COMPLETE MONOTONICITY OF DISCRETE EQUATIONS AND RELATED RESULTS

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Abstract. In this paper we are mainly interested in studying linear discrete Volterra difference equations. Such equations are used within a numerous of different mathematics contexts. We are particularly interested in the complete monotonicity of its solutions. Throughout our study we found very interesting connections between different mathematical fields, so we believe our findings may be useful for researchers acting on subjects such as (but not exclusively) discrete probability theory or numerical anlaysis.

1. Introduction

The theory of fractional difference equations has been evolving rapidly in the past two decades, with novel and interesting results appearing in several articles and books (cf. [5, 6, 9, 10, 11, 18] and the references therein). It is to be mentioned that "fractional" constitutes a generalization to the theory of difference equations [4], essentially due to the fact that one deals with equations depending on a difference operator of arbitrary real order (see for example Definition 2 below).

In this work we study linear Volterra difference equations. We start by highlighting the fact that, for such equations and that depend on a completely monotonic kernel, its solution is also completely monotonic. This strong result is somehow hidden in the work by Li and Wang [14] in which they are mainly concerned to develop numerical schemes for fractional differential equations. We report such an important result in Section 3.1. We then relate these general Volterra difference equations with discrete probability distributions in Section 3.2, in particular, we introduce a general class of probability distributions that include Pillai's et al. distribution [19] as a particular case (the latter mentioned distribution is connected with the discrete Linnik distributions (see [3])). An explicit representation of the probability mass function (see (8) below) is obtained by using the recent results appearing in the literature in [7].

In Section 3.3 we return to the complete monotonicity issue and we deal with the specific *fractional case*. Our approach here is new and uses some known integral representations of the Mittag-Leffler function, from which we extract some boundedness and sign properties of certain Mittag-Leffler functions.

In Section 4 we present a presumably novel identity that emerges from our study (22). Moreover, we dwell into a novel way of showing the notable Mittag-Leffler's formula [21, Théorème 9] when the parameters are real. The formula corresponds to the Laplace transform of the Mittag-Leffler function.

2010 *Mathematics Subject Classification.* 33E12, 39A12, 26A33.

Key words and phrases. Discrete Mittag-Leffler function, completely monotonic, fractional difference equations.

Finally, in Section 5, we close this manuscript presenting an open problem regarding the solutions of linear fractional difference equations. It is astonishing that, contrarily to the continuous case, the geometry of the solutions to such equations is not yet completely known.

2. Definitions, notations and preliminary results

We start with some concepts needed in the sequel.

For $a \in \mathbb{R}$ we define

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots\} \text{ and } \mathbb{N}^a = \{\ldots, a-2, a-1, a\},$$

where we let $\mathbb{N}_1 = \mathbb{N}$.

The *falling function* is defined, for $x, y \in A \subset \mathbb{R}$, by

$$x^{\underline{y}} = \begin{cases} x(x-1)\ldots(x-y+1) \text{ for } y \in \mathbb{N}_1, \\ 1 \text{ for } y = 0, \\ \frac{\Gamma(x+1)}{\Gamma(x+1-y)} \text{ for } x, x-y \notin \mathbb{N}^{-1}, \\ 0 \text{ for } x \notin \mathbb{N}^{-1} \text{ and } x-y \in \mathbb{N}^{-1}, \end{cases}$$

and the *rising function* is defined by

$$x^{\overline{y}} = (x+y-1)^{\underline{y}}. \tag{1}$$

We now present the essential concepts of the discrete Δ fractional calculus and the discrete ∇ fractional calculus. We refer the reader to the monographs [6] and [10] for more on the subject.

Definition 1. *For $a \in \mathbb{R}$ consider a function $f : \mathbb{N}_a \to \mathbb{R}$.*

The forward difference operator is defined by $\Delta[f](t) = f(t+1) - f(t)$, for $t \in \mathbb{N}_a$. Also, we define higher order differences recursively as $\Delta^n[f](t) = \Delta[\Delta^{n-1}f](t)$, $n \in \mathbb{N}_1$, where Δ^0 is the identity operator, i.e. $\Delta^0[f](t) = f(t)$.

The backward difference operator is defined by $\nabla[f](t) = f(t) - f(t-1)$, $t \in \mathbb{N}_{a+1}$. Differences of higher order are defined recursively by $\nabla^n[f](t) = \nabla[\nabla^{n-1}f](t)$, $n \in \mathbb{N}_1$, where ∇^0 is the identity operator, i.e. $\nabla^0[f](t) = f(t)$.

We proceed by introducing the definitions of fractional sum and difference with the Δ operator.

Definition 2. *Consider $a \in \mathbb{R}$ and let $f : \mathbb{N}_a \to \mathbb{R}$. Then we define the* Δ-fractional sum of f *of order $\nu > 0$ to be the function $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu-1} \to \mathbb{R}$ given by*

$$\Delta_a^{-\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-(s+1))^{\underline{\nu-1}} f(s), & \text{if } t \in \mathbb{N}_{a+\nu}, \\ 0 & \text{if } t = a+\nu-1. \end{cases}$$

We also put $\Delta_a^{-0} f(t) = f(t)$.

The Riemann–Liouville Δ-fractional difference of f *of order $0 < \alpha \le 1$ is defined as*

$$\Delta_a^\alpha f(t) = \Delta[\Delta_a^{-(1-\alpha)} f](t), \quad t \in \mathbb{N}_{a+1-\alpha},$$

while the Caputo Δ-fractional difference of f *of order $0 < \alpha \le 1$ is defined by*

$$^C\Delta_a^\alpha f(t) = \Delta_a^{-(1-\alpha)}[\Delta f](t), \quad t \in \mathbb{N}_{a+1-\alpha},$$

The corresponding definitions for the ∇ operator are the following.

Definition 3. *Consider $a \in \mathbb{R}$ and let $f : \mathbb{N}_{a+1} \to \mathbb{R}$. Then we define the* ∇-fractional sum of f *of order $\nu > 0$ by*

$$\nabla_a^{-\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-(s-1))^{\overline{\nu-1}} f(s), & \text{if } t \in \mathbb{N}_{a+\nu}, \\ 0 & \text{if } t = a. \end{cases}$$

We also put $\nabla_a^{-0} f(t) = f(t)$.

The Riemann–Liouville ∇-fractional difference of f *of order $0 < \alpha \le 1$ is defined as*

$$\nabla_a^{\alpha} f(t) = \nabla[\nabla_a^{-(1-\alpha)} f](t), \quad t \in \mathbb{N}_{a+1},$$

while the Caputo ∇-fractional difference of f *of order $0 < \alpha \le 1$ is defined by*

$${}^{C}\nabla_a^{\alpha} f(t) = \nabla_a^{-(1-\alpha)}[\nabla f](t), \quad t \in \mathbb{N}_{a+1}.$$

As is usual we define the (classical) Mittag-Leffler function and the two-parameter Mittag-Leffler function by, respectively,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)} \quad \text{and} \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

As it was mentioned before, we are mainly concerned with the completely monotonicity of sequences. The concept follows.

Definition 4. [25, pag. 108] *We say that a sequence $u : \mathbb{N}_0 \to \mathbb{R}$ is completely monotonic if*

$$(-1)^m \Delta^m u(n) \ge 0, \quad m, n \in \mathbb{N}_0.$$

The complete monotonicity of a sequence is related to the *Hausdorff moment problem.* The following characterization of a completely monotonic sequence is due to Hausdorff.

Theorem 1. [25, Theorem 4a] *A sequence $u : \mathbb{N}_0 \to \mathbb{R}$ is completely monotonic if and only if*

$$u(n) = \int_0^1 t^n d\alpha(t),$$

where $\alpha(t)$ is a bounded and nondecreasing for $t \in [0,1]$.

For example, the sequence $u(n) = \frac{1}{n+1}$ is completely monotonic since $u(n) = \int_0^1 t^n dt$. However, in general, Theorem 1 is of difficult application.

More recently Liu and Pego obtained a novel characterization of a completely monotonic sequence, which is based on generating functions and is of very practical use. In order to enunciate it one needs the concept of Pick function.

Definition 5. *A function f is a Pick function if f is analytic in the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and leaves it invariant, satisfying $\mathrm{Im}(z)\,\mathrm{Im}(f(z)) \ge 0$ for all z in the domain of f.*

Given a sequence $u : \mathbb{N}_0 \to \mathbb{R}$, its generating function is defined by $F_u(z) = \sum_{k=0}^{\infty} u(k) z^k$.

Theorem 2. [15, Theorem 1] *Let $u : \mathbb{N}_0 \to \mathbb{R}$ be a sequence with generating function F_u. Then, u is completely monotonic if and only if F_u is a Pick function that is analytic and nonnegative on $(-\infty, 1)$.*

3. Volterra difference equations

3.1. **Complete monotonicity.** Let $\kappa : \mathbb{N}_0 \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ be such that $1 + \lambda\kappa(0) \neq 0$. Consider the Volterra difference equation

$$u(n) = 1 - \lambda \sum_{\tau=1}^{n} \kappa(n-\tau)u(\tau), \quad n \in \mathbb{N}_0, \tag{2}$$

where $u : \mathbb{N}_0 \to \mathbb{R}$ is the sequence to be found and we are assuming that empty sums are equal to zero, i.e., $\sum_{\tau=1}^{0} g(\tau) = 0$ for any function g. It is well known that (2) has a unique solution $u_{\kappa,\lambda}$. In fact, an explicit representation of the solution of (2) was recently obtained in [7] and we will evidenciate it shortly after.

Put $q_a^{(1)}(n) = q(n)$ and $q_a^{(m)}(n) = (q * q_a^{(m-1)})_a(n)$, $m \in \mathbb{N}_2$, where $*$ denotes the discrete convolution, i.e.,

$$(f * g)_a(n) = \sum_{\tau=a}^{n-1} f(n-\tau-1+a)g(\tau), \quad n \in \mathbb{N}_a.$$

We then have:

Theorem 3. [7, Theorem 5] *Let $\lambda \in \mathbb{R}$ be such that $1 + \lambda\kappa(0) \neq 0$. Then the solution of* (2) *has the following representation,*

$$u_{\kappa,\lambda}(n) = \frac{1}{1+\lambda\kappa(0)} \left(1 + \sum_{k=1}^{n-1} \left(\frac{-\lambda}{1+\lambda\kappa(0)} \right)^k \sum_{s=k}^{n-1} q_1^{(k)}(s) \right), \quad n \in \mathbb{N}. \tag{3}$$

Observe that it follows from (2) that $u(0) = 1$ independently of the kernel κ.

Remark 1. *The previous result was enunciated in* [7] *in a* differential form. *Indeed,* (3) *is the solution of the following initial value problem (cf.* [7, Theorem 5]*)*

$$\begin{aligned} {}^{C}\mathcal{D}_{\{(\tilde{\kappa});0\}} u(n) &= -\lambda u(n), \quad n \in \mathbb{N}, \\ u(0) &= 1, \end{aligned}$$

where κ (in (2)*) and $\tilde{\kappa}$ form a Sonine pair and ${}^{C}\mathcal{D}_{\{(\tilde{\kappa});0\}}$ is the generalized discrete Caputo operator (cf.* [7, Section 2]*).*

Li and Wang introduced in [14] the concept of *complete monotonicity-preserving numerical methods* for fractional ordinary differential equations, in which the discrete convolutional kernels inherit the complete monotonicity property as the continuous equations. In particular they studied the complete monotonicity of equations of the type given in (2). The following result is of interest *per se* and should be highlighted.

Theorem 4. [14, Theorem 4.2] *If κ is completely monotonic and $\lambda > 0$, then the solution of* (2) *is completely monotonic.*

Proof. For completeness we show here that the generating of the solution u of (2) is equal to the generating function given by [14, (4.6)].

We have,

$$\begin{aligned} F_u(z) &= (1-z)^{-1} - \lambda \sum_{r=0}^{\infty} \sum_{\tau=1}^{r} \kappa(r-\tau) u(\tau) z^r \\ &= (1-z)^{-1} - \lambda \sum_{\tau=1}^{\infty} u(\tau) \sum_{r=\tau}^{\infty} \kappa(r-\tau) z^r \\ &= (1-z)^{-1} - \lambda \sum_{\tau=1}^{\infty} u(\tau) z^{\tau} \sum_{r=0}^{\infty} \kappa(r) z^r \\ &= (1-z)^{-1} - \lambda F_u(z) F_\kappa(z) + \lambda F_\kappa(z), \end{aligned}$$

from which we arrive at

$$F_u(z) = \frac{(1-z)^{-1} + \lambda F_\kappa(z)}{1 + \lambda F_\kappa(z)}.$$

In [14, Theorem 4.2] the authors concluded by using Theorem 2 that u is completely monotonic. □

Remark 2. *It is expectable that Theorem 3 may turn useful in the development of numerical schemes for Volterra integral equations, in particular, to fractional differential equations. Indeed,* (3) *furnishes a representation of the solution for the discrete equation used in the numerical scheme* [14].

Remark 3. *We note that Theorem 4 is the discrete analogue of a theorem due to Friedman* [8, Theorem 8]. *Essentially, this result indicates that the solution of a linear Volterra integral equation with a completely monotonic kernel is completely monotonic.*

Example 1. *Let $\lambda > 0$ and consider*

$$\kappa_\alpha(n) = \frac{(n+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}, \quad 0 < \alpha \leq 1, \ n \in \mathbb{N}_0.$$

Using (1) *and* [6, Proposition 1.6] *repeatedly we get*

$$\Delta^m \kappa_\alpha(n) = \frac{(n+\alpha-1)^{\underline{\alpha-m-1}}}{\Gamma(\alpha-m)}, \ \alpha \in (0,1), \quad \Delta^m \kappa_1(n) = 0, \ m \in \mathbb{N}.$$

Therefore, $(-1)^m \Delta^m \kappa_\alpha(n) \geq 0$, i.e., κ_α is completely monotonic. By Theorem 4, the solution of the Volterra equation

$$u(n) = 1 - \lambda \sum_{\tau=1}^{n} \frac{(n-\tau+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(\tau), \quad n \in \mathbb{N}_0,$$

is completely monotonic. In particular, when $\alpha = 1$, the solution is easily seen to be $u_\lambda(n) = \left(\frac{1}{1+\lambda}\right)^n$, and it is well known to be completely monotonic (see for example [25, pag. 108]*).*

3.2. **Probability distributions.** In this section we wish to approach the Volterra difference equation (2) in terms of the theory of probability distributions. In order to motivate it, we start with Pillai and Jayakumar work [19, Section 2] in which they introduced *discrete Mittag-Leffler distributions* based on generating functions (cf. [19, (2.2)]).

Let $0 < p, \alpha < 1$. From [19, Section 3], we know that the probability mass function $a : \mathbb{N}_0 \to \mathbb{R}$ satisfies the recurrence relation,

$$a(0) = p, \tag{4}$$

$$a(n) = (1-p)\sum_{k=0}^{n-1}(-1)^k\binom{\alpha}{k+1}a(n-1-k), \quad n \in \mathbb{N}. \tag{5}$$

We now show that the sequence a defined above satisfies a fractional difference equation.

Proposition 1. *Suppose* $\{\alpha, p\} \in (0,1)$*. Then, the sequence defined by* (4)–(5) *satisfies the following initial value problem (IVP),*

$$\nabla^{\alpha}_{-1}a(n) = \frac{p}{p-1}a(n), \quad n \in \mathbb{N}, \tag{6}$$

$$a(0) = p. \tag{7}$$

Proof. We have, for $n \in \mathbb{N}$,

$$\begin{aligned}
a(n) &= (1-p)\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{\alpha}{n-k}a(k)\\
&= (1-p)\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{\alpha^{\underline{n-k}}}{\Gamma(n-k+1)}a(k)\\
&= (1-p)\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(-\alpha)^{\overline{n-k}}}{\Gamma(n-k+1)}(-1)^{n-k}a(k)\\
&= (1-p)\sum_{k=0}^{n-1}-\frac{(-\alpha)^{\overline{n-k}}}{\Gamma(n-k+1)}a(k)\\
&= (p-1)\sum_{k=0}^{n-1}\frac{\Gamma(n-k-\alpha)}{\Gamma(-\alpha)\Gamma(n-k+1)}a(k)\\
&= (p-1)\sum_{k=0}^{n-1}\frac{(n-k+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}a(k)\\
&= (p-1)\sum_{k=0}^{n}\frac{(n-k+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}a(k) - (p-1)a(n),
\end{aligned}$$

i.e.,

$$pa(n) = (p-1)\sum_{k=0}^{n}\frac{(n-k+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}a(k),$$

and finally,

$$\frac{p}{p-1}a(n) = \nabla^{\alpha}_{-1}a(n),$$

where the previous equality follows from [10, Theorem 3.62]. The proof is done. □

Remark 4. *The previous Proposition provides further "evidence" of Pillai and Jayakumar calling "Mittag-Leffler" to their distribution. Indeed, in the theory of discrete fractional calculus and in analogy with the theory of continuous fractional calculus, the solution of* (6)–(7) *is named as "Discrete Mittag-Leffler function" (see* [10, Definition 3.98 and Theorem 3.100]*).*

We now provide the explicit solution to the IVP (6)–(7), which we emphasize was not given in [19].

Theorem 5. *Under the hypothesis of Proposition 1 we have:*

$$a(n) = (p-1)\sum_{k=0}^{n-1}(-p)^{k+1}\sum_{r=0}^{k}(-1)^r\binom{k+1}{r}\frac{(n+1)^{\overline{(k+1-r)\alpha-1}}}{\Gamma((k+1-r)\alpha)}, \quad n\in\mathbb{N}_1. \tag{8}$$

Proof. In order to prove (8) we make use of the recent result obtained in [7, Corollary 2]. It provides, in particular, the solution to the Caputo initial value problem (recall also Remark 1):

$${}^{C}\nabla^{\alpha}_{-1}u(n) = \frac{p}{p-1}u(n), \quad n\in\mathbb{N}_0, \tag{9}$$

$$u(-1) = 1. \tag{10}$$

Indeed, we have,

$$u(n) = (1-p)\left(1+\sum_{k=0}^{n-1}(-p)^{k+1}\sum_{r=0}^{k}(-1)^r\binom{k+1}{r}\frac{(n+1)^{\overline{(k+1-r)\alpha}}-(k+1)^{\overline{(k+1-r)\alpha}}}{\Gamma((k+1-r)\alpha+1)}\right), \; n\in\mathbb{N}_0.$$

Now, note that by applying the operator ∇ to both sides of (9), we obtain

$$\nabla{}^{C}\nabla^{\alpha}_{-1}u(n) = \frac{p}{p-1}\nabla u(n), \quad n\in\mathbb{N},$$

that is equivalent to (cf. [10, Lemma 3.108]),

$$\nabla^{\alpha}_{-1}[\nabla u](n) = \frac{p}{p-1}\nabla u(n), \quad n\in\mathbb{N}.$$

Define $a(n) = -\nabla u(n)$ for $n\in\mathbb{N}_0$. Then, $a(0) = -(u(0)-u(-1)) = 1-(1-p) = p$ and a obviously satisfies (6), therefore, by the uniqueness of solution to the IVP (6)–(7) we are left to calculate ∇u. Using the Leibniz formula [10, Theorem 3.41] and the nabla power rule [10, Theorem 3.3], we obtain

$$\begin{aligned}\nabla u(n) &= (1-p)\sum_{k=0}^{n-1}(-p)^{k+1}\sum_{r=0}^{k}(-1)^r\binom{k+1}{r}\frac{(k+1-r)\alpha(n+1)^{\overline{(k+1-r)\alpha-1}}}{\Gamma((k+1-r)\alpha+1)}\\ &= (1-p)\sum_{k=0}^{n-1}(-p)^{k+1}\sum_{r=0}^{k}(-1)^r\binom{k+1}{r}\frac{(n+1)^{\overline{(k+1-r)\alpha-1}}}{\Gamma((k+1-r)\alpha)}, \quad n\in\mathbb{N},\end{aligned}$$

hence (8) is shown and the proof is done. □

Remark 5. *We would like to mention that in a paper by Christoph and Schreiber, in which discrete Linnik distributions are studied, they calculate the probabilities referring to the discrete Mittag-Leffler distribution introduced by Pillai and Jayakumar* (4)–(5) *(cf.* [3, Section 1.3]*). The formula presented therein* [3, Theorem 1.3.1]*) is (with $\beta = 1$ and $\gamma = \alpha$):*

$$b(n)=(-1)^n p\sum_{m=0}^{n}(p-1)^m\sum_{j=0}^{m}(-1)^{m+j}\binom{m}{j}\binom{\alpha j}{n},\quad n\in\mathbb{N}_0.$$

The uniqueness of solution to the IVP in Proposition 1 implies that $b(n)=a(n)$ for all $n\in\mathbb{N}_0$, with a given by (7) *and* (8)*.*

Remark 6. *It is easy to show that when $\alpha=1$, we obtain the probability mass function for the geometric distribution:*

$$a(n)=p(1-p)^n,\quad n\in\mathbb{N}_0.$$

Therefore, Pillai and Jayakumar probability function is a generalization of the geometric one.

Motivated by what was seen above, we next introduce a very general probability distribution function. Before we do it, let us observe that for a completely monotonic sequence u, $\lim_{n\to\infty}u(n)=0$ if and only if $\lim_{x\to 1^-}(1-x)F_u(x)=0$ (cf. [16]).

Suppose that $\lambda>0$ and κ is a completely monotonic kernel, and let the sequence $u_{\kappa,\lambda}$ be given by (3) (it is completely monotonic by Theorem 4). We have seen in the proof of Theorem 4 that its generating function is

$$F_{u_{\kappa,\lambda}}(z)=\frac{(1-z)^{-1}+\lambda F_\kappa(z)}{1+\lambda F_\kappa(z)}. \tag{11}$$

Let us assume that

$$\lim_{x\to 1^-}(1-x)F_{u_{\kappa,\lambda}}(x)=0.$$

Then, we define the probability distribution function $D:\mathbb{N}_0\to\mathbb{R}$ associated with $u_{\kappa,\lambda}$ by

$$D(n)=1-u_{\kappa,\lambda}(n).$$

For completeness, let us show that for Pillai's *et al.* distribution, i.e., for when $\kappa_\alpha(n)=\frac{(n+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$, we have $\lim_{x\to 1^-}(1-x)F_{u_{\alpha,\lambda}}(x)=0$, where $u_{\alpha,\lambda}$ is the corresponding solution of (2). Indeed, since $F_{\kappa_\alpha}(x)=(1-x)^{-\alpha}$ we get, by (11),

$$\lim_{x\to 1^-}(1-x)F_{u_{\alpha,\lambda}}(x)=\lim_{x\to 1^-}\frac{1+\lambda(1-x)^{1-\alpha}}{1+\lambda(1-x)^{-\alpha}}=0.$$

3.3. **Fractional difference equations: another approach to complete monotonicity.** In this section we will consider specific kernels leading to the theory of fractional difference equations. We will start with the “nabla (∇) case" and later we deal with the “delta (Δ) case".

It is known[1] that the solution of the initial value problem

$${}^C\nabla_0^\alpha u(n)=-\lambda u(n),\quad n\in\mathbb{N},\ \lambda\neq 1,\ 0<\alpha\leq 1, \tag{12}$$

$$u(0)=1, \tag{13}$$

or, equivalently, of the Volterra difference equation

$$u(n)=1-\lambda\sum_{k=1}^{n}\frac{(n-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(k),\quad n\in\mathbb{N}_0, \tag{14}$$

[1]Probably the first appearence of such functions as solutions of linear fractional difference equations was in the work by Atici and Eloe [1] and this solution was derived in total analogy with the continuous case.

is given, for $|\lambda|<1$, by the series (cf. [10, Lemma 3.141])

$$u_{\alpha,\lambda}(n)=\sum_{k=0}^{\infty}(-\lambda)^k\frac{\Gamma(k\alpha+n)}{\Gamma(n)\Gamma(k\alpha+1)},\quad n\in\mathbb{N}_0. \tag{15}$$

It is understood in (15) that $u_{\alpha,\lambda}(0)=1$. Now, observe that

$$\begin{aligned}u_{\alpha,\lambda}(n)&=\sum_{k=0}^{\infty}(-\lambda)^k\frac{\Gamma(k\alpha+n)}{\Gamma(n)\Gamma(k\alpha+1)}\\&=\sum_{k=0}^{\infty}(-\lambda)^k\frac{\int_0^\infty e^{-t}t^{k\alpha+n-1}dt}{\Gamma(n)\Gamma(k\alpha+1)}\\&=\int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}E_\alpha(-\lambda t^\alpha)dt,\end{aligned} \tag{16}$$

where the interchange of the summation with the integral is justified by the dominated convergence theorem. Lizama [17] obtained the representation (16) for a shifted problem with respect to (12)–(13), but it was deduced in a more general setting than ours and we are not certain if it was obtained as in the equalities leading to (16). In any case, we show next that, as long as the integral in (16) is well defined, then it is the solution of (14).

Lemma 1. *Let $0<\alpha<1$ and $\lambda\in\mathbb{R}$ be such that*

$$u_{\alpha,\lambda}(n)=\int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}E_\alpha(-\lambda t^\alpha)dt,\quad n\in\mathbb{N}_0,$$

is well defined. Then, $u_{\alpha,\lambda}$ solves (14).

Proof. Ite is very well known that (cf. [13, Section 4.1.3]

$$E_\alpha(-\lambda t^\alpha)=1-\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}E_\alpha(-\lambda s^\alpha)ds.$$

Consider $n\in\mathbb{N}$ (recall that we have agreed that $u_{\alpha,\lambda}(0)=1$). We have,

$$\begin{aligned}u_{\alpha,\lambda}(n)&=\int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}dt-\lambda\int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}E_\alpha(-\lambda s^\alpha)dsdt\\&=1-\lambda\int_0^\infty E_\alpha(-\lambda s^\alpha)\int_s^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dtds\\&=1-\lambda\int_0^\infty E_\alpha(-\lambda s^\alpha)\int_0^\infty e^{-s-u}\frac{(s+u)^{n-1}}{\Gamma(n)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}duds\\&=1-\lambda\sum_{k=0}^{n-1}\binom{n-1}{k}\int_0^\infty e^{-s}E_\alpha(-\lambda s^\alpha)\frac{s^{n-1-k}}{\Gamma(n)}ds\int_0^\infty e^{-u}\frac{u^{k+\alpha-1}}{\Gamma(\alpha)}du\\&=1-\lambda\sum_{k=0}^{n-1}\frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)}\int_0^\infty e^{-s}E_\alpha(-\lambda s^\alpha)\frac{s^{n-1-k}}{\Gamma(n-k)}ds\\&=1-\lambda\sum_{k=0}^{n-1}\frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)}u_{\alpha,\lambda}(n-k)\\&=1-\lambda\sum_{k=1}^{n}\frac{(n-k+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u_{\alpha,\lambda}(k),\end{aligned}$$

where at some point we have used the Binomial Theorem. The proof is done. □

Remark 7. *It is nowadays a classic result that the Mittag-Leffler function* $E_\alpha(-x)$ *is completely monotonic for* $x>0$ *and* $\alpha\in(0,1]$ *(cf.* [12, Proposition 3.23]*). Recall that a real valued function defined on* $[0,\infty)$ *is completely monotonic if* $(-1)^m f^{(m)}(x)\geq 0$ *for all* $x>0$ *and* $m\in\mathbb{N}_0$*. Therefore,* $E_\alpha(-\lambda t^\alpha)$ *with* $\lambda>0$ *is nonnegative and bounded, in particular,* (16) *is well defined.*

The following result is crucial for our purposes.

Lemma 2. *Let* $m\in\mathbb{N}_0$ *and* $0<\alpha<1$*. Then, the function* $E_{\alpha,\alpha-m+1}(-x)$ *is bounded on* $[0,\infty)$ *and*

$$E_{\alpha,\alpha-m+1}(-x)\begin{cases}\geq 0 & \text{if } m \text{ is odd,}\\ \leq 0 & \text{if } m \text{ is even.}\end{cases}$$

Proof. We start with $m=0$. By the well known equality [12, (4.2.3)], we may write

$$E_{\alpha,\alpha+1}(-x)=\frac{1-E_\alpha(-x)}{x},$$

from which we immediately infer that

$$0\leq E_{\alpha,\alpha+1}(-x)\leq\frac{1}{x},\quad x>0.$$

Therefore, $\lim_{x\to\infty}E_{\alpha,\alpha+1}(-x)=0$. Moreover, since $E'_\alpha(-x)=-\frac{E_{\alpha,\alpha}(-x)}{\alpha}$, we obtain by L'Hôpital's rule

$$\lim_{x\to 0^+}E_{\alpha,\alpha+1}(-x)=\lim_{x\to 0^+}\frac{E_{\alpha,\alpha}(-x)}{\alpha}=\frac{1}{\Gamma(\alpha+1)}.$$

Therefore $E_{\alpha,\alpha+1}(-x)$ is bounded and nonnegative on $[0,\infty)$.

Now, suppose that $m\in\mathbb{N}$. From [20, (4.10)], we have the following representation for the Mittag-Leffler function:

$$\begin{aligned}E_{\alpha,\alpha-m+1}(-x)&=\frac{\sin(\pi(\alpha-m+1))}{\pi}\int_0^\infty\frac{y^{\alpha+m-1}e^{-y}}{y^{2\alpha}+2xy^\alpha\cos(\pi\alpha)+x^2}dy\\&=\frac{(-1)^{m+1}\sin(\alpha\pi)}{\pi}\int_0^\infty\frac{y^{\alpha+m-1}e^{-y}}{(y^\alpha+x\cos(\alpha\pi))^2+(x\sin(\alpha\pi))^2}dy.\end{aligned}$$

Hence, $|E_{\alpha,\alpha-m+1}(-x)|\leq\frac{\sin(\alpha\pi)}{\pi}\Gamma(m-\alpha)$, i.e., $E_{\alpha,\alpha-m+1}(-x)$ is bounded and its sign is immediately seen. The proof is done. □

Lemma 3. *Let* $m\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$*. Then,*

$$(-1)^m\Delta^m u_{\alpha,\lambda}(n)=\lambda\frac{(-1)^{m+1}}{\Gamma(n+m)}\int_0^\infty e^{-t}t^{n+\alpha-1}E_{\alpha,\alpha-m+1}(-\lambda t^\alpha)dt. \tag{17}$$

Proof. In order to avoid problems with the case $n=0$, we write $u_{\alpha,\lambda}$ in the equivalent way

$$u_{\alpha,\lambda}(n)=1-\lambda\int_0^\infty e^{-t}\frac{t^{n+\alpha-1}}{\Gamma(n)}E_{\alpha,\alpha+1}(-\lambda t^\alpha)dt,\quad n\in\mathbb{N}_0. \tag{18}$$

By Lemma 2 the sequence $u_{\alpha,\lambda}$ given by (18) is well defined.

We now show that (17) holds for $m=1$ and then the result follows easily by induction on m. We have,

$$
\begin{aligned}
-\Delta u(n) &= \lambda \int_0^\infty e^{-t} \frac{t^{n+\alpha-1}}{\Gamma(n+1)} (t-n) E_{\alpha,\alpha+1}(-\lambda t^\alpha) dt \\
&= \lambda \int_0^\infty e^{-t} \frac{t^{n-1}}{\Gamma(n+1)} (t-n) t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) dt \\
&= \lambda \left[-e^{-t} \frac{t^n}{\Gamma(n+1)} t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) \right]_0^\infty + \lambda \int_0^\infty e^{-t} \frac{t^n}{\Gamma(n+1)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) dt \\
&= \lambda \int_0^\infty e^{-t} \frac{t^{n+\alpha-1}}{\Gamma(n+1)} E_{\alpha,\alpha}(-\lambda t^\alpha) dt,
\end{aligned}
$$

where we have used the formula

$$\frac{d}{dt}[t^{b-1} E_{a,b}(xt^a)] = t^{b-2} E_{a,b-1}(xt^a).$$

The proof is now concluded. □

Now it follows immediately the following

Corollary 1. *For $\lambda > 0$ and $0 < \alpha < 1$ the solution of* (14) *is completely monotonic.*

As mentioned at the beginning of this section we now address the “delta Δ case".

Consider the following initial value problem:

$$ {}^C\Delta_{\alpha-1}^\alpha u(n) = -\lambda u(n+\alpha-1), \quad t \in \mathbb{N}_0,\ 0 < \alpha < 1, \tag{19} $$

$$ u(\alpha-1) = 1. \tag{20} $$

In passing we note that, when $\alpha = 1$, the problem above reduces to $\Delta u(n) = -\lambda u(n)$ and $u(0) = 1$ whose solution is $u(n) = (1-\lambda)^n$. For $\lambda \in (0,1)$ it is completely monotonic. Now, with $\alpha < 1$ and $0 < \lambda \le \alpha$ it is known that (19)–(20) has a nonnegative and decreasing solution (cf. [6, Corollary 3.17 and the proof of Theorem 3.22]). However, the solution is not completely monotonic (see Figure 1) being that a considerable difference to the “nabla case".

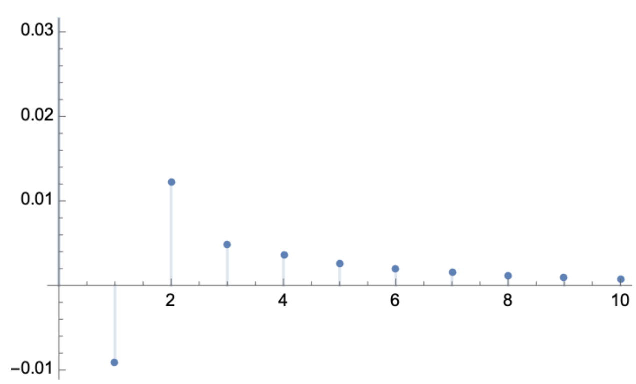

FIGURE 1. $\Delta^2 u(n)$ for $n \in [0,10] \cap \mathbb{N}_0$ with $\alpha = 4/10$ and $\lambda = 3/10$.

4. The Laplace transform of the Mittag-Leffler function

Consider again the initial value problem (12)–(13). Its solution has the following representation (cf. [7, Corollary 2]):

$$u_{\alpha,\lambda}(n)=\frac{1}{1+\lambda}\left(1+\sum_{k=1}^{n-1}\left(\frac{-\lambda}{1+\lambda}\right)^k\sum_{r=0}^{k-1}(-1)^r\binom{k}{r}\frac{n^{\overline{(k-r)\alpha}}-k^{\overline{(k-r)\alpha}}}{\Gamma((k-r)\alpha+1)}\right),\ n\in\mathbb{N}. \tag{21}$$

By Lemma 1 and the uniqueness of solution to (12)–(13), we obtain the interesting equality

$$\int_0^\infty e^{-t}\frac{t^{n-1}}{\Gamma(n)}E_\alpha(-\lambda t^\alpha)dt=\frac{1}{1+\lambda}\left(1+\sum_{k=1}^{n-1}\left(\frac{-\lambda}{1+\lambda}\right)^k\sum_{r=0}^{k-1}(-1)^r\binom{k}{r}\frac{n^{\overline{(k-r)\alpha}}-k^{\overline{(k-r)\alpha}}}{\Gamma((k-r)\alpha+1)}\right),\ n\in\mathbb{N}, \tag{22}$$

provided the left hand side is well defined. Observe in particular for $n=1$ that

$$\int_0^\infty e^{-t}E_\alpha(-\lambda t^\alpha)dt=\frac{1}{1+\lambda}, \tag{23}$$

which is nothing else but the Laplace transform of Mittag-Leffler's function, which was obtained in his remarkable work [21, Théorème 9]. Indeed, Mittag-Leffler uses complex analysis theory to study the left hand side of (23) and determines the exact set (in the complex plane) where the integral is well defined.

In what follows we show that the integral in (23) (and (22)) exists if $\lambda\in(-1,\infty)$. To the best of our knowledge we use a new method to accomplish it, which was probably unknown to Mittag-Leffler himself since we use the fact that his function is the solution of a fractional differential equation. We also emphasize that we do not use here the fact that the Mittag-Leffler function (MLF) is a completely monotonic function as this is usually proved by using contour integration to obtain an integral representation of the MLF. However we could not find an *elementary argument* to show that the integral in (23) diverges for $\lambda\le -1$. In particular, it would be interesting to prove the inequality $E_\alpha(t^\alpha)>e^t$ ($t>0$) without appealing to complex analytic methods and the known integral representation for the Mittag-Leffler function [20, Art. 101].

We proceed using only real analytic tools.

Lemma 4. *Let $\lambda>0$ and $\alpha\in(0,1)$. Then $0<E_\alpha(-\lambda t^\alpha)<1$ for all $t>0$.*

Proof. It is well known that $y(t)=E_\alpha(-\lambda t^\alpha)$ satisfies the initial value problem (cf. [13, Example 4.9])

$${}^C D_{0+}^\alpha y(t)=-\lambda y(t),\quad t>0, \tag{24}$$

$$y(0)=1. \tag{25}$$

The Caputo derivative, being continuous on $[0,\infty)$, has the following representation (cf. [22, Theorem 5.2] and [23, Theorem 4.3 and Remark 4.4]):

$${}^C D_{0+}^\alpha y(t)=\frac{1}{\Gamma(1-\alpha)}\left(\frac{y(t)-y(0)}{t^\alpha}+\alpha\int_0^t(t-s)^{-\alpha-1}(y(t)-y(s))ds\right),\quad t>0.$$

Suppose that $T > 0$ is the first time where $y(T) = 0$. Then, since $y(0) > 0$, we get the contradition

$$0 = {}^C D_{0+}^{\alpha} y(T) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{-y(0)}{T^{\alpha}} + \alpha \int_0^T (T-s)^{-\alpha-1}(-y(s))ds \right) < 0.$$

Therefore, $y(t) > 0$ for all $t > 0$.

Now observe that, by (24)–(25), there exists $\delta > 0$ such that ${}^C D_{0+}^{\alpha} y(t) < 0$ on $[0, \delta)$, from which $y(t) < y(0) = 1$ on $[0, \delta)$. Suppose that $M \geq \delta$ is the first time such that $y(M) = 1$. Then again we get a contradiction:

$$-\lambda = {}^C D_{0+}^{\alpha} y(M) = \frac{1}{\Gamma(1-\alpha)} \left(\alpha \int_0^M (M-s)^{-\alpha-1}(1-y(s))ds \right) > 0.$$

Therefore, $y(t) < 1$ for all $t > 0$ and the proof is done. □

Corollary 2. *If $\lambda \in (-1, \infty)$, then the integral*

$$\int_0^{\infty} e^{-t} \frac{t^{n-1}}{\Gamma(n)} E_{\alpha}(-\lambda t^{\alpha}) dt,$$

is well defined and, therefore, (22) *holds.*

Proof. We already noted in (16) that the integral exists when $|\lambda| < 1$. It is worth mentioning that the convergence of the series therein might be shown by using the elementary inequality on the ratio of gamma functions due to Wendel [24]. The result now follows by Lemma 4. □

5. Open Problem

We have seen that the solution of the initial value problem (12)–(13) is explicitly given by (21). Recall that when $\alpha = 1$ we get,

$$u_{1,\lambda}(n) = \frac{1}{(1+\lambda)^n}, \quad n \in \mathbb{N}_0.$$

It is readily seen that, for $\lambda < -1$, the sequence oscillates in the sense that $u_{1,\lambda}(n) u_{1,\lambda}(n+1) < 0$ for all $n \in \mathbb{N}_0$. We wonder about what is the oscillating behaviour of $u_{\alpha,\lambda}$ when $0 < \alpha < 1$ and $\lambda < -1$. It can be seen in Figure 2 that the sequence might not oscillate at every point, so the pattern as when $\alpha = 1$ might be lost.

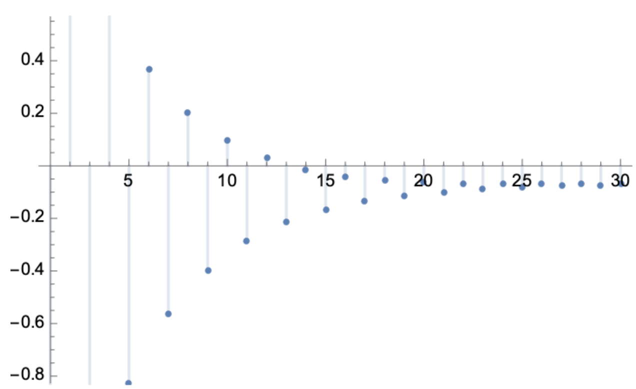

Figure 2. $u_{1/2,-3/2}(n)$ for $n \in [1, 30] \cap \mathbb{N}$.

However, Figure 3 below suggests that $u_{1/2,-7/5}(n) u_{1/2,-7/5}(n+1) < 0$ for all $n \in \mathbb{N}_0$

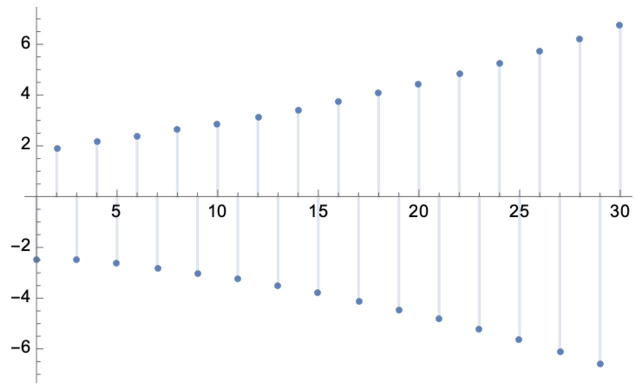

FIGURE 3. $u_{1/2,-7/5}(n)$ for $n \in [1,30] \cap \mathbb{N}$

Numerical simulations provide evidence that one of the two patterns mentioned above will occur, being the turning point: $\lambda = -2^{\alpha}$. It is remarkable that this point is related with the stability of solutions of fractional difference equations (cf. [2]).

We couldn't obtain any result characterizing the oscillatory behaviour of the solution to the initial value problem (12)–(13) and we leave that as an open problem.

Acknowledgments

Rui A. C. Ferreira was supported by the "Fundação para a Ciência e a Tecnologia (FCT)" through the program "Stimulus of Scientific Employment, Individual Support-6th Edition Call" with reference https://doi.org/10.54499/2023.05967.CEECIND/CP2833/CT0001.

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