ON THE COMPLETE MONOTONICITY OF THE SOLUTIONS TO SOME VOLTERRA DIFFERENCE EQUATIONS

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Abstract. We study the complete monotonicity of solutions to a class of linear Volterra difference equations with completely monotonic kernels. Using Hausdorff's moment characterization, we provide direct proofs of complete monotonicity for two relevant kernels: the fractional kernel associated with the Caputo nabla fractional difference equation, and the logarithmic-type kernel given by the reciprocal sequence. In both cases, we derive explicit Hausdorff moment representations for the corresponding solutions. For the fractional kernel, the representation is obtained from an integral formula for the Mittag-Leffler function, while for the reciprocal kernel it is derived through a complex-analytic argument based on the generating function and a keyhole contour around its branch cut. These representations yield alternative proofs of known complete monotonicity results and lead to new integral identities arising from the first Hausdorff moments of the solutions.

1. Preamble

The theory of fractional difference equations has boosted in the XXI century. Several research directions emerged and may be consulted in the books [4, 8] as well as in the references [2, 3, 7, 9, 12], to name a few.. More recently, a general theory of Volterra difference equations, specially with Sonine kernels[1], has gained interest. In particular, the linear problem has been intensively studied [5, 6]. Concretely, let $\kappa : \mathbb{N}_0 \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ be such that $1 + \lambda\kappa(0) \neq 0$. Consider the Volterra difference equation

$$u(n) = 1 - \lambda \sum_{\tau=1}^{n} \kappa(n-\tau)u(\tau), \quad n \in \mathbb{N}_0. \tag{1}$$

Put $q_a^{(1)}(n) = q(n)$ and $q_a^{(m)}(n) = (q * q_a^{(m-1)})_a(n)$, $m \in \mathbb{N}_2$, where $*$ denotes the discrete convolution, i.e.,

$$(f * g)_a(n) = \sum_{\tau=a}^{n-1} f(n-\tau-1+a)g(\tau), \quad n \in \mathbb{N}_a.$$

An explicit representation for the solution of (1) was deduce in [6]. It reads:

Theorem 1. [6, Theorem 5] *Let $\lambda \in \mathbb{R}$ be such that $1 + \lambda\kappa(0) \neq 0$. Then the solution of (1) has the following representation,*

$$u_{\kappa,\lambda}(n) = \frac{1}{1+\lambda\kappa(0)} \left(1 + \sum_{k=1}^{n-1} \left(\frac{-\lambda}{1+\lambda\kappa(0)} \right)^k \sum_{s=k}^{n-1} \kappa_1^{(k)}(s) \right), \quad n \in \mathbb{N}. \tag{2}$$

Another important result was obtained in [5, 10], namely, if $\lambda > 0$ and the kernel κ is completely monotonic (cf. Definition 3 below), then the solution of (1) is completely monotonic. The proof is based on a characterization (based on Pick functions) of complete monotonicity obtained by Liu

2010 *Mathematics Subject Classification.* 33E12, 39A06, 26A33.

Key words and phrases. Discrete Mittag-Leffler function, completely monotonic, Fractional differences.

[1]A pair of sequences are Sonine kernels if their discrete convolution equals 1.

and pego in [11]. We highlight that the result was used in particular to define a general class of probability distributions (see [5] for the details).

Let us remark that the complete monotonicity of a sequence is related to the *Hausdorff moment problem*. The following characterization of a completely monotonic sequence is due to Hausdorff.

Theorem 2. [14, Theorem 4a] *A sequence $u : \mathbb{N}_0 \to \mathbb{R}$ is completely monotonic if and only if*

$$u(n) = \int_0^1 t^n d\alpha(t),$$

where $\alpha(t)$ is bounded and nondecreasing for $t \in [0, 1]$.

In this work we use Theorem 2 to show complete monotonicity of the solution of (1) for two particular kernels (cf. Section 2), that are completely monotonic:

$$\kappa_\alpha(n) = \frac{(n+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}, \quad 0 < \alpha \leq 1 \quad \text{and } \kappa'(n) = \frac{1}{n+1}, \ n \in \mathbb{N}_0$$

Our proof relies on determining explicitly the measure α in Theorem 2, thus being of different nature than proofs known so far in the literature. Since for the first two Hausdorff moments we have

$$u_{\kappa,\lambda}(0) = 1 = u_{\kappa',\lambda}(0) \quad \text{and} u_{\kappa,\lambda}(1) = \frac{1}{1+\lambda} = u_{\kappa',\lambda}(1),$$

we deduce some interesting novel identities, at least to the best of our knowledge (cf. Remark 1).

We organize the rest of this paper as follows: Section 2 contains the definitions needed in the sequel. In Section 2.1 we deal with the "fractional" kernel κ_α, while in Section 2.2 we conclude the manuscript by considering the kernel κ'.

2. Definitions, notations and preliminary results

We start with some concepts needed in the sequel.

For $a \in \mathbb{R}$ we define

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots\}, \ \mathbb{N}_1 = \mathbb{N}.$$

The *rising function* is defined by

$$x^{\overline{y}} = \frac{\Gamma(x+y)}{\Gamma(x)}. \tag{3}$$

We now present the essential concepts regarding the discrete Δ and the discrete ∇ operators. We refer the reader to the monographs [4] and [8] for more on the subject.

Definition 1. *For $a \in \mathbb{R}$ consider a function $f : \mathbb{N}_a \to \mathbb{R}$.*

The forward difference operator is defined by $\Delta[f](t) = f(t+1) - f(t)$, for $t \in \mathbb{N}_a$. Also, we define higher order differences recursively as $\Delta^n[f](t) = \Delta[\Delta^{n-1} f](t)$, $n \in \mathbb{N}_1$, where Δ^0 is the identity operator, i.e. $\Delta^0[f](t) = f(t)$.

The fractional ∇ operators are defined as follows.

Definition 2. *Consider $a \in \mathbb{R}$ and let $f : \mathbb{N}_{a+1} \to \mathbb{R}$. Then we define the* ∇-fractional sum of f *of order $\nu > 0$ by*

$$\nabla_a^{-\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-(s-1))^{\overline{\nu-1}} f(s), & \text{if } t \in \mathbb{N}_{a+\nu}, \\ 0 & \text{if } t = a. \end{cases}$$

We also put $\nabla_a^{-0} f(t) = f(t)$.

The Caputo ∇-fractional difference of f *of order $0 < \alpha \le 1$ is defined by*

$$^C\nabla_a^{\alpha} f(t) = \nabla_a^{-(1-\alpha)}[\nabla f](t), \quad t \in \mathbb{N}_{a+1},$$

where $\nabla f(t) = f(t) - f(t-1)$.

The Mittag-Leffler function id defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0, \; z \in \mathbb{C}.$$

As it was mentioned before, we are mainly concerned with the completely monotonicity of sequences. The concept follows.

Definition 3. [14, pag. 108] *We say that a sequence $u : \mathbb{N}_0 \to \mathbb{R}$ is completely monotonic if*

$$(-1)^m \Delta^m u(n) \ge 0, \quad m, n \in \mathbb{N}_0.$$

Finally, given a sequence $u : \mathbb{N}_0 \to \mathbb{R}$, its generating function is defined by $F_u(z) = \sum_{k=0}^{\infty} u(k) z^k$.

2.1. **Fractional difference equation.** Let $0 < \alpha \le 1$ and $\lambda > 0$. Consider the solution $u_{\alpha,\lambda}$ of the following discrete fractional initial value problem:

$$^C\nabla_0^{\alpha} u(n) = -\lambda u(n), \quad n \in \mathbb{N}, \tag{4}$$

$$u(0) = 1, \tag{5}$$

or, equivalently, of the Volterra difference equation

$$u(n) = 1 - \lambda \sum_{k=1}^{n} \frac{(n-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(k), \quad n \in \mathbb{N}_0. \tag{6}$$

In [5, Section 3.3], the author proved that $u_{\alpha,\lambda}$ is completely monotonic (cf. Definition 3). In this section we revisit that result and provide a novel proof for it.

It is known (cf. [5]) that the function

$$u_{\alpha,\lambda}(n) = \int_0^\infty e^{-t} \frac{t^{n-1}}{\Gamma(n)} E_\alpha(-\lambda t^\alpha) dt, \quad n \in \mathbb{N}_1, \quad u_{\alpha,\lambda}(0) = 1, \tag{7}$$

solves (4)–(5).

Theorem 3. *For $0 < \alpha < 1$ and $\lambda > 0$, the solution of* (4)–(5) *is completely monotonic. Moreover, the solution has the following representation:*

$$u_{\alpha,\lambda}(n) = \lambda \frac{\sin(\pi\alpha)}{\pi} \int_0^1 \frac{r^{\alpha-1}(1-r)^{\alpha-1}}{(1-r)^{2\alpha} + 2\lambda(1-r)^\alpha r^\alpha \cos(\pi\alpha) + \lambda^2 r^{2\alpha}} r^n dr, \quad n \in \mathbb{N}_0. \tag{8}$$

Proof. By [13, equality (4.52)], we have

$$E_\alpha(-\lambda t^\alpha) = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{e^{-s\lambda^{\frac{1}{\alpha}}t}s^{\alpha-1}}{s^{2\alpha}+2s^\alpha\cos(\pi\alpha)+1}ds.$$

Inserting this integral in (7), one gets for $n \in \mathbb{N}_1$,

$$\begin{aligned}
u_{\alpha,\lambda}(n) &= \int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{e^{-s\lambda^{\frac{1}{\alpha}}t}s^{\alpha-1}}{s^{2\alpha}+2s^\alpha\cos(\pi\alpha)+1}dsdt\\
&= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty e^{-(1+s\lambda^{\frac{1}{\alpha}})t}\frac{t^{n-1}}{\Gamma(n)}dt\int_0^\infty \frac{s^{\alpha-1}}{s^{2\alpha}+2s^\alpha\cos(\pi\alpha)+1}ds\\
&= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{s^{\alpha-1}}{s^{2\alpha}+2s^\alpha\cos(\pi\alpha)+1}(1+s\lambda^{\frac{1}{\alpha}})^{-n}ds.
\end{aligned}$$

Making the change of variables $r = 1/(1+s\lambda^{\frac{1}{\alpha}})$, then, after some simplifications, we get

$$u_{\alpha,\lambda}(n) = \lambda\frac{\sin(\pi\alpha)}{\pi}\int_0^1 \frac{r^{\alpha-1}(1-r)^{\alpha-1}}{(1-r)^{2\alpha}+2\lambda(1-r)^\alpha r^\alpha\cos(\pi\alpha)+\lambda^2r^{2\alpha}}r^n dr.$$

Observe that $(1-r)^{2\alpha}+2\lambda(1-r)^\alpha r^\alpha\cos(\pi\alpha)+\lambda^2r^{2\alpha} = ((1-r)^\alpha+\lambda\cos(\pi\alpha)r^\alpha)^2+\lambda^2\sin^2(\pi\alpha)r^{2\alpha} > 0$, for $r > 0$. Moreover, we can verify that the right hand side of (8) is equal to 1 at $n = 0$. Therefore, we conclude that

$$u_{\alpha,\lambda}(n) = \int_0^1 \lambda\frac{\sin(\pi\alpha)}{\pi}\frac{r^{\alpha-1}(1-r)^{\alpha-1}}{(1-r)^{2\alpha}+2\lambda(1-r)^\alpha r^\alpha\cos(\pi\alpha)+\lambda^2r^{2\alpha}}r^n dr, \quad n \in \mathbb{N}_0,$$

is a representation of the solution of (4)–(5). By Theorem 2, we conclude that $u_{\alpha,\lambda}$ is completely monotonic.

□

2.2. **Volterra difference equation with kernel $\kappa(n) = \frac{1}{n+1}$.** Consider the sequence,

$$\kappa(n) = \frac{1}{n+1}, \qquad n \in \mathbb{N}_0,$$

and let u_λ denote the solution of the Volterra difference equation

$$u_\lambda(n) = 1 - \lambda\sum_{\tau=1}^{n}\kappa(n-\tau)u_\lambda(\tau), \qquad n \in \mathbb{N}_0, \quad \lambda > 0.$$

In particular, $u_\lambda(0) = 1$. The generating function of u_λ is [5, Page 8]:

$$F_{u_\lambda}(z) = \frac{\frac{1}{1-z} - \lambda\frac{\log(1-z)}{z}}{1-\lambda\frac{\log(1-z)}{z}},$$

or,

$$F_{u_\lambda}(z) = \frac{\frac{z}{1-z} - \lambda\log(1-z)}{z-\lambda\log(1-z)}.$$

Here log denotes the principal branch of the logarithm. Thus F_{u_λ} is considered in the cut plane

$$\mathbb{C}\setminus[1,\infty).$$

Although the denominator

$$z-\lambda\log(1-z)$$

vanishes at $z = 0$, as is easilly seen this is only a removable singularity of F_{u_λ}.

We now show that there are no non-removable zeros of the denominator in the cut plane. Suppose

$$z - \lambda \log(1-z) = 0,$$

and let $w = 1 - z$. Then

$$1 - w = \lambda \log w,$$

or equivalently,

$$w + \lambda \log w = 1.$$

Write

$$w = re^{i\theta}, \qquad r > 0, \qquad -\pi < \theta < \pi.$$

We get

$$r\cos(\theta) + \lambda \log(r) + i(r\sin(\theta) + \lambda\theta) = 1. \tag{9}$$

Therefore,

$$r \sin\theta + \lambda\theta = 0.$$

Since $\sin\theta$ has the same sign as θ for $-\pi < \theta < \pi$, the only possible solution is

$$\theta = 0.$$

Then, from (9), we obtain

$$r + \lambda \log r = 1.$$

The function

$$h(r) = r + \lambda \log r$$

is strictly increasing on $(0, \infty)$ and, since $h(1) = 1$, we obtain $r = 1$. Therefore $w = 1$ and $z = 0$. Hence the only zero of the denominator in the cut plane is $z = 0$, and this zero is removable for F_{u_λ}.

By Cauchy's coefficient formula, for $0 < \rho < 1$, we have

$$u_\lambda(n) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{F_{u_\lambda}(z)}{z^{n+1}} dz.$$

For $R > 1$ and $\varepsilon > 0$, consider the contour $\Gamma_{\varepsilon,R}$ to be the positively oriented boundary depicted in Figure 1. It consists of four pieces: the large circular arc C_R, the upper side of the cut $L_+(\varepsilon, R)$, the small clockwise circular arc around the branch point C_ε and the lower side of the cut $L_-(\varepsilon, R)$.

Define

$$G_{n,\lambda}(z) = \frac{F_{u_\lambda(z)}}{z^{n+1}}.$$

Since F_{u_λ} is analytic in the cut plane except for the removable singularity at $z = 0$, the only singularity of $G_{n,\lambda}$ inside $\Gamma_{\varepsilon,R}$ is the pole at $z = 0$. By the residue theorem,

$$\frac{1}{2\pi i} \int_{\Gamma_{\varepsilon,R}} G_{n,\lambda}(z)\, dz = \operatorname{Res}_{z=0} G_{n,\lambda}(z)$$

But

$$\operatorname{Res}_{z=0} \frac{F_{u_\lambda(z)}}{z^{n+1}} = u_\lambda(n),$$

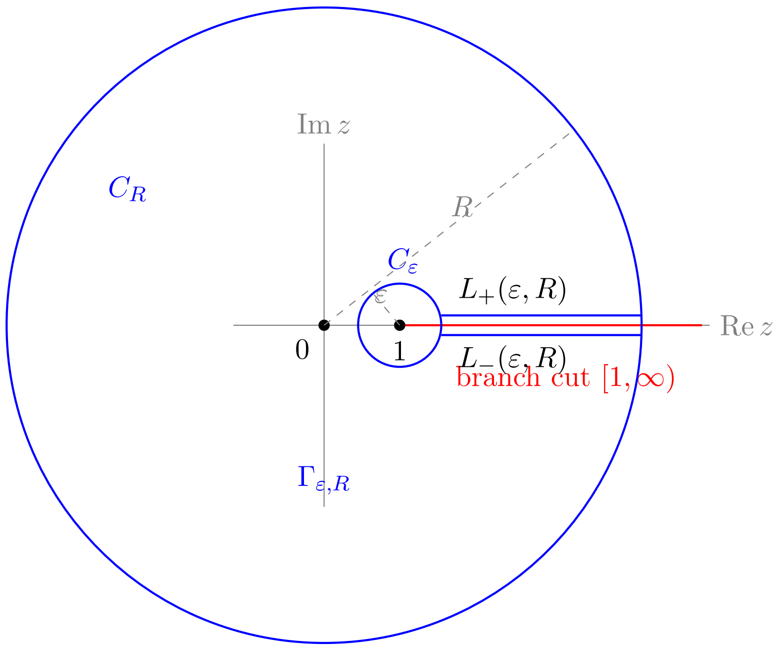

FIGURE 1. Keyhole contour around the branch cut $[1, \infty)$.

hence

$$u_\lambda(n) = \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon,R}} \frac{F_{u_\lambda(z)}}{z^{n+1}} dz.$$

We decompose this contour integral as

$$\int_{\Gamma_{\varepsilon,R}} G_{n,\lambda}(z)dz = \int_{L_+(\varepsilon,R)} G_{n,\lambda}(z)dz + \int_{C_R} G_{n,\lambda}dz + \int_{L_-(\varepsilon,R)} G_{n,\lambda}(z)dz + \int_{C_\varepsilon} G_{n,\lambda}(z)dz.$$

By Theorem 4.3.1 and Theorem 4.3.2 in [1], we obtain

$$\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon} G_{n,\lambda}(z)\, dz = 0, \tag{10}$$

and

$$\lim_{R \to \infty} \int_{C_R} G_{n,\lambda}(z)\, dz = 0. \tag{11}$$

It follows that only the two integrals along the two sides of the branch cut remain. Hence

$$u_\lambda(n) = \frac{1}{2\pi i} \int_1^\infty (F_{\lambda,+}(x) - F_{\lambda,-}(x))\, x^{-n-1} dx,$$

where

$$F_{\lambda,+}(x) := \lim_{\eta \downarrow 0} F_{u_\lambda}(x + i\eta),$$

and

$$F_{\lambda,-}(x) := \lim_{\eta \downarrow 0} F_{u_\lambda}(x - i\eta).$$

For $x > 1$, the upper and lower boundary values of the logarithm are

$$\log(1 - (x + i0)) = \log(x - 1) - i\pi,$$

and

$$\log(1 - (x - i0)) = \log(x - 1) + i\pi.$$

Thus $F_{\lambda,-}(x) = \overline{F_{\lambda,+}(x)}$ and, therefore,

$$F_{\lambda,+}(x) - F_{\lambda,-}(x) = 2i \,\operatorname{Im} F_{\lambda,+}(x).$$

Consequently,

$$u_\lambda(n) = \frac{1}{\pi}\int_1^\infty \operatorname{Im} F_{\lambda,+}(x)x^{-n-1}dx. \tag{12}$$

We now compute $\operatorname{Im} F_{\lambda,+}(x)$.

Put $a = \log(x-1)$. Using $\log(1-(x+i0)) = a - i\pi$, we get

$$F_{\lambda,+}(x) = \frac{\frac{1}{1-x} - \lambda\frac{a}{x} + i\frac{\lambda\pi}{x}}{1 - \lambda\frac{a}{x} + i\frac{\lambda\pi}{x}}.$$

Define

$$A = \frac{1}{1-x} - \lambda\frac{a}{x}, \qquad B = \frac{\lambda\pi}{x}, \qquad C = 1 - \lambda\frac{a}{x}.$$

Then

$$F_{\lambda,+}(x) = \frac{A+iB}{C+iB},$$

and after multiplying numerator and denominator by $C - iB$, we obtain

$$F_{\lambda,+}(x) = \frac{(A+iB)(C-iB)}{C^2+B^2}.$$

Therefore,

$$\operatorname{Im} F_{\lambda,+}(x) = \frac{B(C-A)}{C^2+B^2}.$$

Since

$$C - A = 1 - \frac{1}{1-x} = \frac{x}{x-1},$$

we have

$$\operatorname{Im} F_{\lambda,+}(x) = \frac{\frac{\lambda\pi}{x}\frac{x}{x-1}}{\left(1-\lambda\frac{\log(x-1)}{x}\right)^2 + \left(\frac{\lambda\pi}{x}\right)^2},$$

or

$$\operatorname{Im} F_{\lambda,+}(x) = \frac{\lambda\pi}{(x-1)\left[\left(1-\lambda\frac{\log(x-1)}{x}\right)^2 + \left(\frac{\lambda\pi}{x}\right)^2\right]}.$$

Inserting this into (12) gives

$$u_\lambda(n) = \lambda\int_1^\infty \frac{x^{-n-1}}{(x-1)\left[\left(1-\lambda\frac{\log(x-1)}{x}\right)^2 + \left(\frac{\lambda\pi}{x}\right)^2\right]}dx.$$

Finally, making the change of variables

$$x = \frac{1}{t},$$

we get after some simplifications:

$$u_\lambda(n) = \int_0^1 t^n \frac{\lambda}{(1-t)\left[\left(1-\lambda t\log\frac{1-t}{t}\right)^2 + \pi^2\lambda^2t^2\right]}dt.$$

Since $\lambda > 0$ we conclude that u_λ is completely monotonic by Theorem 2.

The following result is thus proved:

Theorem 4. *The solution of the Volterra difference equation*

$$u(n)=1-\lambda\sum_{\tau=1}^{n}\frac{1}{n-\tau+1}u(\tau),\qquad n\in\mathbb{N}_0,\quad \lambda>0,$$

is given by

$$u_\lambda(n)=\int_0^1 t^n\frac{\lambda}{(1-t)\left[\left(1-\lambda t\log\frac{1-t}{t}\right)^2+\pi^2\lambda^2t^2\right]}dt. \tag{13}$$

In particular, it is completely monotonic.

Remark 1. *By* (1) *we know that* $u(0)=1$*. Moreover, since both kernels in Section 2.1 and Section 2.2 satisfy* $\kappa(0)=1=\kappa'(0)$*, we conclude from* (2) *and* (8) *and* (13)*, the following remarkable identities:*

$$\frac{\sin(\pi\alpha)}{\pi}\int_0^1\frac{r^{\alpha-1}(1-r)^{\alpha-1}}{(1-r)^{2\alpha}+2\lambda(1-r)^\alpha r^\alpha\cos(\pi\alpha)+\lambda^2r^{2\alpha}}dr$$
$$=\int_0^1\frac{1}{(1-t)\left[\left(1-\lambda t\log\frac{1-t}{t}\right)^2+\pi^2\lambda^2t^2\right]}dt,$$

and

$$\frac{\sin(\pi\alpha)}{\pi}\int_0^1\frac{r^{\alpha}(1-r)^{\alpha-1}}{(1-r)^{2\alpha}+2\lambda(1-r)^\alpha r^\alpha\cos(\pi\alpha)+\lambda^2r^{2\alpha}}dr$$
$$=\int_0^1\frac{t}{(1-t)\left[\left(1-\lambda t\log\frac{1-t}{t}\right)^2+\pi^2\lambda^2t^2\right]}dt.$$

Acknowledgments

Rui A. C. Ferreira was supported by the "Fundação para a Ciência e a Tecnologia (FCT)" through the program "Stimulus of Scientific Employment, Individual Support-6th Edition Call" with reference 2023.05967.CEECIND/CP2833/CT0001.

DOI: https://doi.org/10.54499/2023.05967.CEECIND/CP2833/CT0001.

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