Full list of quivers of four and five nodes and mutation-period equal to 2

Inês Cruz* and Jorge Ferreira†

June 25, 2025

Abstract

In 2011, A. Fordy and R. Marsh provided a list of quivers with low number of nodes and mutation-period equal to 2. In the five-node case, they imposed symmetry on the quiver apparently due to the symmetry observed for quivers of four nodes with the same period.

In this paper we give the complete list of quivers of four nodes and mutation-period 2, as a non-symmetric family was left out of their list. For the case of five nodes, we correct one of the families they obtained and prove that the corrected list is exhaustive.

Keywords: cluster algebras, mutation-periodic quivers, cluster maps.

1 Introduction

Cluster algebras were introduced in 2001 by Fomin & Zelevinsky (see [3] for the published version) to study total positivity in semisimple groups. Since then, cluster algebras have been linked to many areas, namely representation theory, pre-symplectic and Poisson geometry, dynamical systems, topology and number theory just to name a few (we refer to the Cluster Algebras Portal by Fomin to have a glimpse on the variety of areas envolved in this subject).

In 2002 Fomin & Zelevinsky proved the so-called *Laurent phenomenon* for cluster algebras (see [4]): every element of a cluster algebra can be written as a Laurent polynomial in the original cluster variables. This phenomenon has been associated to many higher-order sequences (e.g., *Somos sequences*) which exhibit the following behaviour: although defined by rational recurrences, if one starts with the initial data $(1, 1, \ldots, 1)$, all terms in the sequence turn out to be integers.

Motivated by this phenomenon, a new concept was introduced by Fordy & Marsh in [7], that of a *mutation-periodic quiver*. Quivers can be seen as instances

*Centro de Matemática da Universidade do Porto, Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, 4169-007 Porto, Portugal. Email: imcruz@fc.up.pt

†Email: jorgembf_school@outlook.com

of cluster algebras by appropriately defining *mutations* in the direction of the nodes of the quiver. Mutation-periodic quivers of period m naturally give rise to a system of m nonlinear (in fact bi-rational) difference equations, which include some of the Somos sequences. The discrete dynamical systems determined by these systems of difference equations, are commonly referred to as *cluster maps* and are canonically written as a composition of m mutations and a permutation on the nodes of the quiver (this vision of cluster map coincides with the one in [5] in the case of mutation-period 1, but is somewhat different for higher periods and odd number of nodes, in which case we refer to [1]).

In the meantime, Poisson and pre-symplectic geometry had been successfully used in cluster algebra theory by Gekhtman, Shapiro & Vainshtein (see for example [8], [9]), revealing the potential of the use of these areas of geometry both in algebra and dynamical systems. For an account of a dynamical systems point of a view see for example [5], where cluster maps associated to mutation-periodic quivers of period 1 are reduced to symplectic maps by the use of pre-symplectic techniques. For example, reduction of the Somos-4 and of the Somos-5 cluster map leads to integrable maps in dimension 2 (in the sense of [10]).

The notion of mutation-periodicity was proved to be equivalent to a geometric counterpart in [1]. More precisely, a quiver is mutation-periodic (arbitrary period) if and only if the *canonical pre-symplectic form* associated to the quiver is invariant under the corresponding cluster map. As a consequence, cluster maps arising from mutation-periodic quivers of any period can always be reduced to lower dimensional symplectic maps if the “adjacency” matrix of the quiver is singular (which will always be the case for odd number of nodes). Nevertheless, reducing a cluster map to lower dimension by using Poisson structures can turn out to have a negative answer (see [1] for an example). This is the case for all 5-node quivers with mutation-period equal to 2: none of the corresponding cluster maps can be reduced by the use of log-canonical Poisson structures.

2 Preliminaries

2.1 Mutation-periodic quivers

We start by establishing some notation, following the one introduced in [7].

Q will denote a quiver (a directed multi-graph) with N nodes and no loops nor 2-cycles. The nodes will be labelled $1, 2, \ldots, N$ and will be depicted as the vertices of a regular N-gon disposed in clockwise order. To Q we associate a modified version of its adjacency matrix: an integer $N \times N$ skew-symmetric matrix, $B_Q = [b_{ij}]$, whose nonnegative entries b_{ij} are the number of arrows from node i to node j. The fact that there are no loops guarantees that $b_{ii} = 0$. Conversely, given an integer $N \times N$ skew-symmetric matrix $B = [b_{ij}]$, we can define a quiver, Q_B, by defining the number of arrows from node i to node j to be equal to b_{ij}, if this number is positive (no arrows otherwise).

A *skew-symmetric, coefficient-free* cluster algebra can then be associated to

$$B_Q = \begin{bmatrix} 0 & -1 & 2 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 & 2 \\ -1 & 0 & 1 & 0 & -1 \\ -1 & -1 & -2 & 1 & 0 \end{bmatrix}$$

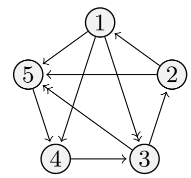

Figure 1: A five node quiver Q and its matrix.

Q by proceeding as follows:

1. attach a variable x_i (cluster variable) to node i of Q, thus getting what is known in cluster algebra theory as *initial seed*:

$$(B, \mathbf{x})$$

with $B = B_Q$ and $\mathbf{x} = (x_1, \ldots, x_N)$;

2. define μ_k, the operation of *mutation at node* $k \in \{1, \ldots, N\}$, acting on a seed $(B, \mathbf{x})$ as:

$$(\mu_k(B), \mu_k(\mathbf{x})) = (B', \mathbf{x}')$$

with

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}\left(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|\right), & \text{otherwise} \end{cases}$$

and

$$x'_i = \begin{cases} \dfrac{\Pi_{b_{kj}\geq 0}\, x_j^{b_{kj}} + \Pi_{b_{kj}\leq 0}\, x_j^{-b_{kj}}}{x_k}, & \text{if } i = k \\ x_i, & \text{otherwise} \end{cases}$$

In cluster algebra theory, the initial seed can be mutated in any direction (at any node) producing another seed, which can again be object of another mutation (note that $\mu_k \circ \mu_k = Id$) and so forth. In our study we will perform mutations in a prescribed order, first at node 1, then at node 2, and so on up to order N.

Remark 1. *In the definition of $B' = \mu_k(B)$, observe that: (a) $b'_{ij} = b_{ij} + b_{ik}b_{kj}$ if b_{ik} and b_{kj} are both positive; (b) $b'_{ij} = b_{ij} - b_{ik}b_{kj}$ if b_{ik} and b_{kj} are both negative and (c) $b'_{ij} = b_{ij}$ whenever $b_{ik}\, b_{kj} \leq 0$. In particular, if all elements in row k have the same sign (zero included in any sign), then only the kth row (or column) of B undergoes some change in the process of mutation.*

In terms of the quiver, mutation at node k is defined so that the matrix associated to the mutated quiver is the mutated matrix of the original quiver:

$$B_{\mu_k(Q)} = \mu_k(B_Q), \tag{1}$$

which translates in the following set of rules:

- reverse all arrows which are incident with node k;
- if $p \geq 0$ is the number of arrows from node i to node j then:
  (a) add qr arrows to p if there are $q > 0$ arrows from node i to node k and $r > 0$ arrows from k to j;
  (b) subtract qr arrows from p if there are $q > 0$ arrows from j to k and $r > 0$ arrows from k to i (a negative result is to be read as $qr - p$ arrows from j to i);
  (c) leave p unchanged in any other situation.

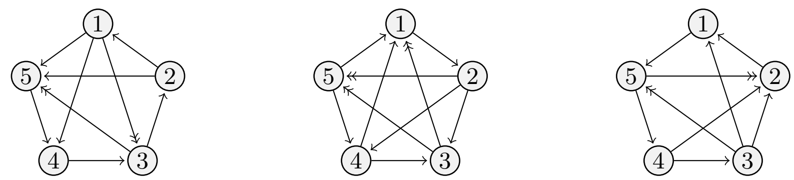

Figure 2: Quivers Q, $\mu_1(Q)$ and $\mu_2 \circ \mu_1(Q)$.

A quiver will be mutation-periodic if, performing the mutation at node 1 followed by the mutation at node 2 and so on, at some point the mutated quiver coincides with the original quiver, up to a certain permutation of the nodes.

To be more precise, we introduce the cyclic permutation[1]

$$\sigma = \begin{pmatrix} 1 & \dots & N-1 & N \\ 2 & \dots & N & 1 \end{pmatrix}$$

and define $\sigma(Q)$ to be the quiver with N nodes where the number of arrows from node $\sigma(i)$ to node $\sigma(j)$ is the number of arrows in Q from node i to node j. One can visualize $\sigma(Q)$ as the N-gon where the arrows remain fixed and the N vertices are rotated counter-clockwise.

It can easily be checked that this operation of permutation translates, in terms of matrices, as

$$B_{\sigma(Q)} = \sigma^{-1} B_Q \, \sigma, \tag{2}$$

where, slightly abusing notation, σ on the right hand side stands for the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3}$$

[1]denoted by ρ^{-1} in [7]

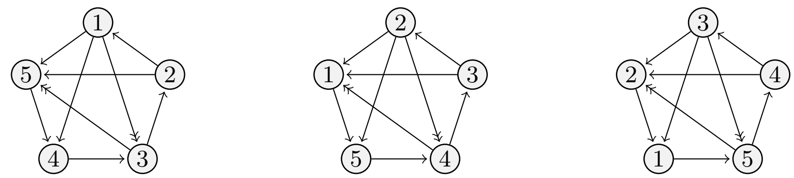

Figure 3: Quivers Q, $\sigma(Q)$ and $\sigma^2(Q)$.

Definition 1. *A quiver Q with N nodes is said to be mutation-periodic with period $m \geq 1$, or m-periodic for short, if*

$$\mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1(Q) = \sigma^m(Q). \tag{4}$$

Example 1. *The quiver Q introduced in figure 1 is 2-periodic (compare the third quivers in figures 2 and 3) but not with period 1 (compare the second quivers in the same figures).*

By using (1) and (2), one can express mutation-periodicity as follows:

$$\mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1(B_Q) = \sigma^{-m} B_Q \, \sigma^m \tag{5}$$

with B_Q denoting the matrix associated to Q.

Example 2. *For the matrix in figure 1 the matrices $\mu_1(B_Q)$ and $\mu_2 \circ \mu_1(B_Q)$ are respectively*

$$\mu_1(B_Q) = \begin{bmatrix} 0 & 1 & -2 & -1 & -1 \\ -1 & 0 & 1 & 1 & 2 \\ 2 & -1 & 0 & -1 & 2 \\ 1 & -1 & 1 & 0 & -1 \\ 1 & -2 & -2 & 1 & 0 \end{bmatrix}, \ \mu_2 \circ \mu_1(B_Q) = \begin{bmatrix} 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & -2 \\ 1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 2 & -2 & 1 & 0 \end{bmatrix}$$

The last matrix agrees with $\sigma^{-2} B_Q \sigma^2$, as can be easily checked.

2.2 Cluster maps

For the sake of completeness, we define the cluster map associated to a mutation-periodic quiver (here our interest lies in period greater than 1, so we follow [1] instead of [5]).

Definition 2. *The cluster map associated to a mutation-period quiver Q with N nodes and strict-period* [2] *m is the map*

$$\varphi(\mathbf{x}) = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1(\mathbf{x}) \tag{6}$$

with $\mathbf{x} = (x_1, \ldots, x_N)$.

[2] m but not n, for any $n < m$

As an example, the cluster map associated to the mutation-period quiver in figure 1 is given by:

$$\varphi(x_1, \dots, x_5) = \left(x_3, x_4, x_5, \frac{x_2 + x_3^2 x_4 x_5}{x_1}, \frac{x_2 + x_3^2 x_4 x_5 + x_1 x_3 x_4 x_5^2}{x_1 x_2} \right)$$

as

$$\mathbf{x}' = \mu_1(\mathbf{x}) = \left(\frac{x_2 + x_3^2 x_4 x_5}{x_1}, x_2, \dots, x_5 \right)$$

and

$$\mu_2(\mathbf{x}') = \left(x_1', \frac{x_1' + x_3' x_4' {x_5'}^2}{x_2'}, x_3', x_4', x_5' \right).$$

The referred Laurent phenomenon implies that, starting with $\mathbf{x} = (1, 1, 1, 1, 1)$ and iterating by φ, one will always obtain integers, which is somewhat surprising as already in the fourth iteration we will be dividing by 3 and 21.

3 Full list of 2-periodic quivers with four nodes

The complete list of 4-node quivers with mutation-period 2 can be found below. Note that mutation-period 1 is also included but can be avoided by restricting the parameters.

We opted not to include the proof of this result as the 4-node case is a lot simpler than the 5-node case and the latter will be carried out with detail in the next section. Still, we can easily provide it if requested.

Proposition 1. *If Q is a 2-periodic 4-node quiver, then B_Q is one of the matrices below or its negative:*

1. $B_I = \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & -r & -s \\ q & r & 0 & -p \\ r & s & p & 0 \end{bmatrix}$ *with* $p, q, r, s \in \mathbb{Z}_0^+$

   *(*sink-type *family);*

2. $B_{II} = \begin{bmatrix} 0 & -p & q & -r \\ p & 0 & -r - pq & q \\ -q & r + pq & 0 & -p \\ r & -q & p & 0 \end{bmatrix}$ *with* $p, r \in \mathbb{Z}_0^+$, $q \in \mathbb{Z}^+$

   (family (10) in [7]);

3. $B_{III} = \begin{bmatrix} 0 & 0 & -p & 0 \\ 0 & 0 & 0 & q \\ p & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \end{bmatrix}$ *with* $p, q \in \mathbb{Z}^+$

   (difference of multiples of primitives*).*

These are all the possible cases (and include matrices for 1-periodic quivers).

Remark 2. *Family B_{III} is missing in [7] and is not symmetric with respect to the counter diagonal* [3]*. Also, family B_I is not symmetric unless $q = s$.*

4 Full list of 2-periodic quivers with five nodes

In the next Proposition we give the complete list of quivers of five nodes and mutation-period strictly 2 (we excluded the cases where the period is 1).

An immediate consequence is that all these quivers possess symmetry as the matrices B_Q are symmetric with respect to the counter diagonal.

Proposition 2. *If Q is a 5-node quiver with mutation-period strictly 2, then B_Q is one of the matrices below or its negative:*

1. $$B_1 = \begin{bmatrix} 0 & -1 & p+1 & 1 & p \\ 1 & 0 & -1 & 0 & 1 \\ -p-1 & 1 & 0 & -1 & p+1 \\ -1 & 0 & 1 & 0 & -1 \\ -p & -1 & -p-1 & 1 & 0 \end{bmatrix} \text{ with } p \in \mathbb{Z}^+$$

   (correct version of family (13) in [7]);

2. $$B_2 = \begin{bmatrix} 0 & p & 1 & p+1 & -1 \\ -p & 0 & -1 & p+1 & p+1 \\ -1 & 1 & 0 & -1 & 1 \\ -p-1 & -p-1 & 1 & 0 & p \\ 1 & -p-1 & -1 & -p & 0 \end{bmatrix} \text{ with } p \in \mathbb{Z}_0^+$$

   (family (12) in [7]);

3. $$B_3 = \begin{bmatrix} 0 & -p & 1 & 1 & -q \\ p & 0 & -p-q & 1-p & 1 \\ -1 & p+q & 0 & -p-q & 1 \\ -1 & p-1 & p+q & 0 & -p \\ q & -1 & -1 & p & 0 \end{bmatrix} \text{ with } p, q \in \mathbb{Z}_0^+ \text{ and } p \neq q$$

   (family (11) in [7]).

These are all the possible cases.

Proof. We are looking for all integer, skew-symmetric matrices satisfying the periodicity condition (5) with $m = 2$.

Clearly the quiver obtained by reverting the arrows retains the mutation-periodicity property. Equivalently $-B_Q$ still satisfies condition (5). This allows

[3]this matrix symmetry translates the graph-symmetry referred in [7]

us to reduce the number of cases to study by a factor of 2. Writing

$$B = B_Q = \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ -b_1 & 0 & b_5 & b_6 & b_7 \\ -b_2 & -b_5 & 0 & b_8 & b_9 \\ -b_3 & -b_6 & -b_8 & 0 & b_{10} \\ -b_4 & -b_7 & -b_9 & -b_{10} & 0 \end{bmatrix}$$

one obtains

$$\sigma^{-2} B \sigma^2 = \begin{bmatrix} 0 & b_{10} & -b_3 & -b_6 & -b_8 \\ -b_{10} & 0 & -b_4 & -b_7 & -b_9 \\ b_3 & b_4 & 0 & b_1 & b_2 \\ b_6 & b_7 & -b_1 & 0 & b_5 \\ b_8 & b_9 & -b_2 & -b_5 & 0 \end{bmatrix}.$$

Before considering the signs needed to perform mutations, note that the first row of $\mu_1(B)$ is the negative of that of B, and the second row of $\mu_2 \circ \mu_1(B)$ is the negative of that of $\mu_1(B)$. This shows that the mutation-periodic condition already implies $b_{10} = b_1$. So we proceed with general B of the form:

$$B = \begin{bmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ -b_1 & 0 & b_5 & b_6 & b_7 \\ -b_2 & -b_5 & 0 & b_8 & b_9 \\ -b_3 & -b_6 & -b_8 & 0 & b_1 \\ -b_4 & -b_7 & -b_9 & -b_1 & 0 \end{bmatrix}$$

and

$$\sigma^{-2} B \sigma^2 = \begin{bmatrix} 0 & b_1 & -b_3 & -b_6 & -b_8 \\ -b_1 & 0 & -b_4 & -b_7 & -b_9 \\ b_3 & b_4 & 0 & b_1 & b_2 \\ b_6 & b_7 & -b_1 & 0 & b_5 \\ b_8 & b_9 & -b_2 & -b_5 & 0 \end{bmatrix}. \tag{7}$$

To perform the first mutation the signs in the first row of B are needed. Using the fact aforementioned, these signs can be reduced to the 8 cases below.

For each case we use the sign information to compute the second row of $\mu_2 \circ \mu_1(B)$, then compare this incomplete matrix with (7) to get information on (the signs of) more entries until we are able to complete $\mu_2 \circ \mu_1(B)$. Then a careful handling of equations (with integer unknowns) leads either to B or the conclusion that B does not exist.

1. $\boxed{b_1, b_2, b_3, b_4 \in \mathbb{Z}_0^+}$ - in this case, mutations at nodes 1 and 2 produce (only the upper triangular part of each matrix is shown)

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 & b_6 & b_7 \\ & & 0 & b_8 & b_9 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & -b_5 & -b_6 & -b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

Comparing with (7) we obtain $b_4 = b_5$, $b_7 = b_6$, $b_9 = b_7$. In particular $b_5 \geq 0$. We argue that $b_6 \geq 0$, otherwise entries in row 1, column 4 would produce

$$0 < -b_6 = -b_3 + b_1 b_6 \leq 0.$$

The signs of b_5 and b_6 are all we need to complete the second mutation:

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & -b_5 & -b_6 & -b_7 \\ & & 0 & b_8 & b_9 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}$$

By comparing again with (7) we arrive at

$$b_3 = b_2, \quad b_4 = b_1, \quad b_5 = b_1, \quad b_6 = b_2, \quad b_7 = b_2, \quad b_8 = b_1, \quad b_9 = b_2$$

leading to the periodic quiver (which turns out to have period 1)

$$B = \begin{bmatrix} 0 & b_1 & b_2 & b_2 & b_1 \\ & 0 & b_1 & b_2 & b_2 \\ & & 0 & b_1 & b_2 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}.$$

2. $\boxed{b_1 \in \mathbb{Z}^-,\, b_2, b_3, b_4 \in \mathbb{Z}_0^+}$ - in this case we obtain

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 - b_1 b_2 & b_6 - b_1 b_3 & b_7 - b_1 b_4 \\ & & 0 & b_8 & b_9 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & b_1 b_2 - b_5 & b_1 b_3 - b_6 & b_1 b_4 - b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}$$

and the comparison with (7) gives:

$$b_4 = b_5 - b_1 b_2, \quad b_7 = b_6 - b_1 b_3, \quad b_9 = b_7 - b_1 b_4.$$

We now argue that $b_7 \geq 0$. In fact, assuming that $b_7 < 0$, entry in row 1, column 4 for $\mu_2 \circ \mu_1(B)$ would be $-b_3$ leading to equality $b_6 = b_3$ and to the contradiction

$$b_7 = b_3(1 - b_1) \geq 0$$

Since $b_7 \geq 0$ we immediately obtain $b_9 \geq 0$ as $b_1 b_4 \leq 0$. With this information we complete the second mutation:

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & -b_2 - b_1 b_4 & -b_3 - b_1 b_7 & -b_4 - b_1 b_9 \\ & 0 & b_1 b_2 - b_5 & b_1 b_3 - b_6 & b_1 b_4 - b_7 \\ & & 0 & b_8 & b_9 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}$$

Then the identity for entries in row 4, column 5 produces $b_5 = b_1$ so that $0 \leq b_4 = b_1(1 - b_2)$, which can only happen with $b_2 - 1 \geq 0$. Equating then entries in row 1, column 3 leads to

$$0 \leq b_3 = b_2 + b_1 b_4 = b_2 + b_1^2(1 - b_2)$$

so finally

$$b_2 \geq b_1^2(b_2 - 1) \tag{8}$$

Due to the sign of $(b_2 - 1)$, there are precisely two possibilities for (8):

i. $b_2 = 1$

ii. $b_1 = -1$ (coming from $b_1^2 \leq \frac{b_2}{b_2 - 1}$ with $b_1 \in \mathbb{Z}^-$)

By comparing the remaining entries of (7) and $\mu_2 \circ \mu_1(B)$ it is easily checked that the first possibility leads to the periodic quiver

$$B_i = \begin{bmatrix} 0 & b_1 & 1 & 1 & 0 \\ & 0 & b_1 & b_1 + 1 & 1 \\ & & 0 & b_1 & 1 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}, \quad b_1 \in \mathbb{Z}^-$$

(included in family B_3 with $q = 0$), whereas the second leads to the quiver

$$B_{ii} = \begin{bmatrix} 0 & -1 & b_2 & 1 & b_2 - 1 \\ & 0 & -1 & 0 & 1 \\ & & 0 & -1 & 0 \\ & & & 0 & -1 \\ & & & & 0 \end{bmatrix}, \quad b_2 > 1$$

(coinciding with family B_1). Both families have period 2 but not 1.

3. $\boxed{b_2 \in \mathbb{Z}^-,\ b_1, b_3, b_4 \in \mathbb{Z}_0^+}$ - this case does not produce any quiver, since

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 + b_1 b_2 & b_6 & b_7 \\ & & 0 & b_8 - b_2 b_3 & b_9 - b_2 b_4 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & -(b_5 + b_1 b_2) & -b_6 & -b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

Because $0 \le b_4 = b_5 + b_1 b_2$, we can compute the entry in row 1, column 3 of $\mu_2 \circ \mu_1(B)$, and by comparing it with that of (7) gives $b_3 = b_2$, which is incompatible with the sign hypothesis.

4. $\boxed{b_3 \in \mathbb{Z}^-,\ b_1, b_2, b_4 \in \mathbb{Z}_0^+}$ - this case is ruled out similarly to case 3, since

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 & b_6 + b_1 b_3 & b_7 \\ & & 0 & b_8 + b_2 b_3 & b_9 \\ & & & 0 & b_1 - b_3 b_4 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & -b_5 & -(b_6 + b_1 b_3) & -b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

As $0 \le b_4 = b_5$ we easily compute the entry in row 1, column 3 of the last matrix, and doing the same comparison leads again to $b_3 = b_2$, which is impossible.

5. $\boxed{b_4 \in \mathbb{Z}^-,\ b_1, b_2, b_3 \in \mathbb{Z}_0^+}$ - start with the mutations:

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 & b_6 & b_7 + b_1 b_4 \\ & & 0 & b_8 & b_9 + b_2 b_4 \\ & & & 0 & b_1 + b_3 b_4 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & -b_5 & -b_6 & -(b_7 + b_1 b_4) \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

Compare with (7) to obtain:

$$b_4 = b_5, \quad b_7 = b_6, \quad b_9 = b_7 + b_1 b_4.$$

We now argue that $b_9 \geq 0$. In fact, assuming $b_9 < 0$, entry in row 3, column 5 for $\mu_2 \circ \mu_1(B)$ would be $b_9 + b_2 b_4$ leading to the contradiction

$$0 \leq b_2 = b_9 + b_2 b_4 < 0$$

Therefore $b_9 \geq 0$ which in turn implies $b_6 = b_7 \geq 0$ as $b_1 b_4 \leq 0$. Now complete the second mutation:

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & -b_2 + b_1 b_4 & -b_3 & -b_4 \\ & 0 & -b_5 & -b_6 & -(b_7 + b_1 b_4) \\ & & 0 & b_8 - b_4 b_6 & b_9 + b_2 b_4 - b_4 b_9 \\ & & & 0 & b_1 + b_3 b_4 \\ & & & & 0 \end{bmatrix}$$

Using again the identity for entries in row 3, column 5 one obtains $b_9 = b_2$. Comparing now the entries in row 4, column 5 and using $b_5 = b_4$ leads to

$$0 \leq b_1 = b_4(1 - b_3)$$

which takes place only if $b_3 - 1 \geq 0$. Finally equality between entries in row 1, column 3 produce

$$0 \leq b_2 = b_3 + b_1 b_4 = b_3 + b_4^2(1 - b_3)$$

that is,

$$b_3 \geq b_4^2(b_3 - 1) \tag{9}$$

Now the argument proceeds along the lines of case 2: due to the sign of $(b_3 - 1)$, there are precisely two possibilities for (9):

iii. $b_3 = 1$

iv. $b_4 = -1$ (coming from $b_4^2 \leq \frac{b_3}{b_3 - 1}$ with $b_4 \in \mathbb{Z}^-$)

The first possibility leads to the periodic quiver

$$B_{iii} = \begin{bmatrix} 0 & 0 & 1 & 1 & b_4 \\ & 0 & b_4 & 1 & 1 \\ & & 0 & b_4 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}, \quad b_4 \in \mathbb{Z}^-$$

(included in family B_3 with $p = 0$), and the second leads to the quiver

$$B_{iv} = \begin{bmatrix} 0 & b_1 & 1 & b_1+1 & -1 \\ & 0 & -1 & b_1+1 & b_1+1 \\ & & 0 & -1 & 1 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}, \quad b_1 \in \mathbb{Z}_0^+$$

(coinciding with family B_2). Again both families have period 2 but not 1.

6. $\boxed{b_1, b_2 \in \mathbb{Z}^-,\ b_3, b_4 \in \mathbb{Z}_0^+}$ - an inconsistence appears fairly quickly as

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 & b_6 - b_1 b_3 & b_7 - b_1 b_4 \\ & & 0 & b_8 - b_2 b_3 & b_9 - b_2 b_4 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & -b_5 & b_1 b_3 - b_6 & b_1 b_4 - b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

Here, as in case 4, we conclude that $0 \le b_4 = b_5$. Equating entries in row 1, column 3 gives $b_3 = b_2 + b_1 b_5$, which is impossible with the signs under consideration.

7. $\boxed{b_1, b_3 \in \mathbb{Z}^-,\ b_2, b_4 \in \mathbb{Z}_0^+}$ - this case requires more care, as there is no easy contradiction.

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 - b_1 b_2 & b_6 & b_7 - b_1 b_4 \\ & & 0 & b_8 + b_2 b_3 & b_9 \\ & & & 0 & b_1 - b_3 b_4 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & b_1 b_2 - b_5 & -b_6 & b_1 b_4 - b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

A first comparison with row 2 of (7) gives

$$b_4 = b_5 - b_1 b_2, \quad b_7 = b_6, \quad b_9 = b_7 - b_1 b_4.$$

Comparing the entries in row 1, column 4 we conclude that $b_6 \leq 0$. In fact, if $b_6 > 0$ we would obtain

$$0 < b_6 = b_3 + b_1 b_6$$

which is impossible with the signs of this case.

Also, using the entries in row 3, column 5 we obtain $b_9 \geq 0$. In fact, assuming $b_9 < 0$ would produce

$$0 \leq b_2 = b_9 + b_4 b_9$$

which is impossible.

The signs of b_6 and b_9 allow us to complete the second mutation:

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & -b_2 - b_1 b_4 & -b_3 & -b_4 - b_1 b_9 \\ & 0 & b_1 b_2 - b_5 & -b_6 & b_1 b_4 - b_7 \\ & & 0 & b_8 + b_2 b_3 + b_4 b_6 & b_9 \\ & & & 0 & b_1 - b_3 b_4 - b_7 b_9 \\ & & & & 0 \end{bmatrix}.$$

and conclude that $b_6 = b_3$ and $b_9 = b_2$.

To see that there are no quivers in this case we proceed to compare entries in row 4, column 5 (using $b_7 = b_6 = b_3$ and $b_9 = b_2$) to arrive at:

$$b_5 = b_1 - b_3 b_4 - b_1 b_2$$

Now using $b_4 = b_5 - b_1 b_2$ and substituting for b_5 in the previous equation we get

$$0 \geq b_4(1 + b_3) = b_1(1 - b_2) - b_2 b_3$$

The only possibilities that respect the signs of this case are $b_2 = 0$ or $b_2 = 1$. Using row 1, column 3 to produce $b_3 = b_2 + b_1 b_4$ we see that

$$b_2 = 0 \quad \Rightarrow \quad \begin{cases} b_4(1 + b_3) = b_1 \\ b_3 = b_1 b_4 \end{cases} \quad \rightsquigarrow \quad b_3 = b_4^2(1 + b_3)$$

which is impossible.

Likewise

$$b_2 = 1 \quad \Rightarrow \quad b_4(1 + b_3) = -b_3$$

which is again impossible.

8. $\boxed{b_1, b_4 \in \mathbb{Z}^-,\ b_2, b_3 \in \mathbb{Z}_0^+}$ - this last case originates family B_3 and a quiver with period 1.

$$\mu_1(B) = \begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ & 0 & b_5 - b_1 b_2 & b_6 - b_1 b_3 & b_7 \\ & & 0 & b_8 & b_9 + b_2 b_4 \\ & & & 0 & b_1 + b_3 b_4 \\ & & & & 0 \end{bmatrix}$$

and

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & * & * & * \\ & 0 & b_1b_2 - b_5 & b_1b_3 - b_6 & -b_7 \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{bmatrix}.$$

Comparing with (7) we get:

$$b_4 = b_5 - b_1b_2, \quad b_7 = b_6 - b_1b_3, \quad b_9 = b_7.$$

Clearly $b_7 \geq 0$, since if $b_7 < 0$ then entry in row 1, column 4 for $\mu_2 \circ \mu_1(B)$ would be $-b_3$ leading to the contradiction

$$0 > b_7 = b_3 - b_1b_3 \geq 0$$

With the sign of b_7 (and recalling $b_9 = b_7$) we complete the second mutation

$$\mu_2 \circ \mu_1(B) = \begin{bmatrix} 0 & b_1 & -b_2 & -b_3 - b_1b_7 & -b_4 - b_1b_7 \\ & 0 & -b_5 & -b_6 & -(b_7 + b_1b_4) \\ & & 0 & b_8 - b_4b_7 & b_9 + b_2b_4 - b_4b_7 \\ & & & 0 & b_1 + b_3b_4 \\ & & & & 0 \end{bmatrix}$$

Now we just have to compare the adequate entries in this matrix and (7) to achieve all solutions. Row 1, column 4 leads to $b_7 = b_3 + b_1(b_7 - b_3)$, so that $b_7 = b_3$. Moreover row 1, column 3 and row 4, column 5 lead to $b_4 = b_1 + b_2(b_4 - b_1)$, and therefore to the following possibilities:

v. $b_2 = 1$

vi. $b_4 = b_1$

Collecting all equalities we conclude that v leads to the periodic quiver

$$B_v = \begin{bmatrix} 0 & b_1 & 1 & 1 & b_4 \\ & 0 & b_1 + b_4 & 1 + b_1 & 1 \\ & & 0 & b_1 + b_4 & 1 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}, \quad b_1, b_4 \in \mathbb{Z}^-$$

(family B_3), which has period 1 if $b_1 = b_4$, whereas vi leads to the quiver

$$B_{vi} = \begin{bmatrix} 0 & b_1 & b_2 & b_2 & b_1 \\ & 0 & b_1(1 + b_2) & b_2(1 + b_1) & b_2 \\ & & 0 & b_1(1 + b_2) & b_2 \\ & & & 0 & b_1 \\ & & & & 0 \end{bmatrix}, \quad b_1 \in \mathbb{Z}^-, b_2 \in \mathbb{Z}_0^+$$

which has period 1.

□

Remark 3. *The inequality* $m_1 > \sqrt{\frac{m_2}{m_2+1}}$ *in page 42 of [7] is wrong: the correct version leads to* $m_1 = 1$, $m_3 = -1$ *and to the family* B_1.

5 Conclusions and comments

In this note we have presented the full list of 2-periodic quivers with both 4 and 5 nodes. The proof was carried out only in the 5-node case, as for quivers with 4 nodes the proof is analogous but simpler. In the case of 4 nodes we included a family which does not have the graph-symmetry mentioned in [7]. As for the 5-node case, our proof shows that all these quivers have graph-symmetry. This is somehow surprising, as the motivation in [7] to impose graph-symmetry on 5-node quivers came from that same property in 4-node quivers, which does not always hold.

In Figure 4 below, we depict new quivers illustrating these conclusions (the first obtained with $(p, q) = (1, 2)$ in family B_{III} and the second with $p = 2$ in family B_1).

Figure 4: New 4-node (non-symmetric) and 5-node (hence symmetric) quivers with strict period 2.

Concerning 5-node quivers with period 2, we note that a common feature to all B_Qs is that all these matrices have rank 4. This means that reduction of the associated cluster map by log-canonical pre-symplectic structures produces a symplectic reduced map in precisely four variables (the reduced symplectic maps for all the families can be found in [2]). Moreover, we claim that it is not possible to use log-canonical Poisson structures to achieve reduction of the cluster map since only the zero Poisson structure is invariant under this map (we refer to [1] for more details and examples). The proof of this statement can also be found in [2].

Acknowledgements. Inês Cruz was partially supported by CMUP – Centro de Matemática da Universidade do Porto, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UID/00144.

References

[1] Cruz, Inês and Sousa-Dias, M. Esmeralda, *Reduction of cluster iteration maps*, J. Geom. Mech., **6**, no. 3, 297–318, (2014).

[2] Ferreira, Jorge M. B., *Classificação de quivers* 2*-periódicos com* 5 *nós (Classification of* 2*-periodic quivers with* 5 *nodes)*, Master thesis, FCUP, University of Porto, (2020). Available at: `https://repositorio-aberto.up.pt/handle/10216/132074`.

[3] Fomin, Sergey and Zelevinsky, Andrei, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15**, no. 2, 497–529, (2002).

[4] Fomin, Sergey and Zelevinsky, Andrei, *The Laurent phenomenon*, Adv. Appl. Math. **28**, 119–144, (2002).

[5] Fordy, Allan P. and Hone, Andrew, *Symplectic maps from cluster algebras*, SIGMA Symmetry Integrability Geom. Methods Appl. **7**, Paper 091, 12, (2011).

[6] Fordy, Allan P. and Hone, Andrew, *Discrete integrable systems and Poisson algebras from cluster maps*, Commun. Math. Phys. **325**, 527–584 (2014).

[7] Fordy, Allan P. and Marsh, Robert J., *Cluster mutation-periodic quivers and associated Laurent sequences*, J. Algebraic Combin. **34**, no. 1, 19–66, (2011).

[8] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Cluster Algebras and Poisson Geometry*, Mathematical Surveys and Monographs, **167**, AMS, (2010).

[9] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. **127**, no. 2, 291–311, (2005).

[10] Quispel, G. R. W., Roberts, J. A. G. and Thompson, C. J., *Integrable mappings and soliton equations*, Phys. Lett. A, **126**, no. 7, 419–421, (1988).