e^{\lambda X_k} \mid \mathcal{G}_k] \leq M_+(\lambda, d_k) := \frac{1}{1-\lambda} + \frac{2\kappa\lambda/d_k}{1-\lambda}. \quad (4.9)$$

A similar argument, using the lower bound in Lemma 4.2, shows that, for all $\lambda > 0$,

$$\begin{aligned} \mathbb{E}[e^{-\lambda X_k} \mid \mathcal{G}_k] &= 1 - \lambda \int_1^\infty \theta^{-\lambda-1} \mathbb{P}[d_{k+1} > \theta d_k \mid \mathcal{G}_k] d\theta \\ &\leq M_-(\lambda, d_k) := \frac{1}{1+\lambda} + \frac{\kappa\lambda/d_k}{2+\lambda}. \end{aligned} \quad (4.10)$$

Iterating the bound (4.9) and using that $d_{k+1} > d_k$ for all $k \in \mathbb{N}$ leads to the bound, for all $\lambda \in (0, 1)$ and all $k > m$,

$$\mathbb{E}\left[\exp\left(\lambda \sum_{j=m}^{k-1} X_j\right) \mid \mathcal{G}_m\right] \leq M_+(\lambda, D)^{k-m}, \text{ on } \{d_m \geq D\}. \quad (4.11)$$

Similarly, iterating (4.10), for all $\lambda > 0$ and all $k > m$,

$$\mathbb{E}\left[\exp\left(-\lambda \sum_{j=m}^{k-1} X_j\right) \mid \mathcal{G}_m\right] \leq M_-(\lambda, D)^{k-m}, \text{ on } \{d_m \geq D\}. \quad (4.12)$$

Observe that

$$\lim_{D \rightarrow \infty} M_+(\lambda, D) = \frac{1}{1-\lambda}, \text{ and } \lim_{D \rightarrow \infty} M_-(\lambda, D) = \frac{1}{1+\lambda}. \quad (4.13)$$

Next we show that sequence d_k quickly exceeds any fixed value D . To this end, let $A_k := \{d_{k+1} \geq 2d_k\}$. Then using the lower bound in Lemma 4.2 we have

$$\mathbb{P}(A_k \mid \mathcal{G}_k) \geq \frac{d_k}{2d_k + \kappa} \geq p_0 := \frac{d_1}{2d_1 + \kappa} > 0. \quad (4.14)$$

Let N_m denote the number of $A_1, \dots, A_{m-1}$ that occur, and write $r := \lceil \log_2(D/d_1) \rceil$. Then if $N_m \geq r$ we have $d_m \geq 2^r d_1 \geq D$, so we have $\mathbb{P}(d_m < D \mid \mathcal{G}_1) \leq \mathbb{P}(N_m < r \mid \mathcal{G}_1)$. Here we have from (4.14), by iterating the conditional expectation, $\mathbb{E}[e^{-tN_m} \mid \mathcal{G}_1] \leq (1 - p_0 + p_0 e^{-t})^{m-1}$. Hence we obtain the Chernoff-type bound $\mathbb{P}(N_m < r \mid \mathcal{G}_1) \leq e^{tr}(1 - p_0 + p_0 e^{-t})^{m-1}$. It follows that, for $C_1 < \infty$ and $c_1 > 0$ (depending on d_1)

$$\mathbb{P}(d_m < D \mid \mathcal{G}_1) \leq C_1 e^{-c_1 m}, \text{ for all } m \in \mathbb{N}. \quad (4.15)$$

Fix $\delta > 0$. We prove that, for $C < \infty$ and $c > 0$ (depending on d_1),

$$\mathbb{P}(\log d_k > (1 + \delta)k \mid \mathcal{G}_1) \leq C e^{-ck}. \quad (4.16)$$

Set $m = \lfloor \eta k \rfloor$ for $\eta \in (0, 1)$ that we will choose to be small later. Write

$$\mathbb{P}(\log d_k > (1 + \delta)k \mid \mathcal{G}_1) \leq \mathbb{P}(\log d_m > \frac{\delta}{4}k \mid \mathcal{G}_1) + \mathbb{P}\left(\sum_{j=m}^{k-1} X_j > (1 + \frac{3\delta}{4})k \mid \mathcal{G}_1\right).$$

To bound the first term on the right-hand side, take $\lambda \in (0, 1)$. It follows from (4.9) that $\mathbb{E}[e^{\lambda X_k} \mid \mathcal{G}_k] \leq M_+(\lambda, d_1) =: M_1 < \infty$. Iterating this gives $\mathbb{E}[e^{\lambda(\log d_m - \log d_1)} \mid \mathcal{G}_1] \leq M_1^{m-1}$. Then, by Markov's inequality,

$$\mathbb{P}(\log d_m > \frac{\delta}{4}k \mid \mathcal{G}_1) \leq d_1^\lambda \exp(-\lambda \frac{\delta}{4}k + (m-1) \log M_1) \leq C_1 e^{-c_1 k}, \quad (4.17)$$

for $C_1 < \infty$ and $c_1 > 0$ (depending on d_1) provided $\eta < \delta/8$ (say) is small enough (depending on δ and d_1). Thus we have, allowing C_1 and c_1 to change from line to line,

$$\begin{aligned} \mathbb{P}(\log d_k > (1 + \delta)k \mid \mathcal{G}_1) &\leq C_1 e^{-c_1 k} + \mathbb{P}\left(\sum_{j=m}^{k-1} X_j > (1 + \frac{3\delta}{4})k \mid \mathcal{G}_1\right) \\ &\leq C_1 e^{-c_1 k} + \mathbb{P}\left(\sum_{j=m}^{k-1} X_j > (1 + \frac{3\delta}{4})k, d_m \geq D \mid \mathcal{G}_1\right), \end{aligned}$$

using (4.15). Then from (4.11) and Markov's inequality, for $\lambda \in (0, 1)$,

$$\begin{aligned} \mathbb{P}\left(\sum_{j=m}^{k-1} X_j > (1 + \frac{3\delta}{4})k, d_m \geq D \mid \mathcal{G}_1\right) &\leq M_+(\lambda, D)^{k-m} \exp\left(-\lambda(1 + \frac{3\delta}{4})k\right) \\ &\leq \exp\left(-k\left(\lambda(1 + \frac{3\delta}{4}) - (1 - \eta) \log M_+(\lambda, D)\right)\right). \end{aligned}$$

We already chose $\eta < \delta/8$; then choose $\lambda > 0$ small enough so that

$$\lambda(1 + \frac{3\delta}{4}) + (1 - \eta) \log(1 - \lambda) = \lambda(\frac{3\delta}{4} + \eta) + O(\lambda^2) > \frac{\delta}{2}\lambda.$$

Then, by (4.13), we can choose D large enough and we verify (4.16).

Finally, we prove for $\delta \in (0, 1)$ that, for $C < \infty$ and $c > 0$ (depending on d_1)

$$\mathbb{P}(\log d_k < (1 - \delta)k \mid \mathcal{G}_1) \leq C e^{-ck}. \quad (4.18)$$

Choose $\eta \in (0, \delta/2)$ and again put $m = \lfloor \eta k \rfloor$. Then

$$\mathbb{P}(\log d_k < (1 - \delta)k \mid \mathcal{G}_1) \leq \mathbb{P}\left(\sum_{j=m}^{k-1} X_j < (1 - \delta)k, d_m \geq D \mid \mathcal{G}_1\right) + \mathbb{P}(d_m < D \mid \mathcal{G}_1),$$

where the final term satisfies (4.15). Using (4.12) and Markov's inequality,

$$\begin{aligned} \mathbb{P}\left(\sum_{j=m}^{k-1} X_j < (1 - \delta)k, d_m \geq D \mid \mathcal{G}_1\right) &\leq M_-(\lambda, D)^{k-m} \exp(\lambda(1 - \delta)k) \\ &\leq \exp\left(-k\left(-\lambda(1 - \delta) - (1 - \eta) \log M_-(\lambda, D)\right)\right). \end{aligned}$$

Here, by (4.13), $\log M_-(\lambda, D) \rightarrow -\log(1 + \lambda)$ as $D \rightarrow \infty$, and $-\lambda(1 - \delta) + (1 - \eta) \log(1 + \lambda) = \lambda(\delta - \eta) + O(\lambda^2)$. So we can choose λ small enough and then D large enough to verify (4.18). $\square$

4.2 Stability and separation for the candidate clouds

In Sinai's regime Theorem 4.6 gives our main structural result, which is a central step in proving Theorems 1.4 and 1.5. The result shows that candidate clouds $[h_k + 1, h_{k+1}]$ are indeed intrinsically stable, and obtains some quantification of associated statistics. Let $u^{(k)}$ and $\tilde{u}^{(k)}$ be the candidates for the normal and the infinite-pressure speeds of the cloud candidate $[h_{k-1} + 1, h_k]$ (computed using the formulas (3.4) and (3.5), respectively).

Theorem 4.6. *Assume Conditions **(N)**, **(UE)**, **(LH)**, and (1.3). For $\mathbb{P}$ -a.a. environments ω there exists $k_0 = k_0(\omega) \in \mathbb{N}$ such that the following hold.*

- (i) *It holds that $\lim_{k \rightarrow \infty} k^{-1} \log d_k = 1$, a.s.*
- (ii) *$[h_k + 1, h_{k+1}]$ is intrinsically stable for each $k \geq k_0$.*
- (iii) *$|u^{(k)}| > |\tilde{u}^{(k+1)}|$ for all $k \geq k_0$.*
- (iv) *It holds that $\lim_{k \rightarrow \infty} d_k^{-1} \log |u^{(k)}| = -1$.*
- (v) *Let $\hat{v} \leq 0$ be the speed of the rightmost stable cloud in $[1, h_{k_0}]$ (considered as an isolated system); then $|\hat{v}| > |\tilde{u}^{(k_0+1)}|$.*

Proof. For part (i), fix $\delta > 0$. Then from Lemma 4.4 and the Borel–Cantelli lemma (conditioned on $\mathcal{F}_{\ell_2}$) we have that $\mathbb{P}(1 - \delta \leq k^{-1} \log d_k \leq 1 + \delta \text{ all but finitely often} \mid \mathcal{F}_{\ell_2}) = 1$, which gives the claimed convergence, since $\delta > 0$ was arbitrary.

For the proofs of parts (ii) and (iii) we make use of some more notation and several families of events defined in terms of the potential. Define $H_k := h_k - h_{k-1}$ to be the size of the candidate cloud $[h_{k-1} + 1, h_k]$. Define the events (with δ_0 from condition **(LH)** and Δ_k from (4.2))

$$\begin{aligned} G_1^{(k)} &:= \{H_k \leq e^{3k}\}, \\ G_2^{(k)} &:= \{d_{k+1} > (1 + k^{-2})d_k\}, \\ G_3^{(k)} &:= \{1 - e^{-\Delta_k} \geq e^{-5\delta_0^{-1}k}\}, \\ G_4^{(k)} &:= \{e^{k/2} \leq d_k \leq e^{2k}\}. \end{aligned}$$

We next show that, a.s., each of the events $G_1^{(k)}$, $G_2^{(k)}$, $G_3^{(k)}$, and $G_4^{(k)}$ occurs for all but finitely many k . For $G_4^{(k)}$, this fact is a consequence of Lemma 4.4. We consider the rest.

Consider $G_1^{(k)}$. Let us prove that there exists $c > 0$ such that

$$\mathbb{P}[H_{k+1} > e^{k/4} d_k^2 \mid \mathcal{F}_{m_k}] \leq e^{-ck}. \quad (4.19)$$

Note that Lemma 4.2 implies that $d_{k+1} \leq e^{\varepsilon k} d_k$ with probability at least $1 - ce^{-\varepsilon k}$. Then (recall (4.1)), define

$$T = \min\{s > 0 : \mathcal{L}(m_k + s) > e^{\varepsilon k} d_k\}.$$

It then holds that $m_k + T > h_{k+1}$ (and therefore $H_{k+1} < T$) on the event $\{d_{k+1} \leq e^{\varepsilon k} d_k\}$; this is because $m_k + T$ has to be in a well of depth greater than d_{k+1} in that case. Then, since on every interval of length $(e^{\varepsilon k} d_k)^2$ the descent $\mathcal{L}$ is at least $e^{\varepsilon k} d_k$ with uniformly positive probability, we also have the estimate

$$\mathbb{P}[T > j(e^{\varepsilon k} d_k)^2 \mid \mathcal{F}_{m_k}] < e^{-c'j},$$

and hence, by taking $\varepsilon < 1/8$, we obtain (4.19). It follows from (4.19) and the Borel–Cantelli lemma that a.s., for all but finitely many $k \in \mathbb{N}$, $H_{k+1} \leq e^{k/4} d_k^2$. But $d_k \leq e^{5k/4}$ for all but finitely many k , by part (i). Hence we verify that

$$\mathbb{P}\text{-a.s., } G_1^{(k)} \text{ occurs for all but finitely many } k \in \mathbb{N}. \quad (4.20)$$

For $G_2^{(k)}$, we have from the lower bound in Lemma 4.2 that

$$\mathbb{P}[(G_2^{(k)})^c \mid \mathcal{F}_{\ell_{k+1}}] \leq \frac{k^{-2}d_k + \kappa}{(1+k^{-2})d_k + \kappa} \leq k^{-2} + \kappa d_k^{-1}. \quad (4.21)$$

Here $\sum_{k \in \mathbb{N}} d_k^{-1} < \infty$ by part (i). Hence, by Lévy's conditional Borel–Cantelli lemma and (4.21),

$$\mathbb{P}\text{-a.s.}, G_2^{(k)} \text{ occurs for all but finitely many } k \in \mathbb{N}. \quad (4.22)$$

Next consider $G_3^{(k)}$. Define

$$\begin{aligned} G_5^{(k)} &:= \{h_k \leq 2e^{3k}\}, \\ G_6^{(k)} &:= \{\text{the first } 3e^{3k} \text{ ladder heights of } V(\cdot) \text{ are each at least } e^{-4\delta_0^{-1}k}\}. \end{aligned}$$

Observe that $G_5^{(k)} \cap G_6^{(k)} \subseteq G_3^{(k)}$, since, on $G_5^{(k)} \cap G_6^{(k)}$, Δ_k is one of the first $2e^{3k}$ ladder heights, and so

$$1 - e^{-\Delta_k} \geq \frac{\Delta_k}{1 + \Delta_k} \geq \frac{e^{-4\delta_0^{-1}k}}{1 + e^{-4\delta_0^{-1}k}} > e^{-5\delta_0^{-1}k},$$

for k sufficiently large. Since $h_k = h_0 + \sum_{j=1}^k H_j$, and $G_1^{(j)}$ occurs for all sufficiently large j , a.s., it follows that $G_5^{(k)}$ occurs for all large enough k , a.s. Now, due to Condition **(LH)**,

$$\mathbb{P}[G_6^{(k)}] \geq 1 - (3e^{3k})\gamma(e^{-4\delta_0^{-1}k})^{\delta_0} = 1 - 3\gamma e^{-k}. \quad (4.23)$$

Then from the Borel–Cantelli lemma with (4.23) and (4.20), we obtain

$$\mathbb{P}\text{-a.s.}, G_3^{(k)} \text{ occurs for all but finitely many } k \in \mathbb{N}. \quad (4.24)$$

Combining (4.20), (4.22) and (4.24) with Lemma 4.4 verifies that $G_1^{(k)}, G_2^{(k)}, G_3^{(k)}$ and $G_4^{(k)}$ occur a.s. for all but finitely many k .

The idea of part (ii), i.e., to show that $[h_k + 1, h_{k+1}]$ is intrinsically stable for large enough k , is to reason in the following way. First, Lemma 3.4 implies that $[h_k + 1, m_{k+1}]$ is intrinsically stable, and Lemma 3.6 shows that its intrinsic speed is about $e^{-d_{k+1}}$. We will then show, roughly speaking, that the particles $[m_{k+1} + 1, h_{k+1}]$ will join that group. Note that the particles $[m_{k+1} + 1, h_{k+1}]$ alone need not form a stable cloud; however, Theorem 2.1 of [27] implies that they can be decomposed into several stable clouds, and the speed of the rightmost one is maximal among these (indeed, since it also has to be negative due to the fact that the unique global maximum of the potential on $[h_k + 1, h_{k+1}]$ is attained at h_{k+1} , the absolute value of the speed of that rightmost cloud should be minimal among these). Next, we use Lemma 4.3 to control the depth of that rightmost cloud (i.e., to assure that it is likely much less than d_{k+1}), and we also have some control for the height increase at h_{k+1} (needed for the lower bound in Lemma 3.6). Then, Lemma 3.6 will show that the rightmost cloud should catch up with $[h_k + 1, m_{k+1}]$. We now give the details.

Assume that k is large, and let us consider the particles $[h_{k-1} + 1, h_k]$ as a separate (finite) system. By Theorem 2.1 of [27] this finite system can be decomposed into a finite number of stable clouds; let $[i_k, h_k]$ be the rightmost of these. Our goal is to prove that $i_k = h_{k-1} + 1$. Note that Lemma 3.4 and Remark 3.2(iii) imply that the particles in $[h_{k-1} + 1, m_k]$ belong to the same stable cloud, so it is enough to prove that $i_k \leq m_k$.

Define

$$G_7^{(k)} := \{\mathcal{D}(m_k, h_k) \leq (1 - k^{-2})d_k\}.$$

Lemma 4.3 implies that

$$\mathbb{P}[(G_7^{(k)})^c \mid \mathcal{F}_{m_k}] \leq 1 - \frac{1 - k^{-2}}{1 + \kappa d_k^{-1}} \leq \kappa d_k^{-1} + k^{-2}, \quad (4.25)$$

which is a.s. summable, by part (i). Hence Lévy's conditional Borel–Cantelli lemma implies that, for a.a. ω , the events $G_7^{(k)}$ occur for all but finitely many $k \in \mathbb{N}$. Assume that k is large enough so that $G_1^{(k)} \cap G_3^{(k)} \cap G_4^{(k)} \cap G_7^{(k)}$ occurs, and assume that $i_k > m_k$. Then, let v' be speed of the leftmost cloud of the isolated system $[h_{k-1} + 1, h_k]$, and v'' be the speed of $[i_k, h_k]$ (when considered as separate systems). Since $v'' < 0$ (as $V(h_k) > V(i_k - 1)$) and cloud speeds are non-decreasing from left to right, we have $v' \leq v'' < 0$, and hence $|v'| \geq |v''|$. We now show that, for all large k , $|v'| < |v''|$, which gives a contradiction. Since $v' < 0$, the leftmost cloud $[h_{k-1} + 1, j]$ has $V(j) > V(h_{k-1})$, so $V(j) = \max_{[h_{k-1}, j]} V$ by Lemma 3.5; moreover, $j \geq m_k$ (this cloud contains $[h_{k-1} + 1, m_k]$), so $\mathcal{D}(h_{k-1}, j) \geq d_k$. Hence Lemma 3.6 implies that $|v'| \leq \varepsilon_0^{-1} e^{-d_k}$ and that, on $G_1^{(k)} \cap G_3^{(k)} \cap G_7^{(k)}$, $|v''| \geq \varepsilon_0 e^{-3k} e^{-5\delta_0^{-1}k} e^{-d_k(1-k^{-2})}$ (apply (3.9) to $[i_k, h_k]$ using $V(h_k) - V(i_k - 1) \geq \Delta_k$, $h_k - i_k + 1 \leq H_k$, and $\mathcal{D}(i_k - 1, h_k) \leq \mathcal{D}(m_k, h_k)$, since $i_k - 1 \geq m_k$). We thus need to check that $e^{k^{-2}d_k} > \varepsilon_0^{-2} e^{(3+5\delta_0^{-1})k}$, which is indeed so on $G_4^{(k)}$ for all large enough k . This gives us a contradiction, and so $[h_{k-1} + 1, h_k]$ is intrinsically stable, proving part (ii).

Roughly speaking, part (iii) says that the candidate stable clouds $[h_k + 1, h_{k+1}]$ (which part (ii) says are intrinsically stable for large enough k) have, for large enough k , strictly ordered speeds. In fact, it will follow from part (iii) that, in the infinite system, all but finitely many of the candidate stable clouds remain distinct stable clouds. The idea of the proof of part (iii) is as follows. Lemma 3.6 tells us that $u^{(k)}$ and $\tilde{u}^{(k+1)}$ are roughly e^{-d_k} and $e^{-d_{k+1}}$, respectively, up to correction factors that we need to control (here Condition **(LH)** is used to make sure $V(h_k) - V(h_{k-1})$ is not too small). Then Lemma 4.2 ensures that d_{k+1} is sufficiently larger than d_k to ensure the claimed inequality in speeds.

Part (iii) follows from the claim

$$\text{for all } k \text{ sufficiently large, } G_1^{(k)} \cap G_2^{(k)} \cap G_3^{(k)} \cap G_4^{(k)} \text{ implies } |u^{(k)}| > |\tilde{u}^{(k+1)}|. \quad (4.26)$$

To verify the claim (4.26), recall from Lemma 4.1(iii) that $d_k = \mathcal{D}(h_{k-1}, h_k)$. Then we have by Lemma 3.6 applied to the system $[h_{k-1} + 1, h_k]$ and the system $[h_k + 1, h_{k+1}]$ we have (since $V(h_k) - V(h_{k-1}) \geq \Delta_k$)

$$|u^{(k)}| \geq \frac{\varepsilon_0}{H_k} (1 - e^{-\Delta_k}) e^{-d_k} \text{ and } |\tilde{u}^{(k+1)}| \leq \varepsilon_0^{-1} e^{-d_{k+1}}. \quad (4.27)$$

Then, we need to prove that $\frac{\varepsilon_0}{H_k} (1 - e^{-\Delta_k}) e^{-d_k} > \varepsilon_0^{-1} e^{-d_{k+1}}$, which is equivalent to $e^{d_{k+1}-d_k} \geq \varepsilon_0^{-2} H_k (1 - e^{-\Delta_k})^{-1}$. But, on $G_2^{(k)} \cap G_4^{(k)}$ we have $e^{d_{k+1}-d_k} \geq \exp(k^{-2}d_k) \geq \exp(k^{-2}e^{k/2})$, and, on $G_1^{(k)} \cap G_3^{(k)}$ we have $\varepsilon_0^{-2} H_k (1 - e^{-\Delta_k})^{-1} \leq \varepsilon_0^{-2} \exp(3k + 5\delta_0^{-1}k)$. Comparing these expressions, we verify the claim (4.26). This proves part (iii).

Next we prove part (iv). Using the bounds in Lemma 3.6 we get, on $G_1^{(k)} \cap G_3^{(k)}$,

$$\varepsilon_0^{-1} e^{-d_k} \geq |u^{(k)}| \geq \varepsilon_0 e^{-3k} e^{-5\delta_0^{-1}k} e^{-d_k},$$

which, together with part (i), yields part (iv).

As for part (v), let $[i, h_k]$ be the rightmost cloud of the isolated system $[1, h_k]$, with speed $\hat{v}$. Since h_k is a strict record, we have $V(h_k) - V(i - 1) \geq \Delta_k$ and $\hat{v} < 0$. Also, $\mathcal{D}(i - 1, h_k) \leq \mathcal{D}(0, h_k) = d_k$ by Lemma 4.1(iii). Hence, by (3.9),

$$|\hat{v}| \geq \varepsilon_0 h_k^{-1} (1 - e^{-\Delta_k}) e^{-d_k},$$

while $|\tilde{u}^{(k+1)}| \leq \varepsilon_0^{-1} e^{-d_{k+1}}$ by (3.8). On $G_2^{(k)} \cap G_4^{(k)} \cap G_5^{(k)} \cap G_6^{(k)}$ we have

$$e^{d_{k+1}-d_k} \geq \exp(k^{-2}e^{k/2}) > 4\varepsilon_0^{-2} e^{(3+4\delta_0^{-1})k} \geq \varepsilon_0^{-2} h_k (1 - e^{-\Delta_k})^{-1}$$

for large k . So (v) holds for all large k ; enlarge k_0 accordingly. $\square$

4.3 Proofs of Theorems 1.4 and 1.5

Basically, Theorem 4.6 is the most complete result, which shows where the stable clouds are located and what are their speeds. Now let us discuss how to obtain the results about convergence in distribution in Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Consider the isolated system $[1, h_{k_0}]$ whose cloud decomposition is into some $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ and whose rightmost stable cloud $\mathcal{C}_\ell$ has speed $\hat{v}$ where $|\hat{v}| > |\tilde{u}^{(k_0+1)}|$, as in Theorem 4.6(v). The isolated system $[1, h_{k_0+1}]$ with infinite pressure from the right then has cloud decomposition into clouds $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ and $\mathcal{C}_{\ell+1} = [h_{k_0} + 1, h_{k_0+1}]$ (this follows from Algorithm 1 in [27]). Since the speed of cloud $\mathcal{C}_\ell$ is strictly less than that of cloud $\mathcal{C}_{\ell+1}$, we may apply Lemma 2.5 to show that there is a coupling of the infinite system and the finite system $[1, h_{k_0+1}]$ with infinite pressure from the right, in which the inter-particle distances corresponding to clouds $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ coincide after some finite time. It follows that $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ are stable clouds in the infinite system, in the sense of Definition 1.2. Similarly, consider the finite system $[1, h_{k+1}]$, $k > k_0$, with infinite pressure from the right. By Theorem 4.6(iii), this finite system has cloud decomposition $\mathcal{C}_1, \dots, \mathcal{C}_\ell, [h_{k_0} + 1, h_{k_0+1}], \dots, [h_k + 1, h_{k+1}]$, with strictly increasing cloud speeds. Another application of Lemma 2.5, using the fact that $k > k_0$ was arbitrary, shows that $[h_{k_0} + 1, h_{k_0+1}], [h_{k_0+1} + 1, h_{k_0+2}], \dots$, are stable clouds in the infinite system. In particular, this means that sequences s_k and $u^{(k)}$ coincide up to a finite (ω -dependent) shift, and the same for sequences S_k and H_k .

To prove part (i) of the theorem, we need to show that each particle in a cloud satisfies a strong law with speed given by the cloud's speed. Consider one of the clouds $\mathcal{C}_k$, with speed s_k , that we now know is stable, and suppose particle j is in cloud $\mathcal{C}_k$. There is a coupling (analogous to the coupling of two finite systems in Proposition 2.4(ii)) of the infinite system and the finite system made up of particles in clouds $\mathcal{C}_1, \dots, \mathcal{C}_k$ in which all particles in the infinite system are no further to the right than in the finite system. In the finite system we have from Theorem 2.1 of [27] that particle j satisfies $t^{-1}X_j(t) \rightarrow s_k$, a.s. So in the infinite system we get

$$\limsup_{t \rightarrow \infty} t^{-1}X_j(t) \leq s_k, \quad \mathbb{P}_\omega^x\text{-a.s., for } \mathbb{P}\text{-a.a. } \omega.$$

On the other hand, consider the finite system made up of particles in clouds $\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{C}_{k+1}$ with infinite pressure from the right. Now there is a coupling (analogous to the coupling of two finite systems in Proposition 2.4(i)) of the infinite system and the finite system in which all particles in the finite system are no further to the right than in the infinite system. We argued above that $\mathcal{C}_k$ is still a stable cloud in this bigger finite system (with infinite pressure). So, again using Theorem 2.1 of [27], in the infinite system we get

$$\liminf_{t \rightarrow \infty} t^{-1}X_j(t) \geq s_k, \quad \mathbb{P}_\omega^x\text{-a.s., for } \mathbb{P}\text{-a.a. } \omega.$$

Thus we obtain the strong law (1.4), proving part (i).

Recall that s_k and $u^{(k)}$ coincide up to a finite shift. Then combining Theorem 4.6(i) and (iv) shows that $u^{(k)} = -\exp(-e^{k(1+o(1))})$, as $k \rightarrow \infty$, $\mathbb{P}$ -a.s. Then part (ii) of the theorem follows.

Next we prove part (iii). For large k , as argued above, $s_k = u^{(k+c)}$ and $S_k = H_{k+c}$ where $c = c(\omega)$ is random. From Theorem 4.6(iv) we have that, a.s., as $k \rightarrow \infty$,

$$\frac{\log |s_{k+1}|}{\log |s_k|} = (1 + o(1)) \frac{d_{k+c+1}}{d_{k+c}}. \quad (4.28)$$

From the upper and lower bounds in Lemma 4.2, and the fact that $d_k \rightarrow \infty$ we have, $\mathbb{P}$ -a.s.,

$$\lim_{k \rightarrow \infty} \mathbb{P} \left[\frac{d_{k+1}}{d_k} > y \mid \mathcal{F}_{\ell_{k+1}} \right] = \frac{1}{y},$$

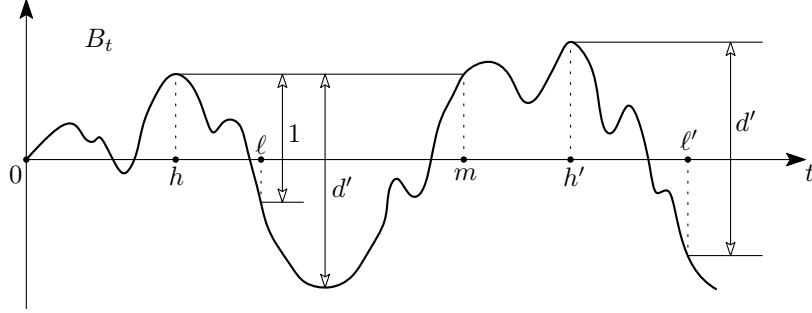


Figure 9: On the proof of the convergence-in-distribution results in Theorem 1.4: definition of the auxiliary random variables on a Brownian trajectory.

and hence, by dominated convergence, $\lim_{k \rightarrow \infty} \mathbb{P}[A \cap \{d_{k+1}/d_k > y\}] = \mathbb{P}(A)/y$ for every $A \in \mathcal{F}'_{\ell_j}$, with fixed $j \in \mathbb{N}$, where $\mathcal{F}'_m := \sigma(a_i, b_i : i \leq m)$. Here we use that Lemma 4.2 holds, with the same proof, with $\mathcal{F}_{\ell_{k+1}}$ replaced by $\mathcal{F}'_{\ell_{k+1}}$. Every event in the environment σ -algebra can be approximated in probability by such events, so the same holds for every such event A . Applying this with $A = \{c = j\}$ and summing over j , we get $\lim_{k \rightarrow \infty} \mathbb{P}[d_{k+c+1}/d_{k+c} > y] = 1/y$. Combined with (4.28) this yields (1.6). Similarly, we have

$$\mathbb{P}\left[\frac{d_{k+1}}{d_k} > y_1, \frac{d_{k+2}}{d_{k+1}} > y_2 \mid \mathcal{F}_{\ell_{k+1}}\right] = \mathbb{E}\left[\mathbb{1}_{\{d_{k+1}/d_k > y_1\}} \mathbb{P}\left[\frac{d_{k+2}}{d_{k+1}} > y_2 \mid \mathcal{F}_{\ell_{k+2}}\right] \mid \mathcal{F}_{\ell_{k+1}}\right],$$

so that, by two applications of dominated convergence,

$$\lim_{k \rightarrow \infty} \mathbb{P}\left[\frac{d_{k+1}}{d_k} > y_1, \frac{d_{k+2}}{d_{k+1}} > y_2\right] = \frac{1}{y_1} \cdot \frac{1}{y_2}.$$

The same idea applies for arbitrary finite sequences. A similar argument to the marginal result, with an indicator 1_A , followed by conditioning on $\{c = j\}$, handles the random shift for the whole finite vector, and hence yields asymptotic independence, completing the proof of part (iii).

To prove part (iv), we will apply Donsker's theorem. To describe the limit random variable, consider a standard Brownian motion $(B_t, t \geq 0)$, and define

$$\begin{aligned} \ell &:= \inf \{t > 0 : \max_{0 \leq s \leq t} B_s - B_t = 1\}, \\ h &:= \arg \max_{t \in [0, \ell]} B_t, \\ m &:= \inf \{t > \ell : B_t = B_h\}, \\ u &:= \arg \min_{t \in [h, m]} B_t, \\ d' &:= \max_{0 \leq s \leq u} B_s - B_u, \\ \ell' &:= \inf \{t > m : \max_{s \leq t} B_s - B_t = d'\}, \\ h' &:= \arg \max_{t \in [m, \ell']} B_t; \end{aligned}$$

see Figure 9. Then define the random variable

$$Y := \frac{h' - h}{(d')^2}. \quad (4.29)$$

Note that if we replace the definition of ℓ by $\inf\{t > 0 : \max_{0 \leq s \leq t} B_s - B_t = d\}$ for $d > 0$, the distribution of Y is unchanged, by Brownian scaling.

Let $\sigma^2 := \mathbb{E}[V(1)^2]$. We define a quantity like Y for an arbitrary continuous trajectory f , which we call $\Phi(f)$. Formally, for continuous $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ set $M_f(t) := \max_{0 \leq s \leq t} f(s)$ and

$$\begin{aligned} \ell_f &:= \inf\{t > 0 : M_f(t) - f(t) = 1\}, \\ h_f &:= \arg \max_{t \in [0, \ell_f]} f(t), \\ m_f &:= \inf\{t > \ell_f : f(t) = f(h_f)\}, \\ u_f &:= \arg \min_{t \in [h_f, m_f]} f(t), \\ d'_f &:= M_f(u_f) - f(u_f), \\ \ell'_f &:= \inf\{t > m_f : M_f(t) - f(t) = d'_f\}, \\ h'_f &:= \arg \max_{t \in [m_f, \ell'_f]} f(t). \end{aligned}$$

Define $\Phi(f) := \frac{h'_f - h_f}{(d'_f)^2}$; then $Y = \Phi(B)$. When $f = B$ all the above quantities are finite and the relevant extrema are a.s. unique. Moreover, passage-time-like quantities are genuine crossings of levels. It follows by the usual continuity of first passage times at genuine crossings and of $\arg \max / \arg \min$ at unique extrema that Φ is a.s. continuous at B . For $c > 0$, define f_c by $f_c(t) := f(ct)$; then $h_{f_c} = h_f/c$, $h'_{f_c} = h'_f/c$ and $d'_{f_c} = d'_f$. Consequently $\Phi(f_c) = c^{-1}\Phi(f)$. Thus Brownian scaling shows that $\Phi(\sigma B)$ has the same distribution as Y/σ^2 .

Now condition on $\mathcal{F}_{m_{k-1}}$. For $t \geq 0$ write

$$j_k(t) = \lfloor d_{k-1}^2 t \rfloor, \text{ and } \theta_k(t) = d_{k-1}^2 t - j_k(t),$$

and define the linearly interpolated process

$$W_k(t) = \frac{1}{d_{k-1}} \left[V(m_{k-1} + j_k(t)) - V(m_{k-1}) + \theta_k(t) \{V(m_{k-1} + j_k(t) + 1) - V(m_{k-1} + j_k(t))\} \right].$$

Since m_{k-1} is a stopping time, the post- m_{k-1} increments are fresh i.i.d. increments. Since d_{k-1} is $\mathcal{F}_{m_{k-1}}$ -measurable and $d_{k-1} \rightarrow \infty$ a.s. by Theorem 4.6(i), the conditional form of Donsker's theorem gives $W_k \Rightarrow \sigma B$ (denoting weak convergence).

For $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in [0, 1]^3$, let Φ_ε be defined as Φ but with the passage levels 1, $f(h_f)$, and d'_f (in ℓ_f, m_f , and ℓ'_f , respectively) replaced by $1 + \varepsilon_1, f(h_f) + \varepsilon_2$, and $d'_f + \varepsilon_3$. Thus $\Phi_0 = \Phi$. The argument above (genuine crossings, unique extrema) shows that, a.s., $\sup_{\varepsilon \in [0, \delta]^3} |\Phi_\varepsilon(f) - \Phi(\sigma B)| \rightarrow 0$ as $f \rightarrow \sigma B$ locally uniformly and $\delta \rightarrow 0$.

The discrete construction starting at m_{k-1} is the same successive well construction. Its first threshold is d_{k-1} , its first maximum is h_{k-1} , the subsequent minimum has depth d_k , and the next maximum is h_k . Since W_k is linear between mesh points, its running descent is monotone between mesh points, and all extrema of W_k are attained at mesh points. Hence each first passage in the discrete construction (which occurs at a mesh point) coincides with the continuous first passage of W_k at the level attained there. By (1.2), this level exceeds the nominal one by at most κ/d_{k-1} . Thus

$$\frac{H_k}{d_k^2} = \Phi_{\varepsilon_k}(W_k),$$

for some $\varepsilon_k \in [0, \kappa/d_{k-1}]^3$, and the continuous mapping theorem gives $H_k/d_k^2 \Rightarrow \Phi(\sigma B) \stackrel{d}{=} Y/\sigma^2$, conditionally on $\mathcal{F}_{m_{k-1}}$. The same argument shows $H_k/d_k^2 \Rightarrow Y/\sigma^2$ conditionally on $\mathcal{F}'_{m_{k-1}}$, and

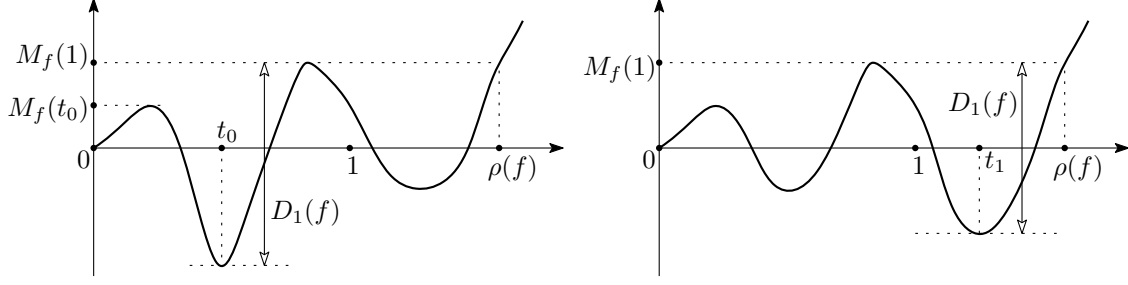


Figure 10: On the definition of $D_1(f)$: on the left picture the supremum is attained at $t_0 < 1$, while on the right picture it is attained at $t_1 > 1$ with $M_f(t_1) = M_f(1)$.

then, as in the proof of part (iii), the conditional form of the convergence handles the shift c , so $H_{k+c}/d_{k+c}^2 \Rightarrow Y/\sigma^2$. Finally, $S_k = H_{k+c}$, and, by Theorem 4.6(iv), $\log |s_k|^{-1}/d_{k+c} \rightarrow 1$. Then Slutsky's theorem completes the proof of (1.8). All that remains is the explicit distribution for Y given at (1.7): we give the proof of this in Lemma 4.7 below. $\square$

Now we turn to the proof of Theorem 1.5, the per-particle speed result.

Proof of Theorem 1.5. First we have from Theorem 1.4(i) that $v_n = \lim_{t \rightarrow \infty} t^{-1}X_n(t) = s_k$ where k is such that $n \in \mathcal{C}_k$, which, as explained in the proof of Theorem 1.4, means that, for all large enough n , $n \in [h_{k'-1} + 1, h_{k'}]$ for k' with $|k - k'|$ bounded by an ω -dependent constant. In particular, the strong law (1.9) follows from the strong law in Theorem 1.4. Theorem 4.6(iv) then implies that $v_n = -e^{-d_{k'}(1+o(1))}$, a.s. Thus to complete the proof of the theorem we must show that $n^{-1/2}d_{k'}$ converges in law, where $d_{k'}$ is the depth of the potential well containing n , and we must identify the limit distribution.

Let B_t be a standard Brownian motion. We want to define a random variable Z as the depth of the well (in trajectory B) that contains time 1. One cannot repeat the construction of quantities associated with successive potential wells from Section 4.1 in the Brownian setting, as there is no “first” well in that sense. Instead, we need a local definition of the depth of the given well. To this end, recall that $M_f(t) = \max_{0 \leq s \leq t} f(s)$, and for continuous $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, define

$$\rho(f) := \inf \left\{ t \geq 1 : f(t) > M_f(1) \right\}$$

and then (see Figure 10)

$$D_1(f) := \sup_{0 \leq t \leq \rho(f)} \{M_f(t) - f(t)\}.$$

Then set

$$Z := D_1(B), \tag{4.30}$$

which we take as our definition of the depth of the Brownian well containing time 1.

Consider the linearly interpolated and scaled potential,

$$\mathcal{V}_n(t) := n^{-1/2} (V(\lfloor nt \rfloor) + (nt - \lfloor nt \rfloor)(V(\lfloor nt \rfloor + 1) - V(\lfloor nt \rfloor))).$$

Almost surely, $B_1 < M_B(1)$, the first post-1 hitting time of $M_B(1)$ is finite, and Brownian motion crosses this level immediately. It follows that $f \mapsto D_1(f)$ is a.s. continuous at $f = B$. Hence, for the linearly interpolated potential $\mathcal{V}_n$, Donsker's theorem and the continuous mapping theorem give $D_1(\mathcal{V}_n) \Rightarrow D_1(\sigma B) = \sigma Z$.

If $h_{k'-1} < n < h_{k'}$, Lemma 4.1 implies that the largest running descent before the first future record maximum after n is exactly $d_{k'}$; since linear interpolation creates no new extrema,

$D_1(\mathcal{V}_n) = n^{-1/2}d_{k'}$. Finally, every h_k is a running-maximum time of the potential, so

$$\{n = h_k \text{ for some } k\} \subseteq \left\{ V(n) = \max_{0 \leq j \leq n} V(j) \right\}.$$

By reversing the increments, the probability of this event equals the probability that a centered finite-variance random walk stays non-negative for n steps, and hence $\mathbb{P}(n = h_k \text{ for some } k) = O(n^{-1/2})$. Therefore if we set $Z_n := -n^{-1/2} \log |v_n|$ we have $Z_n = (1 + o(1))n^{-1/2}d_{k'} \Rightarrow \sigma Z$. It remains to identify the explicit distribution of Z given at (1.10): this is done in Lemma 4.8 below. $\square$

4.4 Explicit limit distributions

Let $\text{BES}(q)$ denote the Bessel process with dimension q .

Lemma 4.7. *The distribution of Y given by (4.29) is the same as the first hitting time of 2 by a $\text{BES}(3)$ process started at 0, and hence has distribution function given by (1.7).*

Proof. Let T_x denote the hitting time of x by a $\text{BES}(3)$ process started at 0, and note that

$$\mathbb{E} e^{-\lambda T_x} = \frac{x\sqrt{2\lambda}}{\sinh(x\sqrt{2\lambda})}; \quad (4.31)$$

see equation 2.0.1 of [7, p. 469]. Recalling the definition of Y from (4.29), and the quantities m, h, h' and d' defined prior to Y , decompose $Y = Y_1 + Y_2$ where

$$Y_1 := \frac{m - h}{(d')^2}, \text{ and } Y_2 := \frac{h' - m}{(d')^2}.$$

Condition on d' . Williams's decomposition of a Brownian excursion at its maximum says that, conditional on the maximum being d' , the part before the maximum and the time-reversal of the part after the maximum are independent $\text{BES}(3)$ processes started from 0 and stopped on first hitting d' . Hence, given d' , $m - h$ has the same law as $T_{d'} + T'_{d'}$ where T'_x is an independent copy of T_x . By Bessel scaling, $T_{d'}$ has the same distribution as $(d')^2 T_1$. Hence Y_1 has the same law as $T_1 + T'_1$ meaning that by (4.31), for $\lambda > 0$,

$$\mathbb{E} e^{-\lambda Y_1} = \frac{2\lambda}{\sinh^2 \sqrt{2\lambda}}.$$

Now, conditional on d' , $(B_{m+t} - B_m)_{t \geq 0}$ is again a Brownian motion, and so $h' - m$ is the left endpoint of the first reflected-Brownian excursion of height at least d' . By Brownian scaling, Y_2 has the same distribution as σ_1 , where, if R_t denotes reflected Brownian motion and $\tau_1 := \inf\{t > 0 : R_t = 1\}$, $\sigma_1 := \sup\{s < \tau_1 : R_s = 0\}$, the left endpoint of the excursion containing τ_1 . The Laplace transform of σ_1 follows from results of Neveu–Pitman [31] (for the explicit statement see e.g. equation (5.9) of [32]): for $\lambda > 0$,

$$\mathbb{E} e^{-\lambda Y_2} = \mathbb{E} e^{-\lambda \sigma_1} = \frac{\tanh \sqrt{2\lambda}}{\sqrt{2\lambda}}. \quad (4.32)$$

Since the law Y_2 conditional on d' is independent of d' , the strong Markov property at time m shows that Y_2 is independent of the Brownian motion up to time m , and, in particular, Y_2 is independent of Y_1 . Then we get

$$\mathbb{E} e^{-\lambda Y} = \frac{2\lambda}{\sinh^2 \sqrt{2\lambda}} \cdot \frac{\tanh \sqrt{2\lambda}}{\sqrt{2\lambda}} = \frac{\sqrt{2\lambda}}{\sinh \sqrt{2\lambda} \cosh \sqrt{2\lambda}} = \frac{2\sqrt{2\lambda}}{\sinh(2\sqrt{2\lambda})}.$$

Comparison with (4.31) shows that Y has the same law as T_2 . The explicit form of the distribution function at (1.7) is given in [17] (in their equation (2.7) take $\nu = 1/2$ so $j_{\nu,k} = k\pi$ and $J_{\nu+1}(j_{\nu,k}) = (-1)^{k+1} \sqrt{2/(\pi^2 k)}$). $\square$

Lemma 4.8. *The random variable Z as defined at (4.30) has the distribution given at (1.10).*

Proof. Let $M_t := \sup_{0 \leq s \leq t} B_s$ and $\rho := \inf\{t \geq 1 : B_t > M_1\}$, so that $Z = \sup_{0 \leq t \leq \rho} (M_t - B_t)$. Here $R_t := M_t - B_t$ is reflected Brownian motion. For $a > 0$, let $\tau_a := \inf\{t \geq 0 : R_t = a\}$ and $\sigma_a := \sup\{s \leq \tau_a : R_s = 0\}$. We relate Z to σ_a . A.s., $B_1 < M_1$, so time 1 lies strictly inside an excursion of R away from 0. Brownian motion immediately crosses level M_1 , and hence the infimum in the definition of ρ is the terminal time of that excursion. We have $Z \geq a$ if and only if an excursion of R whose height reaches a has begun by time 1: either such an excursion started earlier and finished by time 1, or the excursion includes time 1 and finishes at time ρ . In other words, up to events of probability zero, $\{Z < a\} = \{\sigma_a > 1\}$. Brownian scaling implies that σ_a has the same law as $a^2 \sigma_1$. Thus we get, for $z > 0$,

$$\mathbb{P}(Z < z) = \mathbb{P}(\sigma_1 > z^{-2}). \quad (4.33)$$

As at (4.32) above, we have from [31] that, for $\lambda > 0$,

$$\mathbb{E} e^{-\lambda \sigma_1} = \frac{\tanh \sqrt{2\lambda}}{\sqrt{2\lambda}} = \sum_{m=0}^{\infty} \frac{1}{\lambda + \pi^2(2m+1)^2/8}.$$

Since every term is non-negative, Tonelli's theorem shows that the following density has the required Laplace transform

$$f(t) = \sum_{m=0}^{\infty} \exp\left\{-\frac{\pi^2(2m+1)^2 t}{8}\right\},$$

and hence, by uniqueness, f is the density of σ_1 . It follows that, for $t > 0$,

$$\mathbb{P}(\sigma_1 > t) = \frac{8}{\pi^2} \sum_{m=0}^{\infty} (2m+1)^{-2} \exp\left\{-\frac{\pi^2(2m+1)^2 t}{8}\right\}.$$

Combined with (4.33) this yields (1.10). □

5 Proofs in the Kesten–Kozlov–Spitzer regime

Proof of Theorem 1.7. Define a $\mathbb{P}$ -a.s. finite random variable $N = N(\omega)$ in the following way:

$$N := \min\{k \geq 1 : V(j) < V(k-1) \text{ for all } j \geq k\}. \quad (5.1)$$

Note that $N-1$ is the last global maximum of V ; since $V(0) = 0$, this means that

$$\{N > k\} \subseteq \left\{\sup_{j \geq k} V(j) \geq 0\right\}.$$

Write $\xi_k := \log(a_k/b_k)$. Since ξ_1 is bounded (by **(UE)**) and has negative mean (by (1.13)), a Chernoff bound gives $\mathbb{P}(V(j) \geq 0) \leq C e^{-c'j}$ and hence, for some $C' < \infty$ and $c' > 0$,

$$\mathbb{P}(N > k) \leq \sum_{j \geq k} \mathbb{P}(V(j) \geq 0) \leq C' e^{-c'k},$$

as claimed.

Denote the shifted potential by $W(r) := V(N-1+r) - V(N-1)$, $r \geq 0$. Then by definition of N , $W(r) < 0$ for all $r \in \mathbb{N}$ and, by the strong law, $W(r)/r \rightarrow \mathbb{E} \xi_1 < 0$. The zero-speed solution to the general traffic equation for the system $[N, \infty)$ (see Section 1.2 of [28] for definitions) is then $\alpha_r = e^{W(r)}$ satisfying $0 < \alpha_r < 1$ and $\sum_{r \geq 1} \alpha_r < \infty$. Thus the zero-speed solution is

admissible and summable. Theorem 1.12 of [28] then shows that $[N, \infty)$ is intrinsically stable started from any finite configuration.

Since $[N, \infty)$ is intrinsically stable, a coupling argument shows that $[N, \infty)$ must be either an infinite stable cloud in the full system, or a subset of one, since deleting the particles to the left can only increase the internal gap process of $[N, \infty)$, so intrinsic tightness gives the required internal tightness in the full system. If $N = 1$, then the proof is complete. Otherwise, suppose from this point on that $N \geq 2$: to complete the proof we must show (see Definition 1.3) that $X_N(t) - X_{N-1}(t) \rightarrow \infty$ in probability.

First suppose that $N - 1$ is the unique time at which the global maximum for V is attained. Then the rightmost cloud of the isolated finite system $[1, N - 1]$ has strictly negative speed, say $s < 0$. Using the basic couplings (similarly to Proposition 2.4) $X_{N-1}(t) \leq X'_{N-1}(t)$ where X' is the system obtained by deleting all particles to the right of $N - 1$, while $X_N(t) \geq X''_N(t)$ where X'' is obtained by deleting all particles to the left of N . It follows that

$$\limsup_{t \rightarrow \infty} t^{-1} X_{N-1}(t) \leq s < 0,$$

applying the strong law for the finite system X' (Theorem 2.1 of [27]) whereas

$$\liminf_{t \rightarrow \infty} t^{-1} X_N(t) \geq 0,$$

applying the strong law for the stable semi-infinite system X'' (Theorem 1.9 of [28]). Thus $X_N(t) - X_{N-1}(t) \rightarrow \infty$ a.s. (at linear speed) and hence also in probability.

It remains to consider the case where $N - 1$ is *not* the unique global maximum for V . Put $H := V(N - 1)$ and $j_0 := \min\{j \geq 0 : V(j) = H\}$; we are in the case $j_0 < N - 1$. Then $V(k) \leq H$ for $k \geq j_0$ with strict inequality for every $k \geq N$.

We use a truncation and approximation argument. For $M \geq 1$, consider the finite particle system $[1, M]$ with infinite pressure from the right, i.e., with b_M set to 0, and let $\eta^{(M)}(t) := (\eta_j^{(M)}(t))_{j \in [1, M-1]}$ be the associated gap process. Start from the same gaps as the infinite system, so $\eta_j^{(M)}(0) = \eta_j(0)$ for all $j \in [1, M - 1]$. Using the second-class customer coupling from the proof of Lemma 2.5, we may couple the infinite system with the finite system truncated at M with infinite pressure from the right. Starting from the same queues on $[1, M - 1]$, the customers of the truncated system may be regarded as first-class customers in the infinite system, with all additional customers regarded as second class. It follows that $\eta_j^{(M)}(t) \leq \eta_j(t)$ for all $j \in [1, M - 1]$ and all $t \geq 0$.

For $M \geq N$ consider the block $[j_0 + 1, M]$ with infinite pressure at M . Define

$$A_k := \sum_{\ell=j_0+1}^k a_\ell^{-1} \exp\{V(j_0) - V(\ell - 1)\}, \text{ for } k \in [j_0 + 1, M].$$

The infinite-pressure speed of block $[j_0 + 1, M]$ is $\tilde{v}_M = -1/A_M$ (this is the shifted version of the formula (2.8)). The corresponding traffic intensities are

$$\rho_k^{(M)} = e^{V(k) - V(j_0)} + \tilde{v}_M \sum_{\ell=j_0+1}^k a_\ell^{-1} e^{V(k) - V(\ell - 1)} = e^{V(k) - V(j_0)} \left(1 - \frac{A_k}{A_M}\right). \quad (5.2)$$

Here $e^{V(k) - V(j_0)} \leq 1$, while $0 < A_k < A_M$ for $k < M$. Thus it holds that $\rho_k^{(M)} \in (0, 1)$ for all $k \in [j_0 + 1, M - 1]$. Hence $[j_0 + 1, M]$ with infinite pressure at M is intrinsically stable (and $\tilde{v}_M$ is its speed).

At the queue $N - 1$, since $V(N - 1) = V(j_0)$ formula (5.2) reduces to

$$\rho_{N-1}^{(M)} = 1 - \frac{A_{N-1}}{A_M}.$$

Moreover, by hypotheses **(UE)** and (1.13), $A_M \rightarrow \infty$ as $M \rightarrow \infty$. Therefore

$$\tilde{v}_M \uparrow 0, \text{ and } \rho_{N-1}^{(M)} \uparrow 1.$$

We next argue that, for all sufficiently large M , the block $[j_0+1, M]$ is the rightmost stable cloud in the system $[1, M]$ with infinite pressure from the right. If $j_0 = 0$ there is nothing to prove; so suppose $j_0 > 0$. By definition, $V(j) < V(j_0)$ for $0 \leq j < j_0$. In particular, the rightmost cloud in the isolated finite system $[1, j_0]$ has strictly negative speed, $s < 0$, say. The block $[j_0+1, M]$ has speed $\tilde{v}_M \rightarrow 0$, so for all M large enough $s < \tilde{v}_M < 0$. The finite-cloud decomposition algorithm (Algorithm 1 from [27]) implies that the cloud decomposition of $[1, M]$ is given by the decomposition of $[1, j_0]$ followed by the single rightmost cloud $[j_0+1, M]$. Moreover, the queue separating these two parts is transient. Hence from some finite time onwards, the internal gaps of $[j_0+1, M]$ evolve exactly as the gaps in the isolated stable system $[j_0+1, M]$ with infinite pressure at M . Consequently, from Theorem 2.1 of [27] applied to that finite system,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[\eta_{N-1}^{(M)}(t) \leq B \right] = 1 - (\rho_{N-1}^{(M)})^{B+1}, \text{ for } B \in \mathbb{Z}_+.$$

It follows from the coupling described earlier that, in the infinite system,

$$\limsup_{t \rightarrow \infty} \mathbb{P} [\eta_{N-1}(t) \leq B] \leq 1 - (\rho_{N-1}^{(M)})^{B+1}, \text{ for } B \in \mathbb{Z}_+.$$

Take $M \rightarrow \infty$. Since $\rho_{N-1}^{(M)} \rightarrow 1$, we can conclude, for every fixed B , $\mathbb{P}(\eta_{N-1}(t) \leq B) \rightarrow 0$, as was required. $\square$

Remark 5.1. We expect that in most cases the restriction on the initial condition in Theorem 1.7 can be removed, but since there are some subtleties, and the theory of non-finite initial conditions is not so developed [28], we do not pursue this extension, but make some observations here.

Indeed, suppose additionally that $\mathbb{P}[\xi_1 > 0] > 0$. Since $\mathbb{E} \xi_1 < 0$, we have $\mathbb{P}[\xi_1 < 0] > 0$ also; i.e. **(N)** holds. Choose $\delta > 0$ such that $q := \mathbb{P}[\xi_1 \geq \delta] > 0$. For every given $m \in \mathbb{N}$ there are infinitely many $r \in \mathbb{N}$ for which $\xi_{r+1}, \dots, \xi_{r+m}$ are all at least δ . At the endpoint $k = r + m$ of such a run, $V(\ell) \leq V(k)$ for $r + 1 \leq \ell \leq k$. The formula for β_k in (3.2) implies that, under condition **(UE)**,

$$\beta_k \geq \varepsilon_0 \cdot \#\{\ell \leq k : V(\ell) \leq V(k)\},$$

and so $\beta_k \geq \varepsilon_0 m$. Since m was arbitrary we get $\limsup_{k \rightarrow \infty} \beta_k = \infty$. Applying the same argument to the shifted environment on $[N, \infty)$, for every fixed m there are a.s. infinitely many runs of m consecutive non-negative (indeed, uniformly positive) increments. Hence the corresponding shifted β_k is unbounded. Thus no $v > 0$ can give an admissible solution. Nor can $v < 0$, since $\alpha_k \rightarrow 0$ while β_k is bounded below away from zero. Consequently the zero-speed solution is the unique admissible solution. Hence we anticipate the initial condition can be general in that case.

The other case is the *one-sided* case where $\mathbb{P}[\xi_1 > 0] = 0$. Then $\xi_k \leq 0$, a.s. Now V is non-increasing and $N = \min\{k \geq 1 : \xi_k < 0\}$. Here there can be other admissible solutions: the homogeneous case $a_k \equiv a < b \equiv b_k$ already shows this: see Example 3.3 of [28]. Thus this case may be more delicate with regards to extending the initial condition in Theorem 1.7.

Proof of Theorem 1.8. The main idea is the following. For the upper bound on the speed, we argue that one can find a trap of depth essentially $\beta \log k$ to the left of k ; then, even with infinite pressure, the cloud that contains this trap cannot go faster than about $e^{-\beta \log k} = k^{-\beta}$. For the lower bound, we consider the stable cloud containing the k th particle, and, with a suitable refinement of the arguments of Lemma 3.6, prove that the magnitude of its speed cannot be much less than $k^{-\beta}$.

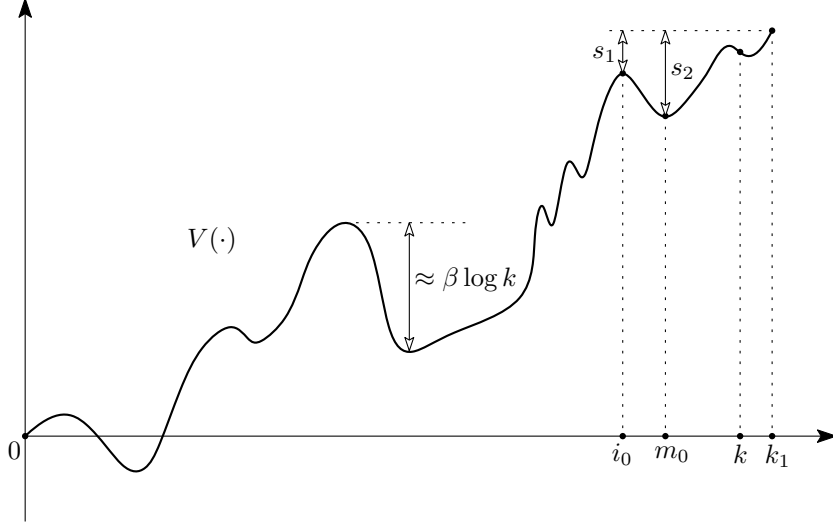


Figure 11: On the proof of Theorem 1.8: the deepest trap in $[0, k]$ has depth about $\beta \log k$.

The key fact about the potential that we need is that the deepest trap in the interval $[0, k]$ has depth roughly $\beta \log k$ (as shown on Figure 11). As shown in Sections 2 and 3 of [13], one can obtain quite precise control on the number of traps of given depth (note that the analysis of [13] is in the case $\mathbb{E} \log \frac{\Theta^-}{\Theta^+} < 0$, but one can obtain the results of our interest by symmetry). In any case, here we only need the following consequence of the results in Section 3 (specifically, see Lemmas 3.4 and 3.5) of [13]: for any $\delta > 0$,

$$(\beta - \delta) \log k \leq \max_{[j_0, j_1] \subset [0, k]} \mathcal{D}(j_0, j_1) \leq (\beta + \delta) \log k, \text{ eventually a.s.} \quad (5.3)$$

It follows from (5.3) applied at $\lfloor k/2 \rfloor$ that $\max_{[j_0, j_1] \subset [0, \lfloor k/2 \rfloor]} \mathcal{D}(j_0, j_1) \geq (\beta - \delta) \log k$ eventually. Choose a time $m \leq \lfloor k/2 \rfloor$ such that

$$\max_{0 \leq i \leq m} V(i) - V(m) = \max_{0 \leq j \leq \lfloor k/2 \rfloor} \max_{0 \leq i \leq j} (V(i) - V(j)) \geq (\beta - \delta) \log k.$$

Then set $a := \max\{i \leq m : V(i) = \max_{0 \leq j \leq m} V(j)\}$. Then $a < m$ and $V(j) < V(a)$ for $a < j \leq m$. The strong law for V , with $\mu := \mathbb{E} \log(\Theta^-/\Theta^+) > 0$ by (1.14), implies that, $\mathbb{P}$ -a.s.,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \max_{0 \leq j \leq \lfloor k/2 \rfloor} V(j) = \frac{\mu}{2}, \text{ and } \lim_{k \rightarrow \infty} \frac{V(k)}{k} = \mu.$$

Thus $V(k) > V(a)$ eventually. Let $b := \min\{j > m : V(j) \geq V(a)\}$. Thus we have shown existence of a, b, m with $a < m < b \leq k$, $V(j) < V(a)$ for $a < j < b$, $V(b) \geq V(a)$, and $V(a) - V(m) \geq (\beta - \delta) \log k$. Thus by Lemma 3.4, we have that $[a + 1, b]$ is intrinsically stable with depth $\mathcal{D}(a, b) \geq (\beta - \delta) \log k$.

Consider the finite system $[1, k']$ with $k' > k$ and infinite pressure at k' . Then in the cloud decomposition of $[1, k']$, either $[a + 1, b]$ and k are in the same cloud, or else k is in a cloud to the right of the cloud containing $[a + 1, b]$. In either case, let w be the speed of the cloud containing $[a + 1, b]$. As in the proof of (3.8), Proposition 2.4 gives a coupling under which the particles $[a + 1, b]$ of $[1, k']$ are no farther to the left than in the system $[a + 1, b]$ with infinite pressure at b . Hence $w \geq \tilde{v}$, where $\tilde{v}$ is the speed of the latter system, and $|\tilde{v}| \leq \varepsilon_0^{-1} e^{-\mathcal{D}(a, b)}$ by (3.8) (note that $V(b) = \max_{[a, b]} V$). Since the cloud speeds in $[1, k']$ are negative and non-decreasing from left to right, the magnitude of the speed of the stable cloud in $[1, k']$ containing k is at most $\varepsilon_0^{-1} k^{-(\beta - \delta)}$.

We next seek a lower bound on the magnitude of the speed. Unfortunately, Lemma 3.6 is not sharp enough in this case, because we are now aiming for polynomial bounds in k , and (3.9) contains a potentially polynomial factor. Therefore, to prove that $|v_k|$ is at least $ck^{-(\beta+\delta)}$, we need to do a finer analysis of the formula (3.4).

Recall $\mu = \mathbb{E} \log(\Theta^-/\Theta^+) > 0$ and notice that there exist $u_0, c_0 > 0$ such that

$$\mathbb{P} \left[\log \frac{\Theta^-}{\Theta^+} > u_0 \right] > c_0. \quad (5.4)$$

Then, we say that a site m is *good* if

$$V(m) \geq u_0 + \max_{0 \leq j < m} V(j).$$

There are typically plenty of good sites; this is a consequence of the following elementary fact about ladder epochs. Denote by $L_1 = \min\{n > 0 : V(n) > 0\}$ the first ladder epoch. Provided that $\mu > 0$, usual large deviation estimates imply that we have for some $c = c(\mu, \varepsilon_0)$

$$\mathbb{P}[L_1 > t] \leq e^{-ct}. \quad (5.5)$$

Then, notice that at each ladder epoch, the next site will be good with probability at least c_0 . From this, it is elementary to obtain that

$$\mathbb{P}\text{-a.s. for all } k \text{ large enough, the interval } [k, k + \log^2 k] \text{ contains a good site.} \quad (5.6)$$

Next, for an interval I , define

$$\Psi(I) := \sum_{j \in I} e^{V(m^I) - V(j)}, \text{ where } m^I := \arg \min_{j \in I} V(j).$$

Let us prove that

$$\mathbb{P}\text{-a.s. for all } k \text{ large enough it holds that } \Psi(I) \leq 2 \log^2 k \text{ for all } I \subset [0, 2k]. \quad (5.7)$$

Indeed, let $I = [n_1, n_2]$, and assume that $n_2 - n_1 \geq 2 \log^2 k$ (otherwise (5.7) trivially holds). First, usual large deviation bounds imply that

$$\mathbb{P} [V(n_1 + \log^2 k) - V(n_1) \geq \frac{2\mu}{3} \log^2 k] \geq 1 - e^{-c_1 \log^2 k}.$$

Then, we also have

$$\mathbb{P} [V(i) - V(n_1 + \log^2 k) \geq -\frac{\mu}{3} \log^2 k \text{ for all } i \in [n_1 + \log^2 k, n_2]] \geq 1 - e^{-c_2 \log^2 k},$$

and so (observe that $V(m^I) \leq V(n_1)$)

$$\mathbb{P} [V(i) - V(m^I) \geq \frac{\mu}{3} \log^2 k \text{ for all } i \in [n_1 + \log^2 k, n_2]] \geq 1 - 2e^{-c_3 \log^2 k}. \quad (5.8)$$

Next, on the event in (5.8), for $I \subset [0, 2k]$ we have that

$$\Psi(I) \leq \log^2 k + |I| \exp(-\frac{\mu}{3} \log^2 k) \leq 2 \log^2 k,$$

and therefore (5.7) follows from (5.8) with Borel–Cantelli and the fact that there are $O(k^2)$ choices for interval $I \subset [0, 2k]$.

Now, we are ready to obtain a lower bound on $|v_k|$. Let $k_1 \geq k$ be the smallest good site one can find in $[k, \infty)$; observe that, by (5.6), we have $k_1 \leq k + \log^2 k$. Then, consider the finite system with particles $\{1, \dots, k_1\}$ and let $[i_0 + 1, k_1]$ be the stable cloud containing k_1 (see Figure 11), let $\hat{v}$ be its speed. Let $m_0 = \arg \min_{j \in [i_0, k_1]} V(j)$, and abbreviate $s_1 = V(k_1) - V(i_0)$,

$s_2 = V(k_1) - V(m_0)$. From (3.4) and using that $s_1 \geq u_0$ (because k_1 is good) together with Condition **(UE)**, we obtain that

$$\begin{aligned} |\hat{v}| &= \frac{e^{V(k_1)-V(i_0)} - 1}{\frac{1}{a_{k_1}}e^{V(k_1)-V(k_1-1)} + \dots + \frac{1}{a_{i_0+1}}e^{V(k_1)-V(i_0)}} \\ &\geq \frac{Ce^{s_1}}{\varepsilon_0^{-1}e^{s_2}\Psi([i_0, k_1])} \\ &\geq C'e^{-(s_2-s_1)}\log^{-2}k. \end{aligned}$$

Since $s_2 - s_1 \leq \mathcal{D}(0, 2k)$, and finite-system cloud speeds are non-decreasing from left to right, this implies the desired lower bound, i.e., that in the system $[1, k_1]$ the cloud containing k has speed of magnitude at least $ck^{-(\beta+\delta)}$.

Next we argue that in the infinite system there must be infinitely many clouds. Let k_1 be the good site used above, and put $M = k^2$. Denote particle speeds in the system $[1, M]$ with infinite pressure by $w_j^{(M)}$. Removing particles to the right of k_1 , and releasing the pressure, can only increase the positions of the retained particles. Hence the lower-magnitude estimate in the ordinary system $[1, k_1]$ gives $w_k^{(M)} \leq -ck^{-(\beta+\varepsilon)}$. On the other hand, the trap estimate at scale M gives

$$w_M^{(M)} \geq -CM^{-(\beta-\varepsilon)} = -Ck^{-2(\beta-\varepsilon)}.$$

Choose $0 < \varepsilon < \beta/3$. Eventually, $w_k^{(M)} < w_M^{(M)}$. Therefore there is a strict increase in cloud speed across some queue with index in $[k, M-1]$. That queue is transient. Since k is arbitrarily large, these transient queues occur arbitrarily far to the right; extracting increasing sequences gives exactly the hypothesis of Corollary 2.6.

To finish the proof, we apply Lemma 2.5 to show that $t^{-1}X_k(t)$ has a limit and the finite-system speed estimates derived above apply to the resulting speed v_k . Indeed, for fixed k , choose one of the finite infinite-pressure truncations having a transient separating queue strictly to the right of k . Lemma 2.5 then identifies the dynamics on the clouds to its left, after finite time, with those of that finite system (up to an irrelevant spatial translation), so the finite-system strong law gives the strong law for particle k . The infinite-pressure truncation gives the lower bound on v_k , while removing particles to the right and using the ordinary $[1, k_1]$ system gives the upper bound on v_k . Hence, for every $\varepsilon > 0$, $-k^{-(\beta-\varepsilon)} \leq v_k \leq -k^{-(\beta+\varepsilon)}$, as claimed. $\square$

A Ladder height condition

An example where Condition **(LH)** trivially holds is when the distribution of $\log(\Theta^-/\Theta^+)$ is lattice, because in that case η will always be at least the mesh of the lattice. Another example is provided under a bounded density hypothesis, as in the following result. Condition **(LH)** can fail, even under **(UE)** and **(N)**, for example by taking $\Theta^- = e^{X/2}$ and $\Theta^+ = e^{-X/2}$ where X has a distribution with a sequence of atoms going rapidly to 0 from above.

Lemma A.1. *Suppose that **(N)** and **(UE)** hold. Then sufficient for **(LH)** to hold is either condition (a) that the distribution of $\log \frac{\Theta^-}{\Theta^+}$ under $\mathbb{P}$ has a bounded density, or (b) that Θ^+ and Θ^- are independent and each has a bounded density under $\mathbb{P}$.*

Proof. Let $\zeta := \log \frac{\Theta^-}{\Theta^+}$. First we prove the lemma under condition (a). Suppose that **(N)** and **(UE)** hold, and that ζ has a bounded density. Let f_n denote the density of V_n . Then $0 < \text{Var } \zeta < \infty$ and (since f_1 is bounded over a bounded support) $f_1 \in L^2(\mathbb{R})$. Let $\varphi(t) := \mathbb{E} e^{it\zeta}$; then $\varphi \in L^2(\mathbb{R})$ by Plancherel's theorem. If ζ' is an independent copy of ζ , then $|\varphi(t)|^2 = \mathbb{E} e^{it(\zeta-\zeta')} = 1 - t^2 \text{Var } \zeta + o(t^2)$ as $t \rightarrow 0$. Hence for some $c, \delta > 0$, $|\varphi(t)| \leq e^{-ct^2}$ for all $|t| < \delta$.

Moreover, $|\varphi(t)| < 1$ for every $t \neq 0$ (see [26, p. 18]) and the Riemann–Lebesgue lemma [26, p. 19] says that $\varphi(t) \rightarrow 0$ as $|t| \rightarrow \infty$. Hence $\rho := \sup_{|t| \geq \delta} |\varphi(t)| < 1$. For $n \geq 2$,

$$\int_{\mathbb{R}} |\varphi(t)|^n dt \leq \int_{|t| \leq \delta} e^{-cnt^2} dt + \rho^{n-2} \int_{|t| > \delta} |\varphi(t)|^2 dt = O(n^{-1/2}) + O(\rho^{n-2}) = O(n^{-1/2}).$$

Thus by Fourier inversion (see Theorem 3.2.2 of [26])

$$\|f_n\|_{\infty} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi(t)|^n dt = O(n^{-1/2}),$$

as $n \rightarrow \infty$. Formula (3) (with $k = 1$) of [1] implies that

$$\mathbb{P}[\eta < y] = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}[0 < V_n \leq \eta, V_n < y] \leq \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}[0 < V_n < y],$$

where, for some $C < \infty$ and all $n \in \mathbb{N}$ (recall $\|f_1\|_{\infty} < \infty$ by hypothesis),

$$\mathbb{P}[0 < V_n < y] = \int_0^y f_n(z) dz \leq y \|f_n\|_{\infty} \leq C y n^{-1/2}.$$

Thus we get $\mathbb{P}[\eta < y] = O(y)$ as $y \downarrow 0$, which verifies **(LH)**.

Finally we show that condition (b) implies (a). Suppose that Θ^- and Θ^+ are independent and have densities f^- and f^+ respectively, and that, for some $B < \infty$, $f^+(x) \leq B$ and $f^-(x) \leq B$ for all $\varepsilon_0 \leq x \leq \varepsilon_0^{-1}$ (recall that **(UE)** is assumed). Then a Jacobian calculation shows that ζ has density f given by

$$f(y) = e^{-y} \int_{\varepsilon_0}^{\varepsilon_0^{-1}} u f^-(u) f^+(u e^{-y}) du \leq e^{-y} B^2 \varepsilon_0^{-2},$$

showing that f is bounded. □

Acknowledgements

The authors thank Vladislav Vysotskiy for discussions about ascending ladder heights and for indicating the paper [1] to them. ChatGPT was used to help identify the explicit distributions of the limit random variables Y and Z , to help with the proof of Lemma A.1, and to identify some errors in earlier drafts; all other mathematical arguments and proofs are entirely human-generated. MM and AW were supported by EPSRC grant EP/W00657X/1. SP was partially supported by CMUP, member of LASI, which is financed by national funds through FCT — Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UID/00144/2025, <https://doi.org/10.54499/UID/00144/2025>. Some of this work was carried out during the programme “Stochastic systems for anomalous diffusion” (July–December 2024) hosted by the Isaac Newton Institute, under EPSRC grant EP/Z000580/1.

References

- [1] L. Alili and R. A. Doney. Wiener-Hopf factorization revisited and some applications. *Stochastics Stochastics Rep.* 66 (1999), pp. 87–102.
- [2] E. D. Andjel, P. A. Ferrari, H. Guiol and C. Landim. Convergence to the maximal invariant measure for a zero-range process with random rates. *Stochastic Process. Appl.* 90 (2000), pp. 67–81.

- [3] C. Bahadoran, T. Mountford, K. Ravishankar and E. Saada. Supercritical behavior of asymmetric zero-range process with sitewise disorder. *Ann. Inst. Henri Poincaré Probab. Stat.* 53 (2017), pp. 766–801.
- [4] C. Bahadoran, T. S. Mountford, K. Ravishankar and E. Saada. ‘Zero-range process in random environment’. In: *From particle systems to partial differential equations*. Vol. 352. Springer Proc. Math. Stat. Springer, Cham, 2021, pp. 51–77.
- [5] C. Bahadoran, T. Mountford, K. Ravishankar and E. Saada. Supercriticality conditions for asymmetric zero-range process with sitewise disorder. *Braz. J. Probab. Stat.* 29 (2015), pp. 313–335.
- [6] I. Benjamini, P. A. Ferrari and C. Landim. Asymmetric conservative processes with random rates. *Stochastic Process. Appl.* 61 (1996), pp. 181–204.
- [7] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Second. Probability and its Applications. Birkhäuser Verlag, Basel, 2002, pp. xvi+672.
- [8] M. R. Evans. Bose-Einstein condensation in disordered exclusion models and relation to traffic flow. *Europhys. Lett.* 36 (1996), pp. 13–18.
- [9] M. R. Evans. Exact steady states of disordered hopping particle models with parallel and ordered sequential dynamics. *J. Phys. A* 30 (1997), pp. 5669–5685.
- [10] A. Faggionato. Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit. *Electron. J. Probab.* 13 (2008), no. 73, 2217–2247.
- [11] P. A. Ferrari and V. V. Sisko. ‘Escape of mass in zero-range processes with random rates’. In: *Asymptotics: particles, processes and inverse problems*. Vol. 55. IMS Lecture Notes Monogr. Ser. Inst. Math. Statist., Beachwood, OH, 2007, pp. 108–120.
- [12] S. Floreani, F. Redig and F. Sau. Hydrodynamics for the partial exclusion process in random environment. *Stochastic Process. Appl.* 142 (2021), pp. 124–158.
- [13] A. Fribergh, N. Gantert and S. Popov. On slowdown and speedup of transient random walks in random environment. *Probab. Theory Related Fields* 147 (2010), pp. 43–88.
- [14] M. Gannon, E. Pechersky, Y. Suhov and A. Yambartsev. Random walks in a queueing network environment. *J. Appl. Probab.* 53 (2016), pp. 448–462.
- [15] G. Gielis, A. Koukkous and C. Landim. Equilibrium fluctuations for zero range processes in random environment. *Stochastic Process. Appl.* 77 (1998), pp. 187–205.
- [16] A. O. Golosov. Limit distributions for random walks in random environments. *Sovet Math. Dokl.* 28 (1983), pp. 18–22.
- [17] Y. Hamana and H. Matsumoto. The probability distributions of the first hitting times of Bessel processes. *Trans. Amer. Math. Soc.* 365 (2013), pp. 5237–5257.
- [18] R. Juhász, L. Santen and F. Iglói. Partially asymmetric exclusion processes with sitewise disorder. *Phys. Rev. E* 74 (2006), p. 061101.
- [19] H. Kesten, M. V. Kozlov and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math.* 30 (1975), pp. 145–168.
- [20] H. Kesten. The limit distribution of Sinai’s random walk in random environment. *Phys. A* 138 (1986), pp. 299–309.
- [21] C. Kipnis. Central limit theorems for infinite series of queues and applications to simple exclusion. *Ann. Probab.* 14 (1986), pp. 397–408.
- [22] R. Krenzler, H. Daduna and S. Otten. Jackson networks in nonautonomous random environments. *Adv. in Appl. Probab.* 48 (2016), pp. 315–331.

- [23] J. Krug and P. A. Ferrari. Phase transitions in driven diffusive systems with random rates. *J. Phys. A: Math. Gen.* 29 (1996), L465–L471.
- [24] H. Lacoïn and S. Yang. Mixing time for the asymmetric simple exclusion process in a random environment. *Ann. Appl. Probab.* 34 (2024), pp. 388–427.
- [25] T. M. Liggett. *Interacting particle systems*. Berlin: Springer-Verlag, 2005, pp. xvi+496.
- [26] E. Lukacs. *Characteristic functions*. 2nd ed. New York: Hafner Publishing Co., 1970, pp. x+350.
- [27] V. Malyshev, M. Menshikov, S. Popov and A. Wade. Dynamics of finite inhomogeneous particle systems with exclusion interaction. *J. Stat. Phys.* 190 (2023), Paper No. 184, 32pp.
- [28] M. Menshikov, S. Popov and A. Wade. Semi-infinite particle systems with exclusion interaction and heterogeneous jump rates. *Probab. Theory Relat. Fields* 195 (2026), pp. 73–107.
- [29] M. Menshikov, S. Popov and A. Wade. Dynamics of the leftmost particle in heterogeneous semi-infinite exclusion systems. *Markov Process. Relat. Fields* 32 (2026), pp. 205–232.
- [30] M. Menshikov, S. Popov and A. Wade. One-dimensional particle clouds with elastic collisions. *Preprint, ArXiv: 2509.08430* (2025).
- [31] J. Neveu and J. Pitman. ‘Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion’. In: *Séminaire de Probabilités, XXIII*. Vol. 1372. Lecture Notes in Math. Berlin: Springer, 1989, pp. 239–247.
- [32] E. Orlandi and P. Picco. One-dimensional random field Kac’s model: weak large deviations principle. *Electron. J. Probab.* 14 (2009), no. 47, 1372–1416.
- [33] P. Révész. *Random walk in random and non-random environments*. 3rd edition. Hackensack, NJ: World Scientific Publishing Co., 2013, pp. xviii+402.
- [34] D. Schmid. Mixing times for the simple exclusion process in ballistic random environment. *Electron. J. Probab.* 24 (2019), Paper No. 22, 25pp.
- [35] Y. G. Sinaï. The limiting behavior of a one-dimensional random walk in a random medium. *Theory Probab. Appl.* 27 (1982), pp. 256–268.
- [36] F. Solomon. Random walks in a random environment. *Ann. Probability* 3 (1975), pp. 1–31.
- [37] O. Zeitouni. ‘Part II: Random Walks in Random Environment’. In: *Lectures on Probability Theory and Statistics: Ecole d’Eté de Probabilités de Saint-Flour XXXI - 2001*. Ed. by J. Picard. Berlin, Heidelberg: Springer, 2004, pp. 190–312.
- [38] M. P. W. Zerner. Lyapounov exponents and quenched large deviations for multidimensional random walk in random environment. *Ann. Probab.* 26 (1998), pp. 1446–1476.
- [39] Y. Zhu. Markovian queueing networks in a random environment. *Oper. Res. Lett.* 15 (1994), pp. 11–17.