FOURIER–MUKAI TRANSFORMS AND NORMALISATION OF NODAL CURVES

EMILIO FRANCO, ROBERT HANSON, JOHANNES HORN, AND ANDRÉ OLIVEIRA

Abstract. We study Arinkin's Poincaré sheaf $\mathcal{P}_C$ on the singular locus of $\overline{\mathsf{Jac}}_C$, the compactified Jacobian of rank one torsion-free sheaves on an integral nodal projective curve C. Each stratum of the singular locus $\mathsf{Sing}(\overline{\mathsf{Jac}}_C)$ is indexed by a partial normalisation $\Sigma \longrightarrow C$. We prove that the Poincaré sheaf $\mathcal{P}_C$ restricted to each stratum can be expressed through the Poincaré sheaf $\mathcal{P}_\Sigma$, obtaining a relation between Fourier–Mukai transforms associated to $\mathcal{P}_C$ and $\mathcal{P}_\Sigma$. Our approach uses an intermediate geometry: the moduli space of parabolic modules of Bhosle and Cook, to intertwine sheaf data over the two curves. In a sequel, our formulae are used to study mirror symmetry in singular loci of Hitchin systems.

Contents

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1. Introduction

In seminal work of Mukai [Muk81], a categorification of the Fourier transform yields a derived equivalence $\Phi^A : D^b(A) \xrightarrow{\cong} D^b(A^\vee)$ between an abelian variety A and its dual $A^\vee := \mathsf{Pic}^0(A)$, with Φ^A satisfying properties that emulate the Fourier analysis of L^2-functions. Equivalences of this type, known as *Fourier–Mukai transforms*, have since become widespread in algebraic geometry.

A classical example of an autodual abelian variety is given by the Jacobian of a smooth projective curve, over which the Fourier–Mukai transform is a non-trivial derived autoequivalence. For certain singular curves C, the Jacobian may be replaced by the compactified Jacobian $\overline{\mathsf{Jac}}_C^0$ of rank one, degree zero torsion-free sheaves on C, the geometry of which is classically studied [AK80; Cap94; DSo79; Est01; EGK02; EK05; MRV17; Sim94]. Extending results of Esteves–Gagné–Kleiman [EGK02], the autoduality of $\overline{\mathsf{Jac}}_C^0$ for integral planar curves was proven by Arinkin [Ari13], and later generalised by Melo–Rapagnetta–Viviani [MRV17; MRV19a; MRV19b] to fine compactified Jacobians of reduced planar curves. These works establish an autoduality isomorphism $\overline{\mathsf{Jac}}_C^0 \cong \overline{\mathsf{Pic}}^0(\overline{\mathsf{Jac}}_C^0)$ and a Fourier–Mukai transform $\Phi^{\mathcal{P}_C} : D^b(\overline{\mathsf{Jac}}_C^0) \xrightarrow{\cong} D^b(\overline{\mathsf{Jac}}_C^0)$, both controlled by the *Poincaré sheaf* $\mathcal{P}_C$ on $\overline{\mathsf{Jac}}_C^0 \times \overline{\mathsf{Jac}}_C^0$; the universal family for autoduality as a moduli problem.

Let C be an integral nodal projective curve. This article and the sequels [FHHOa; FHHOb] describe a filtered approach to computing $\Phi^{\mathcal{P}_C}$ over the singular locus of $\overline{\mathsf{Jac}}_C^0$. The starting point for our work is the observation that the singular locus of $\overline{\mathsf{Jac}}_C$ is stratified by subvarieties of the form

$$\overline{\mathsf{Jac}}_\Sigma^0 \cong \overline{\mathsf{Jac}}_\Sigma^{-k} \hookrightarrow \mathsf{Sing}(\overline{\mathsf{Jac}}_C^0)\,,$$

where $\nu : \Sigma \longrightarrow C$ is a partial normalisation of C that resolves k nodes. These subvarieties have appeared in several contexts in algebraic geometry, for instance: the study of *parabolic modules* [Bho92; Coo93] and *presentation schemes* [EGK00]; the singularities of Hitchin systems via the normalisation of spectral curves [Ngô06; Ngô10; FGOP21; FP23; Hor22; MS21]; and Hitchin's study of 'critical loci' [Hit19], generically modeled on $\overline{\mathsf{Jac}}_\Sigma^{-k} \hookrightarrow \overline{\mathsf{Jac}}_C^0$ studied in families. This article, alongside [FHHOa; FHHOb] (see also Sections 1.2, 1.3), develops an approach that compares, in a systematic way, the interactions between these different appearances of the subvarieties $\overline{\mathsf{Jac}}_\Sigma^{-k} \hookrightarrow \overline{\mathsf{Jac}}_C^0$.

1.1. **Outline of results.** In this article we work over an integral nodal projective curve C with a fixed partial normalisation $\nu : \Sigma \longrightarrow C$ resolving k nodes. We consider two ways of passing between the derived categories of $\overline{\mathsf{Jac}}_\Sigma^0$ and $\overline{\mathsf{Jac}}_C^0$: firstly by pushforward along the map $\check{\nu} : \overline{\mathsf{Jac}}_\Sigma^{-k} \hookrightarrow \overline{\mathsf{Jac}}_C^0$, $M \longmapsto \nu_* M$; and secondly by convolution over an intermediate geometry: the parabolic modules of Bhosle and Cook [Bho92; Coo93], which consist of pairs (M, V) where $M \in \overline{\mathsf{Jac}}_\Sigma^0$ and V is a subsheaf of M restricted to the resolved nodes. The parameter V may be interpreted as gluing data on M, used to construct a point of $\overline{\mathsf{Jac}}_C^0$. The resultant fine moduli space PMod_ν^0 of such objects then fits into a (non-commutative) diagram

$$\begin{array}{ccccc} & & \mathsf{PMod}_\nu^0 & & \\ & \swarrow^{\dot{\nu}} & & \searrow^{\rho} & \\ \overline{\mathsf{Jac}}_\Sigma^0 & \xleftarrow[\tau_{k,y_0}]{} & \overline{\mathsf{Jac}}_\Sigma^{-k} & \xrightarrow[\check{\nu}]{} & \overline{\mathsf{Jac}}_C^0, \end{array} \tag{1.1}$$

such that $\dot{\nu}(M, V) = M$ is a projective fibre bundle and ρ is a finite map that defines a partial resolution of singularities. We take the translation $\tau_{k,y_0} := (\bullet) \otimes \mathcal{O}_\Sigma(k y_0)$, at fixed $y_0 \in \Sigma$, to trivialise Jac_Σ^{-k} as a torsor and correct for degree changes caused by pushforward along ν. The moduli PMod_ν^0 receives a tautological bundle $\mathcal{V}_\Sigma$, universally parameterising the subsheaves V. By treating the arrows $\overline{\mathsf{Jac}}_\Sigma^0 \longleftarrow \mathsf{PMod}_\nu^0 \longrightarrow \overline{\mathsf{Jac}}_C^0$ as a convolution diagram, with convolution kernel $\det(\mathcal{V}_\Sigma) \longrightarrow \mathsf{PMod}_\nu^0$, we define a pair of convolution functors

$$\Theta^{\mathcal{V}_\Sigma} := \rho_*(\det(\mathcal{V}_\Sigma) \otimes \dot{\nu}^*(\bullet)) : D^b(\overline{\mathsf{Jac}}_\Sigma^0) \longrightarrow D^b(\overline{\mathsf{Jac}}_C^0), \tag{1.2}$$

$$\widetilde{\Theta}^{\mathcal{V}_\Sigma} := (\mathrm{id} \times \rho)_*(\mathsf{q}_2^* \det(\mathcal{V}_\Sigma) \otimes (\mathrm{id} \times \dot{\nu})^*(\bullet)) : D^b(\overline{\mathsf{Jac}}_\Sigma^{-k} \times \overline{\mathsf{Jac}}_\Sigma^0) \longrightarrow D^b(\overline{\mathsf{Jac}}_\Sigma^{-k} \times \overline{\mathsf{Jac}}_C^0), \tag{1.3}$$

with $\widetilde{\Theta}^{\mathcal{V}_\Sigma}$ a relative version of $\Theta^{\mathcal{V}_\Sigma}$, and $\mathsf{q}_2 : \overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu^0 \longrightarrow \mathsf{PMod}_\nu^0$ the natural projection.

These functors are then used in the following comparison result, which identifies the relation between the respective Poincaré sheaves and Fourier–Mukai transforms over $\overline{\mathsf{Jac}}^0_\Sigma$ and $\overline{\mathsf{Jac}}^0_C$.

Theorem A. *Let C be an integral nodal projective curve, with partial normalisation $\nu : \Sigma \longrightarrow C$ resolving precisely k nodes.*

- *(Theorem 5.3). The Poincaré sheaves $\mathcal{P}_\Sigma$ on $\overline{\mathsf{Jac}}^0_\Sigma \times \overline{\mathsf{Jac}}^0_\Sigma$ and $\mathcal{P}_C$ on $\overline{\mathsf{Jac}}^0_C \times \overline{\mathsf{Jac}}^0_C$ are related by the isomorphism*
$$(\check{\nu} \times \mathrm{id})^* \mathcal{P}_C \cong \widetilde{\Theta}^{\mathcal{V}_\Sigma}((\tau_{k,y_0} \times \mathrm{id})^* \mathcal{P}_\Sigma).$$
- *(Theorem 5.7). The associated Fourier–Mukai transforms $\Phi^{\mathcal{P}_C} : D^b(\overline{\mathsf{Jac}}^0_C) \longrightarrow D^b(\overline{\mathsf{Jac}}^0_C)$ and $\Phi^{\mathcal{P}_\Sigma} : D^b(\overline{\mathsf{Jac}}^0_\Sigma) \longrightarrow D^b(\overline{\mathsf{Jac}}^0_\Sigma)$ admit, for every object $\mathcal{F}^\bullet \in D^b(\overline{\mathsf{Jac}}^{-k}_\Sigma)$, an isomorphism*
$$\Phi^{\mathcal{P}_C}(\check{\nu}_* \mathcal{F}^\bullet) \cong \Theta^{\mathcal{V}_\Sigma}(\Phi^{\mathcal{P}_\Sigma}(\tau_{k,y_0,*}\mathcal{F}^\bullet)).$$

Theorem A can be thought of as a natural transformation law for composing integral and convolution functors, emulating classical formulae in harmonic analysis. The convolution kernel $\det(\mathcal{V}_\Sigma)$ in our setup universally captures the variation of the subsheaves V in the moduli PMod^0_ν of parabolic modules, thus taking into account the singular geometry of $\overline{\mathsf{Jac}}^0_C$. In this way Theorem A describes the relationship between the three universal families on the moduli under consideration.

Abelian analogue. Theorem A admits an analogue for Fourier–Mukai transforms taken on either side of an embedding $f : A \longrightarrow B$ of abelian varieties. The transforms for sheaves pushed along f can be computed via the dual map $f^\vee : B^\vee \longrightarrow A^\vee$ and the natural transformation

$$\Phi^B \circ f_* \simeq (f^\vee)^* \circ \Phi^A; \tag{1.4}$$

see equation (11.3.3) of [Pol03]. Theorem A can be considered a singular variant of (1.4), where we take $\check{\nu} : \overline{\mathsf{Jac}}^{-k}_\Sigma \longrightarrow \overline{\mathsf{Jac}}^0_C$ to play the role of f. The matching of terms is not exact, stemming from the fact that the dual map $\check{\nu}^\vee = \nu^* : \mathsf{Jac}^0_C \longrightarrow \mathsf{Jac}^0_\Sigma$ does not extend to the compactified Jacobian $\overline{\mathsf{Jac}}^0_C$. Instead, we use $\dot{\nu} : \mathsf{PMod}^0_\nu \longrightarrow \overline{\mathsf{Jac}}^0_\Sigma$, a resolution of the rational map $\nu^* : \overline{\mathsf{Jac}}^0_C \dashrightarrow \overline{\mathsf{Jac}}^0_\Sigma$, and pullback along $\dot{\nu}$ in the convolution construction. This is why the pullback term in (1.4) is replaced by convolution in Theorem A.

Spin-valued formulae. By choosing a spin structure on PMod^0_ν, we derive a spin-valued version of Theorem A, expressed in terms of a so-called *spin-valued Wilson operator*
$$\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)} : D^b(\overline{\mathsf{Jac}}^0_\Sigma) \longrightarrow D^b(\overline{\mathsf{Jac}}^0_\Sigma).$$
See (5.17) for the definition of $\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}$. These functors satisfy $\mathbb{W}_{\Sigma,\otimes\mathsf{Exc}(\nu)} \simeq \mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)} \circ \mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}$ and so are genuine square roots of the usual (abelianised) Wilson functors, defined by a universal tensoral action (as in [DP12, §4.6]). The following is a spin-valued reformulation of Theorem A.

Corollary B. *(Corollary 5.12). Adopt the same hypothesis as in Theorem A. Additionally, let $\omega^{1/2}_{\mathsf{PMod}}$ be one of the spin structures on PMod^0_ν described in Corollary 3.10.*

- *One has isomorphisms of Poincaré sheaves*
$$(\check{\nu} \times \mathrm{id})^* \mathcal{P}_C \cong (\mathrm{id} \times \rho)_* \left(\mathsf{q}_2^* \left(\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^* \mathcal{U}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)} \right) \otimes (\tau_{k,y_0} \times \dot{\nu})^* \mathcal{P}_\Sigma \right).$$
- *For every $\mathcal{F}^\bullet \in D^b(\mathsf{Jac}^{-k}_\Sigma)$, one has isomorphisms of transformed sheaves*
$$\Phi^{\mathcal{P}_C}_{1\to 2}(\check{\nu}_* \mathcal{F}^\bullet) \cong \rho_* \left(\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^* (\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)} \circ \Phi^{\mathcal{P}_\Sigma}_{1\to 2}(\tau_{k,y_0,*}\mathcal{F}^\bullet)) \right).$$

The choice of $\omega^{1/2}_{\mathsf{PMod}}$ is auxiliary to Corollary B, in the sense that both displayed formulae are independent of which spin structure is chosen (see Remark 5.13). The appearance of square roots can be heuristically explained by the fact that $\nu : \Sigma \longrightarrow C$ is two-to-one over the locus of resolved singularities, so performing a 'Wilson loop' over resolved nodes of C lifts to a so-called *'Wilson spin-half loop'* over Σ (see [KW07, §6.1] for the physics of Wilson loops).

Prym-valued formulae. We also establish an analogue of Theorem A over the compactified Prym varieties

$$\overline{\mathsf{Prym}}^0_\Sigma \subset \overline{\mathsf{Jac}}^0_\Sigma, \qquad \overline{\mathsf{Prym}}^0_C \subset \overline{\mathsf{Jac}}^0_C.$$

These are defined as preimages of certain *norm maps*, associated to a pair of finite, ramified and commuting n-coverings $\beta_C : C \longrightarrow X$ and $\beta_\Sigma : \Sigma \longrightarrow X$ over a smooth projective curve X. Our Prymian results are based on the Fourier–Mukai transforms

$$\Psi^{\mathcal{R}_C}_{1\to 2} : D^b(\overline{\mathsf{Prym}}^0_C) \xrightarrow{\cong} D^b(\overline{\mathsf{Prym}}^0_C, \Gamma), \qquad \Psi^{\mathcal{R}_\Sigma}_{1\to 2} : D^b(\overline{\mathsf{Prym}}^0_\Sigma) \xrightarrow{\cong} D^b(\overline{\mathsf{Prym}}^0_\Sigma, \Gamma)\,,$$

established in [FHR25, Thm 4.8] and [GS22, Thm 4.7]. Here Γ is the group of n-torsion points in Jac_X, acting via tensor product, and $D^b(\overline{\mathsf{Prym}}^0_{\beta_\Sigma}, \Gamma)$ denotes the Γ-equivariant derived category. In Lemma 5.14, we construct a Prym variant of the diagram (1.1):

$$\begin{array}{ccccc} & & \mathsf{PMod}^0_\nu \times_{\overline{\mathsf{Jac}}^0_C} \overline{\mathsf{Prym}}^0_C & & \\ & \swarrow^{\mathring{\nu}} & & \searrow^{\mathring{\rho}} & \\ \overline{\mathsf{Prym}}^0_\Sigma & \xleftarrow[\mathring{\tau}]{} & \overline{\mathsf{Prym}}^{-k}_\Sigma & \xrightarrow[\check{\nu}]{} & \overline{\mathsf{Prym}}^0_C, \end{array}$$

compatible with restriction from the ambient compactified Jacobians. The following is the analogue of Theorem A for compactified Prym varieties.

Corollary C. *(Corollary 5.18). Let C be an integral nodal curve with arithmetic genus g and a partial normalisation $\nu : \Sigma \longrightarrow C$. Consider compatible n-coverings $C \longrightarrow X$ and $\Sigma \longrightarrow X$ and their compactified Pryms, as described above. Then, for every $\mathcal{F}^\bullet \in D^b(\overline{\mathsf{Prym}}^{-k}_\Sigma)$, one has the isomorphism*

$$\Psi^{\mathcal{R}_C}_{1\to 2}\left(R\check{\nu}_*\mathcal{F}^\bullet\right) \cong \mathring{\rho}_*\left(\iota_\nu^* \det(\mathcal{V}_\Sigma) \otimes \mathring{\nu}^* \Psi^{\mathcal{R}_\Sigma}_{1\to 2}(\mathring{\tau}_*\mathcal{F}^\bullet)\right).$$

In the remainder of the introduction we discuss future applications of the above results.

1.2. **Varying the normalisation.** By allowing the partial normalisations $\Sigma \longrightarrow C$ to vary, Theorem A can be used to make global statements on the autoduality of $\overline{\mathsf{Jac}}^0_C$. The strategy is to cover $\mathsf{Sing}(\overline{\mathsf{Jac}}^0_C)$ by an increasing filtration of subvarieties

$$\check{\nu}_n(\overline{\mathsf{Jac}}^{-n}_{\Sigma_n}) \subset \cdots \subset \check{\nu}_1(\overline{\mathsf{Jac}}^{-1}_{\Sigma_1}) \subset \mathsf{Sing}(\overline{\mathsf{Jac}}^0_C) \ \subset \overline{\mathsf{Jac}}^0_C, \tag{1.5}$$

obtained from chains of partial normalisation maps $\Sigma_n \xrightarrow{\nu_n} \cdots \longrightarrow \Sigma_1 \xrightarrow{\nu_1} C$. By applying Theorem A inductively along (1.5), one obtains new global descriptions of the Poincaré sheaf $\mathcal{P}_C$ and the associated Fourier–Mukai transforms. This will be taken up in the sequel [FHHOb].

1.3. **Higgs bundles and Langlands duality.** Our work is motivated by the study of Langlands duality on moduli of Higgs bundles, most directly in the form of the *Dolbeault geometric Langlands conjecture* of Donagi and Pantev [DP12]. Via a *classical limit* of the (global, unramified) de Rham geometric Langlands correspondence, the conjecture predicts, for Langlands self-dual group $G = \mathsf{GL}_n$, a derived autoequivalence on the moduli stack $\mathsf{Higgs}(X)$ of GL_n-Higgs bundles on a smooth projective curve X, linear with respect to the *Hitchin fibration* $\mathsf{Higgs}(X) \longrightarrow \mathcal{B}$ discovered in celebrated work of Hitchin [Hit87]. The base $\mathcal{B}$ parametrises *spectral covers* $C \longrightarrow X$, built from the eigenvalues of Higgs fields. The fibre at a point $C \in \mathcal{B}$, known as a *Hitchin fibre*, is isomorphic to $\overline{\mathsf{Jac}}^0_C$.

Arinkin's Fourier–Mukai transform $D^b(\overline{\mathsf{Jac}}^0_C) \longrightarrow D^b(\overline{\mathsf{Jac}}^0_C)$, understood in families, provides a solution to the GL_n-Dolbeault geometric Langlands conjecture over the cuspidal (i.e. elliptic) locus $\mathcal{B}^{cusp} \subset \mathcal{B}$ consisting of integral spectral curves. The conjecture is solved by a relative Fourier–Mukai equivalence

$$\Phi^{cusp} : D^b(\mathsf{Higgs}(X) \times_{\mathcal{B}} \mathcal{B}^{cusp}) \xrightarrow{\cong} D^b(\mathsf{Higgs}(X) \times_{\mathcal{B}} \mathcal{B}^{cusp}).$$

Our work provides a method for computing Φ^{cusp} within a particular locus of singular spectral curves: the equisingular locus $\mathcal{B}^{nodal,k} \subset \mathcal{B}^{cusp}$ of irreducible nodal curves with exactly k nodes. Indeed, our

Theorem A can be understood as a method for computing the functor $\Phi^{nodal,k} = \Phi^{cusp} \times_{\mathcal{B}} \mathcal{B}^{nodal,k}$ over each strata of the singular locus $\mathsf{Sing}(\mathsf{Higgs}(X) \times_{\mathcal{B}} \mathcal{B}^{nodal,k})$, with each strata defined by pushforward along $\nu : \Sigma \longrightarrow C$. Following the ideas outlined in Section 1.2, the sequel [FHHOb] describes how to extract more global conclusions, via an inductive computation of $\Phi^{nodal,k}$ in which all strata participate.

Langlands duality also manifests as mirror symmetry on the hyperkähler moduli space $\mathcal{M}_{\mathsf{Higgs}}(X)$ of semistable GL_n-Higgs bundles on the base curve X. In this context, the Hitchin fibration $\mathcal{M}_{\mathsf{Higgs}}(X) \longrightarrow \mathcal{B}$ is studied as a special Lagrangian torus fibration, where autoduality of the fibre spaces $\overline{\mathsf{Jac}}_C^0$ describes $\mathcal{M}_{\mathsf{Higgs}}(X)$ as SYZ mirror self-dual, in the sense of Strominger–Yau–Zaslow [SYZ01]. A classical heuristic in mirror symmetry is that mirror A-branes and B-branes are interchanged by Fourier–Mukai transforms along dual SYZ fibres. On $\mathcal{M}_{\mathsf{Higgs}}(X)$, this heuristic has been studied by many authors via the Fourier–Mukai transforms $D^b(\overline{\mathsf{Jac}}_C^0) \xrightarrow{\cong} D^b(\overline{\mathsf{Jac}}_C^0)$. Typically, one is constrained to the dense locus $\mathcal{B}^{smth} \subset \mathcal{B}$ of smooth spectral curves, where the transforms are classical. Mirror symmetry over the discriminant locus $\Delta = \mathcal{B} - \mathcal{B}^{smth}$ of singular and possibly non-reduced curves remains poorly understood in many aspects.

The results of this article allow us to run the above-described mirror symmetry heuristic on so-called *'critical loci'* of Hitchin [Hit19], generically modeled on the family of subvarieties $\nu_* : \mathsf{Jac}_\Sigma^{-k} \hookrightarrow \overline{\mathsf{Jac}}_C^0$, with ν the full normalisation and C varying in $\mathcal{B}^{nodal,k} \subset \Delta$. When the spectral cover $C \longrightarrow X$ has the maximal number of nodes, work of the first and fourth named authors with Peon-Nieto and Gothen [FGOP21; FGOP26] gave a description of the mirror pairing using the autoduality $\mathsf{Jac}_C \cong \mathsf{Pic}^0(\overline{\mathsf{Jac}}_C)$. With the Fourier–Mukai technology developed in this article, we give a twofold extension of their results: to any number of nodes, and to a more complete computation of the mirror pairing; using autoduality on the entirety of the Hitchin fibre $\overline{\mathsf{Jac}}_C^0$. This will appear in future work [FHHOa].

1.4. **Structure of the paper.** Section 2 recalls Arinkin's Fourier–Mukai transform for compactified Jacobians, specialised to the case of nodal curves. Section 3 introduces related moduli of divisors and sheaves used throughout the paper, including a non-standard algebraic space we call the *Chow–Hilbert space:* a 'mixture' of symmetric products and Hilbert schemes. We also recall the moduli spaces of parabolic modules, introduced by Bhosle and Cook, adapting their constructions to partial normalisations. Section 4 is dedicated to technical lemmas, comparing several universal families that encode information about Arinkin's Poincaré sheaf. This allows us, in Section 5, to prove our main result: a comparison between Poincaré sheaves and Fourier–Mukai transforms over C and its partial normalisations, as stated above in Theorem A. We then apply this result to derive spin-valued and Prym-valued reformulations, as stated above in Corollaries B and C.

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2. Autoduality for compactified Jacobians of nodal curves

2.1. **Compactified Jacobians.** Let C be an integral projective curve over $\mathbb{C}$ of arithmetic genus g_C, with $|\mathsf{Sing}(C)| \leq g_C$ nodal singularities. Given a sheaf $\mathcal{F}$ on C, we denote its degree by $\deg(\mathcal{F}) = \chi(\mathcal{F}) - \chi(\mathcal{O}_C)$. Consider the Jacobian Jac_C^d, of degree d line bundles on C. Let $\nu : \Sigma \longrightarrow C$ be a partial normalisation resolving $k \leq |\mathsf{Sing}(C)|$ nodes. Σ is again a nodal (possibly smooth) curve of genus $g_\Sigma = g_C - k$, and we have the exact sequence of groups

$$1 \longrightarrow (\mathbb{C}^\times)^k \longrightarrow \mathsf{Jac}_C^d \xrightarrow{\nu^*} \mathsf{Jac}_\Sigma^d \longrightarrow 1.$$

If $k = |\mathsf{Sing}(C)|$, then Jac_Σ^d is an abelian variety and, in this sense, Jac_C^d is said to be a (torsor for a) semi-abelian variety. Note that Jac_C^d is not proper, and the study of its compactifications is a classical problem in algebraic geometry [DSo79; AK80; Est01; Sim94]. We are primarily concerned with a compactification $\overline{\mathsf{Jac}}_C^d$ parametrising rank 1 and degree d torsion-free sheaves on C, rigidified by global scaling. For C smooth, a torsion-free sheaf is automatically locally free, thus one recovers the classical Jacobian as $\overline{\mathsf{Jac}}_C^d = \mathsf{Jac}_C^d$.

Notation 2.1. To streamline notation we abbreviate

$$\mathsf{Jac}_C := \mathsf{Jac}^0_C, \qquad \overline{\mathsf{Jac}}_C := \overline{\mathsf{Jac}}^0_C,$$

and similarly over partial normalisations Σ.

$\overline{\mathsf{Jac}}^d_C$ satisfies the following geometric properties [DSo79; AK80; MRV19a]:

- $\overline{\mathsf{Jac}}^d_C$ is a connected, reduced and irreducible projective scheme of dimension g, with locally complete intersection singularities;
- $\overline{\mathsf{Jac}}^d_C$ has trivial dualising sheaf;
- the smooth locus of $\overline{\mathsf{Jac}}^d_C$ coincides with the dense open subset Jac^d_C.

Let $\mathsf{Hilb}_{n,C}$ denote the punctual Hilbert scheme of C, parameterising zero-dimensional subschemes $D \subset C$ of length n. Associated to a smooth point $x_0 \in C$ is the *Abel–Jacobi map*

$$(2.1) \qquad \begin{array}{rccc} A_C: & \mathsf{Hilb}_{n,C} & \longrightarrow & \overline{\mathsf{Jac}}_C \\ & D & \longmapsto & \mathcal{I}_D^\vee(-nx_0), \end{array}$$

with $\mathcal{I}_D$ the ideal sheaf of the subscheme $D \subset C$ and $\mathcal{I}_D^\vee$ its dual.

We also consider the *curvilinear* Hilbert scheme $\mathsf{Hilb}^{cur}_{n,C}$, parameterising subschemes $D \in \mathsf{Hilb}_{n,C}$ that can be locally embedded into $\mathbb{A}^1$, i.e. D is locally isomorphic to the spectrum of $\mathbb{C}[z]/\langle z^{n_p}\rangle$ for some $n_p \in \mathbb{N}$. $\mathsf{Hilb}^{cur}_{n,C}$ is particularly useful for studying $\overline{\mathsf{Jac}}_C$ over nodal curves, due to the following result of Arinkin.

Proposition 2.2 (Proposition 4.5 of [Ari13])**.** *Let C be an integral nodal projective curve. There exists a sufficiently large integer $n \in \mathbb{N}$ such that the restriction of the associated Abel–Jacobi map*

$$\alpha_C = A_C\big|_{\mathsf{Hilb}^{cur}_{n,C}} : \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C,$$

is surjective.

Both $\overline{\mathsf{Jac}}^d_C$ and $\mathsf{Hilb}^{cur}_{n,C}$ are fine moduli spaces, giving rise to the following universal objects.

- The sheaf $\mathcal{U}_{d,C}$ on $C \times \overline{\mathsf{Jac}}^d_C$, universal for rank 1 and degree d torsion-free sheaves on C. The universal property is as follows: for every $\mathcal{F} \in \overline{\mathsf{Jac}}^d_C$, there is a universal isomorphism $\mathcal{U}_{d,C}|_{C\times\{\mathcal{F}\}} \cong \mathcal{F}$. We normalise $\mathcal{U}_{d,C}$ at a fixed point $x_0 \in C$, i.e. $\mathcal{U}_{d,C}|_{\{x_0\}\times\overline{\mathsf{Jac}}} \cong \mathcal{O}_{\overline{\mathsf{Jac}}^d_C}$. For $d = 0$, we simply write $\mathcal{U}_C = \mathcal{U}_{0,C}$.
- The universal divisor $\mathcal{D}_C \subset C \times \mathsf{Hilb}^{cur}_{n,C}$, universal for curvilinear subschemes.
- Along (a restriction of) the projection $h_C : C \times \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \mathsf{Hilb}^{cur}_{n,C}$, which over $\mathcal{D}$ is a finite map of degree n, we take the pushforward

$$(2.2) \qquad \mathcal{A}_{C,n} := h_{C,*}\mathcal{O}_{\mathcal{D}_C},$$

  which is a locally free sheaf on $\mathsf{Hilb}^{cur}_{n,C}$ of rank n.
- The Poincaré line bundle $P_C \longrightarrow \mathsf{Jac}_C \times \overline{\mathsf{Jac}}_C$, universal for topologically trivial line bundles on $\overline{\mathsf{Jac}}_C$. P_C can be constructed by applying determinant of cohomology to $\mathcal{U}_C$ [EK05].

2.2. **Fourier–Mukai transform of Arinkin.** We now review Arinkin's construction of the *Poincaré sheaf* $\mathcal{P}_C$ on $\overline{\mathsf{Jac}}_C \times \overline{\mathsf{Jac}}_C$ as a universal extension of the Poincaré bundle P_C. We give a simplified construction that suffices for nodal curves [Ari13, § 4.3.]. The simplification originates from the surjectivity of the Abel–Jacobi map stated in Proposition 2.2.

Consider the flag scheme, $\mathsf{Flag}_{n,C}$, parameterising length n filtrations

$$\emptyset = D_0 \subsetneq \cdots \subsetneq D_k \subset \cdots \subsetneq D_n,$$

of finite subschemes $D_i \in \mathsf{Hilb}_{i,C}$ in C. Let $\mathsf{Flag}^{cur}_{n,C}$ be the corresponding curvilinear flag scheme such that each D_i is a point of $\mathsf{Hilb}^{cur}_{i,C}$. $\mathsf{Hilb}^{cur}_{n,C}$ and $\mathsf{Flag}^{cur}_{n,C}$ are open subschemes of $\mathsf{Hilb}_{n,C}$ and

$\mathsf{Flag}_{n,C}$ respectively, as described by Arinkin [Ari13, Section 3]. Moreover, by Lemma 3.9 of [MRV19b], $\operatorname{codim}(\mathsf{Hilb}_{n,C} \setminus \mathsf{Hilb}^{cur}_{n,C}) \geq 2$ in $\mathsf{Hilb}_{n,C}$. The forgetful morphism

$$\psi_C : \mathsf{Flag}^{cur}_{n,C} \longrightarrow \mathsf{Hilb}^{cur}_{n,C}, \quad D_0 \subsetneq \cdots \subsetneq D_n \mapsto D_n,$$

is a finite flat morphism of degree $n!$ [Ari13, Proposition 3.5]. Moreover, we define

$$\sigma_C : \mathsf{Flag}^{cur}_{n,C} \longrightarrow C^n, \quad D_0 \subset \cdots \subset D_n \mapsto \mathsf{supp}(\mathsf{ker}(\mathcal{O}_{D_i} \longrightarrow \mathcal{O}_{D_{i-1}})_{i=1}^n).$$

The curvilinear flag scheme $\mathsf{Flag}^{cur}_{n,C}$ carries a natural action of the symmetric group $\mathfrak{S}_n$. Every divisor in $\mathbb{A}^1$ is the zero locus of some polynomial $(z - t_1) \cdots (z - t_n)$. Hence, there is a natural $\mathfrak{S}_n$-action that permutes t_i. This induces a $\mathfrak{S}_n$-action on flags of curvilinear subschemes. With respect to this action ψ_C is $\mathfrak{S}_n$-invariant and σ_C is a $\mathfrak{S}_n$-equivariant morphism using the permutation action on C^n.

Consider the diagram

$$\begin{array}{ccccc} \mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}^d_C & \xleftarrow{\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & \mathsf{Flag}^{cur}_{n,C} \times \overline{\mathsf{Jac}}^d_C & \xrightarrow{\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & C^n \times \overline{\mathsf{Jac}}^d_C \\ \downarrow{\scriptstyle \mathsf{p}_1} & & & & \\ \mathsf{Hilb}^{cur}_{n,C} & & & & \end{array}.$$

Further, denote by $\mathcal{U}^{\boxtimes_n}_{d,C} \longrightarrow C^n \times \overline{\mathsf{Jac}}^d_C$ the external tensor product with respect to the projections $C^n \times \overline{\mathsf{Jac}}_C \longrightarrow C \times \overline{\mathsf{Jac}}_C$ onto each factor. We then define the Cohen–Macaulay sheaf $\mathcal{G}_{d,C}$ on $\mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}^d_C$ by

$$\mathcal{G}_{d,C} := \left((\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})_* (\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \mathcal{U}^{\boxtimes_n}_{d,C} \right)^{sign} \otimes \mathsf{p}_1^* \det(\mathcal{A}_{C,n})^\vee. \tag{2.3}$$

Note that the pushforward sheaf under ψ_C inherits a $\mathfrak{S}_n$-action on the level of germs of sections. The superscript "sign" denotes the anti-invariant subsheaf, where $\mathfrak{S}_n$ acts by the sign representation.

The main result of Arinkin [Ari13] is as following descent statement for $\mathcal{G}_{0,C}$ along the Abel–Jacobi map.

Theorem 2.3 ([Ari13])**.** *There exists a maximal Cohen–Macaulay sheaf $\mathcal{P}_C$ over $\overline{\mathsf{Jac}}_C \times \overline{\mathsf{Jac}}_C$, called the* Poincaré sheaf, *which restricts to the line bundle P_C under the inclusion $\mathsf{Jac}_C \times \overline{\mathsf{Jac}}_C \cup \overline{\mathsf{Jac}}_C \times \mathsf{Jac}_C \hookrightarrow \overline{\mathsf{Jac}}_C \times \overline{\mathsf{Jac}}_C$. The Poincaré sheaf $\mathcal{P}_C$:*

- *is flat over both factors and symmetric under permutation of these factors;*
- *satisfies*

$$\mathcal{G}_{0,C} \cong (\alpha_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \mathcal{P}_C \otimes \mathsf{p}_2^* N, \tag{2.4}$$

  for some line bundle N on $\overline{\mathsf{Jac}}_C$, where $\mathsf{p}_2 : \mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C \longrightarrow \overline{\mathsf{Jac}}_C$ is the projection;
- *is the universal family for the moduli problem $\overline{\mathsf{Pic}}^0(\overline{\mathsf{Jac}}_C)$ of topologically trivial rank one torsion-free sheaves on $\overline{\mathsf{Jac}}_C$.*

Remark 2.4. *The fact that $\mathcal{G}_{0,C}$ and $(\alpha_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \mathcal{P}_C$ differ by the pullback of a line bundle is clarified in Proposition 4.12, Remark 4.13 and Lemma 4.14 of [MRV19b]. The twist has, however, no influence on Arinkin's results.*

Consider the natural projections

$$\begin{array}{ccccc} & & \overline{\mathsf{Jac}}_C \times \overline{\mathsf{Jac}}_C & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \overline{\mathsf{Jac}}_C & & & & \overline{\mathsf{Jac}}_C. \end{array}$$

The associated Fourier–Mukai transform with kernel $\mathcal{P}_C$ is defined to be

$$\begin{array}{cccc} \Phi^{\mathcal{P}_C}_{1\to 2} : & D^b\left(\overline{\mathsf{Jac}}_C\right) & \longrightarrow & D^b\left(\overline{\mathsf{Jac}}_C\right) \\ & \mathcal{F}^\bullet & \longmapsto & R\pi_{2,*}(\pi_1^* \mathcal{F}^\bullet \otimes \mathcal{P}_C). \end{array} \tag{2.5}$$

Moreover, the dual sheaf $\mathcal{P}_C^\vee$ defines the Fourier–Mukai transform

$$\begin{array}{rccc} \Phi_{2\to 1}^{\mathcal{P}_C^\vee}: & D^b\left(\overline{\mathsf{Jac}}_C\right) & \longrightarrow & D^b\left(\overline{\mathsf{Jac}}_C\right) \\ & \mathcal{F}^\bullet & \longmapsto & R\pi_{1,*}(\pi_2^*\mathcal{F}^\bullet \otimes \mathcal{P}_C^\vee). \end{array} \tag{2.6}$$

The following is a special case of [Ari13, Theorem C].

Theorem 2.5. *Let C be an integral projective curve with nodal singularities and arithmetic genus g. Then, $\Phi_{1\to 2}^{\mathcal{P}_C}$ is a derived equivalence, with quasi-inverse the g-shifted functor $\Phi_{2\to 1}^{\mathcal{P}_C^\vee}[g]$.*

2.3. **Varying the degree.** This section states a relation between the sheaves $\mathcal{G}_{0,C}$ and $\mathcal{G}_{d,C}$ that will be crucial in Section 5. We pass between the two sheaves via twists taken at the fixed smooth point $x_0 \in C$ over which the universal family $\mathcal{U}_{d,C}$ is normalised. Since C is irreducible, the choice of x_0 defines an isomorphism

$$\begin{array}{rccc} \tau_{d,x_0}: & \overline{\mathsf{Jac}}_C^{d'} & \longrightarrow & \overline{\mathsf{Jac}}_C^{d'+d} \\ & \mathcal{F} & \longmapsto & \mathcal{F}\otimes\mathcal{O}_C(dx_0), \end{array} \tag{2.7}$$

with inverse $\tau_{-d,x_0}: \overline{\mathsf{Jac}}_C^{d'+d} \longrightarrow \overline{\mathsf{Jac}}_C^{d'}$. The choice of normalisations on the universal sheaves $\mathcal{U}_{0,C} \longrightarrow C\times\overline{\mathsf{Jac}}_C$ and $\mathcal{U}_{d,C} \longrightarrow C\times\overline{\mathsf{Jac}}_C^d$ provides a universal equivalence

$$\mathcal{U}_{d,C} \cong (\mathrm{id}_C \times \tau_{-d,x_0})^*\mathcal{U}_{0,C}\otimes p_C^*\mathcal{O}_C(dx_0), \tag{2.8}$$

with $p_C : C\times\overline{\mathsf{Jac}}_C \longrightarrow C$ the projection to the first factor.

Given a point $x \in C$, the line bundle $\mathcal{O}_C(x)^{\boxtimes n} = \mathcal{O}_C(x)\boxtimes\cdots\boxtimes\mathcal{O}_C(x)$ over C^n is $\mathfrak{S}_n$-equivariant under permutations in C^n, and since tensorization is invariant under permutations, it follows that $\mathfrak{S}_n$ acts trivially on its fibres over the $\mathfrak{S}_n$-fixed points. By a result of Drézet–Narasimhan [DN89, Theorem 2.3], $\mathcal{O}_C(x)^{\boxtimes n}$ descends to a line bundle L on the symmetric product Sym^n_C;

$$\mathcal{O}_C(x)^{\boxtimes n} \cong \pi_C^* L, \tag{2.9}$$

with $\pi_C : C^n \longrightarrow \mathsf{Sym}^n_C$ the quotient map. To describe L, note that $\mathcal{O}_C(x)$ admits a section vanishing at x, thus L can be endowed with a section vanishing at any tuple containing x, i.e. vanishing over the subvariety of Sym^n_C given by the image of

$$\begin{array}{rccc} f_{C,x}: & \mathsf{Sym}^{n-1}_C & \longrightarrow & \mathsf{Sym}^n_C \\ & \sum_{i=1}^{n-1} x_i & \longmapsto & x+\sum_{i=1}^{n-1} x_i. \end{array}$$

Let

$$\begin{array}{rccc} \mathsf{chow}_C: & \mathsf{Hilb}^{cur}_{n,C} & \longrightarrow & \mathsf{Sym}^n_C \\ & D & \longmapsto & \sum_{p\in C}\mathsf{length}(\mathcal{O}_{D,p})p \end{array} \tag{2.10}$$

be the Chow morphism. On the universal family $\mathcal{D}_C \subset C\times\mathsf{Hilb}^{cur}_{n,C}$ of length n curvilinear subschemes we take the slice

$$\mathcal{D}_{C,x} = \mathcal{D}_C \cap \{x\}\times\mathsf{Hilb}^{cur}_{n,C},$$

parameterising curvilinear finite subschemes containing a point $x \in C$. Then

$$\widetilde{\mathcal{D}}_{C,x} := \mathsf{chow}_C(\mathcal{D}_C \cap \{x\}\times\mathsf{Hilb}^{cur}_{n,C}),$$

is a Cartier divisor in Sym^n_C, and is equal to the image of $f_{C,x}$. Therefore, we may conclude that

$$L \cong \mathcal{O}_{\mathsf{Sym}}(\widetilde{\mathcal{D}}_{C,x}), \tag{2.11}$$

with $\mathsf{Sym} = \mathsf{Sym}^n_C$. From $\mathcal{D}_{C,x} = \mathsf{chow}_C^{-1}\left(\widetilde{\mathcal{D}}_{C,x}\right)$ in $\mathsf{Hilb}^{cur}_{n,C}$ it follows that

$$\mathcal{O}_{\mathsf{Hilb}}(\mathcal{D}_{C,x}) \cong \mathsf{chow}_C^*\mathcal{O}_{\mathsf{Sym}}(\widetilde{\mathcal{D}}_{C,x}). \tag{2.12}$$

In particular, if C is smooth, then chow_C is an isomorphism and we may identify $\mathcal{O}_{\mathsf{Sym}}(\widetilde{\mathcal{D}}_{C,x}) = \mathcal{O}_{\mathsf{Hilb}}(\mathcal{D}_{C,x})$.

Let $p_C : C \times \overline{\mathsf{Jac}}_C \longrightarrow C$ and $h_C : C \times \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \mathsf{Hilb}^{cur}_{n,C}$ be the natural projections. The following universal relation on $C \times \mathsf{Hilb}^{cur}_{n,C}$ is well-known for smooth curves; see for example [Sch63, Proposition 9].

Lemma 2.6. *Let C be an integral projective curve, either nodal or smooth. The universal sheaf $\mathcal{U}_C$ and the dual of the ideal sheaf of the universal divisor $\mathcal{D}_C \subset C \times \mathsf{Hilb}^{cur}_{n,C}$ are related by the isomorphism*

$$\mathcal{I}^{\vee}_{\mathcal{D}_C} \cong (\mathrm{id} \times \alpha_C)^* \left(\mathcal{U}_C \otimes p_C^* \mathcal{O}_C(nx_0)\right) \otimes h_C^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}) \, .$$

Proof. Denote $\mathcal{D}_{C,x_0} = \mathcal{D}_C \cap \{x_0\} \times \mathsf{Hilb}^{cur}_{n,C}$ and recall the Abel-Jacobi map $\alpha_C : \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$ defined in (2.1). Consider the rank 1 torsion-free sheaf

$$(\mathrm{id} \times \alpha_C)^* \left(\mathcal{U}_C \otimes p_C^* \mathcal{O}_C(nx_0)\right),$$

defined on $C \times \mathsf{Hilb}^{cur}_{n,C}$. For fixed $D \in \mathsf{Hilb}^{cur}_{n,C}$, its restriction to $C \times \{D\}$ is given by

$$\mathcal{I}^{\vee}_D(-nx_0) \otimes \mathcal{O}_C(nx_0) = \mathcal{I}^{\vee}_D.$$

Thus $\mathcal{I}^{\vee}_{\mathcal{D}_C}$ and $(\mathrm{id} \times \alpha_C)^* \left(\mathcal{U}_C \otimes p_C^* \mathcal{O}_C(nx_0)\right)$ are equivalent families of rank 1 torsion-free sheaves on $\mathsf{Hilb}^{cur}_{n,C}$. Therefore, they differ by the pullback of a line bundle $\mathcal{L}$ on $\mathsf{Hilb}^{cur}_{n,C}$, i.e.

$$\mathcal{I}^{\vee}_{\mathcal{D}_C} \cong (\mathrm{id} \times \alpha_C)^* \left(\mathcal{U}_C \otimes p_C^* \mathcal{O}_C(nx_0)\right) \otimes h_C^* \mathcal{L}$$

Restricting to $\{x_0\} \times \mathsf{Hilb}^{cur}_{n,C}$, we obtain

$$\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}) \cong \alpha_C^*(\mathcal{U}_C|_{\{x_0\} \times \overline{\mathsf{Jac}}_C}) \otimes \mathcal{L} = \mathcal{L},$$

where we used that the universal sheaf $\mathcal{U}_C$ is normalised at x_0. This proves the formula. □

We now provide a relation between the sheaves $\mathcal{G}_{d,C}$ and $\mathcal{G}_{0,C}$, defined over $\mathsf{Hilb}_{n,C} \times \overline{\mathsf{Jac}}^d_C$ and $\mathsf{Hilb}_{n,C} \times \overline{\mathsf{Jac}}_C$ respectively, that will be used in Proposition 5.1. Let $\mathcal{G}^{cur}_{d,C}$ and $\mathcal{G}^{cur}_{0,C}$ be the restrictions to $\mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}^d_C$ and $\mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C$. Let $\mathcal{O}_{\mathsf{Hilb}}(\mathcal{D}_{C,x_0})^d$ be the line bundle over $\mathsf{Hilb}^{cur}_{n,C}$ obtained by taking d-times the tensor product of the sheaf defined in (2.12) over the point $x_0 \in C$. Let $\mathsf{p}_1 : \mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C \longrightarrow \mathsf{Hilb}^{cur}_{n,C}$ be the projection and recall the tensorization isomorphism $\tau_{-d,x_0} : \overline{\mathsf{Jac}}^d_C \longrightarrow \overline{\mathsf{Jac}}_C$ from (2.7).

Lemma 2.7. *In the above notation, there exists an isomorphism of sheaves*

$$\mathcal{G}^{cur}_{d,C} \cong (\mathrm{id}_{\mathsf{Hilb}_C} \times \tau_{-d,x_0})^* \left(\mathcal{G}^{cur}_{0,C} \otimes \mathsf{p}_1^* \mathcal{O}_{\mathsf{Hilb}}(\mathcal{D}_{C,x_0})^d\right).$$

In particular, $\mathcal{G}^{cur}_{d,C}$ is a maximal Cohen–Macaulay sheaf.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}^d_C & \xleftarrow{\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & \mathsf{Flag}^{cur}_{n,C} \times \overline{\mathsf{Jac}}^d_C & \xrightarrow{\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & C^n \times \overline{\mathsf{Jac}}^d_C \\
{\scriptstyle \mathrm{id}_{\mathsf{Hilb}_C} \times \tau_{-d,x_0}} \downarrow \cong & & \cong \downarrow {\scriptstyle \mathrm{id}_{\mathsf{Flag}} \times \tau_{-d,x_0}} & & \cong \downarrow {\scriptstyle \mathrm{id}_{C^n} \times \tau_{-d,x_0}} \\
\mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C & \xleftarrow[\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}]{} & \mathsf{Flag}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C & \xrightarrow[\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}]{} & C^n \times \overline{\mathsf{Jac}}_C.
\end{array}$$

By plugging (2.8) into (2.3), applying functoriality around the right square and base change around the left square, one has an isomorphism

$$\mathcal{G}^{cur}_{d,C} \cong (\mathrm{id}_{\mathsf{Hilb}_C} \times \tau_{-d,x_0})^* \left((\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})_* (\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \left(\mathcal{U}_{0,C} \otimes p_C^* \mathcal{O}_C(dx_0)\right)^{\boxtimes n} \right)^{sign} \otimes \mathsf{p}_1^* \det(\mathcal{A}_{C,n})^{\vee}.$$

Here we made use of the fact that the pullback under the isomorphism $\mathrm{id}_{\mathsf{Hilb}_C} \times \tau_{-d,x_0}$ is invariant under the symmetric group action. Considering the commutative diagram

$$\begin{array}{ccc}
C^n \times \overline{\mathsf{Jac}}_C & \xrightarrow{\mathsf{q}_i \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & C \times \overline{\mathsf{Jac}}_C \\
{\scriptstyle \mathsf{q}_{C^n}} \downarrow & & \downarrow {\scriptstyle p_C} \\
C^n & \xrightarrow[\mathsf{q}_i]{} & C,
\end{array}$$

and recalling from (2.9) and (2.11) that $\mathcal{O}(dx_0)^{\boxtimes n}$ is isomorphic to $\pi_C^*\mathcal{O}_{\mathsf{Sym}}(\widetilde{\mathcal{D}}^{cur}_{C,x_0})^d$, we obtain

$$\mathcal{G}^{cur}_{d,C} \cong (\mathrm{id}_{\mathsf{Hilb}_C} \times \tau_{-d,x_0})^* \left((\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})_* (\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \left(\mathcal{U}_{0,C}^{\boxtimes n} \otimes \mathsf{q}_{C^n}^* \pi_C^* \mathcal{O}_{\mathsf{Sym}}(\widetilde{\mathcal{D}}^{cur}_{C,x_0})^d \right) \right)^{sign} \otimes \mathsf{p}_1^* \det(\mathcal{A}_{C,n})^\vee.$$

Next, we apply functoriality with respect to

$$\begin{array}{ccc} \mathsf{Flag}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C & \xrightarrow{\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & C^n \times \overline{\mathsf{Jac}}_C \\ \downarrow{\scriptstyle \mathsf{q}_{\mathsf{Flag}}} & & \downarrow{\scriptstyle \mathsf{q}_{C^n}} \\ \mathsf{Flag}^{cur}_{n,C} & \xrightarrow[\sigma_C]{} & C^n, \end{array} \qquad \begin{array}{ccc} \mathsf{Flag}^{cur}_{n,C} & \xrightarrow{\sigma_C} & C^n \\ \downarrow{\scriptstyle \psi_C} & & \downarrow{\scriptstyle \pi_C} \\ \mathsf{Hilb}^{cur}_{n,C} & \xrightarrow[\mathsf{chow}_C]{} & \mathsf{Sym}^n_C, \end{array}$$

and

$$\begin{array}{ccc} \mathsf{Hilb}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C & \xleftarrow{\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}} & \mathsf{Flag}^{cur}_{n,C} \times \overline{\mathsf{Jac}}_C \\ \downarrow{\scriptstyle \mathsf{p}_1} & & \downarrow{\scriptstyle \mathsf{q}_{\mathsf{Flag}}} \\ \mathsf{Hilb}^{cur}_{n,C} & \xleftarrow[\psi_C]{} & \mathsf{Flag}^{cur}_{n,C}, \end{array}$$

to observe that

$$\mathcal{G}^{cur}_{d,C} \cong (\mathrm{id}_{\mathsf{Hilb}_C} \times \tau_{-d,x_0})^* \left((\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})_* \left((\sigma_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \mathcal{U}_{0,C}^{\boxtimes n} \otimes (\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}})^* \mathsf{p}_1^* \mathcal{O}_{\mathsf{Hilb}}(\mathcal{D}_{C,x_0})^d \right) \right)^{sign} \otimes \mathsf{p}_1^* \det(\mathcal{A}_{C,n})^\vee,$$

where we recall that $\mathcal{O}_{\mathsf{Hilb}}(\mathcal{D}_{C,x_0})^d \cong \mathsf{chow}_C^* \mathcal{O}_{\mathsf{Sym}}(\widetilde{\mathcal{D}}^{cur}_{C,x_0})^d$. The statement follows after applying projection formula with respect to $\psi_C \times \mathrm{id}_{\overline{\mathsf{Jac}}}$ followed by (2.3) for $d = 0$. □

3. Chow–Hilbert spaces and parabolic modules

Over a nodal curve C and a partial normalisation $\nu : \Sigma \longrightarrow C$, this section establishes preliminary results on fine moduli spaces of sheaves and divisors. We introduce the moduli spaces of *parabolic modules* associated to ν and discuss the interplay with the curvilinear Hilbert scheme over C. Throughout we refer to the following class of divisors related to the normalised nodes.

Definition 3.1.

- The *resolved singular locus* $\mathsf{RSing}(\nu) \subset \mathsf{Sing}(C)$ is the divisor consisting of nodes resolved by ν.
- The *non-resolved singularities* are denoted by $\mathsf{URSing}(\nu) := \mathsf{Sing}(C) \setminus \mathsf{RSing}(\nu)$.
- The *exceptional divisor* $\mathsf{Exc}(\nu) \subset \Sigma$ is the reduced divisor $\nu^{-1}(\mathsf{RSing}(\nu))$.

It will also be convenient for us to fix an ordering $\mathsf{Sing}(C) = \{b_1, \dots, b_k, \dots, b_n\}$ on the nodes such that

$$\mathsf{RSing}(\nu) = b_1 + \cdots + b_k, \quad \mathsf{Exc}(\nu) = b_1^+ + b_1^- + \cdots + b_k^+ + b_k^-,$$

where, for each $i = 1, \dots, k$, the divisor $b_i^+ + b_i^-$ is given by $\nu^{-1}(b_i)$.

3.1. **Chow–Hilbert spaces.** In this section we introduce the so-called *Chow–Hilbert space* H_ν associated to a partial normalisation $\nu : \Sigma \longrightarrow C$. Roughly speaking, the Chow–Hilbert space resembles the Hilbert scheme away from $\mathsf{RSing}(\nu)$ and the symmetric product over $\mathsf{RSing}(\nu)$. This space will play an important role in our later analysis of partial normalisations. The construction of the space H_ν is as follows. Recall the Chow map $\mathsf{chow}_C : \mathsf{Hilb}_{n,C} \longrightarrow \mathsf{Sym}^n_C$ and consider the subvariety $R_\nu \subset \mathsf{Hilb}_{n,C} \times_{\mathsf{Sym}^n_C} \mathsf{Hilb}_{n,C}$ given by pairs of ideals whose localization outside the resolved singular locus $\mathsf{RSing}(\nu)$ coincide:

$$R_\nu = \{(\mathcal{I}_D, \mathcal{I}_{D'}) \in \mathsf{Hilb}_{n,C} \times_{\mathsf{Sym}^n_C} \mathsf{Hilb}_{n,C} \,|\, \mathcal{I}_{D,x} = \mathcal{I}_{D',x} \text{ for all } x \in C \setminus \mathsf{RSing}(\nu)\}. \tag{3.1}$$

Observe that $\mathcal{I}_{D,x} = \mathcal{I}_{D',x}$ whenever x lies outside the singular locus $\mathsf{Sing}(C)$, so the defining condition of R_ν needs to be imposed only on the finite set $\mathsf{URSing}(\nu)$.

The subvariety R_ν naturally provides an equivalence relation on the Hilbert scheme $\mathsf{Hilb}_{n,C}$ given by $\mathcal{I}_D \simeq \mathcal{I}_{D'}$ if and only if $(\mathcal{I}_D, \mathcal{I}_{D'})$ is a point in R_ν. By identifying $\mathsf{Hilb}_{n,C}$ with its sheaf of points over the étale site, we define the *Chow–Hilbert space associated to* ν to be the quotient sheaf

$$\mathsf{H}_\nu := \mathsf{Hilb}_{n,C} / R_\nu .$$

The Chow-Hilbert space H_ν receives two natural maps: the quotient map

$$c : \mathsf{Hilb}_{n,C} \longrightarrow \mathsf{H}_\nu, \tag{3.2}$$

and a descent of the map $\mathsf{chow}_C : \mathsf{Hilb}_{n,C} \longrightarrow \mathsf{Sym}^n_C$, invariant under R_ν,

$$\mathsf{H}_\nu \longrightarrow \mathsf{Sym}^n_C. \tag{3.3}$$

Lemma 3.2. H_ν *is an algebraic space.*

Proof. Consider the pair of Zariski-open subschemes $C_1 := C \setminus \mathsf{URSing}(\nu)$ and $C_2 := C \setminus \mathsf{RSing}(\nu)$. We define an atlas

$$\bigcup_{k=0}^{n} \mathsf{Sym}^k_{C_1} \times \mathsf{Hilb}_{n-k,C_2} \longrightarrow \mathsf{H}_\nu, \tag{3.4}$$

where each chart is defined on S-points by

$$\mathsf{Sym}^k_{C_1}(S) \times \mathsf{Hilb}_{n-k,C_2}(S) \longrightarrow \mathsf{H}_\nu(S),$$
$$(D_1, D_2) \longmapsto [D_1 \cup D_2].$$

On geometric points, the atlas represents the possible decompositions of a divisor $D \subset C$ into disjoint components $D = D_1 \sqcup D_2$ such that D_1 contains $D \cap \mathsf{RSing}(\nu)$ and D_2 contains $D \cap \mathsf{URSing}(\nu)$. The fibre at $[D] \in \mathsf{H}_\nu(\mathsf{Spec}(\mathbb{C}))$ is finite, of cardinality between 1 and 2^n; the lower bound is obtained when $[D]$ is a single point, and the upper bound when $[D]$ is supported on n points in $C \setminus \mathsf{Sing}(C)$). Each chart is therefore étale and so (3.4) defines an étale presentation on H_ν. □

Remark 3.3. *The sheaf* H_ν *also represents a complex analytic space, where in the analytic topology, one can separate the nodes of* C *by pair-wise disjoint analytic open neighborhoods* $D_b \subset C$*, modeled on the standard open charts* $\{(x, y) \in \mathbb{C}^2 \mid xy = 0, |x|, |y| < \varepsilon\}$ *One can then represent* H_ν *as the analytic space obtained as a gluing of analytic Hilbert schemes over the neighborhoods* D_b.

Remark 3.4. *When* $\nu : \Sigma \longrightarrow C$ *is the full normalisation,* H_ν *is the symmetric product* Sym^n_C *and the quotient* $\mathsf{Hilb}_C \longrightarrow \mathsf{H}_\nu$ *coincides with the chow map in* (2.10).

3.2. **Moduli of parabolic modules.** A key ingredient in our work will be the moduli of *parabolic modules* [Reg80; Bho92; Coo93; Coo98; GO13; FGOP21], used to intertwine sheaf data over Σ and C. The existing literature concerning these objects is still somewhat limited, having been studied over an integral curve C with only simple singularities (i.e. of type ADE). We specialise to the case where C has simple nodal singularities (i.e. of type A_1).

A key reason for introducing parabolic modules is the observation that the map $\hat{\nu} : \mathsf{Jac}^d_C \longrightarrow \mathsf{Jac}^d_\Sigma$ acting via the pullback $L \mapsto \nu^* L$ does not extend to the compactifications by rank one torsion-free sheaves. The moduli of parabolic modules, introduced by Bhosle and Cook [Bho92; Coo93; Coo98], defines a compactification of Jac^d_C where $\hat{\nu}$ does extend. We define a particular case of parabolic modules living over the exceptional divisor $\mathsf{Exc}(\nu) = b_1^\pm + \cdots + b_k^\pm \subset \Sigma$ consisting of nodes resolved by a fixed partial normalisation $\nu : \Sigma \longrightarrow C$.

Definition 3.5. Let $\nu : \Sigma \longrightarrow C$ be as above. A rank 1 and degree d *parabolic module* for ν is a pair (M, V) where M is a degree d rank 1 torsion-free sheaf on Σ and V is a subsheaf of $M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)}$ such that

$$V = V_1 \oplus \cdots \oplus V_k,$$

with V_i a 1-dimensional vector subspace of $M \otimes \mathcal{O}_{\{b_i^+, b_i^-\}} \cong M_{b_i^+} \oplus M_{b_i^-}$, where $M_{b_i^\pm} := M \otimes \mathcal{O}_{b_i^\pm}$.

Remark 3.6.

(1) *The more general notion of parabolic modules associated to other kinds of singularities is defined by Cook [Coo93], where the subspaces V_i are allowed to be higher dimensional. In this situation, one needs to impose that V_i is an $\mathcal{O}_{C,b_i}$-submodule of $M \otimes \mathcal{O}_{\{b_i^+, b_i^-\}}$ via pushforward under ν i.e. via the inclusion $\mathcal{O}_C \hookrightarrow \nu_* \mathcal{O}_\Sigma$.*

(2) *In the case where C has only $k = 1$ node these objects were first considered by Bhosle in [Bho92], who named them 'generalised parabolic bundles'. The case of various simple nodes is an easy generalization of Bhosle's definition, so our setup fits that of [Bho92].*

Denote by PMod_ν^d the fine moduli space of rank 1 and degree d parabolic modules for $\nu : \Sigma \longrightarrow C$. We refer again to [Coo93; Coo98; Bho92] for its construction.

By [Coo98, Theorem 1] or [Bho92, Theorem 3], there is a finite morphism

$$\begin{array}{rccc} \rho: & \mathsf{PMod}_\nu^d & \longrightarrow & \overline{\mathsf{Jac}}_C^d \\ & (M, V) & \longmapsto & \mathcal{F} \end{array}, \tag{3.5}$$

where $\mathcal{F}$ is defined by the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \nu_* M \longrightarrow \nu_* \big(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} / V \big) \longrightarrow 0, \tag{3.6}$$

and the surjection is defined by the composition $\nu_* M \longrightarrow \nu_* \big(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} \big) \longrightarrow \nu_* \big(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} / V \big)$. Note that, for each i, $\dim(\nu_*(M \otimes \mathcal{O}_{\{b_i^+, b_i^-\}} / V_i)) = 1$, hence

$$\nu_* \big(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} / V \big) \cong \mathcal{O}_{\mathsf{RSing}(\nu)}.$$

Since the quotient $\nu_*(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} / V)$ is an $\mathcal{O}_C$-module and $\nu_* M \longrightarrow \nu_* \big(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} / V \big)$ is a morphism of $\mathcal{O}_C$-modules, $\mathcal{F}$ inherits an $\mathcal{O}_C$-module structure too. In addition, $\deg(\nu_* M) = d + k$ and $\dim \big(\nu_* \big(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)} / V \big) \big) = k$, thus indeed $\deg(\mathcal{F}) = d$.

Let ρ_0 denote the restriction of ρ to $\rho^{-1}(\mathsf{Jac}_C^d)$. By [Coo98, Theorem 1] or [Bho92, Theorem 3], we have an isomorphism

$$\rho_0 : \rho^{-1}(\mathsf{Jac}_C^d) \xrightarrow{\cong} \mathsf{Jac}_C^d, \tag{3.7}$$

and so Jac_C^d can be seen as a dense open subspace of PMod_ν^d. Moreover, PMod_ν^d is proper and therefore a compactification of Jac_C^d. It is however different from the compactification $\overline{\mathsf{Jac}}_C^d$, for it admits a natural projection

$$\begin{array}{rccc} \dot{\nu}: & \mathsf{PMod}_\nu^d & \longrightarrow & \overline{\mathsf{Jac}}_\Sigma^d \\ & (M, V) & \longmapsto & M, \end{array} \tag{3.8}$$

extending the pullback morphism $\hat{\nu}$, via the isomorphism (3.7).

We then have the following diagram summarising the previous discussion:

$$\begin{array}{ccccc} & \rho^{-1}(\mathsf{Jac}_C^d) & \hookrightarrow & \mathsf{PMod}_\nu^d & \\ \swarrow^{\cong} & & \swarrow^{\rho} & & \searrow^{\dot{\nu}} \\ \mathsf{Jac}_C^d & \hookrightarrow & \overline{\mathsf{Jac}}_C^d & & \overline{\mathsf{Jac}}_\Sigma^d. \end{array}$$

The geometry of PMod_ν^d and ρ is particularly simple in our case and appears in [Bho92] in the case of a curve with a single node. We provide a proof of the following result for completeness. Recall from Definition 3.1 that we have attached the divisors

$$\mathsf{RSing}(\nu) = b_1 + \cdots + b_k, \qquad \mathsf{Exc}(\nu) = b_1^+ + b_1^- + \cdots + b_k^+ + b_k^-, \qquad \nu^{-1}(b_i) = b_i^+ + b_i^-,$$

over C and Σ, associated to the partial normalisation $\nu : \Sigma \longrightarrow C$.

Lemma 3.7.

(1) *The morphism $\dot{\nu} : \mathsf{PMod}_\nu^d \longrightarrow \overline{\mathsf{Jac}}_\Sigma^d$ from (3.8) is a $(\mathbb{P}^1)^k$-bundle.*

(2) The morphism $\rho : \mathsf{PMod}^d_\nu \longrightarrow \overline{\mathsf{Jac}}^d_C$ *from* (3.5) *is a partial resolution of singularities, i.e. it is a proper birational morphism which is an isomorphism on an open dense subset of* PMod^d_ν*, containing* $\rho^{-1}(\mathsf{Jac}^d_C)$*.*

(3) Two points of PMod^d_ν *have the same image*

$$\rho(M,V) = \rho(M',V'),$$

if and only if

$$M' = M \otimes J, \tag{3.9}$$

where J *is the line bundle*

$$J = \mathcal{O}_\Sigma(b_1^- - b_1^+)^{a_1} \otimes \cdots \otimes \mathcal{O}_\Sigma(b_k^- - b_k^+)^{a_k},$$

with $a_i \in \{-1,0,1\}$*, such that for those* l *with* $a_l = 1$*, those* m *with* $a_m = -1$ *and those* n *with* $a_n = 0$*, we have*

$$V_l = M \otimes \mathcal{O}_{b_l^+} \quad \text{and} \quad V'_l = M' \otimes \mathcal{O}_{b_l^-}, \tag{3.10}$$

$$V_m = M \otimes \mathcal{O}_{b_m^-} \quad \text{and} \quad V'_m = M' \otimes \mathcal{O}_{b_m^+} \tag{3.11}$$

and

$$V_n = V'_n. \tag{3.12}$$

(4) If $\mathcal{F} \in \overline{\mathsf{Jac}}^d_C$ *is not locally free at* $\ell \in \{0,\dots,k\}$ *points of* $\mathsf{RSing}(\nu)$*, then* $\rho^{-1}(\mathcal{F}) \cong \mathbb{Z}_2^\ell$*.*

Proof. (1) Follows directly from the definition of a parabolic module.

For (2), the surjectivity of ρ can be proven as in [FGOP21, Lemma 5.9]. Moreover, as ρ is an isomorphism on the smooth locus $\mathsf{Jac}_C \subset \overline{\mathsf{Jac}}_C$, any two different points (M,V) and (M',V') with the same image under ρ will be mapped to the singular locus of $\overline{\mathsf{Jac}}_C$. Hence, ρ is a partial resolution of singularities.

For (3), let us first consider that the stalk of $\mathcal{F} := \rho(M,V)$ at $b \in \mathsf{Sing}(C)$ is the kernel of a map $\mathcal{O}_{\Sigma,b^+} \oplus \mathcal{O}_{\Sigma,b^-} \longrightarrow \mathcal{O}_b$. If $\mathcal{F}_b$ is not locally-free at b, then

$$\mathcal{F}_b \cong \mathcal{O}_{\Sigma,b^+} \oplus \mathcal{O}_\Sigma(-b^-)_{b^-}, \tag{3.13}$$

if V is locally $\mathcal{O}_{\Sigma,b^+}$, or

$$\mathcal{F}_b \cong \mathcal{O}_\Sigma(-b^+)_{b^+} \oplus \mathcal{O}_{\Sigma,b^-}, \tag{3.14}$$

when V is given by $\mathcal{O}_{\Sigma,b^-}$. The previous distinction is redundant since one can build the isomorphism of $\mathcal{O}_{C,b}$-modules between (3.13) and (3.14), by means of a meromorphic section of $\mathcal{O}_\Sigma(b^- - b^+)$ with divisor $b^- - b^+$.

It then follows that, given (M,V), (M',V') and J satisfying (3.9)–(3.12), one obtains an isomorphism between the sheaves $\mathcal{F} = \rho(M,V) = \ker\left(\nu_* M \longrightarrow \nu_*\left(M \otimes \mathcal{O}_{\mathsf{Exc}(\nu)}/V\right)\right)$ and $\mathcal{F}' := \rho(M',V') = \ker\left(\nu_* M' \longrightarrow \nu_*\left(M' \otimes \mathcal{O}_{\mathsf{Exc}(\nu)}/V'\right)\right)$ by means of a meromorphic section of J having simple zeros on the b_i^-'s and on the b_j^+'s, and having simple poles on the b_i^+'s and on the b_j^-'s.

Assuming that $\mathcal{F} \cong \mathcal{F}'$, one has, for every point $b_i \in \mathsf{RSing}(\nu)$, that $\mathcal{F}$ and $\mathcal{F}'$ are either locally-free or their stalks are described as (3.13) or as (3.14). If both have the same local description we set $a_i = 0$. If not, we set $a_i = 1$ if we need an isomorphism given by a meromorphic section vanishing on b_i^+ or $a_i = -1$ in the remaining case. Repeating this construction for every point of $\mathsf{RSing}(\nu)$, we obtain J satisfying (3.9) and the conditions (3.10), (3.11) and (3.12) hold.

Finally for (4), given a sheaf $\mathcal{F} \in \overline{\mathsf{Jac}}^d_C$ which is not locally free at $\{b_{j_1},\dots,b_{j_\ell}\} \subset \mathsf{RSing}(\nu)$, one immediately sees that its preimages under ρ are of the form (M,V) with V_{j_i} of the form $M \otimes \mathcal{O}_{b^+_{j_i}}$ or $M \otimes \mathcal{O}_{b^-_{j_i}}$. The statement follows from this observation. □

We now study the universal families associated to the fine moduli spaces that have been previously introduced. Fix a smooth geometric point $x_0 \in C$ and the unique preimage $y_0 = \nu^{-1}(x_0) \in \Sigma$. Let

$$\mathcal{U}_\Sigma \longrightarrow \Sigma \times \overline{\mathsf{Jac}}^d_\Sigma,$$

be the universal sheaf for $\overline{\mathsf{Jac}}^d_\Sigma$, normalised at y_0. Thus $\mathcal{U}_\Sigma$ is the universal family such that

$$\mathcal{U}_\Sigma|_{\{y_0\}\times\overline{\mathsf{Jac}}} \cong \mathcal{O}_{\overline{\mathsf{Jac}}}.$$

We denote the restriction of the universal bundle to a slice by $\mathcal{U}_{\Sigma,y} := \mathcal{U}_\Sigma|_{\{y\}\times\overline{\mathsf{Jac}}}$, for any point $y \in \Sigma$. Whenever y is non-singular, $\mathcal{U}_{\Sigma,y}$ is a line bundle. In particular, this is true for $y \in \mathsf{Exc}(\nu)$.

For each of the resolved singular points $\{b_1, \dots, b_k\} = \mathsf{RSing}(\nu)$, consider the rank two vector bundle

$$\dot\nu^*\mathcal{U}_{\Sigma,b_i^+} \oplus \dot\nu^*\mathcal{U}_{\Sigma,b_i^-},$$

over PMod^d_ν, where we recall $\dot\nu : \mathsf{PMod}^d_\nu \longrightarrow \overline{\mathsf{Jac}}^d_\Sigma$ is the forgetful map (3.8). Then, consider the relative tautological line bundle $\mathcal{V}_{\Sigma,i} \longrightarrow \mathsf{PMod}^d_\nu$, that is, the line subbundle

$$\mathcal{V}_{\Sigma,i} \subset \dot\nu^*\mathcal{U}_{\Sigma,b_i^+} \oplus \dot\nu^*\mathcal{U}_{\Sigma,b_i^-}, \tag{3.15}$$

whose fibre over the point $(M, V_1 \oplus \cdots \oplus V_k) \in \mathsf{PMod}^d_\nu$ is V_i, where we recall from the definition of parabolic modules that V_i is a 1-dimensional subspace of $M_{b_i^-} \oplus M_{b_i^+}$. The restriction of this tautological bundle to the preimages $\dot\nu^{-1}(M) \subset \mathsf{PMod}^d_\nu$ is given by

$$\mathcal{V}_{\Sigma,i}|_{\dot\nu^{-1}(M)} \cong \pi_i^*\mathcal{O}_{\mathbb{P}^1}(-1),$$

with $\pi_i : (\mathbb{P}^1)^k \longrightarrow \mathbb{P}^1$ the projection to the i-th factor.

The universal family of parabolic modules is the tuple $((\mathrm{id} \times \dot\nu)^*\mathcal{U}_\Sigma, \mathcal{V}_\Sigma)$, where $\mathcal{V}_\Sigma$ is the vector bundle on PMod^d_ν given by

$$\mathcal{V}_\Sigma = \mathcal{V}_{\Sigma,1} \oplus \cdots \oplus \mathcal{V}_{\Sigma,k}, \tag{3.16}$$

which comes naturally equipped with the inclusion

$$\mathcal{V}_\Sigma = \mathcal{V}_{\Sigma,1} \oplus \cdots \oplus \mathcal{V}_{\Sigma,k} \subset \dot\nu^*\mathcal{U}_\Sigma|_{\nu^{-1}(\mathsf{RSing}(\nu))\times\mathsf{PMod}} \cong \bigoplus_{i=1}^k \dot\nu^*\mathcal{U}_{\Sigma,b_i^+} \oplus \dot\nu^*\mathcal{U}_{\Sigma,b_i^-}.$$

Due to the compatible normalisations of $\mathcal{U}_C$ and $\mathcal{U}_\Sigma$, we obtain a universal version of the exact sequence (3.6) on $C \times \mathsf{PMod}^d_\nu$, given by

$$0 \longrightarrow (\mathrm{id} \times \rho)^*\mathcal{U}_C \longrightarrow (\nu \times \mathrm{id})_*(\mathrm{id} \times \dot\nu)^*\mathcal{U}_\Sigma \longrightarrow \mathcal{T} \longrightarrow 0, \tag{3.17}$$

where

$$\mathcal{T} := \big((\nu \times \mathrm{id})_*\big(\mathrm{id} \times \dot\nu)^*\mathcal{U}_\Sigma|_{\nu^{-1}(\mathsf{Sing}(C))\times\mathsf{PMod}}\big)/\mathcal{V}_\Sigma \tag{3.18}$$

is supported on $\mathsf{Sing}(C) \times \mathsf{PMod}^d_\nu$. The pushforward of $\mathcal{T}$ along the projection $C \times \mathsf{PMod}^d_\nu \longrightarrow \mathsf{PMod}^d_\nu$ recovers the universal quotient bundle

$$\mathcal{Q} := \mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_k := \bigoplus_{i=1}^k (\dot\nu^*\mathcal{U}_{\Sigma,b_i^+} \oplus \dot\nu^*\mathcal{U}_{\Sigma,b_i^-})/\,\mathcal{V}_{\Sigma,i} \tag{3.19}$$

on PMod^d_ν.

We conclude this section by describing the canonical bundle ω_{PMod} of PMod^d_ν.

Lemma 3.8. *Let C be an integral nodal projective curve with partial normalisation $\nu : \Sigma \longrightarrow C$. Let $\mathsf{RSing}(\nu) = \{b_1, \dots, b_k\}$ be the divisor of singularities resolved by ν. Then, there exists an isomorphism of line bundles*

$$\omega_{\mathsf{PMod}} \cong \bigotimes_{i=1}^k \mathcal{V}^2_{\Sigma,i} \otimes \dot\nu^*\mathcal{U}^\vee_{d,b_i^+} \otimes \dot\nu^*\mathcal{U}^\vee_{d,b_i^-} \tag{3.20}$$

Proof. The vector bundle $U_i := \mathcal{U}_{d,b_i^+} \oplus \mathcal{U}_{d,b_i^-}$ gives rise to the projective bundle

$$p : \mathbb{P}(U_i) = \mathbb{P}\left(\mathcal{U}_{d,b_i^+} \oplus \mathcal{U}_{d,b_i^-}\right) \longrightarrow \overline{\mathsf{Jac}}^d_\Sigma.$$

By a standard formula (see [Har86, Ex III, 8.4]) the relative dualising sheaf of the bundle map p is

$$\omega_{\mathbb{P}(U_i)/\overline{\mathsf{Jac}}^d_\Sigma} \cong \mathcal{O}_p(-2) \otimes p^* \det(U_i)^\vee = \mathcal{O}_p(-2) \otimes p^*\mathcal{U}^\vee_{d,b_i^+} \otimes p^*\mathcal{U}^\vee_{d,b_i^-},$$

where $\mathcal{O}_p(-2)$ is the square of the relative tautological line bundle. PMod^d_ν is the $(\mathbb{P}^1)^k$-bundle given by the fibre product

$$\mathbb{P}(U_1) \times_{\overline{\mathsf{Jac}}^d_\Sigma} \cdots \times_{\overline{\mathsf{Jac}}^d_\Sigma} \mathbb{P}(U_k) \longrightarrow \overline{\mathsf{Jac}}^d_\Sigma$$

It follows that the relative dualising sheaf $\omega_{\mathsf{PMod}/\overline{\mathsf{Jac}}^d_\Sigma}$ is isomorphic to (3.20). Moreover $\overline{\mathsf{Jac}}^d_\Sigma$ has trivial canonical bundle, and so we have $\omega_{\mathsf{PMod}} \cong \omega_{\mathsf{PMod}/\overline{\mathsf{Jac}}^d_\Sigma}$ concluding the proof. □

We now take square roots of the expression in Lemma 3.8, which requires the following.

Lemma 3.9. *For each $y \in \Sigma \backslash \mathsf{Sing}(\Sigma)$, the restriction $\mathcal{U}_{\Sigma,y} = \mathcal{U}_\Sigma|_{\{y\}\times\overline{\mathsf{Jac}}_\Sigma}$ admits a square root. Moreover, the choice of square root is parameterised by $H^1(\Sigma, \mathbb{Z}_2)$.*

Proof. The compatibility $A^*_\Sigma\mathcal{P}_\Sigma \cong \mathcal{U}_\Sigma$ between the Poincaré sheaf with the Abel-Jacobi map shows that $\mathcal{U}_{\Sigma,y} \cong \mathcal{P}_\Sigma|_{\{A_\Sigma(y)\}\times\overline{\mathsf{Jac}}_\Sigma}$ is a geometric point of $\mathsf{Pic}^0(\overline{\mathsf{Jac}}_\Sigma)$. The autoduality $\mathsf{Pic}^0(\overline{\mathsf{Jac}}_\Sigma) \cong \mathsf{Jac}_\Sigma$ preserves the group structure, so it is equivalent to describe the square roots of the corresponding geometric point in Jac_Σ. These are parametrised by the 2-torsion points of Jac^0_Σ, which in turn correspond to $H^1(\Sigma, \mathbb{Z}_2)$. □

Corollary 3.10. *Adopt the same hypothesis as Lemma 3.8. At every point $b_i^\pm \in \mathsf{Exc}(\nu) = \nu^{-1}(\mathsf{RSing}(\nu))$, choose a square root $\mathcal{U}^{1/2}_{\Sigma,b_i^\pm}$ of the line bundle $\mathcal{U}_{\Sigma,b_i^\pm}$. Each such family of choices $\{\mathcal{U}^{1/2}_{\Sigma,b_i^\pm}\}_{i=1,\dots,k}$ determines a spin structure on PMod^d_ν given by*

$$\omega^{1/2}_{\mathsf{PMod}} := \bigotimes_{i=1}^{k} \mathcal{V}_{\Sigma,i} \otimes \dot{\nu}^*\mathcal{U}^{-1/2}_{b_i^-} \otimes \dot{\nu}^*\mathcal{U}^{-1/2}_{b_i^+}.$$

4. Comparison results for curvilinear Hilbert schemes and parabolic modules

4.1. **Parabolic modules via curvilinear divisors.** In this section we present some technical results concerning the interplay between the curvilinear Hilbert scheme of C and the moduli space of parabolic modules over $\nu : \Sigma \longrightarrow C$. The results described are preliminaries for Section 5.1. As before, we take C to be an irreducible nodal projective curve with simple nodes and denote by $\nu : \Sigma \longrightarrow C$ its partial normalisation at the nodes $\{b_1, \dots, b_k\} = \mathsf{RSing}(\nu) \subset \mathsf{Sing}(C) \subset C$. Recall that we have chosen a smooth point $x_0 \in C$ with preimage $y_0 = \nu^{-1}(x_0) \in \Sigma$. We specialise to the degree zero case $d = 0$.

Notation 4.1. As in Notation 2.1, when $d = 0$, we will write $\mathsf{PMod}_\nu := \mathsf{PMod}^0_\nu$.

Recall the Chow-Hilbert space introduced in Section 3.1. The natural morphism

$$\mathsf{chow}(\nu) : \mathsf{Sym}^n_{\Sigma\backslash\nu^{-1}(\mathsf{URSing}(\nu))} \longrightarrow \mathsf{Sym}^n_{C\backslash\mathsf{URSing}(\nu)}$$

and the identity isomorphism $\mathsf{Hilb}^n_{\Sigma\backslash\mathsf{Exc}(\nu)} \longrightarrow \mathsf{Hilb}^n_{C\backslash\mathsf{RSing}}$, glue to a morphism

$$\nu^{(n)} : \mathsf{Hilb}^{cur}_{n,\Sigma} \longrightarrow \mathsf{H}_\nu\,.$$

Taking the cartesian square above $\nu^{(n)}$ and the quotient map c from (3.2) we obtain

$$\begin{array}{ccc} \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma} & \xrightarrow{\quad q \quad} & \mathsf{Hilb}^{cur}_{n,\Sigma} \\ \beta\big\downarrow & & \big\downarrow\nu^{(n)} \\ \mathsf{Hilb}^{cur}_{n,C} & \xrightarrow{\quad c \quad} & \mathsf{H}_\nu\,. \end{array} \tag{4.1}$$

Note the pullback is a scheme as the Chow–Hilbert space H_ν is an algebraic space (Lemma 3.2). Recall also the finite morphism ρ of (3.5) and consider also the Cartesian diagram

$$\begin{array}{ccc} \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu & \xrightarrow{\widetilde{\alpha}_C} & \mathsf{PMod}_\nu \\ \widetilde{\rho}\downarrow & & \downarrow\rho \\ \mathsf{Hilb}^{cur}_{n,C} & \xrightarrow[\alpha_C]{} & \overline{\mathsf{Jac}}_C. \end{array} \tag{4.2}$$

Proposition 2.2 allowed us to choose n so that $\alpha_C : \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$ is surjective. However, by [DSo79; AK80], for large enough n, we know that $A_C : \mathsf{Hilb}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$, is actually a projective bundle (see page 425 and Note 4.7 of [DSo79] and [AK80, Theorem 8.6]). We assume such choice of n from now on:

Assumption 4.2. *Take the number $n \in \mathbb{N}$ to be sufficiently large such that $A_C : \mathsf{Hilb}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$ is a projective bundle over $\overline{\mathsf{Jac}}_C$, with fibre over $\mathcal{F}$ equal to $\mathbb{P}(H^0(C, \mathcal{F}(nx_0)))$.*

Under this assumption, $\alpha_C : \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$ is a smooth surjective morphism, such that the complement of its fibres in the (projective) fibres of $A_C : \mathsf{Hilb}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$ is of codimension at least two, by [MRV19b, Lemma 3.9]. In particular, the base change $\widetilde{\alpha}_C$ in (4.2) is also smooth and surjective.

Let $(D, (M, V)) \in \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu$ and let

$$\mathcal{F} := \alpha_C(D) = \rho(M, V).$$

By the definition of α_C, the subscheme $D \subset C$ determines a section $s_D : \mathcal{O}_C \longrightarrow \mathcal{I}_D^\vee = \mathcal{F}(nx_0)$. On the other hand, (M, V) corresponds under ρ to an embedding $\jmath : \mathcal{F} \hookrightarrow \nu_* M$. Hence, the composition $\jmath \circ s_D$ is a section of the sheaf $(\nu_* M)(nx_0) \cong \nu_*(M(ny_0))$. By adjointness, this gives a section of the rank 1 torsion-free sheaf $M(ny_0)$ over Σ. Its vanishing locus determines, in turn, a degree n effective divisor $E \in \alpha_\Sigma^{-1}(M) \subset \mathsf{Hilb}^{cur}_{n,\Sigma}$, where

$$\alpha_\Sigma : \mathsf{Hilb}^{cur}_{n,\Sigma} \longrightarrow \overline{\mathsf{Jac}}_\Sigma, \quad \alpha_\Sigma(\mathcal{I}_E) = \mathcal{I}_E^\vee(-ny_0)$$

is also smooth and projective, whenever Assumption 4.2 holds. As $\jmath$ vanishes nowhere (because the torsion in (3.6) is of rank 1), the vanishing locus of $\jmath \circ s_D$ is still D. Then, again by adjointness, we have $\nu(E) \subset D$, so $c(D) = \nu^{(n)}(E)$ as both have the same length.

The upshot of the preceding discussion is that, taking into account the diagrams (4.1) and (4.2), the morphism

$$j : \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu \longrightarrow \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}, \quad j(D, (M, V)) = (D, E) \tag{4.3}$$

is well-defined. Here E is constructed as above. To describe this map in more detail, we require the following description of points on $\mathsf{Hilb}^{cur}_{n,C}$.

Lemma 4.3. *Let C be an irreducible nodal projective curve with k nodes at $\{b_1, \ldots, b_k\} = \mathsf{RSing}(\nu) \subset C$, with $\nu : \Sigma \longrightarrow C$ the partial normalisation at $\mathsf{RSing}(\nu)$. Decompose the subscheme $D \in \mathsf{Hilb}^{cur}_{n,C}$ as*

$$D = D_0 + D_1 + \cdots + D_k, \tag{4.4}$$

with D_0 being supported on a subset of $C \setminus \mathsf{RSing}(\nu)$ and D_i supported on the nodal point b_i.

For $i \geq 1$, if D_i is non-empty, then one of the following two options holds:

1. *D_i is supported on one of the branches of C at b_i, in which case $\mathcal{I}_{D_i}$ is not principal.*
2. *D_i is D_{b_i,a_i} for $a_i \in \mathbb{C}^*$, where*

$$D_{b_i,a_i} := \mathsf{Spec}\left(\mathbb{C}[y_i^+, y_i^-]/\langle y_i^+ y_i^-, y_i^+ - a_i y_i^- \rangle\right),$$

with y_i^+ and y_i^- being local coordinates of C at the node b_i. In this case, D_i has length 2 and $\mathcal{I}_{D_i}$ is principal.

Proof. Knowing that C is locally isomorphic to $\mathsf{Spec}\left(\mathbb{C}[y_i^+, y_i^-]/\langle y_i^+ y_i^-\rangle\right)$, the proof becomes a simple exercise. If D_i is not contained in $\mathsf{Spec}\left(\mathbb{C}[y_i^{\pm}]\right)$ it must have length at least 2. As, by hypothesis, it is contained in some $\mathbb{A}^1$, then $D_i \subset D_{b_i,a_i}$ for some $a_i \in \mathbb{C}^*$. Observe that the right-hand-side is a length 2 subscheme, hence the latter is identified with D_i itself. It follows from this identification that $\mathcal{I}_{D_i}$ is principal. □

We now describe points $(\mathcal{I}_D, E) \in \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}$ that lie in the image of j. From the description prior to (4.3), such points satisfy the condition $\nu(E) \subset D$. The following is immediate from combining this condition with the description of curvilinear divisors in Lemma 4.3.

Corollary 4.4. *Let C be an irreducible nodal curve and $\nu : \Sigma \longrightarrow C$ a partial normalisation at $\mathsf{RSing}(\nu) = \{b_1, \dots, b_k\}$, with $\mathsf{Exc}(\nu) = \{b_1^+, b_1^-, \dots, b_k^+, b_k^-\}$. Take $(\mathcal{I}_D, \mathcal{I}_E) \in \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}$ so that $\nu(E) \subset D$. Considering the decomposition (4.4) of D, then E decomposes as*

$$E = E_0 + E_1^+ + E_1^- + \cdots + E_k^+ + E_k^-, \tag{4.5}$$

with E_0 being supported on a subset of $\Sigma \setminus \mathsf{Exc}(\nu)$, E_i^+ supported at b_i^+ and E_i^- at b_i^-.

Then $D_0 = \nu(E_0)$ and, for $i \geq 1$,

1. *if D_i is 0 then $E_i^+ = 0$ and $E_i^- = 0$;*
2. *if D_i is supported on one of the branches of C at b_i, then either $D_i = \nu(E_i^+)$ and $E_i^- = 0$, or $D_i = \nu(E_i^-)$ and $E_i^+ = 0$;*
3. *if D_i is D_{b_i,a_i} for $a_i \in \mathbb{C}^*$, then $E_i^+ = b_i^+$ and $E_i^- = b_i^-$, hence $\nu(E_i) = b_i \subset D_i$.*

We are now in a position to complete our description of the map j.

Lemma 4.5. *In the conditions stated above, the morphism*

$$j : \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu \hookrightarrow \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma},$$

as defined in (4.3), is a closed immersion. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}^0_\nu & \xrightarrow{q \circ j} & \mathsf{Hilb}^{cur}_{n,\Sigma} \\ {\scriptstyle \tilde{\alpha}_C}\downarrow & & \downarrow{\scriptstyle \alpha_\Sigma} \\ \mathsf{PMod}^0_\nu & \xrightarrow[\hat{\nu}]{} & \overline{\mathsf{Jac}}_\Sigma, \end{array}$$

where $q : \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma} \longrightarrow \mathsf{Hilb}^{cur}_{n,\Sigma}$ is the projection.

Proof. Let $Z \subset \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}$ denote the closed subset

$$Z := \{(D, E) \in \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma} \,|\, \nu(E) \subset D\}.$$

The strategy of the proof will be to construct, on Z, an explicit inverse map to j. Consider a point $(D, E) \in Z$ and decompose D and E as per (4.4) and (4.5). The geometry we require is an understanding of $\nu^*\mathcal{O}_D$. By Corollary 4.4, one has $\mathcal{O}_{E_0} = \nu^*\mathcal{O}_{D_0}$, alongside also the following case by case analysis of the decomposition:

1. either D_i and E_i are empty;
2. or D_i is supported on one of the branches of C at b_i (so either $\nu^*\mathcal{O}_{D_i} = \mathcal{O}_{E_i^+ + b_i^-}$ and $E_i = E_i^+$, or $\nu^*\mathcal{O}_{D_i} = \mathcal{O}_{b_i^+ + E_i^-}$ and $E_i = E_i^-$);
3. or D_i is D_{b_i,a_i} for $a_i \in \mathbb{C}^*$ (so $\nu^*\mathcal{O}_{D_i} \cong \mathcal{O}_{b_i^+ + b_i^-} = \mathcal{O}_{E_i}$).

In all cases, there exists a divisor F on Σ, supported on a subset of $\mathsf{Exc}(\nu)$, such that $\nu^*\mathcal{O}_D = \mathcal{O}_{E+F}$ and $F \cap E = \varnothing$.

The following homological analysis determines the pullback $\nu^*\mathcal{I}_D^\vee$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{I}_D^\vee \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

and pullback along $\nu : \Sigma \longrightarrow C$ yields a 5-term exact sequence on Σ:

$$0 \longrightarrow \mathcal{T}or_1(\mathcal{I}_D^\vee, \mathcal{O}_\Sigma) \longrightarrow \mathcal{T}or_1(\mathcal{O}_D, \mathcal{O}_\Sigma) \xrightarrow{(*)} \mathcal{O}_\Sigma \longrightarrow \nu^*\mathcal{I}_D^\vee \longrightarrow \nu^*\mathcal{O}_D = \mathcal{O}_{E+F} \longrightarrow 0.$$

Since $\mathcal{T}or_1(\mathcal{O}_D, \mathcal{O}_\Sigma)$ is a torsion sheaf on Σ, the map $(*)$ is 0 and we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \nu^*\mathcal{I}_D^\vee \longrightarrow \nu^*\mathcal{O}_D = \mathcal{O}_{E+F} \longrightarrow 0. \tag{4.6}$$

The coherent sheaf $\nu^*\mathcal{I}_D^\vee$ is torsion-free at $\nu^{-1}(\mathsf{URSing}(\nu))$ and decomposes into a locally free sheaf and a torsion sheaf at the smooth points $\mathsf{Exc}(\nu)$. Hence, globally it decomposes into a torsion-free sheaf - the torsion-free pullback - and a torsion sheaf

$$\nu^*\mathcal{I}_D^\vee = \nu^{\mathrm{tf}}\mathcal{I}_D^\vee \oplus \mathcal{T}_D.$$

Let $\mathsf{RSing}(\nu) = S_{1,3} \cup S_2$ where $S_{1,3}$ contains all nodes of the resolved singular locus where D is principal, i.e. described by case (1) or (3), and S_2 are the nodes where it is not principal, i.e. described by case (2). By a local computation one can see that $\mathcal{T}_D = \mathcal{O}_{\nu^{-1}(S_2)} = \mathcal{O}_{F+\sigma F}$, where $\sigma : \mathsf{Exc}(\nu) \longrightarrow \mathsf{Exc}(\nu)$ is the involution that pairwise interchanges the preimages b_i^+, b_i^-. Hence, equation (4.6) reduces to

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \nu^{\mathrm{tf}}\mathcal{I}_D^\vee \longrightarrow \mathcal{O}_{E-\sigma F} \longrightarrow 0.$$

Therefore, we conclude

$$\nu^{\mathrm{tf}}\mathcal{I}_D^\vee = \mathcal{I}_{E-\sigma F}^\vee. \tag{4.7}$$

Using the previous calculation we can now write down an exact sequence of the form (3.6), therefore defining the parabolic module corresponding to $\mathcal{I}_D^\vee(-nx_0)$. Consider the intermediate partial normalisation $\nu_2 : \Sigma' \longrightarrow C$ at the nodes in S_2. Then ν factors as

$$\nu : \Sigma \xrightarrow{\nu_1} \Sigma' \xrightarrow{\nu_2} C.$$

By Cook [Coo98, Lemma 1], there is a push-pull isomorphism

$$\mathcal{I}_D^\vee \cong \nu_{2*}\nu_2^{\mathrm{tf}}\mathcal{I}_D^\vee,$$

where ν_2^{tf} is the torsion-free pullback. Now, $\nu_2^{\mathrm{tf}}\mathcal{I}_D^\vee$ is a locally free sheaf on Σ' away from $\nu_2^{-1}(\mathsf{Sing}(C) \setminus \mathsf{RSing}(\nu))$, where ν_1 and ν_2 are locally isomorphisms. Hence, there exists an exact sequence

$$0 \longrightarrow \nu_2^{\mathrm{tf}}\mathcal{I}_D^\vee \longrightarrow \nu_{1*}\nu_1^*\nu_2^{\mathrm{tf}}\mathcal{I}_D^\vee \longrightarrow \mathcal{O}_{\mathsf{RSing}(\nu_1)} \longrightarrow 0.$$

Note that $\mathsf{RSing}(\nu_1) = \nu_2^{-1}(S_{1,3})$ and $\nu_1^*\nu_2^{\mathrm{tf}}\mathcal{I}_D^\vee = \nu^{\mathrm{tf}}\mathcal{I}_D^\vee$. Pushing forward this exact sequence to C, we obtain

$$0 \longrightarrow \mathcal{I}_D^\vee \xrightarrow{f_1} \nu_*\nu^{\mathrm{tf}}\mathcal{I}_D^\vee \longrightarrow \mathcal{O}_{S_{1,3}} \longrightarrow 0. \tag{4.8}$$

From equation (4.7) we have the exact sequence

$$0 \longrightarrow \nu^{\mathrm{tf}}\mathcal{I}_D^\vee(-ny_0) = \mathcal{I}_{E-\sigma F}^\vee(-ny_0) \xrightarrow{f_2} \mathcal{I}_E^\vee(E - ny_0) \longrightarrow \mathcal{O}_{\sigma F} \longrightarrow 0.$$

Composing its pushforward with f_1 of (4.8) results in the desired sequence

$$0 \longrightarrow \mathcal{I}_D^\vee(-nx_0) \xrightarrow{\nu_* f_2 \circ f_1} \nu_*\mathcal{I}_E^\vee(-ny_0) \longrightarrow \mathcal{O}_{\mathsf{RSing}(\nu)} \longrightarrow 0. \tag{4.9}$$

Then, since the map $\rho : \mathsf{PMod}_\nu^0 \longrightarrow \overline{\mathsf{Jac}}_C$ is surjective, $\mathcal{F} := \alpha_C(\mathcal{I}_D) = \mathcal{I}_D^\vee(-nx_0)$ and $M := \alpha_\Sigma(\mathcal{I}_E) = \mathcal{I}_E^\vee(-ny_0)$ determine a point (M, V) in PMod_ν. Since we are specifying both $\mathcal{F}$ and M, Lemma 3.7 says that the point (M, V) is uniquely determined by (4.9). This gives rise to a map

$$g : Z \longrightarrow \mathsf{Hilb}_{n,C}^{cur} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu, \quad g(D, E) = (D, (\mathcal{I}_E^\vee(-ny_0), V)).$$

From the definition of j in (4.3), we conclude that g is inverse to j, hence $\mathsf{Hilb}_{n,C}^{cur} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu$ is naturally isomorphic to Z closed in $\mathsf{Hilb}_{n,C}^{cur} \times_{\mathsf{H}_\nu} \mathsf{Hilb}_{n,\Sigma}^{cur}$.

Finally, the identification $M = \mathcal{I}_E^\vee(-ny_0)$ is equivalent to the commutativity of the diagram. □

The next technical result will be applied in Section 5.1. We say that the big diagonal Δ of H_ν is the preimage of the big diagonal in Sym^n_C under the natural morphism $\mathsf{H}_\nu \longrightarrow \mathsf{Sym}^n_C$ from (3.3).

Lemma 4.6. *The complement of the image of* $j : \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}_C}} \mathsf{PMod}_\nu \hookrightarrow \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}$ *maps into the big diagonal* Δ *under the projection* $q : \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma} \longrightarrow \mathsf{H}_\nu$.

Proof. Recalling the decomposition (4.4), if (D, E) is a point of $\mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}$ such that every D_i has length at most 1, after Corollary 4.4 one immediately finds that D_i are contained in the branches of C. In that case $\nu(E)$ embeds into D so the pair lies on the image of j. Therefore, the points that lie outside the big diagonal of H_ν are contained in the image of j, and the last statement is a straightforward consequence of this fact. □

4.2. **Equivalence of universal divisors.** The main result of this section describes a relation between twists of the line bundles $\det \mathcal{A}_{C,n}$ and $\det \mathcal{A}_{\Sigma,n}$ constructed in (2.2). We also recall the tautological line bundles $\mathcal{V}_{\Sigma,i}$ over PMod^d_ν, defined in (3.15), alongside the definition $\mathcal{V}_\Sigma = \mathcal{V}_{\Sigma,1} \oplus \cdots \oplus \mathcal{V}_{\Sigma,k}$ of (3.16).

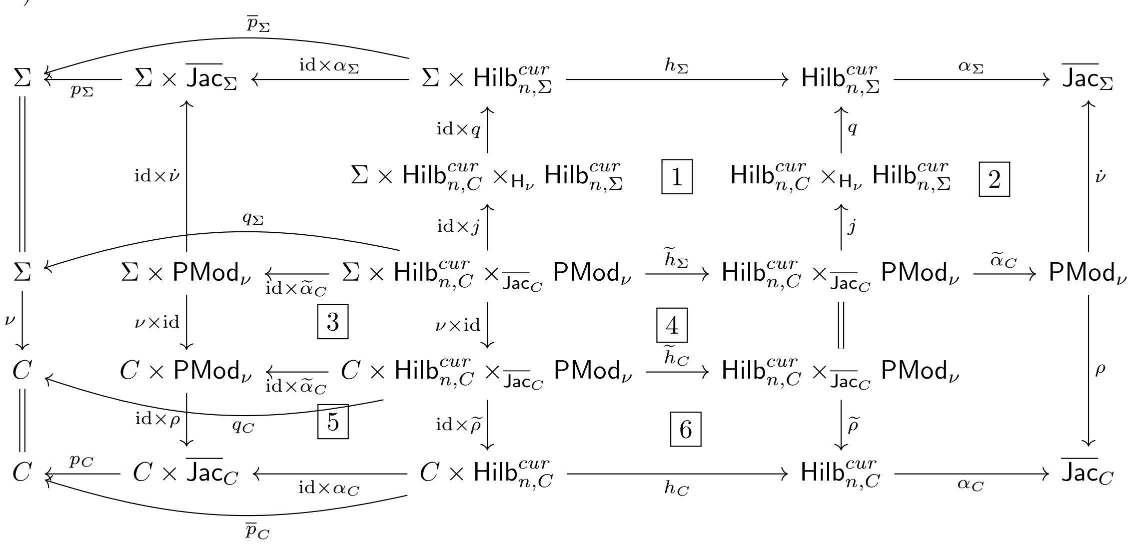
Proposition 4.7. *Let C be an irreducible nodal projective curve and let $\nu : \Sigma \longrightarrow C$ be a partial normalisation at k nodes. Then there exists an isomorphism*

$$j^* q^* (\det(\mathcal{A}_{\Sigma,n}) \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0})^{-k}) \otimes \widetilde{\rho}^* \left(\det(\mathcal{A}_{C,n})\right)^\vee \cong \widetilde{\alpha}_C^* \det(\mathcal{V}_\Sigma), \tag{4.10}$$

as line bundles on $\mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}_{\Sigma}}{}^0_C} \mathsf{PMod}_\nu$.

The remainder of this section is dedicated to the proof of Proposition 4.7. The idea is to relate $\mathcal{A}_{C,n}$ and $\mathcal{A}_{\Sigma,n}$ via a universal exact sequence on $\mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}} \mathsf{PMod}_\nu$ that is a pullback of (3.17). We shall make intensive use of the commutative diagram shown in Figure (4.11) below.

(4.11)



Let us first express $\det(\mathcal{A}_{C,n})$ and $\det(\mathcal{A}_{\Sigma,n})$ in terms of the ideal sheaves.

Lemma 4.8. *We have isomorphisms*

$$\det(\mathcal{A}_{C,n}) \cong \det\left(R^1 h_{C,*}\mathcal{I}_{\mathcal{D}_C}\right), \quad \det(\mathcal{A}_{\Sigma,n}) \cong \det\left(R^1 h_{\Sigma,*}\mathcal{I}_{\mathcal{D}_\Sigma}\right).$$

Proof. Recall that, by definition $\mathcal{A}_C = h_{C,*}\mathcal{O}_{\mathcal{D}_C}$. We use the short exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{D}_C} \longrightarrow \mathcal{O}_{C\times\mathsf{Hilb}_C} \longrightarrow \mathcal{O}_{\mathcal{D}_C} \longrightarrow 0,$$

over $C \times \mathsf{Hilb}^{cur}_{n,C}$. The long exact sequence induced by pushforward along $h_C : C \times \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \mathsf{Hilb}^{cur}_{n,C}$ is given by

$$0 \longrightarrow \mathcal{A}_C \longrightarrow R^1 h_{C,*}\mathcal{I}_{\mathcal{D}_C} \longrightarrow \mathcal{O}^{\oplus g}_{\mathsf{Hilb}_C} \longrightarrow 0,$$

since $h_{C,*}\mathcal{I}_{\mathcal{D}_C} = 0$, $h_{C,*}\mathcal{O}_{C\times\mathsf{Hilb}_C} = \mathcal{O}_{\mathsf{Hilb}_C}$ and so $R^1 h_{C,*}\mathcal{O}_{C\times\mathsf{Hilb}_C} = \mathcal{O}^{\oplus g}_{\mathsf{Hilb}_C}$. Taking determinants yields the formula of the lemma. The same argument applies to the smooth curve Σ. □

Recall that C is an irreducible nodal projective curve equipped with a partial normalisation $\nu : \Sigma \longrightarrow C$ at k (simple) nodes and with a choice of a smooth point $x_0 \in C$ and $y_0 = \nu^{-1}(x_0)$. Recall that we denote the resolved singular locus by $\mathsf{RSing}(\nu) = \{b_1, \dots, b_k\} \subset C$ and exceptional divisor by $\mathsf{Exc}(\nu) = \{b_1^+, b_1^-, \dots, b_k^+, b_k^-\} \subset \Sigma$, where in our notation $\nu^{-1}(b_i) = \{b_i^+, b_i^-\}$. Recall also the universal exact sequence (3.17) involving the universal parabolic module $((\mathrm{id}\times\dot{\nu})^*\mathcal{U}_\Sigma, \mathcal{V}_\Sigma)$ and the torsion sheaf $\mathcal{T}$ defined in (3.18).

Lemma 4.9. *There is an exact sequence,*

$$0 \longrightarrow (\mathrm{id}\times\widetilde{\rho})^*\mathcal{I}^\vee_{\mathcal{D}_C} \longrightarrow (\nu\times\mathrm{id})_*(\mathrm{id}\times(q\circ j))^*\mathcal{I}^\vee_{\mathcal{D}_\Sigma} \longrightarrow (\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{T}\otimes(\mathrm{id}\times\widetilde{\rho})^* h_C^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}) \longrightarrow 0\,,$$

on $C \times \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu$.

Proof. By Assumption 4.2, $\alpha_C : \mathsf{Hilb}^{cur}_{n,C} \longrightarrow \overline{\mathsf{Jac}}_C$ is surjective and smooth. It follows by (4.2) that the same holds for $\widetilde{\alpha}_C$. In particular, $\mathrm{id}\times\widetilde{\alpha}_C : C\times\mathsf{Hilb}^{cur}_{n,C}\times_{\overline{\mathsf{Jac}}_C}\mathsf{PMod}_\nu \longrightarrow C\times\mathsf{PMod}_\nu$ is flat. So pulling back along $\mathrm{id}\times\widetilde{\alpha}_C$ is an exact functor. We pullback the universal exact sequence (3.17) along $\mathrm{id}\times\widetilde{\alpha}_C$. Using commutativity around diagram $\boxed{5}$ and flat base change around diagram $\boxed{3}$ of (4.11), we obtain

$$0 \longrightarrow (\mathrm{id}\times\widetilde{\rho})^*(\mathrm{id}\times\alpha_C)^*\mathcal{U}_C \longrightarrow (\nu\times\mathrm{id})_*(\mathrm{id}\times\widetilde{\alpha}_C)^*(\mathrm{id}\times\dot{\nu})^*\mathcal{U}_\Sigma \longrightarrow (\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{T} \longrightarrow 0. \tag{4.12}$$

We saw in Lemma 4.5 that the diagram $\boxed{2}$ of (4.11) commutes. Hence, we can replace the middle term above by the sheaf

$$(\nu\times\mathrm{id})_*(\mathrm{id}\times q\circ j)^*(\mathrm{id}\times\alpha_\Sigma)^*\mathcal{U}_\Sigma\,.$$

Now, using Lemma 2.6, we have an isomorphism

$$(\mathrm{id}\times\alpha_\Sigma)^*\mathcal{U}_\Sigma \cong \mathcal{I}^\vee_{\mathcal{D}_\Sigma}\otimes\overline{p}_\Sigma^*\mathcal{O}_\Sigma(-ny_0)\otimes h_\Sigma^*\mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0})^\vee,$$

as well as a corresponding isomorphism for $(\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{U}_C$. Inserting this into (4.12) we obtain the exact sequence

$$\begin{aligned}0 &\longrightarrow (\mathrm{id}\times\widetilde{\rho})^*\left(\mathcal{I}^\vee_{\mathcal{D}_C}\otimes\overline{p}_C^*\mathcal{O}_C(-nx_0)\otimes h_C^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^\vee\right)\\ &\longrightarrow (\nu\times\mathrm{id})_*(\mathrm{id}\times q\circ j)^*\left(\mathcal{I}^\vee_{\mathcal{D}_\Sigma}\otimes\overline{p}_\Sigma^*\mathcal{O}_\Sigma(-ny_0)\otimes h_\Sigma^*\mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0})^\vee\right) \longrightarrow (\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{T} \longrightarrow 0.\end{aligned}$$

We saw in Corollary 4.4 that E contains y_0 if and only if D contains x_0. Hence,

$$(q\circ j)^*\mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0}) \cong \widetilde{\rho}^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}). \tag{4.13}$$

In particular,

$$(\mathrm{id}\times(q\circ j))^* h_\Sigma^*\mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0}) = (\nu\times\mathrm{id})^*(\mathrm{id}\times\widetilde{\rho})^* h_C^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})\,.$$

So, using projection formula and tensoring with the line bundle $(\mathrm{id}\times\widetilde{\rho})^* h_C^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})$, we obtain

$$\begin{aligned}0 \longrightarrow (\mathrm{id}\times\widetilde{\rho})^*\left(\mathcal{I}^\vee_{\mathcal{D}_C}\otimes\overline{p}_C^*\mathcal{O}_C(-nx_0)\right) &\longrightarrow (\nu\times\mathrm{id})_*(\mathrm{id}\times(q\circ j))^*\left(\mathcal{I}^\vee_{\mathcal{D}_\Sigma}\otimes\overline{p}_\Sigma^*\mathcal{O}_\Sigma(-ny_0)\right)\\ &\longrightarrow (\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{T}\otimes(\mathrm{id}\times\widetilde{\rho})^* h_C^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}) \longrightarrow 0.\end{aligned}$$

Finally, using projection formula for $\nu^*\mathcal{O}_C(-nx_0) = \mathcal{O}_\Sigma(-ny_0)$ and tensoring by $q_C^*\mathcal{O}_C(nx_0) = (\mathrm{id}\times\widetilde{\rho})^*\overline{p}_C^*\mathcal{O}_C(nx_0)$, yields the exact sequence stated in the lemma. Note that $(\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{T}$ is supported on $\mathsf{RSing}(\nu)\times\mathsf{Hilb}^{cur}_{n,C}\times_{\overline{\mathsf{Jac}}_C}\mathsf{PMod}_\nu$ and hence the tensorization with a line bundle that is a pullback from C yields an isomorphism in the last term. □

To simplify some notation, let us introduce the abbreviations

$$\underline{C} := C\times\mathsf{Hilb}^{cur}_{n,C}\times_{\overline{\mathsf{Jac}}_C}\mathsf{PMod}_\nu, \quad \underline{\Sigma} := \Sigma\times\mathsf{Hilb}^{cur}_{n,C}\times_{\mathsf{H}_\nu}\mathsf{Hilb}^{cur}_{n,\Sigma},$$

for the triple products in the main diagram (4.11). Similarly, we set

$$\overline{\mathcal{I}}_C := (\mathrm{id} \times \widetilde{\rho})^* \mathcal{I}_{\mathcal{D}_C}, \quad \overline{\mathcal{I}}_\Sigma := (\mathrm{id} \times (q \circ j))^* \mathcal{I}_{\mathcal{D}_\Sigma}.$$

Lemma 4.10. *There exists an exact sequence given by*

$$0 \longrightarrow \widetilde{\alpha}_C^* \mathcal{Q}^\vee \otimes \widetilde{\rho}^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^\vee \longrightarrow R^1 \widetilde{h}_{\Sigma *}(\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^* \omega_{\Sigma/C}) \longrightarrow R^1 \widetilde{h}_{C*} \overline{\mathcal{I}}_C \longrightarrow 0,$$

on $\mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu$.

Proof. With the above notation, Lemma 4.9 says that there exists an exact sequence on $\underline{C}$ given by

$$0 \longrightarrow \overline{\mathcal{I}}_C^\vee \longrightarrow (\nu \times \mathrm{id})_* \overline{\mathcal{I}}_\Sigma^\vee \longrightarrow (\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T} \otimes (\mathrm{id} \times \widetilde{\rho})^* h_C^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}) \longrightarrow 0.$$

Now we compute the dual of this sequence. First, since torsion-free sheaves on C are reflexive (see Lemma 1.1 of [Har86]), we have

$$\mathcal{H}om(\overline{\mathcal{I}}_C^\vee, \mathcal{O}_{\underline{C}}) \cong \overline{\mathcal{I}}_C^{\vee\vee} \cong \overline{\mathcal{I}}_C.$$

To dualise the second term, we apply Grothendieck-Verdier duality. If $\omega_{\Sigma/C} := \omega_\Sigma \otimes \nu^* \omega_C^\vee$ and $\omega_{\underline{\Sigma}/\underline{C}} := \omega_{\underline{\Sigma}} \otimes (\nu \times \mathrm{id})^* \omega_{\underline{C}}^\vee$ are the relative dualising sheaves, then $\omega_{\underline{\Sigma}/\underline{C}} = q_\Sigma^* \omega_{\Sigma/C}$, and then

$$\begin{aligned} \mathcal{H}om((\nu \times \mathrm{id})_* \overline{\mathcal{I}}_\Sigma^\vee, \mathcal{O}_{\underline{C}}) &\cong (\nu \times \mathrm{id})_* (\mathcal{H}om(\overline{\mathcal{I}}_\Sigma^\vee, \mathcal{O}_{\underline{\Sigma}}) \otimes \omega_{\underline{\Sigma}/\underline{C}}) \\ &\cong (\nu \times \mathrm{id})_* (\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^* \omega_{\Sigma/C}). \end{aligned}$$

We therefore obtain the dual exact sequence

$$0 \to (\nu \times \mathrm{id})_* (\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^* \omega_{\Sigma/C}) \to \overline{\mathcal{I}}_C \to \mathcal{E}xt^1((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T} \otimes (\mathrm{id} \times \widetilde{\rho})^* h_C^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0}), \mathcal{O}_{\underline{C}}) \to 0\,.$$

By Hom-tensor adjointness and functoriality around diagram $\boxed{6}$ from (4.11), the last term can be rewritten as

$$\begin{aligned} &\mathcal{E}xt^1((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, (\mathrm{id} \times \widetilde{\rho})^* h_C^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^\vee) \\ &= \mathcal{E}xt^1((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, \mathcal{O}_{\underline{C}}) \otimes (\mathrm{id} \times \widetilde{\rho})^* h_C^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^\vee \\ &= \mathcal{E}xt^1((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, \mathcal{O}_{\underline{C}}) \otimes \widetilde{h}_C^* \widetilde{\rho}^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^\vee. \end{aligned}$$

Taking the pushforward along $\widetilde{h}_C : \underline{C} \longrightarrow \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}$, we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow \widetilde{h}_{C,*} \mathcal{E}xt^1((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, \mathcal{O}_{\underline{C}}) \otimes \widetilde{\rho}^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^\vee& \\ \longrightarrow R^1 \widetilde{h}_{C,*} (\nu \times \mathrm{id})_* (\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^* \omega_{\Sigma/C}) \longrightarrow R^1 \widetilde{h}_{C,*} \overline{\mathcal{I}}_C \longrightarrow 0\,.& \end{aligned} \tag{4.14}$$

It remains to compare the first and second terms of this sequence with the statement of the Lemma. To compute the first term, we consider $\widetilde{h}^1_{C,*}((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, \mathcal{O}_{\underline{C}})$. We may apply Grothendieck-Verdier duality for the projection $\widetilde{h}_C$ with relative dualising sheaf $q_C^* \omega_C$. Also recall the definition of $\mathcal{Q}$ from equation (3.19). This allows us to write

$$\begin{aligned} \widetilde{h}_{C,*} \mathcal{E}xt^1((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, \mathcal{O}_{\underline{C}}) &\cong \widetilde{h}_{C,*} R^1 \mathcal{H}om((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T} \otimes q_C^* \omega_C, q_C^* \omega_C), \\ &\cong \mathcal{H}om(\widetilde{h}_{C,*}((\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T} \otimes q_C^* \omega_C), \mathcal{O}_{\mathsf{Hilb} \times \mathsf{PMod}}), \\ &\cong \mathcal{H}om(\widetilde{h}_{C,*}(\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}, \mathcal{O}_{\mathsf{Hilb} \times \mathsf{PMod}}) = \widetilde{\alpha}_C^* \mathcal{Q}^\vee. \end{aligned}$$

Here, as in the proof of the previous lemma, $(\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T} \otimes q_C^* \omega_C \cong (\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}$, since $(\mathrm{id} \times \widetilde{\alpha}_C)^* \mathcal{T}$ is torsion, supported on $\mathsf{RSing}(\nu) \times \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu$. This computation shows that the first term of (4.14) coincides with the statement of the Lemma. Finally, for the second term, commutativity of diagram $\boxed{4}$ from (4.11) yields the isomorphism

$$R^1 \widetilde{h}_{C*} (\nu \times \mathrm{id})_* (\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^* \omega_{\Sigma/C}) \cong R^1 \widetilde{h}_{\Sigma *} (\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^* \omega_{\Sigma/C}),$$

and so (4.14) agrees with the statement of the lemma. This concludes the proof. □

Next, we compute the determinant of the extension term from the previous lemma.

Lemma 4.11. *There exists an isomorphism of line bundles,*

$$\det\left(R^1\widetilde{h}_{\Sigma,*}(\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^*\omega_{\Sigma/C})\right) \cong j^*q^*\left(\det(\mathcal{A}_{\Sigma,n}) \otimes \bigotimes_{i=1}^k \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^+})^\vee \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^-})^\vee\right),$$

on $\mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}_C}} \mathsf{PMod}_\nu$.

Proof. Consider again the exceptional divisor $B = \{b_1^+, b_1^-, \dots, b_k^+, b_k^-\} = \mathsf{Exc}(\nu) \subset \Sigma$. The canonical bundle of a nodal curve C and the partial normalisation Σ are related by the exact sequence

$$0 \longrightarrow \omega_C \longrightarrow \nu_*\omega_\Sigma(B) \xrightarrow{res} \mathcal{O}_{\mathsf{RSing}(\nu)} \longrightarrow 0\,,$$

where res is the sum of the residues; see [HM98, Page 82]. Pulling it back to Σ, and noticing that $\nu^*\omega_C$ is a line bundle (so torsion-free), yields

$$0 \longrightarrow \nu^*\omega_C \longrightarrow \nu^*\nu_*(\omega_\Sigma(B)) \xrightarrow{res} \mathcal{O}_B \longrightarrow 0.$$

But $\nu^*\nu_*(\omega_\Sigma(B)) \cong \omega_\Sigma(B) \oplus \mathcal{O}_B$, hence $\nu^*\omega_C \cong \omega_\Sigma(B)$. The upshot is that the relative canonical bundle is

$$\omega_{\Sigma/C} = \omega_\Sigma \otimes \nu^*\omega_C^\vee \cong \mathcal{O}_\Sigma(-B).$$

Define a divisor $\mathcal{B} \subset \Sigma \times \mathsf{Hilb}^{cur}_{n,\Sigma}$ by

$$\mathcal{B} := \overline{p}_\Sigma^{-1}(B) = B \times \mathsf{Hilb}^{cur}_{n,\Sigma}.$$

Then, we have the isomorphisms

$$\overline{\mathcal{I}}_\Sigma \otimes q_\Sigma^*\omega_{\Sigma/C} \cong (\mathrm{id}\times(q\circ j))^*(\mathcal{I}_{\mathcal{D}_\Sigma} \otimes \overline{p}_\Sigma^*\mathcal{O}_\Sigma(-B)) \cong (\mathrm{id}\times(q\circ j))^*(\mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma - \mathcal{B})).$$

On the other hand, we have an exact sequence on $\Sigma \times \mathsf{Hilb}^{cur}_{n,\Sigma}$ given by

$$0 \longrightarrow \mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma - \mathcal{B}) \longrightarrow \mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma) \longrightarrow \mathcal{O}_\mathcal{B} \otimes \mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma) \longrightarrow 0.$$

Pushing it forward along h_Σ yields

$$0 \to R^0h_{\Sigma,*}(\mathcal{O}_\mathcal{B} \otimes \mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma)) \longrightarrow R^1h_{\Sigma,*}\mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma - \mathcal{B}) \longrightarrow R^1h_{\Sigma,*}\mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma) \to 0.$$

Now we use that $R^0h_{\Sigma,*}(\mathcal{O}_\mathcal{B} \otimes \mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma)) = \bigoplus_{i=1}^k \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^+})^\vee \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^-})^\vee$. So, taking determinants and using Lemma 4.8, we obtain the isomorphisms

$$\begin{aligned}\det(R^1h_{\Sigma,*}(\mathcal{I}_{\mathcal{D}_\Sigma} \otimes \overline{p}_\Sigma^*\omega_{\Sigma/C})) &= \det(R^1h_{\Sigma,*}\mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma - \mathcal{B}))\\ &\cong \det(R^1h_{\Sigma,*}\mathcal{O}_{\Sigma\times\mathsf{Hilb}_\Sigma}(-\mathcal{D}_\Sigma)) \otimes \bigotimes_{i=1}^k \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^+})^\vee \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^-})^\vee\\ &\cong \det(\mathcal{A}_{\Sigma,n}) \otimes \bigotimes_{i=1}^k \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^+})^\vee \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^-})^\vee\end{aligned}$$

Now we obtain the result by the base change around diagram $\boxed{1}$ from (4.11). □

We are now ready to conclude the proof of Proposition 4.7.

Proof of Proposition 4.7. Using Lemmas 4.8 and 4.11, and taking the determinant of the exact sequence from Lemma 4.10, we conclude that

$$\begin{aligned}&j^*q^*\left(\det(\mathcal{A}_{\Sigma,n}) \otimes \bigotimes_{i=1}^k \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^+})^\vee \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,b_i^-})^\vee\right) \otimes \widetilde{\rho}^*\det(\mathcal{A}_{C,n})^\vee\\ &\cong \widetilde{\alpha}_C^*\det(\mathcal{Q})^\vee \otimes \widetilde{\rho}^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^{-k}.\end{aligned}$$

Note that $\widetilde{\alpha}_C^*\mathcal{Q} = (\widetilde{h}_{C,*}(\mathrm{id}\times\widetilde{\alpha}_C)^*\mathcal{T})^\vee$, and thus $\mathcal{Q}$ has rank k. From (4.13), we have $\widetilde{\rho}^*\mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{C,x_0})^k \cong j^*q^*\mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0})^k$. Moreover, by restricting the formula of Lemma 2.6 (applied to the smooth curve Σ with the marked point y_0) to each slice $\{b_i^\pm\} \times \mathsf{Sym}^n_\Sigma$, we get

$$\mathcal{O}_{\mathsf{Sym}}(\mathcal{D}_{\Sigma,b_i^\pm})^\vee \cong \alpha_\Sigma^*\mathcal{U}^\vee_{0,b_i^\pm} \otimes \mathcal{O}_{\mathsf{Sym}}(\mathcal{D}_{\Sigma,y_0})^\vee.$$

Hence, we obtain isomorphisms

$$j^*q^*\Big(\det(\mathcal{A}_{\Sigma,n})\otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0})^{-k}\Big)\otimes \widetilde{\rho}^*\det(\mathcal{A}_{C,n})^\vee \cong \widetilde{\alpha}_C^*\det(\mathcal{Q})^\vee\otimes j^*q^*\alpha_\Sigma^*\Big(\bigotimes_{i=1}^k(\mathcal{U}_{0,b_i^+}\otimes\mathcal{U}_{0,b_i^-})\Big)$$
$$\cong \widetilde{\alpha}_C^*\Big(\det(\mathcal{Q})^\vee\otimes\bigotimes_{i=1}^k(\dot{\nu}^*\mathcal{U}_{0,b_i^+}\otimes\dot{\nu}^*\mathcal{U}_{0,b_i^-})\Big).$$

The second isomorphism uses the commutative square $\boxed{2}$ in (4.11), which commutes by Lemma 4.5. Finally, by the definition of $\mathcal{Q}$ in (3.19), we have, for each i, an exact sequence

$$0\longrightarrow \mathcal{V}_{\Sigma,i}\longrightarrow \dot{\nu}^*\mathcal{U}_{0,b_i^+}\oplus\dot{\nu}^*\mathcal{U}_{0,b_i^-}\longrightarrow \mathcal{Q}_i\longrightarrow 0.$$

Hence, we conclude $\det(\mathcal{Q})^\vee\otimes\bigotimes_{i=1}^k(\dot{\nu}^*\mathcal{U}_{0,b_i^+}\otimes\dot{\nu}^*\mathcal{U}_{0,b_i^-})=\det(\mathcal{V}_\Sigma)$. This completes the proof. □

5. Autoduality and the partial normalisation map

This section computes the relation between two Fourier–Mukai transforms: the one associated to an integral nodal curve C and the one associated to a partial normalisation $\Sigma\longrightarrow C$. We first describe the restriction of the Poincaré sheaf to the locus of the compactified Jacobian described by pushforward under the normalisation map, which is addressed in Section 5.1, and subsequently provide the relation of the associated Fourier–Mukai transforms in Section 5.2.

5.1. **Isomorphism of Poincaré sheaves.** We denote by $\mathcal{P}_C$ and $\mathcal{P}_\Sigma$ the respective Poincaré sheaves on $\overline{\mathsf{Jac}}_C\times\overline{\mathsf{Jac}}_C$ and $\overline{\mathsf{Jac}}_\Sigma\times\overline{\mathsf{Jac}}_\Sigma$. The main result of this section is a comparison between $\mathcal{P}_\Sigma$ and the pullback of $\mathcal{P}_C$ along the closed embedding

$$\check{\nu}\times\mathrm{id}:\overline{\mathsf{Jac}}_\Sigma^{-k}\times\overline{\mathsf{Jac}}_C\hookrightarrow\overline{\mathsf{Jac}}_C\times\overline{\mathsf{Jac}}_C,$$

where

$$\check{\nu}:\overline{\mathsf{Jac}}_\Sigma^{-k}\hookrightarrow\overline{\mathsf{Jac}}_C,\quad \mathcal{L}\longmapsto\nu_*\mathcal{L} \tag{5.1}$$

is the closed embedding defined by pushforward under ν.

We begin with a slice-by-slice comparison on restrictions of the form

$$\mathcal{P}_{\Sigma,\mathcal{F}}:=\mathcal{P}_\Sigma|_{\{\mathcal{F}\}\times\overline{\mathsf{Jac}}_\Sigma},\quad \mathcal{P}_{C,\mathcal{F}'}:=\mathcal{P}_C|_{\{\mathcal{F}'\}\times\overline{\mathsf{Jac}}_C},$$

taken at a compatible pair of geometric points $\mathcal{F}\in\overline{\mathsf{Jac}}_\Sigma$ and $\mathcal{F}'\in\overline{\mathsf{Jac}}_C$. Fix a smooth geometric point $x_0\in C$, with preimage $y_0:=\nu^{-1}(x_0)\subset\Sigma$. Our comparison uses the maps

$$\tau_{k,y_0}:\overline{\mathsf{Jac}}_\Sigma^{-k}\xrightarrow{\cong}\overline{\mathsf{Jac}}_\Sigma,\quad \rho:\mathsf{PMod}_\nu\longrightarrow\overline{\mathsf{Jac}}_C,\quad \dot{\nu}:\mathsf{PMod}_\nu\longrightarrow\overline{\mathsf{Jac}}_\Sigma,$$

as defined in (2.7), (3.5) and (3.8) respectively.

Proposition 5.1. *For every geometric point $\mathcal{L}\in\overline{\mathsf{Jac}}_\Sigma^{-k}$, there exists an isomorphism*

$$\mathcal{P}_{C,\nu_*\mathcal{L}}\cong\rho_*\Big(\det(\mathcal{V}_\Sigma)\otimes\dot{\nu}^*\mathcal{P}_{\Sigma,\mathcal{L}(ky_0)}\Big), \tag{5.2}$$

and similarly for the dual sheaf:

$$\mathcal{P}_{C,\nu_*\mathcal{L}}^\vee\cong\rho_*\Big(\det(\mathcal{V}_\Sigma)^\vee\otimes\omega_{\mathsf{PMod}}\otimes\dot{\nu}^*\mathcal{P}_{\Sigma,\mathcal{L}(ky_0)}^\vee\Big). \tag{5.3}$$

Proof. Since C is nodal and integral, by means of Proposition 2.2, one can choose $n\gg 0$ such that the restriction of the Abel–Jacobi map to the curvilinear Hilbert scheme α_C is surjective. One can assume, without losing generality, that α_Σ is surjective too. In that case, and thanks to equivariance of the Poincaré sheaf under permutation [Ari13, Lemma 6.1.(c)] it is enough to describe the restriction $\mathcal{G}_{0,C,\nu_*\mathcal{L}}:=\mathcal{G}_{0,C}|_{\mathsf{Hilb}\times\{\nu_*\mathcal{L}\}}$ of $\mathcal{G}_{0,C}$ to the slice associated to $\nu_*\mathcal{L}\in\overline{\mathsf{Jac}}_C$.

As an application of proper base change and projection formula, one has the isomorphism $(\nu_*\mathcal{L})^{\boxtimes n}\cong\nu_*^n\left(\mathcal{L}^{\boxtimes n}\right)$, where $\nu^n:\Sigma^n\longrightarrow C^n$ is induced by ν. Specializing formula (2.3) to the slice henceforth results in

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \left(\psi_{C,*}\sigma_C^*\nu_*^n\mathcal{L}^{\boxtimes_n}\right)^{sign} \otimes \det(\mathcal{A}_{C,n})^\vee.$$

Consider the following Cartesian diagram

$$\begin{CD} \mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n @>{\widetilde{\sigma}}>> \Sigma^n \\ @V{\widetilde{\nu}}VV @VV{\nu^n}V \\ \mathsf{Flag}^{cur}_{n,C} @>>{\sigma_C}> C^n. \end{CD} \tag{5.4}$$

Observe that the data of a length 1 extension of ideal sheaves $\mathcal{I}_D \subset \mathcal{I}_{D'}$ over C, an ideal sheaf $\mathcal{I}_E$ such that $\mathsf{chow}_C(\mathcal{I}_D) = \mathsf{chow}_\Sigma(\mathcal{I}_E)$ and a point $y \in \Sigma$ whose image $\nu(y)$ coincides with $D' \setminus D$, determines naturally a length 1 extension of ideal sheaves $\mathcal{I}_E \subset \mathcal{I}_{E'}$ over Σ. To prove this claim note that either the partial normalisation $\nu : \Sigma \longrightarrow C$ is a local isomorphism around y or Σ is smooth at y. In the first case, the extension $\mathcal{I}_D \subset \mathcal{I}_{D'}$ determines naturally the extension $\mathcal{I}_E \subset \mathcal{I}_{E'}$ and in the second case, there is only one possible extension of ideal sheaves supported at a y as it is a smooth point. This implies that a point $((\mathcal{I}_{D_1}, \dots, \mathcal{I}_{D_n}), (y_1, \dots, y_n))$ in $\mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n$ determines naturally a point $(\mathcal{I}_{E_1}, \dots, \mathcal{I}_{E_n})$, so one obtains a morphism

$$s : \mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n \longrightarrow \mathsf{Flag}^{cur}_{n,\Sigma}$$

fitting in a commutative diagram

$$\begin{array}{ccc} \mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n & \xrightarrow{\;s\;} & \mathsf{Flag}^{cur}_{n,\Sigma} \\ & \searrow^{\widetilde{\sigma}} & \downarrow{\scriptstyle \sigma_\Sigma} \\ & & \Sigma^n \end{array} . \tag{5.5}$$

Making use of σ_C, σ_Σ and s defined above, one can easily construct the morphism $\widetilde{\psi}$ fitting in the commutative square

$$\begin{CD} \mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n @>{\widetilde{\psi}}>> \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma} \\ @V{\widetilde{\nu}}VV @VV{\beta}V \\ \mathsf{Flag}^{cur}_{n,C} @>>{\psi_C}> \mathsf{Hilb}^{cur}_{n,C} \end{CD} . \tag{5.6}$$

Recalling that ν^n is proper (because ν is so), we use proper base change around (5.4), and functoriality around (5.5) and (5.6), to get

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \left(\beta_*\widetilde{\psi}_* s^*\sigma_\Sigma^*\mathcal{L}^{\boxtimes_n}\right)^{sign} \otimes \det(\mathcal{A}_{C,n})^\vee,$$

where the permutation action lifts to $\mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n$ by pullback under $\widetilde{\nu}$. By invariance of β with respect to this action we obtain,

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \beta_*\left(\widetilde{\psi}_* s^*\sigma_\Sigma^*\mathcal{L}^{\boxtimes_n}\right)^{sign} \otimes \det(\mathcal{A}_{C,n})^\vee.$$

We shall see next that the previous construction factors through the closed embedding $j : \mathsf{Hilb}^{cur}_{n,C} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu \hookrightarrow \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Hilb}^{cur}_{n,\Sigma}$ of (4.3).

We claim that the support of $\left(\widetilde{\psi}_* s^*\sigma_\Sigma^*\mathcal{L}^{\boxtimes_n}\right)^{sign}$ lies in the image of j. This will be proved in Lemma 5.2 below. Consequently, one has that

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \widetilde{\rho}_* j^*\left(\widetilde{\psi}_* s^*\sigma_\Sigma^*\mathcal{L}^{\boxtimes_n}\right)^{sign} \otimes \det(\mathcal{A}_{C,n})^\vee,$$

where we observe, as in (4.2), that $\widetilde{\rho} = \beta \circ j$.

The projection $\psi_C : \mathsf{Flag}^{cur}_{n,C} \longrightarrow \mathsf{Hilb}^{cur}_{n,C}$, gives rise to $\mathsf{Flag}^{cur}_{n,C} \times_{C^n} \Sigma^n \longrightarrow \mathsf{Hilb}^{cur}_{n,C} \times_{\mathsf{H}_\nu} \mathsf{Flag}^{cur}_{n,\Sigma}$. By definition, the elements of $\mathsf{Flag}^{cur}_{n,C}$ and $\mathsf{Hilb}^{cur}_{n,C}$ are locally contained in smooth curves. Hence, starting from $D \in \mathsf{Hilb}^{cur}_{n,C}$, one determines uniquely a filtration of D out of a filtration on $\mathsf{Flag}_{n,\Sigma}$. This naturally provides an inverse to the previous morphism, so

$$\mathsf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n \cong \mathsf{Hilb}_{n,C}^{cur} \times_{\mathsf{H}_\nu} \mathsf{Flag}_{n,\Sigma}^{cur}.$$

The statement above, followed by the isomorphism

$$\mathsf{Hilb}_{n,C}^{cur} \times_{\mathsf{H}_\nu} \mathsf{Flag}_{n,C}^{cur} \cong (\mathsf{Hilb}_{n,C}^{cur} \times_{\mathsf{H}_\nu} \mathsf{Hilb}_{n,\Sigma}^{cur}) \times_{\mathsf{Hilb}_{n,\Sigma}^{cur}} \mathsf{Flag}_{n,\Sigma}^{cur},$$

provides us with the Cartesian diagram,

$$\begin{array}{ccc} \mathsf{Flag}_{n,C}^{cur} \times_{C^n} \Sigma^n & \xrightarrow{\widetilde{\psi}} & \mathsf{Hilb}_{n,C}^{cur} \times_{\mathsf{H}_\nu} \mathsf{Hilb}_{n,\Sigma}^{cur} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{Flag}_{n,\Sigma}^{cur} & \xrightarrow[\psi_\Sigma]{} & \mathsf{Hilb}_{n,\Sigma}^{cur}, \end{array}$$

where q denotes the obvious projection. Since π_Σ is proper, then proper base change with respect to it gives

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \widetilde{\rho}_* j^* \left(q^* \psi_{\Sigma,*} \sigma_\Sigma^* \mathcal{L}^{\boxtimes_n} \right)^{sign} \otimes \det(\mathcal{A}_{C,n})^\vee,$$

and by equivariance with respect to the action of the symmetric group,

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \widetilde{\rho}_* j^* q^* \left(\psi_{\Sigma,*} \sigma_\Sigma^* \mathcal{L}^{\boxtimes_n} \right)^{sign} \otimes \det(\mathcal{A}_{C,n})^\vee. \tag{5.7}$$

We now turn our attention to the sheaf $\mathcal{G}_{-k,\Sigma}$. After (2.3) (applied to Σ), we have that the restriction $\mathcal{G}_{-k,\Sigma}|_{\mathsf{Hilb}_{n,\Sigma} \times \{\mathcal{L}\}}$ to slice associated to $\mathcal{L} \in \mathsf{Jac}_\Sigma^{-k}$ is

$$\mathcal{G}_{-k,\Sigma,\mathcal{L}} \cong (\psi_{\Sigma,*} \sigma_\Sigma^* (\mathcal{L}^{\boxtimes_n}))^{sign} \otimes \det(\mathcal{A}_{\Sigma,n})^\vee.$$

Recalling Lemma 2.7 (again applied to Σ and to the point y_0),

$$\mathcal{G}_{-k,\Sigma,\mathcal{L}} \cong \mathcal{G}_{0,\Sigma,\mathcal{L}(ky_0)} \otimes \mathcal{O}_{\mathsf{Hilb}_\Sigma}(\mathcal{D}_{\Sigma,y_0})^{-k}.$$

Using the two previous isomorphisms in (5.7) yields

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \widetilde{\rho}_* j^* q^* \left(\mathcal{G}_{0,\Sigma,\mathcal{L}(ky_0)} \otimes \det(\mathcal{A}_{\Sigma,n}) \otimes \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{\Sigma,y_0})^{-k} \right) \otimes \det(\mathcal{A}_{C,n})^\vee.$$

Statement (2.4) and functoriality with respect to the commutative diagram

$$\begin{array}{ccc} \mathsf{Hilb}_{n,C}^{cur} \times_{\overline{\mathsf{Jac}}_C} \mathsf{PMod}_\nu & \xrightarrow{\tilde{\alpha}_C} & \mathsf{PMod}_\nu \\ {\scriptstyle q\circ j}\downarrow & & \downarrow{\scriptstyle \dot{\nu}} \\ \mathsf{Hilb}_{n,\Sigma}^{cur} & \xrightarrow[\alpha_\Sigma]{} & \overline{\mathsf{Jac}}_\Sigma, \end{array}$$

gives rise to the isomorphism

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \widetilde{\rho}_* \left(\widetilde{\alpha}_C^* \dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(ky_0)} \otimes j^* q^* \det(\mathcal{A}_{\Sigma,n}) \otimes j^* q^* \mathcal{O}_{\mathsf{Hilb}_C}(\mathcal{D}_{\Sigma,y_0})^{-k} \right) \otimes \det(\mathcal{A}_{C,n})^\vee.$$

An application of the projection formula together with Proposition 4.7, allows us to write

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \widetilde{\rho}_* \widetilde{\alpha}_C^* \Big(\dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(ky_0)} \otimes \det(\mathcal{V}_\Sigma) \Big).$$

Proper base change with respect to the Cartesian diagram (4.2) provides us with the conclusive isomorphism

$$\mathcal{G}_{0,C,\nu_*\mathcal{L}} \cong \alpha_C^* \rho_* \left(\dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(ky_0)} \otimes \det(\mathcal{V}_\Sigma) \right). \tag{5.8}$$

Note that $\rho_* \left(\dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(ky_0)} \otimes \det(\mathcal{V}_\Sigma) \right)$ is maximal Cohen–Macaulay as $\rho : \mathsf{PMod}_\nu \longrightarrow \overline{\mathsf{Jac}}_C$ is finite and $\dot{\nu}^* \mathcal{P}_{\Sigma,\mathcal{L}(ky_0)} \otimes \det(\mathcal{V}_\Sigma)$ is maximal Cohen–Macaulay. Therefore, both sides of (5.8) extend along $\mathsf{Hilb}_{C,n}^{cur} \subset \mathsf{Hilb}_{C,n}$ to isomorphic maximal Cohen–Macaulay sheaves using [Ari13, Lemma 2.2] and [MRV19b, Lemma 3.9]. Now, the faithfulness of the pullback along the projective bundle $A_C : \mathsf{Hilb}_{C,n} \longrightarrow \overline{\mathsf{Jac}}_C$ (recall Assumption 4.2) implies the first statement of the proposition, namely (5.2).

We address the second statement making use of Grothendieck-Verdier duality in (5.2), recalling that ρ is proper and that the dualising sheaf of $\overline{\mathsf{Jac}}_C$ is trivial:

$$\begin{aligned}\mathcal{P}^{\vee}_{C,\nu_*\mathcal{L}} &\cong \mathcal{H}om(\rho_*\left(\dot{\nu}^*\mathcal{P}_{\Sigma,\mathcal{L}(ky_0)}\otimes\det(\mathcal{V}_\Sigma)\right),\mathcal{O}_{\overline{\mathsf{Jac}}_C})\\ &\cong \rho_*(\mathcal{H}om(\dot{\nu}^*\mathcal{P}_{\Sigma,\mathcal{L}(ky_0)}\otimes\det(\mathcal{V}_\Sigma),\omega_{\mathsf{PMod}}))\\ &\cong \rho_*(\dot{\nu}^*\mathcal{P}^{\vee}_{\Sigma,\mathcal{L}(ky_0)}\otimes\det(\mathcal{V}_\Sigma)^{\vee}\otimes\omega_{\mathsf{PMod}}).\end{aligned}$$

Now, the second statement (5.3) follows naturally from (5.2) and Lemma 2.1 of [Ari13]. This concludes the proof of Proposition 5.1. □

The above proof relied on the following Lemma, whose proof we now address.

Lemma 5.2. *The support of the sheaf* $\left(\widetilde{\psi}_*\widetilde{\sigma}^*\mathcal{L}^{\boxtimes_n}\right)^{sign}$ *is contained in the image of the map*

$$j:\mathsf{Hilb}^{cur}_{n,C}\times_{\overline{\mathsf{Jac}}_C}\mathsf{PMod}_\nu\longrightarrow\mathsf{Hilb}^{cur}_{n,C}\times_{\mathsf{H}_\nu}\mathsf{Hilb}^{cur}_{n,\Sigma},$$

as defined in (4.3).

Proof. Recall that the action of the symmetric group $\mathfrak{S}_n$ on $\mathsf{Flag}^{cur}_{n,C}$ is induced by its action on C^n. It follows that, for $\gamma\in\mathfrak{S}_n$, the fixed point set $\left(\mathsf{Flag}^{cur}_{n,C}\right)^\gamma$ coincides with the preimage of $(C^n)^\gamma$ inside $\mathsf{Flag}^{cur}_{n,C}$. Furthermore, since we consider the action of $\mathfrak{S}_n$ on $\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n$ to be the one obtained by pullback under $\widetilde{\nu}:\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\longrightarrow\mathsf{Flag}^{cur}_{n,C}$, and if $\left(\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\right)^\gamma$ denotes the locus fixed by γ, then

$$\left(\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\right)^\gamma\cong\left(\mathsf{Flag}^{cur}_{n,C}\right)^\gamma\times_{(C^n)^\gamma}(\nu^n)^{-1}(C^n)^\gamma. \tag{5.9}$$

By Lemma 4.5, the image of the immersion j is closed in $\mathsf{Hilb}^{cur}_{n,C}\times_{\mathsf{H}_\nu}\mathsf{Sym}^n_\Sigma$, and by Lemma 4.6 lies over the big diagonal $\Delta\subset\mathsf{H}_\nu$, so it is contained in $\mathsf{chow}^{-1}(\Delta)\times_\Delta(\nu^{(n)})^{-1}(\Delta)\subset\mathsf{Hilb}^{cur}_{n,C}\times_{\mathsf{H}_\nu}\mathsf{Sym}^n_\Sigma$. Observe that Δ is the image under $\pi_C:C^n\longrightarrow\mathsf{H}_\nu$ of the union of $(C^n)^\gamma$, with γ varying in $\mathfrak{S}_n\setminus\{0\}$. It follows, using (5.9), that the complement of the image of j is contained in

$$\bigcup_{\gamma\in\mathfrak{S}_n\setminus\{0\}}\widetilde{\psi}\left(\left(\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\right)^\gamma\right).$$

Consider $\gamma\in\mathfrak{S}_n$ odd , since γ is the identity on $\left(\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\right)^\gamma$, it follows that the restriction of $\left(\widetilde{\psi}_*\widetilde{\sigma}^*\mathcal{L}^{\boxtimes_n}\right)^{sign}$ to $\widetilde{\psi}\left(\left(\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\right)^\gamma\right)$ vanishes as the sections must satisfy $-s=\gamma^*s=s$. For any non-trivial even $\gamma\in\mathfrak{S}_n$, there always exists an odd permutation γ' acting trivially on $\left(\mathsf{Flag}^{cur}_{n,C}\times_{C^n}\Sigma^n\right)^\gamma$, the locus fixed by γ and $\left(\widetilde{\psi}_*\widetilde{\sigma}^*\mathcal{L}^{\boxtimes_n}\right)^{sign}$ vanishes there. Hence, $\left(\widetilde{\psi}_*\widetilde{\sigma}^*\mathcal{L}^{\boxtimes_n}\right)^{sign}$ vanishes on the complement of $\mathsf{im}(j)$ □

With the fibrewise comparison between $\mathcal{P}_C$ and $\mathcal{P}_\Sigma$ at hand, we now pass to the global comparison result, which describes how the Poincaré sheaves interact with the partial normalisation map.

Theorem 5.3. *Let C be an integral nodal curve. Fix a partial normalisation $\nu:\Sigma\longrightarrow C$ that resolves precisely k nodes. Pick $y_0\in\Sigma$ such that $\nu(y_0)$ is a smooth point of C. Then, there exists an isomorphism*

$$(\check{\nu}\times\mathrm{id})^*\mathcal{P}_C\cong(\mathrm{id}\times\rho)_*\left(\mathsf{q}_2^*\det(\mathcal{V}_\Sigma)\otimes(\tau_{k,y_0}\times\dot{\nu})^*\mathcal{P}_\Sigma\right), \tag{5.10}$$

over $\overline{\mathsf{Jac}}_\Sigma^{-k}\times\overline{\mathsf{Jac}}_C$, *where* $\mathsf{q}_2:\overline{\mathsf{Jac}}_\Sigma^{-k}\times\mathsf{PMod}_\nu\longrightarrow\mathsf{PMod}_\nu$ *is the natural projection.*

For the dual sheaves, we similarly have

$$(\mathrm{id}\times\check{\nu})^*\mathcal{P}_C^{\vee}\cong(\rho\times\mathrm{id})_*\left(\mathsf{q}_1^*\det(\mathcal{V}_\Sigma)^{\vee}\otimes\mathsf{q}_1^*\omega_{\mathsf{PMod}}\otimes(\dot{\nu}\times\tau_{k,y_0})^*\mathcal{P}_\Sigma^{\vee}\right). \tag{5.11}$$

where $\mathsf{q}_1:\mathsf{PMod}_\nu\times\overline{\mathsf{Jac}}_\Sigma^{-k}\longrightarrow\mathsf{PMod}_\nu$ *is the natural projection.*

Remark 5.4. *In Theorem 5.3, the term* $\det(\mathcal{V}_\Sigma)$ *universally reflects the possible variations of the subspaces* V *that define points* $(M,V) \in \mathsf{PMod}^d_\nu$ *on the moduli of parabolic modules (see Definition 3.5).*

Remark 5.5. *The Poincaré sheaf* $\mathcal{P}_C$ *is the unique maximal Cohen-Macaulay extension of the Poincaré line bundle on* $\overline{\mathsf{Jac}}_C \times \mathsf{Jac}_C \cup \mathsf{Jac}_C \times \overline{\mathsf{Jac}}_C$ *defined via determinant of cohomology [Ari13, Theorem A]. Moreover, this Poincaré line bundle in degree* 0 *is independent of the choice of normalisation of the universal sheaf* $\mathcal{U}_C$ *[MRV19b, Remark 4.2]. Hence the left-hand-side of formula (5.10) is independent of the choice of the smooth point* y_0*.*

That the right-hand-side of (5.10) is independent of this choice can be seen a posteriori by Theorem 5.3, but also a priori as follows. Changing the fixed point to $y_1 \in \Sigma \setminus \nu^{-1}(\mathsf{Sing}(C))$ *acts on the Poincaré sheaf by*

$$(\tau_{k,y_1} \times \mathrm{id})^* \mathcal{P}_\Sigma \cong \pi_2^* \mathcal{P}_{\Sigma, \mathcal{O}(y_1-y_0)^k} \otimes (\tau_{k,y_0} \times \mathrm{id})^* \mathcal{P}_\Sigma$$

as proven by Arinkin in [Ari13, Lemma 6.5]. Here π_2 *is the projection onto the second factor of* $\overline{\mathsf{Jac}}_\Sigma \times \overline{\mathsf{Jac}}_\Sigma$*. Moreover, by compatibility with the Abel-Jacobi map one has* $\mathcal{P}_{\Sigma,\mathcal{O}(y_1-y_0)^k} \cong \mathcal{P}^k_{\Sigma,\mathcal{O}(y_1-y_0)} \cong \mathcal{U}^k_{\Sigma,y_1}$*, and thus* $(\tau_{k,y_1} \times \dot\nu)^* \mathcal{P}_\Sigma \cong \pi_2^* \dot\nu^* \mathcal{U}^k_{\Sigma,y_1} \otimes (\tau_{k,y_0} \times \dot\nu)^* \mathcal{P}_\Sigma$*. On the other hand,* $\det(\mathcal{V}_\Sigma)$ *depends on the choice of* y_0 *and changes to* $\det(\mathcal{V}_\Sigma) \otimes \dot\nu^* \mathcal{U}^{-k}_{\Sigma,y_1}$ *upon choosing* y_1*. A cancellation of terms allows us to conclude the right-hand-side is independent of choosing* y_0*. The same conclusion holds for (5.11).*

We need to recall the *see-saw* principle before addressing the proof of Theorem 5.3. Here, we reproduce the statement as in [MRV19b, Lemma 5.5], adapting the hypothesis to our case.

Lemma 5.6 (See-saw principle)**.** *Let* Z *and* T *be two reduced locally Noetherian schemes with* Z *proper and connected. Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be two sheaves on* $T \times Z$*, flat over* T*, such that*

(i) $\mathcal{F}|_{\{t\}\times Z} \cong \mathcal{E}|_{\{t\}\times Z}$*, for all* $t \in T$*;*

(ii) $\mathcal{F}|_{\{t\}\times Z}$ *is simple for every* $t \in T$*;*

(iii) there exists $z_0 \in Z$ *and an isomorphism of line bundles* $\mathcal{F}|_{T\times\{z_0\}} \cong \mathcal{E}|_{T\times\{z_0\}}$*.*

Then $\mathcal{E}$ *and* $\mathcal{F}$ *are isomorphic.*

Proof of Theorem 5.3. It suffices to show that the right and left-hand-sides of (5.10) and (5.11) satisfy the hypothesis of the see-saw principle.

First we check the flatness hypothesis. Let $\mathcal{H} := \mathsf{q}_2^* \det(\mathcal{V}_\Sigma) \otimes (\tau_{k,y_0} \times \dot\nu)^* \mathcal{P}_\Sigma$. This is a sheaf over $\overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu$ that is flat over $\overline{\mathsf{Jac}}_\Sigma^{-k}$. Thus pushforward $(\mathrm{id} \times \rho)_* \mathcal{H}$ along the affine map $\mathrm{id} \times \rho$ is flat over $\overline{\mathsf{Jac}}_\Sigma^{-k}$ by [Sta26, Lemma 29.25.4]. Hence the right-hand side of (5.10) is a flat sheaf over the first factor. Similarly, starting with the sheaf $\mathsf{q}_1^* \omega_{\mathsf{PMod}} \otimes \mathcal{H}^\vee$, one can show that the right-hand side of (5.11) is a flat sheaf over the second factor. For the left-hand sides of (5.10) and (5.11), flatness follows by noting that $\mathcal{P}_C$ and $\mathcal{P}_C^\vee$ are flat over the second factor, and so $(\check\nu \times \mathrm{id})^* \mathcal{P}_C$ and $(\mathrm{id} \times \check\nu)^* \mathcal{P}_C^\vee$ are flat over $\overline{\mathsf{Jac}}_\Sigma^{-k}$.

Now we deal with (i)-(iii). Proposition 5.1 implies that condition (i) holds for (5.10) and (5.11). Condition (ii) is automatic since torsion-free rank one sheaves on irreducible varieties are simple. Since ρ is locally an isomorphism at $\mathcal{O}_C \in \overline{\mathsf{Jac}}_C$ and $\mathsf{q}_2^* \det(\mathcal{V}_\Sigma)$ is pullback from PMod_ν the right-hand-side restricted to $\overline{\mathsf{Jac}}_\Sigma^{-k} \times \{\mathcal{O}_C\}$ is trivial as the Poincaré bundle of Σ restricts to the trivial bundle on $\overline{\mathsf{Jac}}_\Sigma \times \{\mathcal{O}_\Sigma\}$. Therefore, hypothesis (iii) holds for (5.10). Similarly, $\mathcal{P}_C^\vee$ and $\mathcal{P}_\Sigma^\vee$ are trivial when restricted to $\{\mathcal{O}_C\} \times \overline{\mathsf{Jac}}_C$ and $\{\mathcal{O}_\Sigma\} \times \overline{\mathsf{Jac}}_\Sigma$, implying (iii) in the case of (5.11). □

5.2. **Equivalence of Fourier–Mukai transforms.** Using the description of the Poincaré sheaves obtained in Section 5.1, we provide here a relation between the associated Fourier–Mukai transforms.

Theorem 5.7. *Let* C *be an integral nodal curve. Fix a partial normalisation* $\nu : \Sigma \longrightarrow C$ *that resolves precisely* k *nodes. Pick* $y_0 \in \Sigma$ *such that* $\nu(y_0)$ *is a smooth point of* C*. Then, for every object*

$\mathcal{F}^\bullet \in D^b(\overline{\mathsf{Jac}}_\Sigma^{-k})$, *one has an isomorphism*

$$\Phi^{\mathcal{P}_C}_{1\to 2}\left(\check{\nu}_*\mathcal{F}^\bullet\right) \cong \rho_*\left(\det(\mathcal{V}_\Sigma)\otimes \dot{\nu}^*\Phi^{\mathcal{P}_\Sigma}_{1\to 2}(\tau_{k,y_0,*}\mathcal{F}^\bullet)\right), \tag{5.12}$$

and, similarly, for the inverse transform one has

$$\Phi^{\mathcal{P}_C^\vee}_{2\to 1}\left(\check{\nu}_*\mathcal{F}^\bullet\right) \cong \rho_*\left(\det(\mathcal{V}_\Sigma)^\vee\otimes \omega_{\mathsf{PMod}}\otimes \dot{\nu}^*\Phi^{\mathcal{P}_\Sigma^\vee}_{2\to 1}(\tau_{k,y_0,*}\mathcal{F}^\bullet)\right). \tag{5.13}$$

Remark 5.8. *Theorem 5.7 can be interpreted as a rule for composing Fourier–Mukai transforms with the convolution functor*

$$\Theta^{\mathcal{V}_\Sigma} := \rho_*\big(\det(\mathcal{V}_\Sigma)\otimes\dot{\nu}^*(\bullet)\big) : D^b(\overline{\mathsf{Jac}}_\Sigma)\longrightarrow D^b(\overline{\mathsf{Jac}}_C), \tag{5.14}$$

where $\det(\mathcal{V}_\Sigma)$ *is a tautological choice of convolution kernel, universal for the variation of the vector spaces* V *for points* (M,V) *in the moduli* PMod_ν *of parabolic modules. Indeed, Theorem 5.7 provides a natural equivalence*

$$\Phi^{\mathcal{P}_C}_{1\to 2}\circ\check{\nu}_*\simeq\Theta^{\mathcal{V}_\Sigma}\circ\Phi^{\mathcal{P}_\Sigma}_{1\to 2}\circ\tau_{k,y_0},$$

emulating formulae for composing integration and convolution in Fourier analysis.

Remark 5.9. *Recall that* ρ *is a finite morphism and hence the pushforward* ρ_* *is exact (see [Gro61, Corollaire 5.2.2]). Thus, from Theorem 5.7 we can deduce that the pushforward* $\check{\nu}_*\mathcal{F}$ *will be WIT for* $\Phi^{\mathcal{P}_C}$ *for every coherent sheaf* $\mathcal{F}$ *on* $\overline{\mathsf{Jac}}_\Sigma^0$ *that is WIT for* $\Phi^{\mathcal{P}_\Sigma}$.

Remark 5.10. *Similarly to Remark 5.5, we can see that the right-hand-side of (5.12) and (5.13) do not depend on the choice of* y_0. *We will see that for (5.12), the case of (5.13) being the same. If we choose* $y_1\in\Sigma\setminus\nu^{-1}(\mathsf{Sing}(C))$ *instead, then* $\det(\mathcal{V}_\Sigma)$ *changes to* $\det(\mathcal{V}_\Sigma)\otimes\dot{\nu}^*\mathcal{U}^{-k}_{\Sigma,y_1}$. *Now, we look at* $\Phi^{\mathcal{P}_\Sigma}_{1\to 2}(\tau_{k,y_1,*}\mathcal{F}^\bullet)$. *Let* $\widetilde{\tau}:\mathsf{Jac}^0_\Sigma\longrightarrow\mathsf{Jac}^0_\Sigma$ *given by* $\widetilde{\tau}(M)=M\otimes\mathcal{O}_\Sigma(ky_1-ky_0)$, *so that* $\tau_{k,y_1}=\widetilde{\tau}\circ\tau_{k,y_0}$. *Then, using base change, projection formula and [Ari13, Lemma 6.5]*

$$\begin{aligned}
\Phi^{\mathcal{P}_\Sigma}_{1\to 2}(\tau_{k,y_1,*}\mathcal{F}^\bullet) &= R\pi_{2,*}\big(\pi_1^*\widetilde{\tau}_*\tau_{k,y_0,*}\mathcal{F}^\bullet\otimes\mathcal{P}_\Sigma\big)\\
&\cong R\pi_{2,*}(\mathrm{id}\times\widetilde{\tau})_*\big(\pi_1^*\tau_{k,y_0,*}\mathcal{F}^\bullet\otimes(\mathrm{id}\times\widetilde{\tau})^*\mathcal{P}_\Sigma\big)\\
&\cong R\pi_{2,*}\big(\pi_1^*\tau_{k,y_0,*}\mathcal{F}^\bullet\otimes(\mathrm{id}\times\tau_{-k,y_0})^*(\pi_2^*\mathcal{P}_{\Sigma,\mathcal{O}(y_1-y_0)^k}\otimes(\mathrm{id}\times\tau_{k,y_0})^*\mathcal{P}_\Sigma)\big)\\
&\cong R\pi_{2,*}\big(\pi_1^*\tau_{k,y_0,*}\mathcal{F}^\bullet\otimes\mathcal{P}_\Sigma\big)\otimes\mathcal{P}_{\Sigma,\mathcal{O}(y_1-y_0)^k}\\
&\cong \Phi^{\mathcal{P}_\Sigma}_{1\to 2}(\tau_{k,y_0,*}\mathcal{F}^\bullet)\otimes\mathcal{U}^k_{\Sigma,y_1}.
\end{aligned}$$

This proves the independence of the choice of y_0.

Proof of Theorem 5.7. Recall the construction of $\Phi^{\mathcal{P}_C}_{1\to 2}$ described in (2.5). Following the Cartesian diagram

$$\begin{array}{ccc}
\overline{\mathsf{Jac}}_\Sigma^{-k}\times\overline{\mathsf{Jac}}_C & \xrightarrow{\ \mathrm{t}_1\ } & \overline{\mathsf{Jac}}_\Sigma^{-k}\\
{\scriptstyle(\check{\nu}\times\mathrm{id})}\big\downarrow & & \big\downarrow{\scriptstyle\check{\nu}}\\
\overline{\mathsf{Jac}}_C\times\overline{\mathsf{Jac}}_C & \xrightarrow[\ \pi_1\ ]{} & \overline{\mathsf{Jac}}_C,
\end{array}$$

one has

$$\Phi^{\mathcal{P}_C}_{1\to 2}(\check{\nu}_*\mathcal{F}^\bullet)\cong R\mathrm{t}_{2,*}\left(\mathrm{t}_1^*\mathcal{F}^\bullet\otimes(\check{\nu}\times\mathrm{id})^*\mathcal{P}_C\right), \tag{5.15}$$

where we have applied flat base change, projection formula and functoriality with respect to

$$\begin{array}{ccc}
\overline{\mathsf{Jac}}_\Sigma^{-k}\times\overline{\mathsf{Jac}}_C & & \\
{\scriptstyle(\check{\nu}\times\mathrm{id})}\big\downarrow & \searrow^{\mathrm{t}_2} & \\
\overline{\mathsf{Jac}}_C\times\overline{\mathsf{Jac}}_C & \xrightarrow[\ \pi_2\ ]{} & \overline{\mathsf{Jac}}_C.
\end{array}$$

We now substitute (5.10) into (5.15) and apply the projection formula with respect to t_2 and $(\rho \times \mathrm{id})$, and functoriality with respect to the commutative diagrams

$$\begin{array}{ccc} \overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu & & \\ \big\downarrow{\scriptstyle (\mathrm{id}\times\rho)} & \searrow^{\mathsf{q}_1} & \\ \overline{\mathsf{Jac}}_\Sigma^{-k} \times \overline{\mathsf{Jac}}_C & \xrightarrow[\mathsf{t}_1]{} & \overline{\mathsf{Jac}}_\Sigma^{-k}, \end{array} \qquad \begin{array}{ccc} \overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu & \xrightarrow{\mathsf{q}_2} & \mathsf{PMod}_\nu \\ \big\downarrow{\scriptstyle (\mathrm{id}\times\rho)} & & \big\downarrow{\scriptstyle \rho} \\ \overline{\mathsf{Jac}}_\Sigma^{-k} \times \overline{\mathsf{Jac}}_C & \xrightarrow[\mathsf{t}_2]{} & \overline{\mathsf{Jac}}_C\,, \end{array}$$

which allows us to write

$$\Phi^{\mathcal{P}_C}_{1\to 2}(\check{\nu}_* \mathcal{F}^\bullet) \cong \rho_* \left(\det(\mathcal{V}_\Sigma) \otimes R\mathsf{q}_{2,*}\left(\mathsf{q}_1^* \mathcal{F}^\bullet \otimes (\tau_{y_0} \times \dot{\nu})^* \mathcal{P}_\Sigma\right)\right) . \tag{5.16}$$

Due to functoriality with respect to the diagram on the left, and flat base change for the diagram on the right,

$$\begin{array}{ccc} \overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu & & \\ \big\downarrow{\scriptstyle (\mathrm{id}\times\dot{\nu})} & \searrow^{\mathsf{q}_1} & \\ \overline{\mathsf{Jac}}_\Sigma^{-k} \times \overline{\mathsf{Jac}}_\Sigma & \xrightarrow[\mathsf{r}_1]{} & \overline{\mathsf{Jac}}_\Sigma^{-k}, \end{array} \qquad \begin{array}{ccc} \overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu & \xrightarrow{\mathsf{q}_2} & \mathsf{PMod}_\nu \\ \big\downarrow{\scriptstyle (\mathrm{id}\times\dot{\nu})} & & \big\downarrow{\scriptstyle \dot{\nu}} \\ \overline{\mathsf{Jac}}_\Sigma^{-k} \times \overline{\mathsf{Jac}}_\Sigma & \xrightarrow[\mathsf{r}_2]{} & \overline{\mathsf{Jac}}_\Sigma\,, \end{array}$$

one observes that (5.16) becomes

$$\Phi^{\mathcal{P}_C}_{1\to 2}(\check{\nu}_* \mathcal{F}^\bullet) \cong \rho_* \left(\det(\mathcal{V}_\Sigma) \otimes \dot{\nu}^* R\mathsf{r}_{2,*}\left(\mathsf{r}_1^* \mathcal{F}^\bullet \otimes (\tau_{k,y_0} \times \mathrm{id})^* \mathcal{P}_\Sigma\right)\right) .$$

This proves the statement (5.12) after applying projection formula with respect to $\tau_{k,y_0} \times \mathrm{id}$.

Since the definition of $\Phi^{\mathcal{P}_C^\vee}_{2\to 1}$ given in (2.6) is analogous to that of (2.5) and the relation between $\mathcal{P}_C^\vee$ and $\mathcal{P}_\Sigma^\vee$ of (5.11) is analogous to (5.10), the proof (5.13) follows verbatim from the proof of (5.12). □

Example 5.11. We provide here an example where the formula of Theorem 5.7 can be reconfirmed by direct calculation. Suppose that C is a nodal elliptic curve, hence $k = 1$, and let the node be $b \in C$. Then $\mathsf{Jac}^0_C \cong C \setminus \{b\}$ and $\overline{\mathsf{Jac}}_C \cong C$ [Kas13, Section 3]. In addition, $\Sigma = \mathbb{P}^1$, $\mathsf{Jac}^0_{\mathbb{P}^1} = \{\mathcal{O}_{\mathbb{P}^1}\}$ and $\mathsf{PMod}_\nu = \mathbb{P}^1$. Under these identifications, the maps ν and ρ coincide and $\dot{\nu}$ is the constant map. Furthermore, $\mathcal{V}_\Sigma = \det(\mathcal{V}_\Sigma) = \mathcal{O}_{\mathbb{P}^1}(-1)$. Now, take $\mathcal{F}^\bullet = \mathcal{O}_{\mathsf{Jac}^{-1}_{\mathbb{P}^1}}$ the trivial bundle over the point $\mathsf{Jac}^{-1}_{\mathbb{P}^1} = \{\mathcal{O}_{\mathbb{P}^1}(-1)\}$. Then,

$$\rho_*\big(\det(\mathcal{V}_\Sigma) \otimes \dot{\nu}^* \Phi^{\mathcal{P}_{\mathbb{P}^1}}_{1\to 2}(\tau_{1,y_0,*}\mathcal{O}_{\mathsf{Jac}^{-1}_{\mathbb{P}^1}})\big) = \rho_* \mathcal{O}_{\mathbb{P}^1}(-1).$$

On the other hand, $\Phi^{\mathcal{P}_C}_{1\to 2}\big(\check{\nu}_* \mathcal{O}_{\mathsf{Jac}^{-1}_{\mathbb{P}^1}}\big) = \Phi^{\mathcal{P}_C}_{1\to 2}(\mathcal{O}_{\tilde{b}})$ where $\tilde{b}$ is the point in $\overline{\mathsf{Jac}}_C \cong C$ corresponding to the node $b \in C$. By the universal property of $\mathcal{P}_C$, $\Phi^{\mathcal{P}_C}_{1\to 2}(\mathcal{O}_{\tilde{b}})$ is the unique rank 1 and degree zero torsion-free sheaf on $\overline{\mathsf{Jac}}_C$ which is not a line bundle. With the isomorphism $\overline{\mathsf{Jac}}_C \cong C$ this corresponds to the unique such object but now on C, this being $\nu_* \mathcal{O}(-1)$. Hence,

$$\Phi^{\mathcal{P}_C}_{1\to 2}(\mathcal{O}_{\tilde{b}}) \cong \rho_* \mathcal{O}_{\mathbb{P}^1}(-1).$$

We then see that the claimed isomorphism holds.

5.3. **Spin-valued Wilson operators.** We explain how a spin structure on PMod_ν and a spin-valued Wilson operator leads to a more symmetric presentation of our Fourier–Mukai transform formulae from Theorem 5.7. This form of our results will be applied to mirror symmetry for branes in the Hitchin system in a sequel article [FHHOa]. The appearance of a spin structure compensation for the fact that $\nu : \Sigma \longrightarrow C$ is two-to-one over the locus $\mathsf{Exc}(\nu) \subset \Sigma$ of resolved singularities, so a Wilson loop over C (as per [KW07, §6.1]) lifts to a so-called *'Wilson spin-half loop'* over Σ.

Precisely, the Wilson operators in question are defined as follows. Given a point $y \in \mathsf{Exc}(\nu) \subset \Sigma$, we choose a square root $\mathcal{U}^{1/2}_{\Sigma,y}$ of the line bundle $\mathcal{U}_{\Sigma,y} := \mathcal{U}_\Sigma|_{\{y\}\times \overline{\mathsf{Jac}}_\Sigma}$. The restriction $\mathcal{U}_{\Sigma,y}$ is locally free, even though $\mathcal{U}_\Sigma$ is not, because $y \in \Sigma$ is by assumption a smooth point. Then, to any $y \in \mathsf{Exc}(\nu)$ and any subset $I = \{y_1, \dots, y_l\} \subset \mathsf{Exc}(\nu)$, let us introduce the notation

$$\mathbb{W}^{1/2}_{\Sigma,y} := (\bullet) \otimes \mathcal{U}^{1/2}_{\Sigma,y} : D^b(\overline{\mathsf{Jac}}_\Sigma) \longrightarrow D^b(\overline{\mathsf{Jac}}_\Sigma), \quad \mathbb{W}^{1/2}_{\Sigma,\otimes I} := \mathbb{W}^{1/2}_{\Sigma,y_1} \circ \dots \circ \mathbb{W}^{1/2}_{\Sigma,y_l}, \tag{5.17}$$

and refer to these as the *half-twisted Wilson operators*. Note that the definition of $\mathbb{W}^{1/2}_{\Sigma,\otimes I}$ is independent of the labeling of the elements of I as tensor products commute. These functors naturally satisfy the relations $\mathbb{W}_{\Sigma,y} \simeq \mathbb{W}^{1/2}_{\Sigma,y} \circ \mathbb{W}^{1/2}_{\Sigma,y}$ and $\mathbb{W}_{\Sigma,I} \simeq \mathbb{W}^{1/2}_{\Sigma,I} \circ \mathbb{W}^{1/2}_{\Sigma,I}$. The divisor I of interest for us is the entire resolved locus $I = \mathsf{Exc}(\nu)$. For convenience, let us fix orderings $\mathsf{RSing}(\nu) = \{b_1, \dots, b_k\}$ and $\mathsf{Exc}(\nu) = \{b_1^\pm, \dots, b_k^\pm\}$. The construction of $\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}$ depends on the family of choices $\{\mathcal{U}^{1/2}_{\Sigma,b_i^\pm}\}_{b_i^\pm \in \mathsf{Exc}(\nu)}$, and we recall from Corollary 3.10 that such choices define a spin structure on PMod_ν given by

$$\omega^{1/2}_{\mathsf{PMod}} := \bigotimes_{i=1}^{k} \mathcal{V}_{\Sigma,i} \otimes \dot{\nu}^*\mathcal{U}^{-1/2}_{\Sigma,b_i^-} \otimes \dot{\nu}^*\mathcal{U}^{-1/2}_{\Sigma,b_i^+}. \tag{5.18}$$

Note that, by construction, there exists a natural equivalence $\det(\mathcal{V}_\Sigma) \otimes (\bullet) \simeq \omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^*\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}$, which we use to give the following immediate restatement of Theorems 5.3 and 5.7.

Corollary 5.12. *Let C be an integral nodal curve. Fix a partial normalisation $\nu : \Sigma \longrightarrow C$ that resolves precisely k nodes. Pick $y_0 \in \Sigma$ such that $\nu(y_0)$ is a smooth point of C. Moreover, fix square roots $\mathcal{U}^{1/2}_{\Sigma,b_i^\pm}$ and let $\omega^{1/2}_{\mathsf{PMod}}$ be the corresponding square root on PMod_ν as per (5.18).*

- *One has isomorphisms of Poincaré sheaves*
$$(\check{\nu} \times \mathrm{id})^*\mathcal{P}_C \cong (\mathrm{id} \times \rho)_* \left(\mathsf{q}_2^* \left(\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^*\mathcal{U}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}\right) \otimes (\tau_{k,y_0} \times \dot{\nu})^*\mathcal{P}_\Sigma\right),$$
$$(\mathrm{id} \times \check{\nu})^*\mathcal{P}_C^\vee \cong (\rho \times \mathrm{id})_* \left(\mathsf{q}_1^* \left(\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^*\mathcal{U}^{-1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}\right) \otimes (\dot{\nu} \times \tau_{k,y_0})^*\mathcal{P}_\Sigma^\vee\right),$$
where $\mathsf{q}_i : \overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu \longrightarrow \mathsf{PMod}_\nu$ are the natural projections.
- *For every $\mathcal{F}^\bullet \in D^b(\overline{\mathsf{Jac}}_\Sigma^{-k})$, one has isomorphisms of transformed sheaves*
$$\Phi^{\mathcal{P}_C}_{1\to 2}(\check{\nu}_*\mathcal{F}^\bullet) \cong \rho_* \left(\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^*\left(\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)} \circ \Phi^{\mathcal{P}_\Sigma}_{1\to 2}(\tau_{k,y_0,*}\mathcal{F}^\bullet)\right)\right),$$
$$\Phi^{\mathcal{P}_C^\vee}_{2\to 1}(\check{\nu}_*\mathcal{F}^\bullet) \cong \rho_* \left(\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^*\left(\mathbb{W}^{-1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)} \circ \Phi^{\mathcal{P}_\Sigma^\vee}_{2\to 1}(\tau_{k,y_0,*}\mathcal{F}^\bullet)\right)\right).$$

Remark 5.13. *A priori, the formulae in Corollary 5.12 appear to depend on the choice of square roots $\mathcal{U}^{1/2}_{\Sigma,b_i^\pm}$ and the resultant choice of spin structure $\omega^{1/2}_{\mathsf{PMod}}$. However, the functor*

$$\omega^{1/2}_{\mathsf{PMod}} \otimes \dot{\nu}^*\mathbb{W}^{1/2}_{\Sigma,\otimes\mathsf{Exc}(\nu)}(\bullet) \simeq \det(\mathcal{V}_\Sigma) \otimes (\bullet),$$

and therefore the formulae of Corollary 5.12, are independent of these choices.

5.4. **Restriction to compactified Prym varieties.** Motivated by the study of mirror symmetry between moduli spaces of $\mathsf{SL}(m,\mathbb{C})$ and $\mathsf{PGL}(m,\mathbb{C})$-Higgs bundles [HT03; DP12], this section shows that our main constructions and results are well-behaved upon restriction to compactified Prym varieties. In [FHR25; GS22], an equivalence is established between the derived category of a compactified Prym variety and a Γ-equivariant incarnation of the latter, where Γ is a group of m-torsion points of a base smooth curve. This can be understood as a duality statement between cuspidal $\mathsf{SL}(m,\mathbb{C})$ and $\mathsf{PGL}(m,\mathbb{C})$-Hitchin fibres, generalizing the autoduality of cuspidal $\mathsf{GL}(m,\mathbb{C})$-Hitchin fibres obtained by Arinkin [Ari13].

We begin by describing the compactified Prym varieties and their properties. Consider a partial normalisation map $\nu : \Sigma \longrightarrow C$ between irreducible nodal curves together with two ramified $m:1$-coverings $\beta_C : C \longrightarrow X$ and $\beta_\Sigma : \Sigma \longrightarrow X$ over a smooth projective curve X. We assume β_C and β_Σ fit into the commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \nu\ } & C \\ & {\scriptstyle\beta_\Sigma}\searrow \quad \swarrow{\scriptstyle\beta_C} & \\ & X. & \end{array} \tag{5.19}$$

Let us define the *norm map* associated to β_C to be the degree preserving morphism

$$\mathsf{Nm}_{\beta_C}(\bullet) := \det(\beta_{C,*}\mathcal{O}_C)^{-1} \otimes \det \circ \beta_{C,*}(\bullet) : \overline{\mathsf{Jac}}^d_C \longrightarrow \mathsf{Jac}^d_X,$$

and $\mathsf{Nm}_{\beta_\Sigma} : \overline{\mathsf{Jac}}^d_\Sigma \longrightarrow \mathsf{Jac}^d_X$ is defined identically. For $d = 0$, the associated *compactified Prym varieties* are defined to be the preimage under the norm map of the trivial line bundle,

$$\overline{\mathsf{Prym}}_C := \mathsf{Nm}^{-1}_{\beta_C}(\mathcal{O}_X) \quad \text{and} \quad \overline{\mathsf{Prym}}_\Sigma := \mathsf{Nm}^{-1}_{\beta_\Sigma}(\mathcal{O}_X)\,.$$

For the non-zero degree $d = -k$ we give a twisted version of the Prym construction, compatible with the normalisation $\nu : \Sigma \longrightarrow C$. It is convenient to consider the geometric point $\mathcal{O}(-\beta_C(\mathsf{RSing}(\nu)) \in \mathsf{Jac}^{-k}_X$ defined by the resolved singularities $\mathsf{RSing}(\nu) \subset C$ and take the preimage

$$\overline{\mathsf{Prym}}_\Sigma^{-k} := \mathsf{Nm}^{-1}_{\beta_\Sigma}(\mathcal{O}(-\beta_C(\mathsf{RSing}(\nu))))\,.$$

The compactified Prym varieties $\overline{\mathsf{Prym}}_C$, $\overline{\mathsf{Prym}}_\Sigma$ and $\overline{\mathsf{Prym}}_\Sigma^{-k}$ are Gorenstein varieties with trivial dualising sheaf (see [FHR25, Corollary 3.2] for instance). They come equipped with closed embeddings

$$\jmath_C : \overline{\mathsf{Prym}}_C \hookrightarrow \overline{\mathsf{Jac}}_C, \quad \jmath_\Sigma : \overline{\mathsf{Prym}}_\Sigma \hookrightarrow \overline{\mathsf{Jac}}_\Sigma \quad \text{and} \quad \jmath'_\Sigma : \overline{\mathsf{Prym}}_\Sigma^{-k} \hookrightarrow \overline{\mathsf{Jac}}_\Sigma^{-k}.$$

The choice of preimage in the construction of $\overline{\mathsf{Prym}}_\Sigma^{-k}$ is designed for the following purpose. The translation isomorphism

$$\begin{array}{rccc} \tau_{k,y_0} : & \overline{\mathsf{Jac}}_\Sigma^{-k} & \longrightarrow & \overline{\mathsf{Jac}}_\Sigma \\ & \mathcal{F} & \longmapsto & \mathcal{F} \otimes \mathcal{O}_\Sigma(ky_0), \end{array}$$

associated to a smooth point $y_0 \in \Sigma$, can be shown, after an additional twist, to restrict to $\overline{\mathsf{Prym}}_\Sigma$, simultaneously trivialising the torsors $\overline{\mathsf{Jac}}_\Sigma^{-k}$ and $\overline{\mathsf{Prym}}_\Sigma^{-k}$. We prove this in parallel with the following compatibility results between normalisation maps and Prym constructions. Let

$$\Gamma := \mathsf{Jac}_X[m]\,,$$

denote the subgroup of Jac_X consisting of n-torsion line bundles.

Lemma 5.14. *Let C be an integral nodal curve. Let $\nu : \Sigma \longrightarrow C$ be a partial normalisation resolving the divisor $\mathsf{RSing}(\nu) \subset \mathsf{Sing}(C)$ of cardinality k. Then, the following hold:*

(1) The tensoral action of Γ on $\overline{\mathsf{Jac}}_C$, $\overline{\mathsf{Jac}}_\Sigma$ and $\overline{\mathsf{Jac}}_\Sigma^{-k}$ restricts to $\overline{\mathsf{Prym}}_C$, $\overline{\mathsf{Prym}}_\Sigma$ and $\overline{\mathsf{Prym}}_\Sigma^{-k}$.

(2) The morphism $\check{\nu}$ restricts to the compactified Prym varieties, giving rise to a closed embedding

$$\check{\nu} : \overline{\mathsf{Prym}}_\Sigma^{-k} \hookrightarrow \overline{\mathsf{Prym}}_C,$$

which then lies in the commutative square

$$\begin{array}{ccc} \overline{\mathsf{Prym}}_\Sigma^{-k} & \xrightarrow{\check{\nu}} & \overline{\mathsf{Prym}}_C \\ {\scriptstyle \jmath'_\Sigma}\downarrow & & \downarrow{\scriptstyle \jmath_C} \\ \overline{\mathsf{Jac}}_\Sigma^{-k} & \xrightarrow[\check{\nu}]{} & \overline{\mathsf{Jac}}_C. \end{array} \tag{5.20}$$

(3) Let $\mathcal{L}$ be an m-th root of $\mathcal{O}_X(\beta_C(\mathsf{RSing}(\nu) - kx_0))$ and $\tau_{\mathcal{L}}$ be the translation

$$\tau_{\mathcal{L}} : \overline{\mathsf{Jac}}_\Sigma \xrightarrow{\cong} \overline{\mathsf{Jac}}_\Sigma, \quad \mathcal{F} \longrightarrow \mathcal{F} \otimes \beta_\Sigma^* \mathcal{L}\,. \tag{5.21}$$

Then the following diagram commutes:

$$\begin{array}{ccccc} \overline{\mathsf{Prym}}_\Sigma^{-k} & \xrightarrow[\cong]{\mathring{\tau}_{k,y_0}} & \mathsf{Nm}^{-1}_{\beta_\Sigma}(\mathcal{O}_X(k\beta_\Sigma(y_0) - \beta_C(\mathsf{RSing}(\nu)))) & \xrightarrow[\cong]{\mathring{\tau}_{\mathcal{L}}} & \overline{\mathsf{Prym}}_\Sigma \\ {\scriptstyle \jmath'_\Sigma}\downarrow & & \downarrow & & \downarrow{\scriptstyle \jmath_\Sigma} \\ \overline{\mathsf{Jac}}_\Sigma^{-k} & \xrightarrow[\tau_{k,y_0}]{\cong} & \overline{\mathsf{Jac}}_\Sigma & \xrightarrow[\tau_{\mathcal{L}}]{\cong} & \overline{\mathsf{Jac}}_\Sigma\,. \end{array} \tag{5.22}$$

Here $\mathring{\tau}_{k,y_0}$, $\mathring{\tau}_{\mathcal{L}}$ denote the restricted translation isomorphisms.

(4) In the following Cartesian square

$$\begin{array}{ccc} \mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C & \xrightarrow{\mathring{\rho}} & \overline{\mathsf{Prym}}_C \\ \imath_\nu \downarrow & & \downarrow \jmath_C \\ \mathsf{PMod}_\nu & \xrightarrow[\rho]{} & \overline{\mathsf{Jac}}_C , \end{array} \tag{5.23}$$

the composition of the inclusion $\imath_\nu$ with $\dot{\nu}$ factors through $\overline{\mathsf{Prym}}_\Sigma$, giving rise to the commutative diagram

$$\begin{array}{ccc} \mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C & \xrightarrow{\mathring{\nu}} & \overline{\mathsf{Prym}}_\Sigma \\ \imath_\nu \downarrow & & \downarrow \jmath_\Sigma \\ \mathsf{PMod}_\nu & \xrightarrow[\dot{\nu}]{} & \overline{\mathsf{Jac}}_\Sigma . \end{array} \tag{5.24}$$

Proof. As a direct consequence of the projection formula, we have, for any $\mathcal{L} \in \mathsf{Jac}^0_X$;

$$\mathsf{Nm}_{\beta_C}(\beta_C^* \mathcal{L} \otimes \bullet) \cong \mathcal{L}^m \otimes \mathsf{Nm}_{\beta_C}(\bullet) \quad \text{and} \quad \mathsf{Nm}_{\beta_\Sigma}(\beta_\Sigma^* \mathcal{L} \otimes \bullet) \cong \mathcal{L}^m \otimes \mathsf{Nm}_{\beta_\Sigma}(\bullet). \tag{5.25}$$

This immediately implies (1). For (2), we make use of the determinant formula

$$\det(R\beta_{C,*}\mathcal{O}_C) \cong \det(R\beta_{\Sigma,*}\mathcal{O}_\Sigma) \otimes \mathcal{O}_X(-\beta_C(\mathsf{RSing}(\nu))),$$

derived from pushforward of the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\mathsf{RSing}(\nu)} \longrightarrow 0.$$

Given a point $\mathcal{F} \in \overline{\mathsf{Prym}}_\Sigma^{-k}$, we then obtain the following isomorphisms:

$$\begin{aligned} \mathsf{Nm}_{\beta_C}(\nu_*\mathcal{F}) &= \det(\beta_{C,*}\mathcal{O}_C)^{-1} \otimes \det(\beta_{C,*}\nu_*\mathcal{F}) \\ &\cong \det(\beta_{\Sigma,*}\mathcal{O}_\Sigma)^{-1} \otimes \mathcal{O}_X(\beta_C(\mathsf{RSing}(\nu))) \otimes \det(\beta_{\Sigma,*}\mathcal{F}) \\ &\cong \mathsf{Nm}_{\beta_\Sigma}(\mathcal{F}) \otimes \mathcal{O}_X(\beta_C(\mathsf{RSing}(\nu))) \\ &\cong \mathcal{O}_X, \end{aligned}$$

and thus $\check{\nu}(\mathcal{F}) = \nu_*\mathcal{F}$ lands in $\overline{\mathsf{Prym}}_C$. This proves (2). For (3), by definition of the $\overline{\mathsf{Prym}}_\Sigma^{-k}$ we have

$$\tau_{k,y_0}\left(\overline{\mathsf{Prym}}_\Sigma^{-k}\right) \cong \mathsf{Nm}_{\beta_\Sigma}^{-1}\left(\mathcal{O}_X(-\beta_C \mathsf{RSing}(\nu) + k\beta_\Sigma(y_0))\right), \tag{5.26}$$

and it follows from (5.25) that composing with $\tau_{\mathcal{L}} = (\bullet) \otimes \beta_\Sigma^*\mathcal{L}$ lands in $\overline{\mathsf{Prym}}_\Sigma^0$.

For (4), we push forward under β_C the short exact sequence (3.6) associated to a point in $\mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}} \overline{\mathsf{Prym}}_C$, followed by taking determinants. We observe that the composition of the inclusion $\imath_\nu$ with the morphism $\dot{\nu} : \mathsf{PMod}_\nu \longrightarrow \overline{\mathsf{Jac}}_\Sigma$ factors through $\overline{\mathsf{Prym}}_\Sigma$ (see [GO13] for more details). □

Lemma 5.15. *There is a Γ-action on PMod_ν, such that the morphisms*

$$\overline{\mathsf{Jac}}_C \xleftarrow{\rho} \mathsf{PMod}_\nu \xrightarrow{\dot{\nu}} \overline{\mathsf{Jac}}_\Sigma$$

are Γ-equivariant. In particular, the universal bundle $\mathcal{V}_\Sigma \longrightarrow \mathsf{PMod}_\nu$ inherits a Γ-action.

Proof. Given a parabolic module (M, V), with $M \in \overline{\mathsf{Jac}}_\Sigma$, a n-torsion line bundle $L \in \Gamma$ acts by

$$L \cdot (M, V) = (\beta_\Sigma^* L \otimes M, W),$$

where

$$W \;:=\; V \otimes \beta_\Sigma^* L \;=\; \bigoplus_{i=1}^k V_i \otimes (\beta_\Sigma^* L)_{b_i^\pm} \;\subset\; \bigoplus_{i=1}^k (M_{b_i^-} \oplus M_{b_i^+}) \otimes (\beta_\Sigma^* L)_{b_i^\pm}.$$

Note that there is a natural isomorphism of stalks $(\beta_\Sigma^* L)_{b_i^+} \cong (\beta_\Sigma^* L)_{b_i^-} \cong L_{\beta_C(b)}$. Hence, W is well-defined. By definition of the action $\dot{\nu}$ is Γ-equivariant. Furthermore, the exact sequence (3.6) is compatible with the Γ-action on $\overline{\mathsf{Jac}}_C$ and the action on PMod_ν just defined. In particular, ρ is Γ-equivariant. □

Let us now restrict the Poincaré sheaves to our compactified Prym varieties. Consider the pullback the Poincaré sheaf along the closed embeddings of the Prym varieties,

$$\mathcal{R}_C := (\jmath_C \times \jmath_C)^*\mathcal{P}_C, \quad \mathcal{R}_\Sigma := (\jmath_\Sigma \times \jmath_\Sigma)^*\,\mathcal{P}_\Sigma, \tag{5.27}$$

supported on $\overline{\mathsf{Prym}}_C \times \overline{\mathsf{Prym}}_C$ and $\overline{\mathsf{Prym}}_\Sigma \times \overline{\mathsf{Prym}}_\Sigma$ respectively. These sheaves can be related via the following Prymian version of Theorem 5.3.

Proposition 5.16. *With the notation introduced above, one has*

$$(\breve{\nu} \times \mathrm{id})^*\mathcal{R}_C \cong (\mathrm{id} \times \mathring{\rho})_*(\mathring{q}_2^*\imath_\nu^* \det(\mathcal{V}_\Sigma) \otimes ((\mathring{\tau}_{\mathcal{L}} \circ \mathring{\tau}_{k,y_0}) \times \mathring{\nu})^*\mathcal{R}_\Sigma)\,, \tag{5.28}$$

where $\mathring{q}_2$ is the natural projection from $\overline{\mathsf{Prym}}_\Sigma^{-k} \times \left(\mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C\right)$ to $\mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C$.

Proof. Applying $(\jmath'_\Sigma \times \jmath_C)^*$ to (5.10), one gets the identification

$$(\breve{\nu} \times \mathrm{id})^*(\jmath_C \times \jmath_C)^*\mathcal{P}_C \cong (\mathrm{id} \times \mathring{\rho})_* \left(\mathring{q}_2^*\imath_\nu^* \det(\mathcal{V}_\Sigma) \otimes ((\mathring{\tau}_{\mathcal{L}} \circ \mathring{\tau}_{k,y_0}) \times \mathring{\nu})^* \,(\jmath_\Sigma \times \jmath_\Sigma)^*\,(\tau_{\mathcal{L}^{-1}} \times \mathrm{id})^*\mathcal{P}_\Sigma\right)\,, \tag{5.29}$$

after an iteration of base change under (5.23) and functoriality under (5.20), (5.24) and

$$\begin{array}{ccc}
\overline{\mathsf{Prym}}_\Sigma^{-k} \times \left(\mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C\right) & \xrightarrow{\ \mathring{q}_2\ } & \mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C \\
{\scriptstyle \jmath'_\Sigma \times \imath_\nu}\big\downarrow & & \big\downarrow{\scriptstyle \imath_\nu} \\
\overline{\mathsf{Jac}}_\Sigma^{-k} \times \mathsf{PMod}_\nu & \xrightarrow[\ q_2\ ]{} & \mathsf{PMod}_\nu\,.
\end{array}$$

Thanks to [Ari13, Lemma 6.5] and [FHR25, Proposition 4.5], one has for any line bundle $N \in \mathsf{Jac}_X$ and τ_N defined as in (5.21),

$$(\tau_N \times \mathrm{id})^*\mathcal{P}_\Sigma \cong \mathsf{p}_2^*\mathcal{P}_\Sigma|_{\{\beta_\Sigma^* N\}\times\overline{\mathsf{Jac}}} \otimes \mathcal{P}_\Sigma \cong \mathsf{p}_2^*\mathsf{Nm}^*\mathcal{P}_X|_{\{N\}\times\mathsf{Jac}_X} \otimes \mathcal{P}_\Sigma,$$

where we recall that p_2 is the projection to the second factor $\overline{\mathsf{Jac}}_\Sigma \times \overline{\mathsf{Jac}}_\Sigma \longrightarrow \overline{\mathsf{Jac}}_\Sigma$. If further, $\mathring{\mathsf{p}}_2$ denotes the projection to the second factor $\overline{\mathsf{Prym}}_\Sigma \times \overline{\mathsf{Prym}}_\Sigma \longrightarrow \overline{\mathsf{Prym}}_\Sigma$,

$$(\jmath_\Sigma \times \jmath_\Sigma)^*\,(\tau_N \times \mathrm{id})^*\mathcal{P}_\Sigma \cong \mathring{\mathsf{p}}_2^*\jmath_\Sigma^*\mathsf{Nm}^*\mathcal{P}_X|_{\{N\}\times\mathsf{Jac}_X} \otimes \mathcal{R}_\Sigma \cong \mathcal{R}_\Sigma, \tag{5.30}$$

since $\jmath_\Sigma$ amounts to the inclusion of the central fibre of the Norm map Nm. We obtain (5.28) after substituting into (5.29) the identifications (5.27) and (5.30) when $N = \mathcal{L}^{-1}$. □

We consider the Γ-action on the products $\overline{\mathsf{Prym}}_C \times \overline{\mathsf{Prym}}_C$ and $\overline{\mathsf{Prym}}_\Sigma^{-k} \times \overline{\mathsf{Prym}}_\Sigma$ given by the standard Γ-action on the second factor and the trivial one on the first. Observe that the morphisms $\jmath_C$, $\jmath_\Sigma$, ρ and $\dot{\nu}$ are Γ-equivariant by Lemma 5.15. Then, so are $\mathring{\rho}$, $\imath_\nu$ and $\mathring{\nu}$ by construction, hence (5.23) and (5.24) are, respectively, Cartesian and commutative diagrams in the Γ-equivariant category. It follows from Lemma 5.15 that $\imath_\nu^* \det(\mathcal{V}_\Sigma)$ is Γ-equivariant.

Crucially, the sheaves in (5.27) are Γ-equivariant [FHR25, Proposition 4.5]. Furthermore, they can be equipped with compatible Γ-equivariant structures.

Proposition 5.17. *A Γ-equivariant structure on $\mathcal{R}_C$ and $\imath_\nu^* \det(\mathcal{V}_\Sigma)$ naturally induce a Γ-equivariant structure on $\mathcal{R}_\Sigma$.*

Proof. Given a certain $L \in \Gamma$, denote by $\tau_{L,C}$ and $\tau_{L,\Sigma}$ the shifts provided by tensoring under β_C^*L and β_Σ^*L. An isomorphism between $\mathcal{R}_C$ and $(\mathrm{id} \times \tau_{L,C})^*\mathcal{R}_C$ induces an isomorphism between $(\mathrm{id} \times \mathring{\rho})^*(\breve{\nu} \times \mathrm{id})^*\mathcal{R}_C$ and $(\mathrm{id} \times \tau_{L,\Sigma})^*(\mathrm{id} \times \mathring{\rho})^*(\breve{\nu} \times \mathrm{id})^*\mathcal{R}_C$, where we made use of the Γ-equivariance of $\mathring{\rho}$. Under the identification (5.28) and the counit transformation, this further provides an isomorphism between $(\mathring{q}_2^*\imath_\nu^* \det(\mathcal{V}_\Sigma) \otimes ((\mathring{\tau}_{\mathcal{L}} \circ \mathring{\tau}_{k,y_0}) \times \mathring{\nu})^*\mathcal{R}_\Sigma)$ and $(\mathrm{id} \times \tau_{L,\Sigma})^*(\mathring{q}_2^*\imath_\nu^* \det(\mathcal{V}_\Sigma) \otimes ((\mathring{\tau}_{\mathcal{L}} \circ \mathring{\tau}_{k,y_0}) \times \mathring{\nu})^*\mathcal{R}_\Sigma)$. Recalling that $\mathring{\nu}$ is Γ-equivariant, the previous isomorphism together with the one provided by the Γ-equivariant structure on $\imath_\nu^* \det(\mathcal{V}_\Sigma)$, induces an isomorphism between $\mathcal{R}_\Sigma$ and $(\mathrm{id} \times \tau_{L,\Sigma})^*\mathcal{R}_\Sigma$. □

Let us equip $\mathcal{R}_C$ and $\mathcal{R}_\Sigma$ with compatible Γ-equivariant structures, as per Proposition 5.17. One can then consider the following integral functors in the category of Γ-equivariant sheaves, with Γ acting trivially on the first factor,

$$\Psi^{\mathcal{R}_C}_{1\to 2} : D^b\left(\overline{\mathsf{Prym}}_C\right) \longrightarrow D^b\left(\overline{\mathsf{Prym}}_C, \Gamma\right), \qquad \Psi^{\mathcal{R}_\Sigma}_{1\to 2} : D^b\left(\overline{\mathsf{Prym}}_\Sigma\right) \longrightarrow D^b\left(\overline{\mathsf{Prym}}_\Sigma, \Gamma\right).$$

Both functors are shown to be equivalences of categories in [FHR25, Thm 4.8], [GS22, Thm 4.7]. The following Prymian analogue of Theorem 5.7 describes the relation between them.

Corollary 5.18. *Let C be an integral nodal curve with arithmetic genus g and a partial normalisation $\nu : \Sigma \longrightarrow C$. Consider compatible m-coverings $C \longrightarrow X$ and $\Sigma \longrightarrow X$ and their compactified Pryms, as described above. Pick $y_0 \in \Sigma$ such that $\nu(y_0)$ lies in the smooth locus of C. Then, for every $\mathcal{F}^\bullet \in D^b(\overline{\mathsf{Prym}}^{-k}_\Sigma)$, one has the isomorphism*

$$\Psi^{\mathcal{R}_C}_{1\to 2}\left(R\check{\nu}_*\mathcal{F}^\bullet\right) \cong \mathring{\rho}_*\left(\imath_\nu^* \det(\mathcal{V}_\Sigma) \otimes \mathring{\nu}^* \Psi^{\mathcal{R}_\Sigma}_{1\to 2}(\mathring{\tau}_{\mathcal{L}*}\mathring{\tau}_{k,y_0,*}\mathcal{F}^\bullet)\right). \tag{5.31}$$

Proof. Starting from (5.28), the proof is analogous to that of Theorem 5.7. □

Remark 5.19. *Note that Theorem 5.18 does not depend on the choice of m-th root $\mathcal{L}$ of $\mathcal{O}_X(k\beta_\Sigma(y_0) - \beta_C(\mathsf{RSing}(\nu)))$. Indeed, choosing a different m-th root $L \otimes \mathcal{L}$ for any $L \in \Gamma$, provides*

$$\Psi^{\mathcal{R}_\Sigma}_{1\to 2}(\mathring{\tau}_{L\otimes\mathcal{L},*}\mathring{\tau}_{k,y_0,*}\mathcal{F}^\bullet) \cong \Psi^{\mathcal{R}_\Sigma}_{1\to 2}\left(\mathring{\tau}_{L,*}(\mathring{\tau}_{\mathcal{L},*}\mathring{\tau}_{k,y_0,*}\mathcal{F}^\bullet)\right) \cong \left[\mathcal{R}_\Sigma|_{\{L\}\times\overline{\mathsf{Jac}}}\right] \otimes \Psi^{\mathcal{R}_\Sigma}_{1\to 2}(\mathring{\tau}_{\mathcal{L},*}\mathring{\tau}_{k,y_0,*}\mathcal{F}^\bullet).$$

After (5.30), one has

$$\mathcal{R}_\Sigma|_{\{L\}\times\overline{\mathsf{Jac}}} \cong \mathcal{O}_{\overline{\mathsf{Jac}}}$$

naturally equipped with the trivial Γ-equivariant structure. Hence,

$$\Psi^{\mathcal{R}_\Sigma}_{1\to 2}(\mathring{\tau}_{L\otimes\mathcal{L},*}\mathring{\tau}_{k,y_0,*}\mathcal{F}^\bullet) \cong \Psi^{\mathcal{R}_\Sigma}_{1\to 2}(\mathring{\tau}_{\mathcal{L},*}\mathring{\tau}_{k,y_0,*}\mathcal{F}^\bullet),$$

giving the same description of (5.31).

We dedicate the rest of the section to discuss the role of endoscopy. Inspired by [Ngô06], [FW08] and [HP12], we say that the finite flat covering $\beta_C : C \longrightarrow X$ is of *endoscopic type* if the action of Γ on the corresponding compactified Prym variety is not free. Otherwise, we say that β_C is of *non-endoscopic type*. By [FHR25, Lemma 3.3 and Proposition 3.4], whenever C is irreducible, β_C is of endoscopic type if and only if its composition with the (full) normalisation of C provides an unramified cover of X.

In the framework of (5.19), as both C and Σ have the same (full) normalisation, one has that β_C is of the same type (endoscopic or non-endoscopic) as β_Σ. In the non-endoscopic case, the quotient maps

$$\xi_C : \overline{\mathsf{Prym}}_C \longrightarrow \overline{\mathsf{Prym}}_C/\Gamma, \qquad \xi_\Sigma : \overline{\mathsf{Prym}}_\Sigma \longrightarrow \overline{\mathsf{Prym}}_\Sigma/\Gamma,$$

are étale, and the quotients are connected reduced projective Gorenstein schemes of finite type [FHR25, Proposition 3.6]. Furthermore, $\mathcal{R}_C$ and $\mathcal{R}_\Sigma$ descend along ξ_C and ξ_Σ [FHR25, Corollary 5.1]; so there exists sheaves $\mathcal{Q}_C$ and $\mathcal{Q}_\Sigma$ over $\overline{\mathsf{Prym}}_\Sigma \times \overline{\mathsf{Prym}}_C/\Gamma$ and $\mathring{\tau}^{-1}_{\mathcal{L}}\left(\overline{\mathsf{Prym}}_\Sigma\right) \times \overline{\mathsf{Prym}}_\Sigma/\Gamma$ respectively, such that

$$\mathcal{R}_C \cong (\mathrm{id} \times \xi_C)^*\mathcal{Q}_C, \qquad \mathcal{R}_\Sigma \cong (\mathrm{id} \times \xi_\Sigma)^*\mathcal{Q}_\Sigma.$$

Taking $\mathcal{Q}_C$ and $\mathcal{Q}_\Sigma$ as integral kernels, we define the integral functors

$$\Phi^{\mathcal{Q}_C}_{1\to 2} : D^b\left(\overline{\mathsf{Prym}}_C\right) \longrightarrow D^b\left(\overline{\mathsf{Prym}}_C/\Gamma\right), \qquad \Phi^{\mathcal{Q}_\Sigma}_{1\to 2} : D^b\left(\overline{\mathsf{Prym}}_\Sigma\right) \longrightarrow D^b\left(\overline{\mathsf{Prym}}_\Sigma/\Gamma\right). \tag{5.32}$$

Theorem 5.20 (Theorem 5.2 of [FHR25])**.** *Let C be an integral nodal curve with arithmetic genus g and a partial normalisation $\nu : \Sigma \longrightarrow C$ resolving precisely k nodes. Suppose that C and Σ are equipped with degree m-coverings $\beta_C : C \longrightarrow X$ and $\beta_\Sigma : \Sigma \longrightarrow X$ of non-endoscopic type. The associated integral functors $\Phi^{\mathcal{Q}_C}_{1\to 2}$ and $\Phi^{\mathcal{R}_\Sigma}_{1\to 2}$ are equivalences of categories fitting in the commutative diagrams*

$$\begin{array}{ccc} D^b(\overline{\mathsf{Prym}}_C) & \xrightarrow{\Phi^{\mathcal{Q}_C}_{1\to 2}} & D^b(\overline{\mathsf{Prym}}_C/\Gamma) \\ & \searrow^{\Psi^{\mathcal{R}_C}_{1\to 2}} & \xi_C^* \downarrow \uparrow \xi^\Gamma_{C,*} \\ & & D^b(\overline{\mathsf{Prym}}_C, \Gamma) \end{array}, \qquad \begin{array}{ccc} D^b\left(\overline{\mathsf{Prym}}_\Sigma\right) & \xrightarrow{\Phi^{\mathcal{Q}_\Sigma}_{1\to 2}} & D^b(\overline{\mathsf{Prym}}_\Sigma/\Gamma) \\ & \searrow^{\Psi^{\mathcal{R}_\Sigma}_{1\to 2}} & \xi_\Sigma^* \downarrow \uparrow \xi^\Gamma_{\Sigma,*} \\ & & D^b(\overline{\mathsf{Prym}}_\Sigma, \Gamma) \end{array},$$

where $\xi^\Gamma_{C,*}$ and $\xi^\Gamma_{\Sigma,*}$ amount to the Γ-invariant part of their corresponding push-forwards.

In the non-endoscopic case, the action of Γ is faithful. Denote by

$$\widetilde{\xi}_\Sigma : \mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C \longrightarrow \mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C/\Gamma$$

the quotient of the Prymian version of the moduli space of parabolic modules by the m-torsion points. By Proposition 5.17 $\imath_\nu^* \det(\mathcal{V}_\Sigma)$ can be equipped with a Γ-equivariant structure. By the Drézet–Narasimhan–Kempf descent criterion [DN89, Theorem 2.3], it descends along $\widetilde{\xi}_\Sigma$ to a sheaf $[\imath_\nu^* \det(\mathcal{V}_\Sigma)]$ on the quotient $\mathsf{PMod}_\nu \times_{\overline{\mathsf{Jac}}_C} \overline{\mathsf{Prym}}_C/\Gamma$, such that

$$\imath_\nu^* \det(\mathcal{V}_\Sigma) \cong \widetilde{\xi}_\Sigma^* [\imath_\nu^* \det(\mathcal{V}_\Sigma)]. \tag{5.33}$$

Given all of the above, one can naturally derive the following from Theorem 5.20 and Theorem 5.18.

Corollary 5.21. *Let C be an integral nodal curve with arithmetic genus g and partial normalisation $\nu : \Sigma \longrightarrow C$, which resolves k nodes. Suppose that C and Σ are equipped with m-coverings $\beta_C : C \longrightarrow X$ and $\beta_\Sigma : \Sigma \longrightarrow X$ of non-endoscopic type. Denote by $[\mathring{\rho}]$, $[\mathring{\nu}]$, $[\imath_\nu]$ the corresponding morphisms between Γ-orbits induced by $\mathring{\rho}$, $\mathring{\nu}$, $\imath_\nu$. Then, for $\mathcal{F}^\bullet \in D^b(\overline{\mathsf{Prym}}_\Sigma^{-k})$, one has the isomorphism*

$$\Phi^{\mathcal{Q}_C}_{1\to 2}(\check{\nu}_* \mathcal{F}^\bullet) \cong [\mathring{\rho}]_* \left([\imath_\nu^* \det(\mathcal{V}_\Sigma)] \otimes [\mathring{\nu}]^* \Phi^{\mathcal{Q}_\Sigma}_{1\to 2}(\mathring{\tau}_{\mathcal{L},*} \mathring{\tau}_{k,y_0,*} \mathcal{F}^\bullet) \right).$$

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E. Franco,
Universidad Autónoma de Madrid
and
Instituto de Ciencias Matemáticas (CSIC–UAM–UCM–UC3M)
Campus de Cantoblanco 28049, Madrid, España.

Email address: emilio.franco@uam.es

R. Hanson,
Imperial College London,
Exhibition Rd, South Kensington, London SW7 2AZ

Email address: robert.hanson@imperial.ac.uk

J. Horn,
Goethe-Universität Frankfurt,
Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main

Email address: horn@math.uni-frankfurt.de

A. Oliveira,
Centro de Matemática da Universidade do Porto,
Departamento de Matemática, Faculdade de Ciências, Universidade do Porto,
Rua do Campo Alegre s/n, 4169-007 Porto, Portugal

Email address: andre.oliveira@fc.up.pt