

A Characterization of Local Nilpotence for Derivations of Ore Extensions

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Abstract

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. A recent theorem of I. Pan characterizes the nonzero locally nilpotent derivations of $\mathbb{k}[x, y]$ in terms of their isotropy groups: such a derivation is locally nilpotent if and only if its isotropy group contains automorphisms of arbitrarily large degree. We study analogues of this characterization for several noncommutative algebras related to the affine plane. Our main result shows that Pan's characterization extends to the differential Ore extensions $A_h = \mathbb{k}[x][t; h(x)\partial_x]$, where $h \in \mathbb{k}[x] \setminus \mathbb{k}$. For the first Weyl algebra A_1 , we establish the same equivalence for every nonzero locally finite derivation. For the quantum plane and the first quantum Weyl algebra, known results imply that the corresponding equivalence holds vacuously for nonzero derivations. In contrast, the converse fails for the free associative algebra $\mathbb{k}\langle x, y \rangle$: we construct a non-locally-nilpotent inner derivation whose isotropy group has unbounded degree. We further analyze this failure through the behavior of automorphisms and derivations under abelianization.

1 Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. For a $\mathbb{k}$ -algebra A , we denote by $\text{Aut}(A)$ and $\text{Der}(A)$ the group of $\mathbb{k}$ -automorphisms and the space of $\mathbb{k}$ -derivations of A , respectively. The group $\text{Aut}(A)$ acts by conjugation on $\text{Der}(A)$ and, for $D \in \text{Der}(A)$, the *isotropy group* of D is

$$\text{Aut}_D(A) = \{\rho \in \text{Aut}(A) \mid \rho D = D\rho\}.$$

A derivation D is said to be *locally finite* if, for every $a \in A$, the $\mathbb{k}$ -vector space $\text{span}_{\mathbb{k}}\{D^n(a) \mid n \geq 0\}$ is finite-dimensional. It is said to be *locally nilpotent* if, for every $a \in A$, there exists $n \geq 1$ such that $D^n(a) = 0$. Thus, every locally nilpotent derivation is locally finite. We denote by $\text{LFD}(A)$ and $\text{LND}(A)$ the sets of locally finite and locally nilpotent derivations of A , respectively.

A classical theorem of Rentschler [16] states that every locally nilpotent derivation of $\mathbb{k}[x, y]$ is, after a suitable polynomial change of coordinates, of the form $f(x)\partial_y$, where $f(x) \in \mathbb{k}[x]$. Locally finite derivations of $\mathbb{k}[x, y]$ are also classified, up to conjugation by polynomial automorphisms; see [18, Corollary

4.7]. For example, Van den Essen used this classification to obtain an algorithm for deciding whether a polynomial vector field in dimension two has a polynomial flow; see [18, Section 5.].

A central motivation for this work is the following theorem of I. Pan [15], which characterizes local nilpotence of plane polynomial derivations in terms of their isotropy groups. The proof of this characterization relies strongly on earlier work of R. Baltazar and I. Pan [6] concerning the structure of isotropy groups of polynomial derivations.

Theorem 1 ([15, Theorem 1.1]). *Let D be a nonzero derivation of $\mathbb{k}[x, y]$. The following conditions are equivalent:*

- (a) *D is locally nilpotent;*
- (b) *for every $d \geq 1$, there exists $\rho = (f, g) \in \text{Aut}_D(\mathbb{k}[x, y])$ such that $\deg(f) \geq d$ or $\deg(g) \geq d$;*
- (c) *$\text{Aut}_D(\mathbb{k}[x, y])$ is not an algebraic subgroup of $\text{Aut}(\mathbb{k}[x, y])$;*
- (d) *$\text{Aut}_D(\mathbb{k}[x, y])$ is an infinite-dimensional algebraic subgroup of $\text{Aut}(\mathbb{k}[x, y])$ in the sense of Shafarevich–Kambayashi.*

A natural question suggested by Pan’s theorem is whether the same characterization of local nilpotence survives in noncommutative deformations of the affine plane. In this direction, we consider the differential Ore extensions

$$A_h = \mathbb{k}[x][t; h(x)\partial_x] = \mathbb{k}\langle x, t \mid tx - xt = h(x) \rangle,$$

where $h \in \mathbb{k}[x] \setminus \mathbb{k}$. Our main result shows that Pan’s criterion extends completely to this family: for every $D \in \text{Der}(A_h)$. More precisely, our main result, Theorem 22, shows that a derivation of A_h is locally nilpotent if and only if its isotropy group contains automorphisms of arbitrarily large degree.

The two extremal cases of the family A_h recover two classical algebras. If $h = 0$, then $A_h = \mathbb{k}[x, t]$, and Pan’s theorem applies. If $h \in \mathbb{k}^*$, then, after a nonzero scalar rescaling, A_h is isomorphic to the first Weyl algebra A_1 . If $h \in \mathbb{k}^*$, then, after a nonzero scalar rescaling, one obtains the first Weyl algebra

$$A_1 = \mathbb{k}\langle p, q \mid pq - qp = 1 \rangle.$$

The first Weyl algebra requires a separate analysis. It is not known whether, for an arbitrary nonzero derivation D , the existence of automorphisms of arbitrarily large degree in $\text{Aut}_D(A_1)$ necessarily implies that D is locally nilpotent. However, using Dixmier’s results on the first Weyl algebra together with explicit computations of isotropy groups, we prove in Theorem 8 that every nonzero locally finite derivation D of A_1 is locally nilpotent if and only if its isotropy group $\text{Aut}_D(A_1)$ contains automorphisms of arbitrarily large degree.

These results fit naturally into the classification of Ore extensions of $\mathbb{k}[x]$. By results of Awami, Van den Bergh and Van Oystaeyen [4, Section 2.1] and Alev and Dumas [2, Proposition 3.2], every Ore extension

$$S = \mathbb{k}[x][y; \sigma, \delta],$$

with $\sigma \in \text{Aut}(\mathbb{k}[x])$ and δ a σ -derivation, is isomorphic to a polynomial algebra, a quantum plane, a first quantum Weyl algebra, or a differential Ore extension. The remaining quantum cases are discussed in Section 5. By collecting well-known results from the literature concerning their automorphism groups and locally nilpotent derivations, we make explicit how the same Pan-type equivalence behaves for the quantum plane and the first quantum Weyl algebra.

Beyond the class of Ore extensions, we also consider the free associative algebra $F = \mathbb{k}\langle x, y \rangle$, where the relation between local nilpotence and the isotropy group exhibits a different behavior. The implication from local nilpotence to unbounded degree of the isotropy group still holds, as a consequence of the triangularization of locally nilpotent derivations of F . The converse, however, fails: we exhibit an inner derivation which is not locally nilpotent but whose isotropy group contains automorphisms of arbitrarily large degree.

Abelianization provides a conceptual explanation for this failure. By the classical results of Czerniakiewicz and Makar-Limanov (see [9, 14]), automorphisms of F are completely detected by their images in $\mathbb{k}[x, y]$. Derivations behave differently: the induced map $\pi_* : \text{Der}(F) \rightarrow \text{Der}(\mathbb{k}[x, y])$ has a nontrivial kernel, so this map is not injective. We show in Corollary 28 that its kernel contains no nonzero locally nilpotent derivations:

$$\ker(\pi_*) \cap \text{LND}(F) = \{0\}.$$

Although automorphisms of F are completely determined by their abelianizations, local nilpotence of the induced derivation on the commutative quotient does not imply local nilpotence of the original derivation. Our inner counterexample in Proposition 27 has zero abelianization. Furthermore, Proposition 29 shows that the converse to Pan's criterion can fail even when the induced derivation is nonzero and locally nilpotent.

2 The first Weyl algebra

Let $A_1 = \mathbb{k}\langle p, q \mid pq - qp = 1 \rangle$ be the first Weyl algebra: its center is $Z(A_1) = \mathbb{k}$ [10, p. 210]. Every element of A_1 has a unique PBW expression $u = \sum_{i,j \geq 0} c_{ij} p^i q^j$ and we define $\deg(u) = \max\{i + j \mid c_{ij} \neq 0\}$, and, for $\rho \in \text{Aut}(A_1)$,

$$\deg(\rho) = \max\{\deg(\rho(p)), \deg(\rho(q))\}.$$

It is well known that every derivation of A_1 is inner; see [10, p. 210]. Then, every $D \in \text{Der}(A_1)$ has the form $D = \text{ad}_w := [w, _]$, for some $w \in A_1$.

The following simple consequence will be useful throughout this section.

Lemma 2. *Let $w \in A_1$. Then*

$$\text{Aut}_{\text{ad}_w}(A_1) = \{\rho \in \text{Aut}(A_1) \mid \rho(w) - w \in \mathbb{k}\}.$$

Proof. For every $\rho \in \text{Aut}(A_1)$, $\rho \text{ad}_w \rho^{-1} = \text{ad}_{\rho(w)}$. Then, $\rho \in \text{Aut}_{\text{ad}_w}(A_1)$ if and only if $\text{ad}_{\rho(w)} = \text{ad}_w$, which is equivalent to $\rho(w) - w \in Z(A_1)$. Since $Z(A_1) = \mathbb{k}$, the result follows. $\square$

We recall the results of Dixmier [10] that are relevant to locally finite derivations. For $w \in A_1$, let

$$F_w := \{u \in A_1 \mid \dim_{\mathbb{k}} \text{span}_{\mathbb{k}}\{u, \text{ad}_w(u), \text{ad}_w^2(u), \dots\} < \infty\},$$

and

$$N_w := \{u \in A_1 \mid \text{ad}_w^n(u) = 0 \text{ for some } n \geq 1\}.$$

For $\lambda \in \mathbb{k}$, let

$$E_w(\lambda) := \{u \in A_1 \mid \text{ad}_w(u) = \lambda u\},$$

and denote

$$E_w = \bigoplus_{\lambda \in \mathbb{k}} E_w(\lambda).$$

Dixmier [10, Corollary 6.6] proves that

$$F_w = N_w \quad \text{or} \quad F_w = E_w.$$

Therefore, if ad_w is locally finite, then $F_w = A_1$ and there are only two possibilities: either $N_w = A_1$, in which case ad_w is locally nilpotent, or $E_w = A_1$, in which case ad_w is locally finite but not locally nilpotent. Thus, we obtain the following statement (see [10, Corollary 6.6 and Theorems 9.1, 9.2] and the proof of [10, Corollary 9.3]).

Proposition 3 ([10, Corollary 6.6, Theorems 9.1 - 9.2]). *Let $0 \neq D = \text{ad}_w \in \text{LFD}(A_1)$. Then, exactly one of the following occurs:*

- i) *D is locally nilpotent, and there exists $\varphi \in \text{Aut}(A_1)$ such that $\varphi(w) = f(p)$, for some $f(p) \in \mathbb{k}[p] \setminus \mathbb{k}$;*
- ii) *D is locally finite but not locally nilpotent, and there exist $\varphi \in \text{Aut}(A_1)$, $\lambda \in \mathbb{k}^*$ and $\mu \in \mathbb{k}$ such that $\varphi(w) = \lambda p q + \mu$.*

Proof. Since $D = \text{ad}_w$ is locally finite, $F_w = A_1$. By [10, Cor. 6.6], $F_w = N_w$ or $F_w = E_w$.

If $F_w = N_w = A_1$, then ad_w is locally nilpotent. By [10, Thm. 9.1], there exists $\varphi \in \text{Aut}(A_1)$ such that $\varphi(w) \in \mathbb{k}[p]$.

If $F_w = E_w = A_1$, then D is locally finite and, since $D \neq 0$, it is not locally nilpotent. By [10, Thm. 9.2], together with the reduction used in the proof of [10, Cor. 9.3] over an algebraically closed field, one may assume $\varphi(w) = \lambda p q + \mu$, for some $\lambda \in \mathbb{k}^*$ and $\mu \in \mathbb{k}$. $\square$

Let $f(p) \in \mathbb{k}[p] \setminus \mathbb{k}$ and consider $D = \text{ad}_{f(p)} \in \text{Der}(A_1)$. The following lemma determines the centralizer of $f(p)$ in A_1 .

Lemma 4. *If $f(p) \in \mathbb{k}[p] \setminus \mathbb{k}$, then $C_{A_1}(f(p)) = \mathbb{k}[p]$.*

Proof. The inclusion $\mathbb{k}[p] \subseteq C_{A_1}(f(p))$ is clear. Conversely, let

$$u = \sum_{j=0}^m u_j(p)q^j, \quad u_m(p) \neq 0.$$

Suppose that $m \geq 1$. Since $[p, q] = 1$, the coefficient of q^{m-1} in $[f(p), u]$ is $mu_m(p)f'(p)$. Since $\text{char}(\mathbb{k}) = 0$ and f is nonconstant, $f'(p) \neq 0$. Then, $[f(p), u] \neq 0$, a contradiction. Thus, $m = 0$ and $u \in \mathbb{k}[p]$. $\square$

Proposition 5. *Let $D = \text{ad}_{f(p)}$, where $f(p) \in \mathbb{k}[p] \setminus \mathbb{k}$. Then*

$$\text{Aut}_D(A_1) = \{\rho_{a,b,g} \in \text{Aut}(A_1) \mid f(ap + b) - f(p) \in \mathbb{k}\},$$

where $\rho_{a,b,g}(p) = ap + b$ and $\rho_{a,b,g}(q) = a^{-1}q + g(p)$, with $a \in \mathbb{k}^*$, $b \in \mathbb{k}$ and $g(p) \in \mathbb{k}[p]$. In particular, $\text{Aut}_D(A_1)$ contains automorphisms of arbitrarily large degree.

Proof. Let $\rho \in \text{Aut}_D(A_1)$. By Lemma 2, $\rho(f(p)) - f(p) \in \mathbb{k}$. Then, $\rho(f(p)) = f(p) + c$, for some $c \in \mathbb{k}$. Since adding a scalar does not change the centralizer,

$$\rho(C_{A_1}(f(p))) = C_{A_1}(\rho(f(p))) = C_{A_1}(f(p)).$$

By Lemma 4, $\rho(\mathbb{k}[p]) = \mathbb{k}[p]$. Thus, ρ restricts to an automorphism of the polynomial algebra $\mathbb{k}[p]$, and consequently $\rho(p) = ap + b$, for some $a \in \mathbb{k}^*$ and $b \in \mathbb{k}$. Using the Weyl relation and $\rho(p) = ap + b$, we obtain

$$1 = [\rho(p), \rho(q)] = [ap + b, \rho(q)] = a[p, \rho(q)].$$

Then, $[p, \rho(q)] = a^{-1} = [p, a^{-1}q]$ and, therefore, $[p, \rho(q) - a^{-1}q] = 0$. Since $C_{A_1}(p) = \mathbb{k}[p]$, there exists $g(p) \in \mathbb{k}[p]$ such that $\rho(q) = a^{-1}q + g(p)$. Finally, the condition of Lemma 2 becomes $f(ap + b) - f(p) \in \mathbb{k}$.

Conversely, every map of this form is an automorphism of A_1 and satisfies $\rho(f(p)) - f(p) \in \mathbb{k}$, then belongs to $\text{Aut}_D(A_1)$. Additionally, let $a = 1$ and $b = 0$, we obtain automorphisms $p \mapsto p$, $q \mapsto q + g(p)$, with $g(p) \in \mathbb{k}[p]$ arbitrary: their degrees are unbounded. $\square$

Let $H := pq$. Then, $[H, p] = -p$ and $[H, q] = q$. More generally,

$$[H, p^i q^j] = (j - i)p^i q^j.$$

Thus, every PBW monomial is an eigenvector of ad_H , and then ad_H is locally finite.

Lemma 6. *Let $H = pq$. Then,*

$$E_H(-1) = p\mathbb{k}[H] \quad \text{and} \quad E_H(1) = \mathbb{k}[H]q.$$

Proof. Since $[H, p^i q^j] = (j - i)p^i q^j$, the eigenspace associated to -1 is spanned by the monomials $p^{n+1} q^n$, $n \geq 0$. Using $pH = (H + 1)p$, we obtain inductively,

$$p^n q^n = H(H + 1) \cdots (H + n - 1).$$

Therefore, $p^{n+1} q^n \in p\mathbb{k}[H]$. The reverse inclusion follows from $[H, p] = -p$ and $[H, H] = 0$. Thus, $E_H(-1) = p\mathbb{k}[H]$. The second claim is analogous. $\square$

Proposition 7. *Let $H = pq$. Then,*

$$\text{Aut}_{\text{ad}_H}(A_1) = \{\rho_a \mid \rho_a(p) = ap, \rho_a(q) = a^{-1}q, a \in \mathbb{k}^*\}.$$

In particular, $\text{Aut}_{\text{ad}_H}(A_1) \simeq \mathbb{k}^$ and every automorphism in this isotropy group has degree one.*

Proof. Let $\rho \in \text{Aut}_{\text{ad}_H}(A_1)$. Since $\text{ad}_H(\rho(p)) = \rho(\text{ad}_H(p)) = -\rho(p)$, Lemma 6 gives $\rho(p) = pf(H)$, for some $f(H) \in \mathbb{k}[H]$. Similarly, $\rho(q) = g(H)q$, for some $g(H) \in \mathbb{k}[H]$.

On the other hand, Lemma 2 gives $\rho(H) = H + c$, for some $c \in \mathbb{k}$. Let $S(H) = f(H)g(H)$. Using $pS(H) = S(H + 1)p$, we obtain

$$\rho(H) = \rho(p)\rho(q) = pS(H)q = S(H + 1)H.$$

Then, $S(H + 1)H = H + c$. Since the left-hand side belongs to $H\mathbb{k}[H]$, its constant term is zero: $c = 0$. Consequently, $S(H + 1) = 1$. It follows that $f(H)g(H) = 1$. Since the only units of $\mathbb{k}[H]$ are the nonzero scalars, there exists $a \in \mathbb{k}^*$ such that $f(H) = a$ and $g(H) = a^{-1}$. Thus,

$$\rho(p) = ap, \quad \rho(q) = a^{-1}q.$$

Conversely, every automorphism of this form fixes H and, therefore, commutes with ad_H . $\square$

Theorem 8. *Let $0 \neq D \in \text{LFD}(A_1)$. Then, the following conditions are equivalent:*

- (i) *D is locally nilpotent;*
- (ii) *$\text{Aut}_D(A_1)$ contains automorphisms of arbitrarily large degree.*

Proof. Let $D = \text{ad}_w$, with $w \in A_1$. Suppose first that D is locally nilpotent. By Proposition 3, after conjugation by a fixed automorphism of A_1 , we may assume $w = f(p)$, for some $f(p) \in \mathbb{k}[p] \setminus \mathbb{k}$. By Proposition 5, the isotropy contains all automorphisms

$$p \mapsto p, \quad q \mapsto q + g(p),$$

and consequently has unbounded degree.

Conversely, suppose that D is locally finite but not locally nilpotent. By Proposition 3, after conjugation and multiplication by a nonzero scalar, we may assume $D = \text{ad}_H$, where $H = pq$. By Proposition 7,

$$\text{Aut}_D(A_1) = \{p \mapsto ap, q \mapsto a^{-1}q \mid a \in \mathbb{k}^*\},$$

and hence its degree is bounded by one. Since conjugation by a fixed automorphism preserves the distinction between bounded and unbounded degree, the result follows. $\square$

Problem 9. *Let $w \in A_1 \setminus \mathbb{k}$. If $\{\rho \in \text{Aut}(A_1) \mid \rho(w) - w \in \mathbb{k}\}$ has unbounded degree, must ad_w be locally nilpotent? Theorem 8 gives an affirmative answer whenever ad_w is locally finite.*

3 Differential Ore extensions A_h

Throughout this section, we consider $A_h = \mathbb{k}[x][t; h(x)\partial_x]$, with $h \in \mathbb{k}[x] \setminus \mathbb{k}$. Equivalently, A_h is generated by x and t subject to $tx = xt + h(x)$. We write $N = \deg(h) \geq 1$. For $g(x) \in \mathbb{k}[x]$, let $D_{g(x)}$ be the derivation of A_h defined by

$$D_{g(x)}(x) = 0, \quad D_{g(x)}(t) = g(x).$$

We recall the following result:

Proposition 10 ([13, Prop. 2]). *Let $h \in \mathbb{k}[x] \setminus \mathbb{k}$. Then*

$$\text{LND}(A_h) = \{D_{g(x)} \mid g(x) \in \mathbb{k}[x]\}.$$

Moreover, $\text{ML}(A_h) = \mathbb{k}[x]$.

The automorphism group was described by G. Benkart, S. A. Lopes and M. Ondrus [7].

Theorem 11 ([7, Thm. 8.3]). *Let $\deg(h) = N \geq 1$ and*

$$\mathbb{P}_h = \{(a, b) \in \mathbb{k}^* \times \mathbb{k} \mid h(ax + b) = a^N h(x)\}.$$

Then every $\rho \in \text{Aut}(A_h)$ can be written as $\rho = \sigma_{r(x)} \circ \tau_{a,b}$, where $(a, b) \in \mathbb{P}_h$, $r(x) \in \mathbb{k}[x]$,

$$\sigma_{r(x)}(x) = x, \quad \sigma_{r(x)}(t) = t + r(x),$$

and

$$\tau_{a,b}(x) = ax + b, \quad \tau_{a,b}(t) = a^{N-1}t.$$

We use the following normalization obtained in [5].

Proposition 12 ([5, Prop. 13]). *Let $h(x) \in \mathbb{k}[x]$, with $\deg(h) = N \geq 1$. There exists a monic polynomial $h^* \in \mathbb{k}[x]$ such that $A_h \simeq A_{h^*}$, the coefficient of degree $N - 1$ of h^* is zero, and*

$$\text{Aut}(A_{h^*}) = \{\sigma_{r(x)} \circ \tau_a \mid h^*(ax) = a^N h^*(x)\},$$

where $\tau_a = \tau_{a,0}$.

From now on, we replace A_h by this isomorphic normalized and write again h for h^* . Thus, h is monic and its coefficient of degree $N - 1$ is zero. Every automorphism can be written as $\rho = \sigma_{r(x)} \circ \tau_a$, where

$$\rho(x) = ax, \quad \rho(t) = a^{N-1}(t + r(x)),$$

and

$$h(ax) = a^N h(x).$$

Since this normalization is obtained by an affine change of the base variable together with a nonzero scalar rescaling, both local nilpotence and the distinction between bounded and unbounded degree are preserved.

Proposition 13. *If $D \in \text{LND}(A_h)$, then $\text{Aut}_D(A_h)$ contains automorphisms of arbitrarily large degree.*

Proof. By Proposition 10, there exists $g(x) \in \mathbb{k}[x]$ such that $D(x) = 0$ and $D(t) = g(x)$. For every $r(x) \in \mathbb{k}[x]$, consider

$$\sigma_{r(x)}(x) = x, \quad \sigma_{r(x)}(t) = t + r(x).$$

Then, $D(\sigma_{r(x)}(x)) = 0 = \sigma_{r(x)}(D(x))$, and

$$D(\sigma_{r(x)}(t)) = D(t + r(x)) = g(x) = \sigma_{r(x)}(D(t)).$$

Therefore, $\sigma_{r(x)} \in \text{Aut}_D(A_h)$, for every $r(x) \in \mathbb{k}[x]$. $\square$

We recall some well-known elementary relations in the algebra A_h that will be used throughout this section. Proofs can be found, for instance, in [5]. Every element $u \in A_h$ can be written uniquely as $u = \sum_{i=0}^m u_i(x)t^i$, where $u_i(x) \in \mathbb{k}[x]$. If $u_m(x) \neq 0$, we write $\deg_t(u) = m$, and we denote $\deg_t(0) = -\infty$.

Lemma 14. *For every $f(x) \in \mathbb{k}[x]$, we have $tf(x) = f(x)t + f'(x)h(x)$.*

Corollary 15. *Let $g = \sum_{i=0}^n g_i(x)t^i \in A_h$. Then, $tg = gt + h(x)\tilde{g}$, where $\tilde{g} = \sum_{i=0}^n g'_i(x)t^i$ and $\deg_t(\tilde{g}) \leq \deg_t(g)$.*

Lemma 16. *For every $i \geq 1$, there exists $u_i \in A_h$ such that*

$$[t^i, x] = ih(x)t^{i-1} + h(x)u_i$$

and $\deg_t(u_i) < i - 1$.

Lemma 17. *Let $r(x) \in \mathbb{k}[x]$. For every $i \geq 1$, there exists $g_i \in A_h$ such that*

$$(t + r(x))^i = t^i + ir(x)t^{i-1} + g_i$$

and $\deg_t(g_i) \leq i - 2$.

We also use the following elementary description of the centralizer of x in A_h .

Lemma 18. *Let $h(x) \neq 0$. Then, $C_{A_h}(x) = \mathbb{K}[x]$. In particular, $Z(A_h) = \mathbb{K}$.*

For $u = \sum_{i,j} c_{ij} x^i t^j \in A_h$, we consider the total degree $\deg(u) = \max\{i+j \mid c_{ij} \neq 0\}$ with the convention $\deg_t(0) = \deg(0) = -\infty$. For $\rho \in \text{Aut}(A_h)$, we also define $\deg(\rho) = \max\{\deg(\rho(x)), \deg(\rho(t))\}$. If $\rho = \sigma_{r(x)} \circ \tau_a$, then

$$\rho(x) = ax, \quad \rho(t) = a^{N-1}(t + r(x)),$$

and, consequently, $\deg(\rho) = \max\{1, \deg(r(x))\}$. Thus, we immediately obtain the following observation.

Lemma 19. *Let $G \subseteq \text{Aut}(A_h)$. Then, G contains automorphisms of arbitrarily large degree if and only if there is a sequence*

$$\rho_n = \sigma_{r_n(x)} \circ \tau_{a_n} \in G$$

such that $\deg(r_n) \rightarrow \infty$.

The following two results will play a central role in the proof of the main theorem of this section.

Lemma 20. *Let $D \in \text{Der}(A_h)$ and suppose that $\deg_t(D(x)) \geq 1$. Then, there exists $M_D \geq 0$ such that, for every $\rho = \sigma_{r(x)} \circ \tau_a \in \text{Aut}_D(A_h)$, we have $\deg(r(x)) \leq M_D$.*

Proof. Write $D(x) = u = \sum_{i=0}^m u_i(x) t^i$, where $u_m(x) \neq 0$ and $m \geq 1$. Let $\rho = \sigma_{r(x)} \circ \tau_a \in \text{Aut}_D(A_h)$. Since $\rho(x) = ax$, the state $\rho D = D\rho$ gives

$$\rho(D(x)) = D(\rho(x)) = D(ax) = aD(x) = au.$$

On the other hand, since $\rho(t) = a^{N-1}(t + r(x))$, we have

$$\rho(u) = \sum_{i=0}^m u_i(ax) a^{i(N-1)} (t + r(x))^i.$$

By Lemma 17, the coefficient of t^{m-1} in $(t + r(x))^m$ is $mr(x)$. Then, notice that the coefficient of t^{m-1} in $\rho(u)$ is

$$ma^{m(N-1)} u_m(ax) r(x) + a^{(m-1)(N-1)} u_{m-1}(ax).$$

Comparing this with the coefficient of t^{m-1} in au , we obtain

$$ma^{m(N-1)} u_m(ax) r(x) = au_{m-1}(x) - a^{(m-1)(N-1)} u_{m-1}(ax). \quad (*)$$

If $u_{m-1} = 0$, then the right-hand side of $(*)$ is zero. Since $m \neq 0$, $a \neq 0$, and $u_m(ax) \neq 0$, it follows that $r(x) = 0$.

Suppose that $u_{m-1} \neq 0$ and $r(x) \neq 0$. Since $\deg(u_m(ax)) = \deg(u_m)$, the left-hand side of $(*)$ has degree $\deg(u_m) + \deg(r)$. On the other hand, each term on the right-hand side of $(*)$ has degree $\deg(u_{m-1})$, and therefore

$$\deg(au_{m-1}(x) - a^{(m-1)(N-1)} u_{m-1}(ax)) \leq \deg(u_{m-1}).$$

Consequently, $\deg(r) \leq \deg(u_{m-1}) - \deg(u_m)$. Therefore, the degree of $r(x)$ is bounded independently of ρ . In fact, we can choose

$$M_D = \max\{0, \deg(u_{m-1}) - \deg(u_m)\}.$$

□

Lemma 21. *Let $D \in \text{Der}(A_h)$ and suppose that $D(x) = p(x) \in \mathbb{k}[x] \setminus \{0\}$. Then, there exists $M_D \geq 0$ such that, for every $\rho = \sigma_{r(x)} \circ \tau_a \in \text{Aut}_D(A_h)$, we have $\deg(r(x)) \leq M_D$.*

Proof. Write $D(t) = v \in A_h$. Applying D to the relation $[t, x] = h(x)$ gives $[D(t), x] + [t, D(x)] = D(h(x))$. Since $D(x) = p(x) \in \mathbb{k}[x]$, we have also $D(h(x)) = h'(x)p(x)$. Moreover, by Lemma 14, $[t, p(x)] = h(x)p'(x)$. Therefore,

$$[v, x] = h'(x)p(x) - h(x)p'(x) \in \mathbb{k}[x]. \quad (1)$$

We claim that v has t -degree at most one. Suppose that $v = \sum_{i=0}^s v_i(x)t^i$, with $v_s(x) \neq 0$ and $s \geq 2$. Since

$$[v, x] = \sum_{i=0}^s v_i(x)[t^i, x],$$

Lemma 16 shows that the coefficient of t^{s-1} coming from the term $v_s(x)t^s$ is $sv_s(x)h(x)$. For $i \leq s-1$, the commutator $[t^i, x]$ has t -degree at most $i-1 \leq s-2$, so no other term contributes to the coefficient of t^{s-1} . Thus, this coefficient in $[v, x]$ is $sv_s(x)h(x)$. Since $\text{char}(\mathbb{k}) = 0$, $v_s(x) \neq 0$ and $h(x) \neq 0$, this coefficient is nonzero, contradicting (1). Then, $D(t) = b(x)t + c(x)$, for some $b(x), c(x) \in \mathbb{k}[x]$. Substituting into (1), we obtain

$$b(x)h(x) = h'(x)p(x) - h(x)p'(x). \quad (2)$$

Let $\rho = \sigma_{r(x)} \circ \tau_a \in \text{Aut}_D(A_h)$. Since $\rho(x) = ax$ and $\rho(t) = a^{N-1}(t + r(x))$, the condition $\rho D = D\rho$, applied to t , gives

$$a^{N-1}b(ax)(t + r(x)) + c(ax) = a^{N-1}(b(x)t + c(x) + p(x)r'(x)),$$

Comparing the coefficients of t , we obtain $b(ax) = b(x)$. Also, comparing the terms of t -degree zero, we obtain

$$p(x)r'(x) - b(x)r(x) = a^{1-N}c(ax) - c(x). \quad (3)$$

Since $p(x)$, $b(x)$, and $c(x)$ depend only on D , they are fixed. Moreover, since $a \in \mathbb{k}^*$, $\deg(c(ax)) = \deg(c)$, whenever $c \neq 0$. Then, we have that $\deg(a^{1-N}c(ax) - c(x)) \leq \deg(c)$, so the degree of the right-hand side of (3) is bounded independently of ρ . We show that there exists a bound for $\deg(r(x))$ depending only on D .

Suppose that $b(x) = 0$. Constant polynomials $r(x)$ already have bounded degree, so assume that $r(x)$ is nonconstant. Denote $d = \deg(r)$ and $n = \deg(p)$.

Thus, $\deg(p(x)r'(x)) = n + d - 1$. If $c \neq 0$, the right-hand side of (3) has degree at most $\deg(c)$. Then, $n + d - 1 \leq \deg(c)$, and so $d \leq \deg(c) - n + 1$. If $c = 0$, then (3) gives $p(x)r'(x) = 0$, and then $r'(x) = 0$, contradicting the assumption that $r(x)$ is nonconstant. Thus, in either case, the degree of $r(x)$ is bounded independently of ρ .

Suppose that $b(x) \neq 0$, as in the previous case, $n = \deg(p)$, $s = \deg(b)$, $d = \deg(r)$, with $r(x)$ nonconstant. Then, $\deg(p(x)r'(x)) = n + d - 1$ and $\deg(b(x)r(x)) = s + d$. If $s \neq n - 1$, these two degrees are different. Then, no cancellation of their leading terms is possible, and $\deg(p(x)r'(x) - b(x)r(x)) = d + \max\{n - 1, s\}$. Since the right-hand side of (3) has bounded degree, d is bounded. It remains to consider the case $s = n - 1$. Let p_n , b_s , and r_d denote the leading coefficients of $p(x)$, $b(x)$, and $r(x)$, respectively. The coefficient of x^{n+d-1} in $p(x)r'(x) - b(x)r(x)$ is $r_d(dp_n - b_s)$. Thus, the leading term can cancel only if $d = \frac{b_s}{p_n}$. If this does not hold, then

$$\deg(p(x)r'(x) - b(x)r(x)) = n + d - 1,$$

and again the bound on the degree of the right-hand side of (3) gives a bound on d . If $d = \frac{b_s}{p_n}$, then d is fixed, since p_n and b_s depend only on D .

Therefore, in every case, the degree of $r(x)$ is bounded by a constant depending only on D . Therefore, there exists $M_D \geq 0$ such that

$$\deg(r(x)) \leq M_D,$$

for every $\rho \in \text{Aut}_D(A_h)$. □

Theorem 22. *Let $h \in \mathbb{k}[x] \setminus \mathbb{k}$. For every $D \in \text{Der}(A_h)$, the following conditions are equivalent:*

- (i) *D is locally nilpotent;*
- (ii) *$\text{Aut}_D(A_h)$ contains automorphisms of arbitrarily large degree.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 13.

Conversely, suppose that $\text{Aut}_D(A_h)$ contains automorphisms of arbitrarily large degree. By Lemma 19, there exists a sequence $\rho_n = \sigma_{r_n(x)} \circ \tau_{a_n} \in \text{Aut}_D(A_h)$ such that $\deg(r_n) \rightarrow \infty$.

We claim that $D(x) = 0$. Indeed, if $D(x)$ contains a positive power of t , then $\deg_t(D(x)) \geq 1$, and Lemma 20 implies that the degrees of the polynomials $r_n(x)$ are bounded, a contradiction. Then, $D(x) \in \mathbb{k}[x]$. If $D(x) \neq 0$, then Lemma 21 again implies that the degrees of the polynomials $r_n(x)$ are bounded, which is impossible. Therefore, $D(x) = 0$.

Applying D to the defining relation $[t, x] = h(x)$, we obtain

$$[D(t), x] = D(h(x)) = h'(x)D(x) = 0.$$

By Lemma 18, $C_{A_h}(x) = \mathbb{k}[x]$, and therefore $D(t) \in \mathbb{k}[x]$. Thus,

$$D(x) = 0, \quad D(t) = g(x),$$

for some $g(x) \in \mathbb{k}[x]$. Then, $D = D_{g(x)}$, and Proposition 10 gives $D \in \text{LND}(A_h)$. $\square$

Remark 23 (Comparison with Pan's argument). *Pan [15] uses in the proof of Theorem 1 the invariant $e(D)$, already considered in [6], which counts the D -stable reduced principal ideals of height one in $\mathbb{k}[x, y]$. This invariant organizes the proof according to the invariant curves of D . The case $0 < e(D) < \infty$ in Pan's proof is particularly similar to our argument: after suitable reductions, unbounded isotropy is encoded by a polynomial parameter $P(x)$ of arbitrarily large degree, and the commutation equations rule out a nonzero ∂_x -component of D . For A_h , the explicit description of the whole automorphism group makes such a case distinction unnecessary. After normalization,*

$$\text{Aut}(A_h) = \{\sigma_{r(x)} \circ \tau_a \mid h(ax) = a^N h(x)\},$$

and unbounded degree is exactly reflected in the polynomial parameter $r(x)$. The two boundedness lemmas above show that polynomials $r(x)$ of arbitrarily large degree cannot occur when $D(x) \neq 0$. Then, unbounded isotropy forces $D(x) = 0$, and the defining relation then gives $D(t) \in \mathbb{k}[x]$.

Remark 24 (The degree filtration). *The description of $\text{Aut}(A_h)$ provides a natural analogue of the degree filtration considered in the polynomial case. For $d \geq 0$, denote*

$$\mathcal{A}_d(A_h) = \{\sigma_{r(x)} \circ \tau_a \in \text{Aut}(A_h) \mid \deg(r(x)) \leq d\}.$$

Then,

$$\text{Aut}(A_h) = \bigcup_{d \geq 0} \mathcal{A}_d(A_h).$$

Moreover, since $\deg(\sigma_{r(x)} \circ \tau_a) = \max\{1, \deg(r(x))\}$, for every $d \geq 1$ we have

$$\mathcal{A}_d(A_h) = \{\rho \in \text{Aut}(A_h) \mid \deg(\rho) \leq d\}.$$

Notice that $\mathcal{A}_0(A_h)$ consists of the automorphisms for which $r(x)$ is constant, and these automorphisms have degree one.

Let $G_h = \{a \in \mathbb{k}^ \mid h(ax) = a^N h(x)\}$. Writing $r(x) = r_0 + r_1x + \dots + r_dx^d$, the set $\mathcal{A}_d(A_h)$ is naturally parametrized by $\mathbb{k}^{d+1} \times G_h$. For a fixed $D \in \text{Der}(A_h)$, the conditions $\rho D(x) = D\rho(x)$ and $\rho D(t) = D\rho(t)$, after writing these identities in PBW form, their coefficients give polynomial equations in $r_0, \dots, r_d$ and a . Then, $\text{Aut}_D(A_h) \cap \mathcal{A}_d(A_h)$ is Zariski closed in this finite-dimensional parameter space. Furthermore, Theorem 22 is equivalent to saying that D is locally nilpotent if and only if $\text{Aut}_D(A_h)$ is not contained in any $\mathcal{A}_d(A_h)$.*

4 The free associative algebra

Let $F = \mathbb{k}\langle x, y \rangle$ be the free associative algebra on two generators. The behavior of locally nilpotent derivations of F is closely related to the corresponding theory for $\mathbb{k}[x, y]$. In fact, their triangulability can be obtained from

Rentschler's theorem together with the classical results of Czerniakiewicz and Makar-Limanov on automorphisms of the free associative algebra of rank two.

We recall the noncommutative analogue of Rentschler's theorem. The following result is due to Drensky and Limanov [11]; see also [3, 8].

Proposition 25 ([11, Proposition 3.2]). *Let $D \in \text{LND}(F)$. Then, there exists a free generating system x', y' of F such that*

$$D(y') = 0, \quad D(x') = f(y')$$

for some $f(y') \in \mathbb{k}[y']$.

Proposition 26. *If $0 \neq D \in \text{LND}(F)$, then $\text{Aut}_D(F)$ contains automorphisms of arbitrarily large degree.*

Proof. By Proposition 25, choose free generators x', y' such that

$$D(y') = 0, \quad D(x') = f(y').$$

For every $g(y') \in \mathbb{k}[y']$, define $\sigma_g(x') = x' + g(y')$ and $\sigma_g(y') = y'$. Since x', y' freely generate F , the map σ_g is an automorphism, with inverse $x' \mapsto x' - g(y')$ and $y' \mapsto y'$. Moreover,

$$D(\sigma_g(x')) = D(x' + g(y')) = f(y') = \sigma_g(D(x')),$$

and the condition on y' is immediate. Then, $\sigma_g \in \text{Aut}_D(F)$. Taking g of arbitrarily large degree gives automorphisms of arbitrarily large degree in the coordinates x', y' . Since the change of free generators is fixed, boundedness or unboundedness of degree is preserved under conjugation. Therefore, $\text{Aut}_D(F)$ has unbounded degree. $\square$

In contrast with Theorem 1, the converse implication fails in the free associative algebra.

Proposition 27. *Let $F = \mathbb{k}\langle x, y \rangle$, and define $D \in \text{Der}(F)$ by*

$$D(y) = 0, \quad D(x) = [x, y].$$

Then, D is not locally nilpotent, while $\text{Aut}_D(F)$ contains automorphisms of arbitrarily large degree.

Proof. We have $D = -\text{ad}_y$. We claim that, for every $n \geq 1$,

$$D^n(x) = \sum_{i=0}^n (-1)^i \binom{n}{i} y^i x y^{n-i}.$$

The formula is clear for $n = 1$. If it holds for some n , then, since $D(y) = 0$,

$$D(y^i x y^{n-i}) = y^i x y^{n-i+1} - y^{i+1} x y^{n-i},$$

and the formula for $n+1$ follows from Pascal's identity. The monomials $y^i x y^{n-i}$, with $0 \leq i \leq n$, are pairwise distinct. Then, $D^n(x) \neq 0$, for every $n \geq 1$, and therefore D is not locally nilpotent.

For every $g(y) \in \mathbb{k}[y]$, consider the automorphism

$$\sigma_g(x) = x + g(y), \quad \sigma_g(y) = y.$$

Since $\sigma_g(y) = y$, $\sigma_g \operatorname{ad}_y \sigma_g^{-1} = \operatorname{ad}_y$, and therefore $\sigma_g D = D \sigma_g$. Thus, $\sigma_g \in \operatorname{Aut}_D(F)$. Finally, $\deg(\sigma_g) = \max\{1, \deg(g)\}$, and then $\operatorname{Aut}_D(F)$ contains automorphisms of arbitrarily large degree. $\square$

4.1 Abelianization: automorphisms and derivations

Let $\pi : F = \mathbb{k}\langle x, y \rangle \longrightarrow \mathbb{k}[x, y]$ be the natural abelianization map, and denote $I = \ker(\pi)$. Thus I is the commutator ideal of F , which in rank two is the two-sided ideal generated by $[x, y]$.

The behavior of automorphisms and derivations under abelianization is rather different. Every automorphism of F preserves I and therefore induces an automorphism of $\mathbb{k}[x, y]$. This gives a homomorphism

$$\gamma : \operatorname{Aut}(F) \longrightarrow \operatorname{Aut}(\mathbb{k}[x, y]).$$

By [9, Theorem 1], the homomorphism γ is injective. On the other hand, the Jung-van der Kulk theorem [12, 19] implies that every automorphism of $\mathbb{k}[x, y]$ is tame. Since affine and triangular automorphisms lift naturally to F , the map γ is also surjective. Consequently,

$$\operatorname{Aut}(\mathbb{k}\langle x, y \rangle) \simeq \operatorname{Aut}(\mathbb{k}[x, y]).$$

This is the classical result of Czerniakiewicz and Makar-Limanov; see also [14]. Equivalently, every automorphism of $\mathbb{k}[x, y]$ has a unique lift to F . In particular, $\gamma(\rho) = \operatorname{id}$, then $\rho = \operatorname{id}$.

The situation for derivations is different. For every $D \in \operatorname{Der}(F)$,

$$D([a, b]) = [D(a), b] + [a, D(b)].$$

Since I is generated by commutators, it follows that $D(I) \subseteq I$. Then, D induces a derivation $\pi_*(D) \in \operatorname{Der}(\mathbb{k}[x, y])$. In this way, abelianization defines a linear map

$$\pi_* : \operatorname{Der}(F) \longrightarrow \operatorname{Der}(\mathbb{k}[x, y]).$$

The map π_* is not injective. Indeed, for every $w \in F$, $\pi_*(\operatorname{ad}_w) = 0$. Thus, a nonzero derivation of F may become trivial in the commutative quotient.

Corollary 28. *If $D \in \operatorname{LND}(F)$ and $\pi_*(D) = 0$, then $D = 0$. Equivalently,*

$$\ker(\pi_*) \cap \operatorname{LND}(F) = \{0\}.$$

Proof. By Proposition 25, there exist free generators x', y' of F such that

$$D(y') = 0, \quad D(x') = f(y'),$$

for some $f(y') \in \mathbb{k}[y']$.

Let $\overline{x'} = \pi(x')$ and $\overline{y'} = \pi(y')$. Since x', y' form a free generating system of F , there exists $\alpha \in \text{Aut}(F)$ such that $\alpha(x) = x'$ and $\alpha(y) = y'$. Passing to the abelianization, α induces an automorphism of $\mathbb{k}[x, y]$ sending

$$x \mapsto \overline{x'}, \quad y \mapsto \overline{y'}.$$

Then, $\overline{x'}$ and $\overline{y'}$ form a system of polynomial coordinates of $\mathbb{k}[x, y]$. In particular, $\overline{y'}$ is transcendental over $\mathbb{k}$. Since $\pi_*(D) = 0$, we have

$$0 = \pi_*(D)(\overline{x'}) = \pi(D(x')) = \pi(f(y')) = f(\overline{y'}).$$

As $\overline{y'}$ is transcendental over $\mathbb{k}$, it follows that $f = 0$. Therefore, $D(x') = D(y') = 0$. Since x', y' generate F , we conclude that $D = 0$. $\square$

The preceding corollary shows that a nonzero locally nilpotent derivation of F cannot vanish under abelianization. However, local nilpotence of the induced derivation $\pi_*(D)$ does not imply local nilpotence of D . The following example shows that the converse to Pan's criterion can fail even when $\pi_*(D)$ is nonzero and locally nilpotent.

Proposition 29. *Let $F = \mathbb{k}\langle x, y \rangle$, and let $D \in \text{Der}(F)$ be defined by*

$$D(y) = 0, \quad D(x) = 1 + [x, y].$$

Then D is not locally nilpotent, whereas $\pi_(D) = \partial_x$ is nonzero and locally nilpotent. Moreover, $\text{Aut}_D(F)$ contains automorphisms of arbitrarily large degree.*

Proof. Notice that

$$\pi_*(D)(x) = 1, \quad \pi_*(D)(y) = 0.$$

Thus, $\pi_*(D) = \partial_x$, which is nonzero and locally nilpotent. For every $g(y) \in \mathbb{k}[y]$, consider the automorphism

$$\sigma_g(x) = x + g(y), \quad \sigma_g(y) = y.$$

Since $D(y) = 0$, we have $D(g(y)) = 0$, and then $D(\sigma_g(x)) = 1 + [x, y]$. On the other hand,

$$\sigma_g(D(x)) = 1 + [x + g(y), y] = 1 + [x, y].$$

The condition $D(\sigma_g(y)) = \sigma_g(D(y))$ is immediate. Therefore, $\sigma_g \in \text{Aut}_D(F)$. Taking $g(y) = y^m$, for $m \geq 1$, gives $\deg(\sigma_g) = m$, so the degrees are unbounded.

It remains to show that D is not locally nilpotent. For $u \in F$, the identity $D(y) = 0$ gives $D([u, y]) = [D(u), y]$. In particular, $D^2(x) = [[x, y], y]$. It follows inductively that, for every $n \geq 2$,

$$D^n(x) = (-\text{ad}_y)^n(x) = \sum_{i=0}^n (-1)^i \binom{n}{i} y^i x y^{n-i}.$$

The elements $y^i xy^{n-i}$, with $0 \leq i \leq n$, are pairwise distinct, and the coefficient of xy^n is 1. Consequently, $D^n(x) \neq 0$ for every $n \geq 2$. Therefore, D is not locally nilpotent. $\square$

In this example, abelianization removes the commutator term $[x, y]$ and retains the nonzero locally nilpotent derivation ∂_x . Nevertheless, the original derivation is not locally nilpotent and its isotropy group has unbounded degree. Thus, Pan's criterion can fail for derivations of F with nonzero abelianization.

5 The quantum cases

M. Awami, M. Van den Bergh and F. Van Oystaeyen [4, Section 2.1], and J. Alev and F. Dumas [2, Proposition 3.2], proved that, given any $\mathbb{k}$ -linear automorphism σ of $\mathbb{k}[x]$ and δ a σ -derivation, the Ore extension $S = \mathbb{k}[x][y; \sigma, \delta]$ is isomorphic to one of the following algebras: $\mathbb{k}[x, y]$, a quantum plane, a quantum Weyl algebra, or a differential Ore extension. Since any automorphism of $\mathbb{k}[x]$ is such that $\sigma(x) = qx + b$, for some $q \in \mathbb{k}^*$ and $b \in \mathbb{k}$, following the proofs of [4, 2], we have:

- If $q \neq 1$, then $S \simeq \mathbb{k}[x'][y; \sigma', \delta]$, where $x' = x + b(q-1)^{-1}$ and $\sigma'(x') = qx'$. Now, if $p(x) \in \mathbb{k}[x]$ and $r \in \mathbb{k}$ are such that $\delta(x') = p(x')(1-q)x' + r$, then $(y - p(x'))x' = qx'(y - p(x')) + r$. If $r = 0$, it is easy to see that $S \simeq \mathbb{k}_q[x', y'] = \mathbb{k}\langle x', y' \mid y'x' = qx'y' \rangle$ for a suitable change of variables. If $r \neq 0$, with $y'' = r^{-1}(y - p(x'))$, we obtain $S \simeq A_1^q(\mathbb{k}) := \mathbb{k}\langle x', y'' \mid y''x' = qx'y'' + 1 \rangle$.
- If $q = 1$ and $b = 0$, then S is either $\mathbb{k}[x, y]$ or a differential Ore extension, $S = \mathbb{k}[x][y; \delta]$.
- If $q = 1$ and $b \neq 0$, then $S \simeq \mathbb{k}[x'][y; \sigma', \delta']$ where $x' = b^{-1}x$, $\sigma'(x') = x' + 1$, and $\delta'(x') = b^{-1}\delta(x)$. Since $(y + b^{-1}\delta(x))x' = (x' + 1)(y + b^{-1}\delta(x))$, it follows that $S \simeq \mathbb{k}[y'][x'; -y'\partial_{y'}]$.

Let $A_q = \mathbb{k}\langle x, y \mid xy = qyx \rangle$, where $q \in \mathbb{k}^*$ and $q \neq 1$. The automorphism computation of Alev and Chamarie [1, Proposition 1.4.4] gives, for $q \neq \pm 1$,

$$\text{Aut}(A_q) = \{x \mapsto \alpha x, y \mapsto \beta y\} \simeq (\mathbb{k}^*)^2.$$

For $q = -1$ one has instead $\text{Aut}(A_{-1}) \simeq (\mathbb{k}^*)^2 \rtimes \mathbb{Z}_2$, where the nontrivial element interchanges x and y . Thus, for every $q \neq 1$, all automorphisms of A_q have degree one. The same description of the automorphism group also shows that A_q admits no nonzero locally nilpotent derivations. Indeed, suppose that $D \in \text{LND}(A_q)$. Since D is locally nilpotent, the exponential

$$\exp(sD) = \sum_{n \geq 0} \frac{s^n D^n}{n!}$$

defines an automorphism of A_q for every $s \in \mathbb{k}$, and the map

$$\mathbb{G}_a \longrightarrow \text{Aut}(A_q), \quad s \longmapsto \exp(sD),$$

is a homomorphism of algebraic groups. Since $\mathbb{G}_a$ is connected and $\exp(0D) = \text{id}$, its image is contained in the identity component of $\text{Aut}(A_q)$, which is a torus. A torus admits no nontrivial algebraic group homomorphism from $\mathbb{G}_a$. Consequently, $\exp(sD) = \text{id}$, for every $s \in \mathbb{k}$. Comparing the coefficient of s in $\exp(sD)(a) = a$, for $a \in A_q$, gives $D(a) = 0$. Therefore, $D = 0$.

Let $A_q^1(\mathbb{k}) = \mathbb{k}\langle X, Y \mid YX - qXY = 1 \rangle$, where $q \in \mathbb{k}^* \setminus \{1\}$. This algebra can be realized as the quantum generalized Weyl algebra $A(\mathbb{k}[h], q, h-1)$. Since the defining polynomial $h-1$ is not a monomial, [17, Proposition 2.3] gives

$$\text{LFD}(A_q^1(\mathbb{k})) = \mathbb{k}H,$$

where $H(X) = X$ and $H(Y) = -Y$. Consequently, $\text{LND}(A_q^1(\mathbb{k})) = \{0\}$, because, for $c \neq 0$, $(cH)^n(X) = c^n X \neq 0$, for every $n \geq 1$. By [17, Theorem B], if $q \neq -1$, every automorphism of $A_q^1(\mathbb{k})$ has the form

$$X \longmapsto aX, \quad Y \longmapsto a^{-1}Y, \quad a \in \mathbb{k}^*.$$

If $q = -1$, every automorphism is either of this form or of the form

$$X \longmapsto aY, \quad Y \longmapsto a^{-1}X, \quad a \in \mathbb{k}^*.$$

These descriptions also apply when q is a root of unity. Thus every automorphism of $A_q^1(\mathbb{k})$ has degree one. In particular, for every derivation D of $A_q^1(\mathbb{k})$, the degrees of the automorphisms in $\text{Aut}_D(A_q^1(\mathbb{k}))$ are bounded.

Therefore, for either the quantum plane $A = A_q$ or the first quantum Weyl algebra $A = A_q^1(\mathbb{k})$, with $q \in \mathbb{k}^* \setminus \{1\}$, we have

$$\text{LND}(A) = \{0\}, \quad \deg(\rho) = 1 \quad \text{for every } \rho \in \text{Aut}(A).$$

Consequently, for every nonzero derivation D of A , neither local nilpotence nor the existence of automorphisms of arbitrarily large degree in $\text{Aut}_D(A)$ occurs. Hence the equivalence in Pan's criterion holds vacuously for nonzero derivations of these algebras. The restriction $D \neq 0$ is essential: the zero derivation is locally nilpotent, whereas $\text{Aut}_0(A) = \text{Aut}(A)$ has bounded degree.

Finally, together with Pan's theorem 1 and Theorem 22, these observations establish the equivalence for all nonzero derivations of the Ore extensions of $\mathbb{k}[x]$, except for the classical first Weyl algebra A_1 . For A_1 , recall that Theorem 8 establishes the equivalence under the additional assumption that the derivation is locally finite.

Acknowledgements

S. Lopes was partially supported by CMUP – Centro de Matemática da Universidade do Porto, member of LASI, which is financed by national funds

through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UID/00144/2025, doi: <https://doi.org/10.54499/UID/00144/2025>. R. Baltazar was partially supported by Rio Grande do Sul Research Foundation – FAPERGS, project n. 24/2551-0001560-4.

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