Marsden-Weinstein reduction as an instance of Marsden-Ratiu reduction

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Abstract

The process of reduction of a symplectic manifold by a Hamiltonian Lie group action, is commonly referred to as *Marsden-Weinstein reduction* and was published in 1974. About a decade later, in 1986, Marsden and Ratiu developed a theory of reduction of Poisson manifolds by distributions, fitting several constructions of symplectic and Poisson geometry in this framework.

In this note we prove that Marsden-Weinstein reduction is indeed an instance of Poisson reduction by distributions, as claimed by Marsden and Ratiu, filling what we believe is a gap in the literature.

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Declarations

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1 Introduction

Poisson manifolds are a generalization of symplectic manifolds, which can be thought of as "everywhere nondegenerate" Poisson manifolds. As stated by Marsden and Ratiu in their paper *Reduction of Poisson manifolds* ([5]), the context of reduction by distributions was chosen to "include the usual theorems on reduction of symplectic manifolds, as well as ...". In their example B, the

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authors explain why all assumptions they make are satisfied in the setting of Marsden and Weinstein's *Reduction of Symplectic Manifolds with Symmetry* ([4]). Still, one of the assumptions does not follow in the way they seem to argue. We try to explain the problem with an example (subsection 3.1).

We remark that the reduction of the dynamics in the symplectic setting also fits in the reduction of the dynamics in the setting of Marsden and Ratiu, but this proof presents no problems (see [7] for all details).

2 Preliminaries

In this section we will settle the notation, terminology and sign conventions to be followed throughout this note. Unimportant as these may be, we will be comparing two papers with different notation, so statements can be made easier to read if some care is taken in uniformization. For example, we decided to keep M for a Poisson manifold (and use P to denote the Poisson tensor on M), as some confusion would arise if, when moving from the symplectic case to the Poisson case, M would be replaced by P and a submanifold N would be replaced by M. Regarding sign conventions, which differ from author to author and are often a matter of convenience, our choice fell on [1], although the organization in [3] is probably more friendly to a newcomer. Still the sign conventions in [3] lead to the wrong sign in Hamilton's equations, which are historically foundational in Symplectic Geometry.

2.1 Setting for symplectic reduction by symmetries

The notation for this setting is as follows: (M, ω) denotes a connected symplectic manifold, G a connected Lie group acting (on the left) on M by symplectomorphims Ψ_g (with $g \in G$), and $\mathcal{G}$ the Lie algebra of G. We assume both objects are finite-dimensional.

If p is a point in M, then G_p and $\mathcal{O}_p$ denote, respectively, the isotropy subgroup and the G-orbit of p:

$$G_p = \{g \in G : \Psi_g(p) = p\}, \quad \mathcal{O}_p = \{\Psi_g(p) : g \in G\}.$$

We assume that the action is (strongly) Hamiltonian ([3], [6]), meaning that: (i) for any $\xi \in \mathcal{G}$, the fundamental vector field ξ_M is Hamiltonian with Hamiltonian function J_ξ; (ii) the map $\xi \longrightarrow J_\xi$ is a Lie algebra-homomorphism, in particular the moment map $J : M \longrightarrow \mathcal{G}^*$ is Ad^*-equivariant, where

$$\langle J(p), \xi \rangle = J_\xi(p), \quad \forall p \in M, \xi \in \mathcal{G}.$$

The requirements for symplectic reduction ([4]) are as follows:

S1. $\mu \in \mathcal{G}^*$ - regular value of J;

S2. $N = J^{-1}(\{\mu\}) \neq \emptyset$ - (embedded) submanifold of M;

S3. $G_\mu = \{g \in G : Ad_g^*(\mu) = \mu\}$ (isotropy subgroup of μ for the co-adjoint action) - assumed to act[1] freely and properly on N. The orbit of $p \in N$ by this restricted action will be denoted by $\mathcal{O}_p^\mu$.

Under these assumptions, the space of G_μ-orbits, $M_\mu = N/G_\mu$, has a differentiable manifold structure for which the canonical projection

$$\pi_\mu : N \longrightarrow M_\mu$$

is a submersion.

We start with a lemma (used in the proof of the reduction theorem), which we will use in section 3.

Lemma 1 ([4]). *Under assumptions S1–S3, and for any $p \in N$, the following hold:*

1. $T_p\mathcal{O}_p^\mu = T_p\mathcal{O}_p \cap T_pN$;
2. $T_pN = (T_p\mathcal{O}_p)^{\omega_p}$,

where $(T_p\mathcal{O}_p)^{\omega_p}$ is the symplectic complement of $T_p\mathcal{O}_p$ in T_pM.

With the notation just established, the Marsden-Weinstein reduction theorem can be stated as below.

Theorem 1 (Marsden-Weinstein [4]). *Under assumptions S1–S3, there is a unique symplectic structure ω_μ on M_μ satisfying*

$$i^*\omega = \pi_\mu^*\omega_\mu,$$

where $i : N \hookrightarrow M$ is the inclusion.

2.2 Setting for Poisson reduction by distributions

In this setting we will consider Poisson, not necessarily symplectic, manifolds. The pair $(M, \{\,,\})$ will denote a Poisson manifold, P the Poisson tensor associated to $\{\,,\}$, that is

$$P_p(df_p, dg_p) = \{f, g\}(p), \quad p \in M,$$

and $P^\sharp : T^*M \longrightarrow TM$ the bundle map defined by

$$\langle \alpha, P^\sharp(\beta)\rangle = P(\alpha, \beta), \quad \alpha, \beta \in T^*M.$$

N will be an embedded submanifold of M with $i : N \hookrightarrow M$ denoting the inclusion map.

Poisson reduction will be performed using $E \subset TM_{|N}$, a sub-bundle of TM restricted to N, with the following requirements:

[1] by restricting Ψ both to G_μ and N, which is possible by the Ad*-equivariance of J

P1. $F := E \cap TN$ is an integrable distribution (with constant rank) on N;

P2. the foliation Φ associated to F is simple, so the space of leaves, $\overline{N} = N/\Phi$, has a differentiable manifold structure for which the canonical projection

$$\pi : N \longrightarrow \overline{N}$$

is a submersion;

P3. the Poisson bracket $\{\,,\}$ *is E-invariant*, that is, for any $f, g \in C^\infty(M)_E$, their Poisson bracket $\{f, g\}$ also belongs to $C^\infty(M)_E$, where

$$C^\infty(M)_E = \{f \in C^\infty(M) : df_p \in E_p^\circ, \forall p \in N\}$$

and E_p° denotes, as usual, the annihilator of E_p in T_p^*M.

Definition 1. *Under the requirements P1–P3, the quadruple $(M, \{\,,\}, N, E)$ is* Poisson-reducible *if there exists a Poisson bracket $\{\,,\}_{\overline{N}}$ on $\overline{N}$ with the following property: for any $\bar{f}$ and $\bar{g} \in C^\infty(\overline{N})$, and for any extensions f and g of $\bar{f} \circ \pi$ and $\bar{g} \circ \pi$ to $C^\infty(M)_E$, respectively, the following identity holds:*

$$\{\bar{f}, \bar{g}\}_{\overline{N}} \circ \pi = \{f, g\} \circ i.$$

Remark 1. *In the definition, $\bar{f}, \bar{g}$ and f, g need only be* locally defined *smooth functions.*

The main theorem in [5] establishes, under the requirements P1–P3, a necessary and sufficient condition for $(M, \{\,,\}, N, E)$ to be Poisson-reducible.

Theorem 2 (Marsden and Ratiu, [5])**.** *Under the requirements P1–P3, the quadruple $(M, \{\,,\}, N, E)$ is Poisson-reducible if and only if $P^\sharp(E^\circ) \subset E + TN$.*

Remark 2. *If $E = 0$, then P3 is trivially satisfied since $C^\infty(M)_E = C^\infty(M)$. In such case, $(M, \{\,,\}, N, E = 0)$ being Poisson-reducible means that $\overline{N} = N$ is a Poisson submanifold of $(M, \{\,,\})$.*

If $E \neq 0$, Falceto and Zambon ([2]) showed that condition P3 already implies $P^\sharp(E^\circ) \subset TN$. In particular, assuming P1–P3, the quadruple $(M, \{\,,\}, N, E)$ is always Poisson-reducible.

3 Reduction by symmetries is an instance of reduction by distributions

Given the data $(M, \omega, G, \Psi, \mu)$ satisfying assumptions S1–S3 of 2.1, consider the sub-bundle $E \subset TM_{|N}$ defined by

$$E_p = T_p\mathcal{O}_p, \quad p \in N. \tag{1}$$

Marsden and Ratiu claim that $(M, \{\,,\}, N, E)$ satisfies both P1–P3 and the reducibility condition $P^\sharp(E^\circ) \subset E + TN$, where $\{\,,\}$ is the Poisson structure associated to ω.

Remark 3. *The relations between* ω *and* $\{\,,\}$ *are common knowledge and the sign conventions are actually not important, still we state our choice here:*

$$\{f, g\} = \omega(X_f, X_g) = \langle \omega^\flat(X_f), X_g\rangle = \langle df, X_g\rangle,$$

with X_f *denoting the Hamiltonian vector field of* f *with respect to* ω.

With this choice we obtain the usual identity: $\omega^\flat = (P^\sharp)^{-1}$. *Recall that we are using the sign conventions in [1] so* $\omega^\flat : TM \longrightarrow T^*M$ *is given by* $\omega^\flat(X) = i_X\omega$.

As expected (see remark 2) the main difficulty lies in proving P3. Concerning P3, the example in subsection 3.1 below, shows that functions in $C^\infty(M)_E$ are not necessarily G-invariant functions (for the action Ψ), which seems to invalidate the argument in example B of [5].

In subsection 3.2, we will present an alternative proof of P3, and in subsection 3.3 we prove that the reduced Poisson structure on $\overline{N} = M_\mu$ is precisely the Poisson structure determined by the reduced symplectic form ω_μ on M_μ.

3.1 An example

An obvious observation is that G-invariant functions on M belong to $C^\infty(M)_E$ with E as in (1), that is, for any $f \in C^\infty(M)$, the implication below holds:

$$f \circ \Psi_g = f \quad \Longrightarrow \quad f \in C^\infty(M)_E.$$

Moreover, the Poisson bracket of G-invariant functions on M is again G-invariant, as the action is symplectic.

Nevertheless, a function $f \in C^\infty(M)_E$ need not be G-invariant, even though $(f \circ \Psi_g)|_N = f|_N$.

In fact consider $M = \mathbb{R}^2$, $\omega = dx \wedge dy$ the canonical symplectic form, and $G = (\mathbb{R}, +)$ acting on M by diagonal translations:

$$\Psi_a(x, y) = (x + a, y + a).$$

This is a (strong) Hamiltonian action with moment map $J : M \longrightarrow \mathbb{R} \simeq \mathcal{G}^*$ given by

$$J(x, y) = y - x.$$

All reals are regular values of J so we choose $N = J^{-1}(0) = \{(x, x) : x \in \mathbb{R}\}$.

We observe that G_0, isotropy subgroup of 0 for the co-adjoint action, is precisely $\mathbb{R}$, which acts freely and properly on N, and that all conditions for Marsden-Weinstein reduction are met.

Now consider $f \in C^\infty(M)$ given by $f(x, y) = x(x - y)$. Then:

- $f \in C^\infty(M)_E$ since $E_{(x,x)} = \{\xi_M(x, x) : \xi \in \mathcal{G}\}$ and

$$df_{(x,x)}\left(\xi_M(x, x)\right) = \frac{d}{dt} f(x + t\xi, x + t\xi)\bigg|_{t=0} = 0$$

- f is not G-invariant as, for example, $f \circ \Psi_a(1, 0) \neq f(1, 0)$, for $a \neq 0$.

3.2 Proof - part I

This section is devoted to the proof that $(M, \{\,,\}, N, E)$, with E given by (1), satisfies both P1–P3 and $P^\sharp(E^\circ) \subset E + TN$.

Clearly E is a distribution with constant rank equal to $\dim M - \dim N$. We proceed to check all the required conditions.

- We prove P1 and P2 simultaneously. By lemma 1, and for all $p \in N$,

$$F_p = T_p\mathcal{O}_p \cap T_pN = T_p\mathcal{O}_p^\mu,$$

so F is an integrable distribution with constant rank. In fact, the leaves of the foliation Φ are precisely the G_μ-orbits

$$\mathcal{O}_p^\mu = \mathcal{O}_p \cap N,$$

which are embedded submanifolds of N by the property of π_μ. This argument also proves that M_μ, the reduced symplectic manifold, coincides with $\overline{N}$, the reduced Poisson manifold.

- To prove P3 we follow three steps.

Step 1: characterize the set $C^\infty(M)_E$. Note that

$$E_p = T_p\mathcal{O}_p = \{\xi_M(p) : \xi \in \mathcal{G}\}$$

and that ξ_M is the Hamiltonian vector field of J_ξ. Then f belongs to $C^\infty(M)_E$ if and only if, for all $p \in N$, the following holds:

$$0 = \langle df_p, X_{J_\xi}(p)\rangle = \omega_p\left(X_f(p), \xi_M(p)\right), \ \forall \xi \in \mathcal{G},$$

that is, $X_f(p)$ belongs to $(T_p\mathcal{O}_p)^{\omega_p}$. Using lemma 1 we arrive at

$$C^\infty(M)_E = \{f \in C^\infty(M) : X_f(p) \in T_pN, \forall p \in N\} \tag{2}$$

(equivalently $C^\infty(M)_E$ is the set of functions whose Hamiltonian vector field is tangent to N, at all points of N).

Step 2: show that, in the symplectic setting:

$$P_p^\sharp(E_p^\circ) = T_pN, \quad \forall p \in N. \tag{3}$$

By lemma 1, $(T_p\mathcal{O}_p)^{\omega_p} = T_pN$, so that

$$\forall \xi \in \mathcal{G}, \forall u \in T_pN,\ \omega_p\left(\xi_M(p), u\right) = 0$$

or, equivalently, $\omega_p^\flat(T_pN) = T_p^\circ\mathcal{O}_p = E_p^\circ$. Using the fact that $\omega^\flat$ and $P^\sharp$ are inverse maps, the equality (3) follows.

Step 3: consider the characterization of $T_p^\circ N$ (see, for example, lemma 1.1.9. in [8]):

$$T_p^\circ N = \{dh_p : h \in C^\infty(M) \text{ with } h_{|N} = 0\}. \tag{4}$$

With this last ingredient we can now prove P3. To this end, consider f and g in $C^\infty(M)_E$, that is:

$$X_f(p) \in T_pN \quad \text{and} \quad X_g(p) \in T_pN, \quad \forall p \in N.$$

Consider $h \in C^\infty(M)$, arbitrary, with $h_{|N} = 0$. By (4), all we need to prove is that

$$\langle dh_p, X_{\{f,g\}}(p)\rangle = 0, \quad \forall p \in N,$$

or, rephrasing in terms of $\{\,,\}$,

$$\{h, \{f, g\}\}(p) = 0, \quad \forall p \in N.$$

This last equality follows directly from Jacobi identity for $\{\,,\}$ and from the fact that

$$\langle dh_p, X_f(p)\rangle = 0 \quad \text{and} \quad \langle dh_p, X_g(p)\rangle = 0, \quad \forall p \in N.$$

- The fact that the reducibility condition $P^\sharp(E^\circ) \subset E+TN$ holds is a direct consequence of (3).

3.3 Proof - part II

The proof that the reduced Poisson structure on $\overline{N} = M_\mu$ coincides with the Poisson structure determined by ω_μ on M_μ, follows the lines described in example B of [5], but we need to replace the term *G-invariant functions* by *functions in $C^\infty(M)_E$*. We start with a lemma which relates the Hamiltonian vector fields on N and on $\overline{N}$.

Lemma 2. *Under the conditions of section 3, let $\pi : N \longrightarrow \overline{N}$ denote the canonical projection. Then, for all $\bar{f} \in C^\infty(\overline{N})$ and for any extension f of $\bar{f} \circ \pi$ to $C^\infty(M)_E$, the Hamiltonian vector fields of $\bar{f}$ and f are π-related:*

$$\overline{X}_{\bar{f}}(\pi(p)) = d\pi_p(X_f(p)), \quad \forall p \in N$$

where $\overline{X}_{\bar{f}}$ is the Hamiltonian vector field of $\bar{f}$ with respect to ω_μ and X_f is the Hamiltonian vector field of f with respect to ω.

Proof. We have to show that:

$$(\omega_\mu^\flat)(d\pi_p(X_f(p)) = d\bar{f}_{\pi(p)}, \quad \forall p \in N.$$

Now, for any $p \in N$ and $\overline{u} \in T_{\pi(p)}\overline{N}$, the following equality holds

$$(\omega_\mu)_{\pi(p)}(d\pi_p(X_f(p), \overline{u})) = (\pi^*\omega_\mu)_p\,(X_f(p), u)$$

with $u \in T_pN$, so by the definition of ω_μ we obtain

$$(\omega_\mu)_{\pi(p)}(d\pi_p(X_f(p)), \overline{u}) = (i^*\omega)_p\,(X_f(p), u)\,.$$

Since, by (2), $X_f(p) \in T_pN$ and u is arbitrary, the right-hand-side equals

$$df_p(u) = d(\bar{f} \circ \pi)_p(u) = d\bar{f}_{\pi(p)}(\overline{u})$$

completing the proof of the lemma. □

The rest of the proof of part II is straightforward. Take $\bar{f}, \bar{g}$ in $C^\infty(\overline{N})$ and $\pi(p) \in \overline{N}$. By definition of the reduced Poisson bracket:

$$\{\bar{f}, \bar{g}\}_{\overline{N}} \circ \pi(p) = \{f, g\} \circ i(p),$$

where f and g are arbitrary extensions of $\bar{f} \circ \pi$ and $\bar{g} \circ \pi$ to $C^\infty(M)_E$, or

$$\{\bar{f}, \bar{g}\}_{\overline{N}} \circ \pi(p) = \omega_p(X_f(p), X_g(p)).$$

On the other hand, the Poisson bracket determined by ω_μ, denoted momentarily by $\{\, ,\}_\mu$, satisfies:

$$\{\bar{f}, \bar{g}\}_\mu(\pi(p)) = (\omega_\mu)_{\pi(p)} \left(\overline{X}_{\bar{f}}(\pi(p)), \overline{X}_{\bar{g}}(\pi(p))\right),$$

so lemma 2, the definition of ω_μ and (2) lead to:

$$\begin{aligned} \{\bar{f}, \bar{g}\}_\mu(\pi(p)) &= (\pi^* \omega_\mu)_p \left(X_f(p), X_g(p)\right) \\ &= (i^* \omega)_p \left(X_f(p), X_g(p)\right) \\ &= \omega_p(X_f(p), X_g(p)) \end{aligned}$$

and to the conclusion that $\{\, ,\}_{\overline{N}}$ coincides with $\{\, ,\}_\mu$.

4 Comments

In this note we presented a proof that Marsden-Weinstein reduction of a symplectic manifold by a (strong) Hamiltonian action can be obtained by Marsden-Ratiu reduction of the same manifold using the natural distribution consisting of the tangent space to the G-orbits (N is level set of a regular value of the moment map).

Although such claim is present in other papers, they point to the explanation given in [5]. Our example in subsection 3.1 explains why we believe the argument in [5] is misleading.

Although it is probably possible to use the results in [2] to obtain an alternative proof, we could not devise a natural way to do it.

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References

[1] Abraham, Ralph and Marsden, Jerrold E., *Foundations of mechanics*, 2nd edition, Benjamin/Cummings Publishing Co. Inc., (1978).

[2] F. Falceto and M. Zambon, An Extension of the Marsden-Ratiu Reduction for Poisson Manifolds, *Lett. Math. Phys.*, **85** (2008), pp. 203-219.

[3] Libermann, Paulette and Marle, Charles-Michel, *Symplectic Geometry and Analytical Mechanics*, Mathematics and its Applications, **35**, D. Reidel Publishing Co., Dordrecht, (1987).

[4] J. Marsden and A. Weinstein, Reduction of Symplectic Manifolds with Symmetry, *Rep. Math. Phys.*, **5** (1974), pp. 121-130.

[5] J. Marsden and T. Ratiu, Reduction of Poisson Manifolds, *Lett. Math. Phys.*, **11** (1986), pp. 161-169.

[6] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Graduate Text in Mathematics, 3rd edition. 2017.

[7] Oliveira, Vanessa A. S., *Redução de Variedades de Poisson (Reduction of Poisson Manifolds)*, Master thesis, FCUP, University of Porto, (2023). Available at: `https://hdl.handle.net/10216/154017`.

[8] J. Ortega, T. Ratiu. *Momentum Maps and Hamiltonian Reduction.* Progress in Mathematics, **222**. Springer Science+Business Media New York, (2004).