An analogue of a formula of Popov II

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Abstract

Let $r_k(n)$ denote the number of representations of the positive integer n as the sum of k squares. We prove a generalization of a summation formula already proved by us [Advances in Applied Mathematics, 175 (2026) 103201], which involves the arithmetical function $r_k(n)$ and the Bessel functions of the first kind. We extend the Bessel functions in the aforementioned formula to Whittaker functions, and our proof of this generalization is drastically different from the proof of the particular case presented in [Advances in Applied Mathematics, 175 (2026) 103201].

1 Introduction and Main results

In a fairly unknown paper [7], Popov states the following beautiful result. If $\text{Re}(x)>0$ and $z\in\mathbb{C}$, then

$$\frac{z^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}x^{\frac{k}{4}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\,e^{z^2/8}+\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}\,J_{\frac{k}{2}-1}(\sqrt{\pi n x}z)$$
$$=\frac{z^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}x^{-\frac{k}{4}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\,e^{-z^2/8}+\frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}}z\right), \tag{1.1}$$

where $r_k(n)$ denotes the number of representations of the positive integer n as a sum of k squares and, as usual, $J_\nu(z)$ and $I_\nu(z)$ respectively denote the Bessel and modified Bessel functions of the first kind. A couple of reasons why this identity is fascinating are already provided by Berndt, Dixit, Kim and Zaharescu [[1], pp. 3795-3796]. For the purposes of our discussion, we repeat verbatim the reasons already explained in [9].

1. If we construct the Dirichlet series attached to $r_k(n)$,

$$\zeta_k(s)=\sum_{n=1}^{\infty}\frac{r_k(n)}{n^s},\quad \text{Re}(s)>\frac{k}{2}, \tag{1.2}$$

then $\zeta_k(s)$ can be continued to the complex plane as a meromorphic function possessing only a simple pole at $s=\frac{k}{2}$ with residue $\pi^{k/2}/\Gamma(k/2)$. Moreover, it satisfies the functional equation

$$\pi^{-s}\Gamma\left(s\right)\zeta_k(s)=\pi^{s-\frac{k}{2}}\Gamma\left(\frac{k}{2}-s\right)\zeta_k\left(\frac{k}{2}-s\right). \tag{1.3}$$

Note that, when $k=1$, $r_1(n)=2$ if and only if n is a perfect square and zero otherwise. Therefore, (1.2) reduces to

$$\zeta_1(s):=\sum_{n=1}^{\infty}\frac{r_1(n)}{n^s}=2\sum_{n=1}^{\infty}\frac{1}{n^{2s}}=2\zeta(2s),\quad \text{Re}(s)>\frac{1}{2}. \tag{1.4}$$

Furthermore, (1.3) with $k=1$ gives the functional equation for Riemann's $\zeta-$function

$$\pi^{-s}\Gamma(s)\,\zeta(2s)=\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}-s\right)\zeta\left(1-2s\right). \tag{1.5}$$

The first point highlighted in [1] is that the powers of n in the denominators of both sides of (1.1) are remindful of the functional equation (1.3).

2. Riemann's second proof of the functional equation for $\zeta(s)$, (1.5), employs the transformation formula for Jacobi's $\theta-$function,

$$\theta(x):=\sum_{n\in\mathbb{Z}}e^{-\pi n^2 x}=\frac{1}{\sqrt{x}}\sum_{n\in\mathbb{Z}}e^{-\frac{\pi n^2}{x}}:=\frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right),\quad \text{Re}(x)>0. \tag{1.6}$$

The theta transformation formula associated to the Dirichlet series $\zeta_k(s)$ can be obtained by taking the k^{th} power on both sides of (1.6). This results in the transformation

$$\sum_{n=0}^{\infty}r_k(n)\,e^{-\pi n x}=x^{-\frac{k}{2}}\sum_{n=0}^{\infty}r_k(n)\,e^{-\frac{\pi n}{x}},\quad \text{Re}(x)>0. \tag{1.7}$$

Of course, the exponential factors on both sides of (1.1) remind us the theta transformation formula (1.7). In fact, (1.7) is a particular case of (1.1) when we let $z\rightarrow 0$, due to the limiting relations for the Bessel functions [[6], p. 223, eq. (10.7.3)]

$$\lim_{y\rightarrow 0}y^{-\nu}J_{\nu}(y)=\lim_{y\rightarrow 0}y^{-\nu}I_{\nu}(y)=\frac{2^{-\nu}}{\Gamma(\nu+1)}. \tag{1.8}$$

3. Chandrasekharan and Narasimhan [[4], p. 19, eq. (65)] proved yet another equivalent identity to (1.3) and (1.7). If $x>0$ and $q>\frac{k-1}{2}$, then

$$\frac{1}{\Gamma(q+1)}\sum_{0\leq n\leq x}{}^{\prime}\,r_k(n)\,(x-n)^q=\frac{\pi^{\frac{k}{2}}x^{\frac{k}{2}+q}}{\Gamma\left(q+1+\frac{k}{2}\right)}+\pi^{-q}\sum_{n=1}^{\infty}r_k(n)\left(\frac{x}{n}\right)^{\frac{k}{4}+\frac{q}{2}}J_{\frac{k}{2}+q}\left(2\pi\sqrt{nx}\right), \tag{1.9}$$

where the Bessel series on the right-hand side converges absolutely. The prime on the summation sign indicates that, if $q=0$ and x is an integer, then the last contribution in this Riesz sum is just $\frac{1}{2}r_k(x)$. The appearance of the Bessel functions in (1.1) reminds us of (1.9).

In [9] we have proved two interesting summation formulas that extend the transformation formula for Jacobi's theta function (1.6). These formulas are, respectively,

$$\begin{aligned}&\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}J_{\frac{k}{4}-\frac{1}{2}}\left(\pi n y\right)\\&=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2+y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n x}{x^2+y^2}}J_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi n y}{x^2+y^2}\right)\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}&\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}I_{\frac{k}{4}-\frac{1}{2}}\left(\pi n y\right)\\&=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2-y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2-y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n x}{x^2-y^2}}I_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi n y}{x^2-y^2}\right).\end{aligned} \tag{1.11}$$

Before proceeding to the main result of this paper, let us remark that (1.10) and (1.11) contain some interesting analogues and particular cases. First, if we take $k=4$ in (1.11), we obtain the curious identity

$$2\pi y+\sum_{n=1}^{\infty}\frac{r_4(n)}{n}\left\{e^{-\pi n(x-y)}-e^{-\pi n(x+y)}\right\}=\frac{2\pi y}{x^2-y^2}+\sum_{n=1}^{\infty}\frac{r_4(n)}{n}\left\{e^{-\frac{\pi n}{x+y}}-e^{-\frac{\pi n}{x-y}}\right\},$$

valid for $x>y>0$. Also, when $k=1$, it is simple to see that (1.10) and (1.11) imply the beautiful formulas

$$\sum_{n\in\mathbb{Z}}\sqrt{|n|}\,e^{-\pi n^2 x}J_{-\frac{1}{4}}\left(\pi n^2 y\right)=\frac{1}{\sqrt{x^2+y^2}}\sum_{n\in\mathbb{Z}}\sqrt{|n|}\,e^{-\frac{\pi n^2 x}{x^2+y^2}}\,J_{-\frac{1}{4}}\left(\frac{\pi n^2 y}{x^2+y^2}\right), \tag{1.12}$$

$$\sum_{n\in\mathbb{Z}}\sqrt{|n|}\,e^{-\pi n^2 x}I_{-\frac{1}{4}}\left(\pi n^2 y\right)=\frac{1}{\sqrt{x^2-y^2}}\sum_{n\in\mathbb{Z}}\sqrt{|n|}\,e^{-\frac{\pi n^2 x}{x^2-y^2}}\,I_{-\frac{1}{4}}\left(\frac{\pi n^2 y}{x^2-y^2}\right), \tag{1.13}$$

where $x>y>0$. Of course, when $y\to 0^+$, (1.12) and (1.13) both reduce to the theta transformation formula (1.6).

Although the proof of (1.10) and (1.11) employed the interesting symmetries of the Gauss hypergeometric function, ${}_2F_1(a,b;c;z)$, the reader may find their scope a bit limited. First of all, the condition $x>y>0$ seems to be unnecessary for the first formula, (1.10), to be valid. However, the proof developed in [9] only seems to work under this assumption. These limitations beg the question whether it is possible to obtain a generalization of (1.10) and (1.11) that lifts the restrictive conditions on x and y imposed by the identities (1.10) and (1.11).

Since we have already studied summation formulas for indices of Whittaker functions, we thought it would be suitable to complement the results from both [12] and [9] by presenting a generalization of the formulas (1.10) and (1.11) involving the Whittaker function of the first kind, $M_{\mu,\nu}(z)$. As it is well known, this function arises in the study of the Kummer confluent hypergeometric function, ${}_1F_1(a;c;z)$, which is usually defined by the power series [[6], p. 322, eq. (13.2.2)],

$${}_1F_1\left(a;c;z\right):=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\,\frac{z^k}{k!}, \tag{1.14}$$

where $(a)_k:=\Gamma(a+k)/\Gamma(a)$ denotes the Pochhammer symbol. From the power series (1.14), one can define the Whittaker function of the first kind [[6], p. 334, eq. (13.14.2)]

$$M_{\mu,\nu}(z)=z^{\nu+\frac{1}{2}}e^{-z/2}\,{}_1F_1\left(\nu-\mu+\frac{1}{2};\,2\nu+1;\,z\right). \tag{1.15}$$

This function has several interesting integral representations. One representation that could be interesting to employ in a generalization of (1.10) and (1.11) is [[3], p. 458, eq. (3.30.2.1)]

$$e^{-ax}M_{\rho,\nu}(2bx)=\left(\frac{2b}{a+b}\right)^{\nu}\frac{\sqrt{2bx}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\nu\right)\,{}_2F_1\left(\nu+\frac{1}{2}-\rho,s+\nu;2\nu+1;\frac{2b}{a+b}\right)\left(x\left(a+b\right)\right)^{-s}ds, \tag{1.16}$$

whose integrand has the right kind of symmetries compatible with the functional equation of $\zeta_k(s)$ and Euler's formula for Gauss' hypergeometric function. Applying the same ideas as the ones presented in [9], one can rightfully come to the following result.

Theorem. *Assume that x, y are two real numbers such that $x > y > 0$ and that $-\frac{k}{4} < \rho < \frac{k}{4}$. Then the following summation formula holds*

$$(2\pi y)^{\frac{k}{4}} + \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}}} e^{-\pi n x} M_{\rho, \frac{k}{4}-\frac{1}{2}} (2\pi n y)$$

$$= \left(\frac{2\pi y}{x^2 - y^2}\right)^{\frac{k}{4}} \left(\frac{x-y}{x+y}\right)^{\rho} + \left(\frac{x-y}{x+y}\right)^{\rho} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}}} e^{-\frac{\pi n x}{x^2-y^2}} M_{-\rho, \frac{k}{4}-\frac{1}{2}} \left(\frac{2\pi n y}{x^2 - y^2}\right). \tag{1.17}$$

Just like the conditions giving the formulas (1.10) and (1.11), the previous result offers significant obstacles to a wide extension of (1.10) and (1.11). Note that the condition $x > y > 0$ is indeed crucial to apply Watson-type estimates for the Gauss' hypergeometric function (cf. [[9], Lemma 2.1.]), which are vital to prove (1.17) using the Mellin-Barnes integral (1.16). However, this condition on x and y is too restrictive because, when $\rho = 0$, (1.17) can only be reduced to (1.11) and not to (1.10)!

Therefore, we need an argument that provides a slightly more general version of (1.17) and implies, at the same time, both (1.10) and (1.11) Finding such an identity requires us to present a drastically different argument from the one used in [9], not requiring at all the implementation of the Mellin-Barnes integral (1.16). What is truly curious about this different proof is that it uses nothing more than Popov's classical formula itself (1.1), which served as the main motivation of the investigation of (1.10) and (1.11) in the first place! Without any further delay, we present the main result of this paper.

Theorem 1.1. *Assume that x, y are two complex numbers such that $Re(x) > |Re(y)|$ and that ρ is a complex number satisfying $-\frac{k}{4} < Re(\rho) < \frac{k}{4}$. Then the following summation formula holds*

$$(2\pi y)^{\frac{k}{4}} + \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}}} e^{-\pi n x} M_{\rho, \frac{k}{4}-\frac{1}{2}} (2\pi n y)$$

$$= \left(\frac{2\pi y}{x^2 - y^2}\right)^{\frac{k}{4}} \left(\frac{x-y}{x+y}\right)^{\rho} + \left(\frac{x-y}{x+y}\right)^{\rho} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}}} e^{-\frac{\pi n x}{x^2-y^2}} M_{-\rho, \frac{k}{4}-\frac{1}{2}} \left(\frac{2\pi n y}{x^2 - y^2}\right), \tag{1.18}$$

where $M_{\mu,\nu}(z)$ denotes the Whittaker function of the first kind (1.15).

To end our introduction, let us remark that it comes as no surprise that we can extend Theorem 1.1 to a more general class of Dirichlet series satisfying Hecke's functional equation. For example, in [10], the author of this paper and Yakubovich proved a generalization of (1.1) with the role of $r_k(n)$ being replaced by a generic arithmetical function, $a(n)$. To work in this setting, we introduced the following general class of Dirichlet series (cf. [[10], Def. 1.1.]).

Definition 1.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers strictly increasing to ∞ and $(a(n))_{n\in\mathbb{N}}$ and $(b(n))_{n\in\mathbb{N}}$ two sequences of complex numbers not identically zero. Consider the functions $\phi(s)$ and $\psi(s)$ representable as Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s} \tag{1.19}$$

with finite abscissas of absolute convergence σ_a and σ_b, respectively. We say that $\phi(s)$ and $\psi(s)$ satisfy the functional equation

$$\Gamma(s)\,\phi(s)=\Gamma(r-s)\,\psi(r-s),\quad r>0, \tag{1.20}$$

if there exists a meromorphic function $\chi(s)$ with the following properties:

1. $\chi(s)=\Gamma(s)\,\phi(s)$ for $\mathrm{Re}(s)>\sigma_a$ and $\chi(s)=\Gamma(r-s)\,\psi(r-s)$ for $\mathrm{Re}(s)<r-\sigma_b$;

2. $\lim_{|\mathrm{Im}(s)|\to\infty}\chi(s)=0$ uniformly in every interval $-\infty<\sigma_1\leq\mathrm{Re}(s)\leq\sigma_2<\infty$.

3. $\phi(s)$ and $\psi(s)$ have analytic continuations to the entire complex plane and are analytic on $\mathbb{C}$ except for possible simple poles located at $s=r$ with residues ρ and $\rho^{\star}$, respectively.

Under the scope of the previous definition, we have been able to establish the following generalization of Popov's formula (1.1) [[10], p. 38, eq. (2.64)],

$$\begin{aligned}&-\phi(0)e^{z^2/8}+2^{r-1}\Gamma(r)\,x^{\frac{1-r}{2}}z^{1-r}\,e^{z^2/8}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n x}\,J_{r-1}\left(\sqrt{\lambda_n x}\,z\right)\\&=\frac{\rho\Gamma(r)}{x^r}e^{-z^2/8}+2^{r-1}\Gamma(r)\,z^{1-r}x^{-\frac{r+1}{2}}e^{-z^2/8}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\frac{\mu_n}{x}}\,I_{r-1}\left(\sqrt{\frac{\mu_n}{x}}\,z\right),\end{aligned} \tag{1.21}$$

which is valid for $\mathrm{Re}(x)>0$, $z\in\mathbb{C}$ and the general Dirichlet series and arithmetical functions given in Definition 1.1. Since (1.21) generalizes (1.1), one may ask about the possibility of obtaining a generalization of (1.18) with $r_k(n)$ replaced by a general arithmetical function. By following the same lines as in the proof of Theorem 1.1, it is possible to obtain such generalization. In particular, if $\phi(s)$ is a Dirichlet series satisfying Definition 1.1 and $\mathrm{Re}(x)>|\mathrm{Re}(y)|$, $-\frac{r}{2}<\mu<\frac{r}{2}$, then the following summation formula holds

$$\begin{aligned}&-\phi(0)\ (2y)^{\frac{r}{2}}+\sum_{n=1}^{\infty}a(n)\lambda_n^{-\frac{r}{2}}e^{-\lambda_n x}\,M_{\mu,\frac{r-1}{2}}\left(2\lambda_n y\right)\\&=\rho\,\frac{(2y)^{\frac{r}{2}}}{(x^2+y^2)^{\frac{r}{2}}}\Gamma(r)+\left(\frac{x-y}{x+y}\right)^{\mu}\sum_{n=1}^{\infty}b(n)\,\mu_n^{-\frac{r}{2}}\,e^{-\frac{\mu_n x}{x^2-y^2}}M_{-\mu,\frac{r-1}{2}}\left(\frac{2\mu_n y}{x^2-y^2}\right).\end{aligned} \tag{1.22}$$

Note that, as $y\to0^{+}$ in (1.22), the limiting relation [[6], p. 335, eq. (13.14.14)] $M_{\mu,\nu}(z)=z^{\nu+\frac{1}{2}}\,\left(1+O(z)\right),\ \ 2\nu\notin\mathbb{Z}^{-}$, $z\to0$, yields the well-known formula

$$-\phi(0)+\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n x}=\rho\Gamma(r)\,x^{-r}+x^{-r}\sum_{n=1}^{\infty}b(n)\,e^{-\frac{\mu_n}{x}},\quad\mathrm{Re}(x)>0, \tag{1.23}$$

due to Bochner [2], who proved for the first time the equivalence of (1.23) and the functional equation for $\phi(s)$, (1.20). Note that (1.23) gives the theta transformation formula (1.6) when $\phi(s)=2\,\zeta(2s)$.

The interested reader can now find the particular summation formulas arising from the very general identity (1.22). To get a neat formula as an appetizer for such explorations, one can note that, when $a(n)$ is Ramanujan's $\tau-$function, $\tau(n)$, then the following formula holds

$$\sum_{n=1}^{\infty}\frac{\tau(n)}{n^6}\,e^{-2\pi nx}M_{\mu,\frac{11}{2}}\left(4\pi ny\right)=\left(\frac{x-y}{x+y}\right)^{\mu}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^6}\,e^{-\frac{2\pi nx}{x^2-y^2}}\,M_{-\mu,\frac{11}{2}}\left(\frac{4\pi ny}{x^2-y^2}\right),$$

whenever $-6<\mathrm{Re}(\mu)<6$ and $\mathrm{Re}(x)>|\mathrm{Re}(y)|$.

2 Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we need the following integral representation for the Whittaker function of the first kind, which can be found in [[6], p. 337, eq. (13.16.3) and (13.16.4)].

Lemma 2.1. *For the confluent hypergeometric function $M_{\rho,\nu}(z)$, the following integral representations hold*[1]

$$M_{\rho,\nu}(z)=\frac{\Gamma(1+2\nu)\sqrt{z}}{\Gamma\left(\frac{1}{2}+\rho+\nu\right)}\,e^{\frac{z}{2}}\int_0^{\infty}e^{-t}t^{\rho-\frac{1}{2}}\,J_{2\nu}\left(2\sqrt{zt}\right)\,dt,\quad Re\,(\rho+\nu)+\frac{1}{2}>0,\quad z\in\mathbb{C},\tag{2.1}$$

$$M_{\rho,\nu}(z)=\frac{\Gamma(1+2\nu)\sqrt{z}}{\Gamma\left(\frac{1}{2}+\rho-\nu\right)}\,e^{-\frac{z}{2}}\int_0^{\infty}e^{-t}t^{-\rho-\frac{1}{2}}I_{2\nu}\left(2\sqrt{zt}\right)\,dt,\quad Re\,(\rho-\nu)-\frac{1}{2}<0,\quad z\in\mathbb{C}.\tag{2.2}$$

Our departure point will be the first integral representation (2.1), as well as the ubiquitous Popov's formula (1.1). As remarked above, the advantage of this proof is that we no longer need Watson type estimates for the Gauss hypergeometric function, nor the somewhat restrictive conditions that these entail, i.e., the condition that x and y are positive real numbers such that $x>y$. Throughout this argument we shall assume that x and y are complex numbers such that $\mathrm{Re}(x)>|\mathrm{Re}(y)|$ and also that $-\frac{k}{4}<\mathrm{Re}(\rho)<\frac{k}{4}$. Under these conditions, (2.1) yields

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}}}e^{-\pi nx}\,M_{\rho,\frac{k}{4}-\frac{1}{2}}\left(2\pi ny\right)&=\frac{\Gamma\left(\frac{k}{2}\right)\sqrt{2\pi y}}{\Gamma\left(\frac{k}{4}+\rho\right)}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}e^{-\pi n(x-y)}\int_0^{\infty}e^{-t}t^{\rho-\frac{1}{2}}J_{\frac{k}{2}-1}\left(2\sqrt{2\pi ny\,t}\right)\,dt\\&=\frac{\Gamma\left(\frac{k}{2}\right)\sqrt{2\pi y}}{\Gamma\left(\frac{k}{4}+\rho\right)}\int_0^{\infty}e^{-t}t^{\rho-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}e^{-\pi n(x-y)}J_{\frac{k}{2}-1}\left(2\sqrt{2\pi ny\,t}\right)\,dt,\end{aligned}\tag{2.3}$$

where the interchange of the integration and summation orders is due to the simple inequalities

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}e^{-\pi n\mathrm{Re}(x-y)}\int_0^{\infty}e^{-t}t^{\mathrm{Re}(\rho)-\frac{1}{2}}\left|J_{\frac{k}{2}-1}\left(2\sqrt{2\pi ny\,t}\right)\right|\,dt\\&\leq\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}e^{-\pi n\mathrm{Re}(x-y)}\int_0^{\infty}e^{-t}t^{\mathrm{Re}(\rho)-\frac{1}{2}}I_{\frac{k}{2}-1}\left(2\sqrt{2\pi n|y|t}\right)\,dt\\&\leq\frac{(2\pi|y|)^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{n=1}^{\infty}r_k(n)e^{-\pi n\mathrm{Re}(x-y)}\int_0^{\infty}e^{-\left(t-2\sqrt{2\pi n|y|t}\right)}t^{\mathrm{Re}(\rho)+\frac{k}{4}-1}\,dt\\&=\frac{2\,(2\pi|y|)^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{n=1}^{\infty}r_k(n)e^{-\pi n\mathrm{Re}(x-y)}\int_0^{\infty}e^{-\left(u^2-2\sqrt{2\pi n|y|}\,u\right)}u^{2\mathrm{Re}(\rho)+\frac{k}{2}-1}\,du\\&=\frac{2\,(2\pi|y|)^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\frac{\Gamma\left(2\mathrm{Re}(\rho)+\frac{k}{2}\right)}{2^{\mathrm{Re}(\rho)+\frac{k}{4}-\frac{1}{2}}}\sum_{n=1}^{\infty}r_k(n)e^{-\pi n\mathrm{Re}(x-y)}\,e^{\pi n|y|}\,D_{-2\mathrm{Re}(\rho)-\frac{k}{2}}\left(2\sqrt{\pi n|y|}\right),\end{aligned}\tag{2.4}$$

where the second inequality is due to the integral representation for the modified Bessel function [[6], p.252, eq. (10.32.2)],

$$\left(\frac{z}{2}\right)^{-\nu}I_{\nu}(z)=\frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,e^{zt}dt,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\quad z\in\mathbb{C},\tag{2.5}$$

[1]Note that the condition given in [6] for the integral (2.2) contains a typo, which is corrected in the NIST webpage.

from which one can immediately obtain the bound (with real $\nu > -\frac{1}{2}$ and $z \in \mathbb{C}$)[2]

$$\left|\left(\frac{z}{2}\right)^{-\nu} I_{\nu}(z)\right| \leq \frac{e^{|\mathrm{Re}(z)|}}{\Gamma(\nu+1)}. \tag{2.6}$$

Moreover, the last step of (2.4) can be justified by the integral representation for the parabolic cylinder function [[3], p. 20, 2.2.1.6] (see also relation 2.3.15.3 on page 343 of vol. I of [8]),

$$\int_0^{\infty} u^{\mu-1} e^{-xu^2-yu} du = \frac{\Gamma(\mu)}{(2x)^{\mu/2}}\, e^{\frac{y^2}{8x}}\, D_{-\mu}\left(\frac{y}{\sqrt{2x}}\right), \quad \mathrm{Re}(\mu),\ \mathrm{Re}(x) > 0,\ y \in \mathbb{C}, \tag{2.7}$$

which can be applied because $\mathrm{Re}(\rho) > -\frac{k}{4}$ by hypothesis. The last series in (2.4) converges absolutely, as $D_{\mu}(x)$ satisfies the asymptotic formula (cf. [[6], p. 309, 12.9 (i)], [[5], Vol. II, p. 122, eq. 8.4(1)]),

$$D_{\mu}(x) \sim x^{\mu} e^{-\frac{x^2}{4}}, \quad x \to \infty. \tag{2.8}$$

Looking now at the right-hand side of (2.3), we are able to apply Popov's formula (1.1) to the infinite series

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\pi n(x-y)} J_{\frac{k}{2}-1}\left(2\sqrt{2\pi n y\, t}\right)$$

because, by hypothesis, $\mathrm{Re}(x) > |\mathrm{Re}(y)|$. Employing (1.1) and replacing there x by $x-y$ and z by $\sqrt{\frac{8yt}{x-y}}$, we obtain the equivalent identity

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\pi n(x-y)} J_{\frac{k}{2}-1}\left(2\sqrt{2\pi n y\, t}\right) &= \frac{2^{\frac{k}{4}-\frac{1}{2}} \pi^{\frac{k}{4}-\frac{1}{2}} y^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)(x-y)^{\frac{k}{2}}}\, t^{\frac{k}{4}-\frac{1}{2}} e^{-\frac{2yt}{x-y}} \\ &- \frac{\pi^{\frac{k}{4}-\frac{1}{2}} 2^{\frac{k}{4}-\frac{1}{2}} y^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\, t^{\frac{k}{4}-\frac{1}{2}} + \frac{e^{-\frac{2yt}{x-y}}}{x-y} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, e^{-\frac{\pi n}{x-y}}\, I_{\frac{k}{2}-1}\left(\frac{\sqrt{8\pi n y t}}{x-y}\right).\end{aligned}$$

The previous sibling of Popov's formula (1.1) yields the equality

$$\begin{aligned}&\frac{\Gamma\left(\frac{k}{2}\right)\sqrt{2\pi y}}{\Gamma\left(\frac{k}{4}+\rho\right)} \int_0^{\infty} e^{-t} t^{\rho-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\pi n(x-y)} J_{\frac{k}{2}-1}\left(2\sqrt{2\pi n y\, t}\right) dt \\ =& \frac{(2\pi y)^{\frac{k}{4}}}{\Gamma\left(\frac{k}{4}+\rho\right)} \frac{1}{(x-y)^{\frac{k}{2}}} \int_0^{\infty} e^{-\frac{x+y}{x-y}t}\, t^{\rho+\frac{k}{4}-1}\, dt - \frac{(2\pi y)^{\frac{k}{4}}}{\Gamma\left(\frac{k}{4}+\rho\right)} \int_0^{\infty} t^{\rho+\frac{k}{4}-1} e^{-t}\, dt \\ +& \frac{\Gamma\left(\frac{k}{2}\right)\sqrt{2\pi y}}{\Gamma\left(\frac{k}{4}+\rho\right)(x-y)} \int_0^{\infty} e^{-\frac{x+y}{x-y}t}\, t^{\rho-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, e^{-\frac{\pi n}{x-y}}\, I_{\frac{k}{2}-1}\left(\frac{\sqrt{8\pi n y t}}{x-y}\right) dt \\ =& \frac{(2\pi y)^{\frac{k}{4}} (x-y)^{\rho-\frac{k}{4}}}{(x+y)^{\rho+\frac{k}{4}}} - (2\pi y)^{\frac{k}{4}} \\ +& \frac{\Gamma\left(\frac{k}{2}\right)\sqrt{2\pi y}}{\Gamma\left(\frac{k}{4}+\rho\right)(x-y)} \left(\frac{x-y}{x+y}\right)^{\rho+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\frac{\pi n}{x-y}} \int_0^{\infty} e^{-u}\, u^{\rho-\frac{1}{2}}\, I_{\frac{k}{2}-1}\left(\frac{2\sqrt{2\pi n y u}}{\sqrt{x^2-y^2}}\right) du,\end{aligned}$$

[2]despite the fact that the Poisson integral (2.5) holds for $\mathrm{Re}(\nu) > -\frac{1}{2}$, which covers the case where $k \geq 2$, when $\nu = -\frac{1}{2}$ ($k=1$), the identity $I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x)$ actually serves the same purpose.

where the evaluation of the first two terms can be truly justified by the hypothesis $\text{Re}(\rho) > -\frac{k}{4}$. The reason for the interchange of the orders of summation and integration in the last step is analogous to that provided for (2.4). Indeed,

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}e^{-\pi n\,\text{Re}\left(\frac{1}{x-y}\right)}\int_0^{\infty}e^{-u}\,u^{\text{Re}(\rho)-\frac{1}{2}}\left|I_{\frac{k}{2}-1}\left(\frac{2\sqrt{2\pi nyu}}{\sqrt{x^2-y^2}}\right)\right|du$$

$$\leq\frac{(2\pi)^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left|\frac{y}{x^2-y^2}\right|^{\frac{k}{4}-\frac{1}{2}}\sum_{n=1}^{\infty}r_k(n)\,e^{-\pi n\,\text{Re}\left(\frac{1}{x-y}\right)}\int_0^{\infty}u^{\text{Re}(\rho)+\frac{k}{4}-1}e^{-u}\,e^{2\sqrt{2\pi n}\left|\text{Re}\left(\sqrt{\frac{y}{x^2-y^2}}\right)\right|\sqrt{u}}\,du$$

$$=\frac{2\,(2\pi)^{\frac{k}{4}-\frac{1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left|\frac{y}{x^2-y^2}\right|^{\frac{k}{4}-\frac{1}{2}}\sum_{n=1}^{\infty}r_k(n)\,e^{-\pi n\,\text{Re}\left(\frac{1}{x-y}\right)}\int_0^{\infty}t^{2\text{Re}(\rho)+\frac{k}{2}-1}e^{-t^2}\,e^{2\sqrt{2\pi n}\left|\text{Re}\left(\sqrt{\frac{y}{x^2-y^2}}\right)\right|t}\,dt$$

$$=\frac{2\,(2\pi)^{\frac{k}{4}-\frac{1}{2}}\,\Gamma\left(-2\text{Re}(\rho)-\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\,2^{\text{Re}(\rho)+\frac{k}{4}}}\left|\frac{y}{x^2-y^2}\right|^{\frac{k}{4}-\frac{1}{2}}\sum_{n=1}^{\infty}r_k(n)\,\exp\left(-\pi n\,\text{Re}\left(\frac{1}{x-y}\right)+\pi n\left|\text{Re}\left(\sqrt{\frac{y}{x^2-y^2}}\right)\right|^2\right)$$
$$\times D_{-2\text{Re}(\rho)-\frac{k}{2}}\left(2\sqrt{\pi n}\,\left|\text{Re}\left(\sqrt{\frac{y}{x^2-y^2}}\right)\right|\right),$$

which converges absolutely by the asymptotic formula (2.8) for the cylinder function $D_{\nu}(x)$, holding when $x\to\infty$. On the other hand, using the hypothesis that $-\frac{k}{4} < \text{Re}(\rho) < \frac{k}{4}$, we see from (2.2) that

$$\int_0^{\infty}e^{-u}\,u^{\rho-\frac{1}{2}}\,I_{\frac{k}{2}-1}\left(2\sqrt{\frac{2\pi nyu}{x^2-y^2}}\right)du=\frac{\Gamma\left(\rho+\frac{k}{4}\right)}{\Gamma\left(\frac{k}{2}\right)}\sqrt{\frac{x^2-y^2}{2\pi ny}}\,e^{\frac{\pi ny}{x^2-y^2}}\,M_{-\rho,\frac{k}{4}-\frac{1}{2}}\left(\frac{2\pi ny}{x^2-y^2}\right),$$

yielding

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}}}e^{-\pi nx}\,M_{\rho,\frac{k}{4}-\frac{1}{2}}(2\pi ny)=\left(\frac{2\pi y}{x^2-y^2}\right)^{\frac{k}{4}}\left(\frac{x-y}{x+y}\right)^{\rho}-(2\pi y)^{\frac{k}{4}}$$
$$+\left(\frac{x-y}{x+y}\right)^{\rho}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}}}\,e^{-\frac{\pi nx}{x^2-y^2}}\,M_{-\rho,\frac{k}{4}-\frac{1}{2}}\left(\frac{2\pi ny}{x^2-y^2}\right),\tag{2.9}$$

which is precisely (1.18). ■

As promised at the beginning of our paper, we can now derive a much stronger version of formulas (1.10) and (1.11), totally erasing its restrictive conditions on x and y.

Corollary 2.1. *If x, y are two complex numbers such that $Re(x) > |Im(y)|$, then the following transformation formula holds*

$$\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi nx}J_{\frac{k}{4}-\frac{1}{2}}(\pi ny)$$
$$=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2+y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi nx}{x^2+y^2}}J_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi ny}{x^2+y^2}\right).\tag{2.10}$$

Moreover, if x, y are two complex numbers such that $Re(x) > |Re(y)|$, the analogous formula is valid

$$\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi nx}I_{\frac{k}{4}-\frac{1}{2}}(\pi ny)$$
$$=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)(x^2-y^2)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2-y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi nx}{x^2-y^2}}I_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi ny}{x^2-y^2}\right).\tag{2.11}$$

Proof. Clearly, (2.11) implies (2.10) under the substitution $y \leftrightarrow iy$. Since the conditions over x and y in the formula (2.11) are actually the same as the conditions in our general Theorem 1.1, we just need to check that (1.18) reduces to (2.11). Indeed, if we take $\rho = 0$ in (1.18) and use the reduction formula [[6], p. 338, eq. (13.18.8)],

$$M_{0,\nu}(2z) = 2^{2\nu+\frac{1}{2}}\Gamma(\nu+1)\sqrt{z}\, I_\nu(z),$$

we can easily get (2.11).

□

Remark 2.1. We can generalize the previous corollary for any complex numbers x, y such that $\mathrm{Re}(x) > |\mathrm{Im}(y)|$. In fact, such generalization reads

$$-\frac{\phi(0)}{\Gamma\left(\frac{r+1}{2}\right)}\left(\frac{y}{2}\right)^{\frac{r-1}{2}} + \sum_{n=1}^{\infty} a(n)\,\lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x} J_{\frac{r-1}{2}}\left(\lambda_n y\right)$$
$$= \frac{\rho\Gamma\left(\frac{r}{2}\right)}{\sqrt{\pi}}\,\frac{(2y)^{\frac{r-1}{2}}}{(x^2+y^2)^{\frac{r}{2}}} + \frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{1-r}{2}}\,\exp\left\{-\frac{\mu_n x}{x^2+y^2}\right\} J_{\frac{r-1}{2}}\left(\frac{\mu_n y}{x^2+y^2}\right). \tag{2.12}$$

Analogously, under the same conditions, one can find the generalization of (2.11),

$$-\frac{\phi(0)}{\Gamma\left(\frac{r+1}{2}\right)}\left(\frac{y}{2}\right)^{\frac{r-1}{2}} + \sum_{n=1}^{\infty} a(n)\,\lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x} I_{\frac{r-1}{2}}\left(\lambda_n y\right)$$
$$= \frac{\rho\Gamma\left(\frac{r}{2}\right)}{\sqrt{\pi}}\,\frac{(2y)^{\frac{r-1}{2}}}{(x^2-y^2)^{\frac{r}{2}}} + \frac{1}{\sqrt{x^2-y^2}}\sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{1-r}{2}}\,\exp\left\{-\frac{\mu_n x}{x^2-y^2}\right\} I_{\frac{r-1}{2}}\left(\frac{\mu_n y}{x^2-y^2}\right). \tag{2.13}$$

We can also present a very interesting formula involving the Laguerre function. The next corollary is a very curious case of Theorem 1.1 happening when $k = 2$.

Corollary 2.2. *For $-\frac{1}{2} < Re(\rho) < \frac{1}{2}$ and complex x, y such that $Re(x) > |Re(y)|$, the following summation formula holds*

$$1 + \sum_{n=1}^{\infty} r_2(n) e^{-\pi n(x+y)}\, L_{\rho-\frac{1}{2}}\left(2\pi n y\right)$$
$$= \frac{1}{\sqrt{x^2-y^2}}\left(\frac{x-y}{x+y}\right)^{\rho} + \frac{1}{\sqrt{x^2-y^2}}\left(\frac{x-y}{x+y}\right)^{\rho}\sum_{n=1}^{\infty} r_2(n) e^{-\frac{\pi n}{x-y}}\, L_{-\rho-\frac{1}{2}}\left(\frac{2\pi n y}{x^2-y^2}\right), \tag{2.14}$$

where $L_\mu(z)$ denotes the Laguerre function.

Proof. The proof is just an application of (1.18) with $k = 2$ and the reduction formula for the Whittaker $M-$function

$$M_{\rho,0}(z) = \sqrt{z}\, e^{-\frac{z}{2}}\, L_{\nu-\frac{1}{2}}\left(z\right).$$

□

We end this paper by presenting yet another interesting corollary, whose formal shape is strikingly similar to (1.11).

Corollary 2.3. *Let x, y be two complex numbers such that $Re(x) > |Re(y)|$ and $k \geq 3$. Then the following transformation formula holds*

$$(2\pi y)^{\frac{k}{4}} \sqrt{\frac{x-y}{x+y}} + 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{4}\right) \pi y \sqrt{\frac{x-y}{x+y}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-1}} e^{-\pi n x} \left(I_{\frac{k}{4}}(\pi n y) + I_{\frac{k}{4}-1}(\pi n y)\right)$$
$$= \left(\frac{2\pi y}{x^2-y^2}\right)^{\frac{k}{4}} + 2^{\frac{k}{2}-1}\pi\,\Gamma\left(\frac{k}{4}\right) \frac{y}{x^2-y^2} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-1}} e^{-\frac{\pi n x}{x^2-y^2}} \left(I_{\frac{k}{4}}\left(\frac{\pi n y}{x^2-y^2}\right) + I_{\frac{k}{4}-1}\left(\frac{\pi n y}{x^2-y^2}\right)\right). \tag{2.15}$$

Proof. Taking $\rho = -\frac{1}{2}$ in the general formula (1.18) and using the reduction formulas for the Whittaker function,

$$M_{-\frac{1}{2},\nu}(z) = 2^{2\nu-1}\Gamma\left(\nu+\frac{1}{2}\right) z \left(I_{\nu+\frac{1}{2}}\left(\frac{z}{2}\right) + I_{\nu-\frac{1}{2}}\left(\frac{z}{2}\right)\right),$$

$$M_{\frac{1}{2},\nu}(z) = 2^{2\nu-1}\Gamma\left(\nu+\frac{1}{2}\right) z \left(I_{\nu-\frac{1}{2}}\left(\frac{z}{2}\right) - I_{\nu+\frac{1}{2}}\left(\frac{z}{2}\right)\right),$$

we get, for $k \geq 3$,

$$(2\pi y)^{\frac{k}{4}} \sqrt{\frac{x-y}{x+y}} + 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{4}\right) \pi y \sqrt{\frac{x-y}{x+y}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-1}} e^{-\pi n x} \left(I_{\frac{k}{4}}(\pi n y) + I_{\frac{k}{4}-1}(\pi n y)\right)$$
$$= \left(\frac{2\pi y}{x^2-y^2}\right)^{\frac{k}{4}} + 2^{\frac{k}{2}-1}\pi\,\Gamma\left(\frac{k}{4}\right) \frac{y}{x^2-y^2} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-1}} e^{-\frac{\pi n x}{x^2-y^2}} \left(I_{\frac{k}{4}}\left(\frac{\pi n y}{x^2-y^2}\right) + I_{\frac{k}{4}-1}\left(\frac{\pi n y}{x^2-y^2}\right)\right).$$

□

Remark 2.2. It is actually possible to get analogues of the summation formula (2.15) by specifying $\rho = -\frac{\ell}{2}$, $\ell \in \mathbb{N}$, in (1.18). But the resulting formula is somewhat cumbersome, so we leave the details of this to the reader.

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