Homological properties of some quantum Heisenberg algebras

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Abstract

In this paper we study the properties Koszul, Artin-Schelter regular and (skew) Calabi-Yau of some special types of quantum and generalized Heisenberg algebras and also analyze relations between these algebras, (graded) iterated Ore extensions and (graded) skew PBW extensions. The first-named author and Razavinia [17] introduced the quantum generalized Heisenberg algebras, which depend on a parameter q and two polynomials $f, g \in K[t]$. We prove that under certain conditions for f, g these algebras are Koszul, Artin-Shelter regular, Calabi-Yau and graded Calabi-Yau.

Key words and phrases. Quantum generalized Heisenberg algebras, graded skew PBW extensions, Calabi-Yau algebras, Artin-Schelter regular, Koszul algebras.

2020 *Mathematics Subject Classification.* 16E65, 16S36, 16S37, 16S80, 16W50, 17B81.

1 Introduction

Quantum generalized Heisenberg algebras were introduced by the first-named author and Razavinia in [17] where they classified their finite-dimensional irreducible representations. Also in [18], they made their classification by isomorphism, the description of their automorphism groups and the study of ring-theoretical properties like Gelfand-Kirillov dimension and being noetherian. Skew PBW extensions were defined in [7], and recently studied in many papers (see for example [22, 23, 26, 27, 29]). In [25], the

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second-named author defined the graded skew PBW extensions for an algebra R as a generalization of graded iterated Ore extensions. Some properties of graded skew PBW extensions have been studied in [25, 24, 28].

Throughout this paper, K is an arbitrary field, $K^* = K \setminus \{0\}$, every ring (algebra) is associative with identity and algebras are K-algebras, $K\langle x_1, \dots, x_n\rangle$ is the free associative algebra with n generators, unless otherwise stated. We will denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}^+$ the sets of all integers, nonnegative integers, and positive integers, respectively. A graded algebra $B = \bigoplus_{p\geq 0} B_p$ is called *connected* if $B_0 = K$. A connected graded algebra $B = K \oplus B_1 \oplus B_2 \oplus \cdots$ is said to be *Artin-Schelter regular* of dimension d if B has finite global dimension d and

$$Ext^i_B(K, B) \cong \begin{cases} K(l), & i = d; \\ 0, & i \neq d, \end{cases}$$

for some $l \in \mathbb{Z}$. Commutative Artin-Schelter regular algebras are polynomial rings. Artin-Schelter regular algebras of global dimension two and global dimension three were classified in [1], but there are also many open questions about these algebras; for example, the classification of Artin-Schelter regular algebras of global dimension greater than three. Different authors have focused on studying Artin-Schelter regular algebras, especially those of global dimension four and global dimension five (see for example [30, 32, 33, 34]). Zhang and Zhang in [32] introduced a special class of algebras called double Ore extensions. They proved that a connected graded double Ore extension of an Artin-Schelter regular algebra is Artin-Schelter regular. The same authors in [33] constructed 26 families of Artin-Schelter regular algebras of global dimension four, using double Ore extensions.

The *enveloping algebra* of an algebra B is the tensor product $B^e = B \otimes B^{op}$, where B^{op} is the opposite algebra of B. Calabi-Yau algebras were defined by Ginzburg in [8] and the skew Calabi-Yau algebras were defined as a generalization of those. A graded algebra B is called *graded skew Calabi-Yau* of dimension d if, as a B^e-module, B has a projective resolution that has finite length and such that each term in the projective resolution is finitely generated and there exists an algebra automorphism ν of B such that

$$Ext^i_{B^e}(B, B^e) \cong \begin{cases} 0, & i \neq d; \\ B^{\nu}(l), & i = d; \end{cases}$$

as B^e -modules, for some integer l. If ν is the identity, then B is said to be *Calabi-Yau*. Ungraded Calabi-Yau algebras are defined similarly but without degree shift. The automorphism ν is called the *Nakayama* automorphism of B, and is unique up to inner automorphisms of B. Note that a skew Calabi-Yau algebra is Calabi-Yau if and only if its Nakayama automorphism is inner. Reyes, Rogalski and Zhang in [21] proved that for connected algebras, the skew Calabi-Yau property is equivalent to the Artin-Schelter regular property. Calabi-Yau algebras are skew Calabi-Yau, but some skew Calabi-Yau algebras are not Calabi-Yau, for example the Jordan plane. Gómez and Suárez in [9] gave necessary and sufficient conditions for a graded (trimmed) double Ore extension to be a graded (quasi-commutative) skew PBW extension and using this fact, they proved that a graded skew PBW extension $A = \sigma(R)\langle x_1, x_2\rangle$ of an Artin-Schelter regular algebra R is Artin-Schelter regular and that every graded skew PBW extension $A = \sigma(R)\langle x_1, x_2\rangle$ of a connected skew Calabi-Yau algebra R of dimension d is skew Calabi-Yau of dimension $d + 2$.

A graded algebra $B = K \oplus B_1 \oplus B_2 \oplus \cdots$ is called *Koszul* if there exists a graded projective resolution of K

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to K \to 0,$$

such that for any $i \geq 0$, P_i is generated in degree i. In [25, Theorem 5.5], the second-named author proved that if A is a graded skew PBW extension of a finitely presented Koszul algebra R, then A is Koszul.

Definition 1.1 ([7], Definition 1)**.** Let R and A be rings. We say that A is a *skew PBW extension over* R (the ring of coefficients), denoted $A = \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same identity element.

(ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\mathrm{Mon}(A) := \{x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1.1}$$

(iv) For $1 \leq i, j \leq n$, there exists $d_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - d_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n, \tag{1.2}$$

i.e. there exist elements $r_0^{(i,j)}, r_1^{(i,j)}, \dots, r_n^{(i,j)} \in R$ with $x_j x_i - d_{i,j} x_i x_j = r_0^{(i,j)} + \sum_{k=1}^{n} r_k^{(i,j)} x_k$.

From the definition it follows that every non-zero element $f \in A$ can be uniquely expressed as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R$ and $X_i \in \mathrm{Mon}(A)$, for $0 \leq i \leq m$ [7, Remark 2]. For $X = x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathrm{Mon}(A)$, $\deg(X) = |\alpha| := \alpha_1 + \cdots + \alpha_n$.

The next result establishes the importance of the (injective) endomorphisms of the ring of coefficients and later on we will make these endomorphisms interact in different aspects.

Proposition 1.2 ([7], Proposition 3). *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension, then there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $1 \le i \le n$, where $r \in R$.*

Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension over a ring R and σ_i, δ_i as in Proposition 1.2. A is said to be *bijective* if σ_i is bijective for each $1 \le i \le n$ and $d_{i,j}$ is invertible for any $1 \le i < j \le n$. If δ_i is zero for every i, then A is called a skew PBW extension of *endomorphism type*.

Proposition 1.3 ([25], Proposition 2.7). *Let $R = \bigoplus_{m\ge 0} R_m$ be an $\mathbb{N}$-graded algebra and let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R satisfying the following two conditions:*

(i) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation for all $1 \le i \le n$, where σ_i and δ_i are as in Proposition 1.2.*

(ii) *$x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (1.2) and $c_{i,j} \in R_0$.*

For $p \ge 0$, let A_p be the K-space generated by the set

$$\Big\{ r_t x^\alpha \mid t + |\alpha| = p,\ r_t \in R_t \text{ and } x^\alpha \in \mathrm{Mon}(A) \Big\}.$$

Then A is an $\mathbb{N}$-graded algebra with graduation

$$A = \bigoplus_{p\ge 0} A_p. \tag{1.3}$$

Definition 1.4 ([25], Definition 2.6). Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of an $\mathbb{N}$-graded algebra $R = \bigoplus_{m\ge 0} R_m$. We say that A is a *graded skew PBW extension* if A satisfies the conditions (i) and (ii) in Proposition 1.3.

2 Quantum Heisenberg algebras

The *enveloping algebra of the 3-dimensional Heisenberg Lie algebra* (or *Heisenberg algebra*, for short) is the unital associative algebra $\mathcal{H}$ generated by the variables x, y, t, with relations

$$tx = xt, \quad yt = ty, \quad yx - xy = t. \tag{2.1}$$

According to the relations (2.1) we have that $\mathcal{H}$ is a skew PBW extension of $K[t]$, i.e., $\mathcal{H} = \sigma(K[t])\langle x, y\rangle$. Also, $\mathcal{H}$ is the Ore extension of derivation type $K[t, x][y; \delta]$, where $\delta(t) = 0$ and $\delta(x) = t$.

Note that the Heisenberg algebra is Calabi-Yau of dimension 3 [11, Proposition 4.6] and Artin-Schelter regular [1, Page 172].

Kirkman and Small in [13] studied a q-analogue to $\mathcal{H}$, known as quantum Heisenberg algebra. For $q \in K^*$ the *quantum Heisenberg algebra* $\mathcal{H}_q$ is the algebra generated by t, x, y subject to the relations

$$yx - qxy = t, \quad xt = qtx, \quad yt = q^{-1}ty. \tag{2.2}$$

Note that $\mathcal{H}_q$ is a skew PBW extension of $K[t]$, i.e., $\mathcal{H}_q = \sigma(K[t])\langle x, y\rangle$. Also $\mathcal{H}_q$ is the iterated Ore extension

$$K[t][x; \sigma_x][y; \sigma_y, \delta_y],$$

where $\sigma_x(t) = qt$, $\sigma_y(t) = q^{-1}t$, $\sigma_y(x) = qx$, $\delta_y(t) = 0$, $\delta_y(x) = t$.

For $q \in K^*$ and $n > 1$, the *q-Heisenberg algebra* $\mathcal{H}_{q(n)}$ is the algebra generated by the variables $x_1, \dots, x_n$, $y_1, \dots, y_n$, $t_1, \dots, t_n$ subject to the relations:

$$x_jx_i = x_ix_j, \qquad t_jt_i = t_it_j, \qquad y_jy_i = y_iy_j, \quad 1 \leq i, j \leq n, \tag{2.3}$$

$$y_it_j = t_jy_i, \qquad x_it_j = t_jx_i, \qquad y_jx_i = x_iy_j, \quad i \neq j, \tag{2.4}$$

$$y_ix_i = qx_iy_i + t_i, \qquad x_it_i = qt_ix_i, \qquad y_it_i = q^{-1}t_iy_i, \ 1 \leq i \leq n. \tag{2.5}$$

Note that $\mathcal{H}_{q(n)}$ is a skew PBW extension of $K[t_1, \dots, t_n]$, i.e.,

$$\mathcal{H}_{q(n)} = \sigma(K[t_1, \dots, t_n])\langle x_1, \dots, x_n; y_1, \dots, y_n\rangle,$$

although $\mathcal{H}_{q(n)}$ is also a bijective skew PBW extension of K. Moreover, $\mathcal{H}_{q(n)}$ is the iterated Ore extension

$$K[t_1, \dots, t_n][x_1; \sigma_1] \cdots [x_n; \sigma_n][y_1; \theta_1, \delta_1] \cdots [y_n; \theta_n, \delta_n],$$

where

$$\begin{aligned}
&\theta_j(t_i) = t_i, && \delta_j(t_i) = 0, && \sigma_j(t_i) = t_i, && 1 \leq i < j \leq n, \\
&\theta_j(x_i) = x_i, && \delta_j(x_i) = 0, && \theta_j(t_i) = t_i, && \delta_j(t_i) = 0, \quad \sigma_j(t_i) = t_i, \ i \neq j, \\
&\theta_i(x_i) = qx_i, && \delta_i(x_i) = t_i, && \theta_i(t_i) = q^{-1}t_i, && \delta_i(t_i) = 0, \quad \sigma_i(t_i) = qt_i, 1 \leq i \leq n.
\end{aligned}$$

Gaddis in [6] introduced and studied a two-parameter analog of the Heisenberg algebra $\mathcal{H}$. For $p, q \in K^*$, the *two-parameter quantum Heisenberg algebra* $\mathcal{H}_{p,q}$ is the algebra generated by t, x, y subject to the relations

$$yx - qxy = t, \quad xt = ptx, \quad yt = p^{-1}ty. \tag{2.6}$$

Note that if $p = q$, the two-parameter quantum Heisenberg algebra $\mathcal{H}_{p,q}$ becomes the quantum Heisenberg algebra $\mathcal{H}_q$.

In spite of the seemingly lack of symmetry between p and q in the definition of the two-parameter quantum Heisenberg algebra $\mathcal{H}_{p,q}$, we actually have $\mathcal{H}_{p,q} \simeq \mathcal{H}_{q,p}$, as shown below, a result which we believe hadn't been noticed before.

Proposition 2.1. *Let $p, q \in K^*$. We have the following algebra isomorphisms:*

$$\mathcal{H}_{p,q} \simeq \mathcal{H}_{p^{-1},q^{-1}} \simeq \mathcal{H}_{q,p}. \tag{2.7}$$

Proof. First we show that there is an isomorphism $\phi : \mathcal{H}_{p,q} \longrightarrow \mathcal{H}_{p^{-1},q^{-1}}$. For the sake of clarity, we denote the defining generators of $\mathcal{H}_{p,q}$ by x, y, t and those of $\mathcal{H}_{p^{-1},q^{-1}}$ by X, Y, T. Consider the free associative algebra $K\langle x, y, t\rangle$ and define $\phi : K\langle x, y, t\rangle \longrightarrow \mathcal{H}_{p^{-1},q^{-1}}$ by $\phi(x) = Y$, $\phi(y) = X$ and $\phi(t) = -qT$. Then, computing in $\mathcal{H}_{p^{-1},q^{-1}}$, we have

$$\begin{aligned}\phi(yx - qxy - t) &= XY - qYX + qT = -q(YX - q^{-1}XY - T) = 0;\\ \phi(xt - ptx) &= Y(-qT) - p(-qT)Y = -q(YT - pTY) = 0;\\ \phi(yt - p^{-1}ty) &= X(-qT) - p^{-1}(-qT)X = -q(XT - p^{-1}TX) = 0.\end{aligned}$$

Thus, ϕ factors through $\mathcal{H}_{p,q}$ and defines an algebra homomorphism $\phi : \mathcal{H}_{p,q} \longrightarrow \mathcal{H}_{p^{-1},q^{-1}}$, which is clearly bijective, proving the isomorphism.

Next, we show that there is an isomorphism $\phi : \mathcal{H}_{p,q} \longrightarrow \mathcal{H}_{q^{-1},p^{-1}}$. Proceeding as before, we denote the defining generators of $\mathcal{H}_{q^{-1},p^{-1}}$ by X, Y, T and set $\phi(x) = X$, $\phi(y) = Y$ and $\phi(t) = T + (p^{-1} - q)XY$. Then, computing in $\mathcal{H}_{q^{-1},p^{-1}}$, we have

$$\begin{aligned}\phi(yx - qxy - t) &= YX - qXY - T - (p^{-1} - q)XY\\ &= p^{-1}XY + T - qXY - T - (p^{-1} - q)XY = 0;\\ \phi(xt - ptx) &= X(T + (p^{-1} - q)XY) - p(T + (p^{-1} - q)XY)X\\ &= XT + (p^{-1} - q)X^2Y - pTX + (pq - 1)X(p^{-1}XY + T)\\ &= XT + (p^{-1} - q)X^2Y - pqXT + (pq - 1)XT + (q - p^{-1})X^2Y = 0;\\ \phi(yt - p^{-1}ty) &= Y(T + (p^{-1} - q)XY) - p^{-1}(T + (p^{-1} - q)XY)Y\\ &= YT + (p^{-1} - q)YXY - p^{-1}TY - (p^{-2} - qp^{-1})XY^2\\ &= qTY + (p^{-1} - q)p^{-1}XY^2 - qTY - (p^{-2} - qp^{-1})XY^2 = 0.\end{aligned}$$

So, ϕ factors through $\mathcal{H}_{p,q}$ and defines an isomorphism.

To conclude, we apply the latter isomorphism to $\mathcal{H}_{p^{-1},q^{-1}}$ to get $\mathcal{H}_{p^{-1},q^{-1}} \simeq \mathcal{H}_{q,p}$. □

Proposition 2.2. *$\mathcal{H}_{p,q}$ is a bijective skew PBW extension of endomorphism type of $K[t]$ and an iterated Ore extension.*

Proof. The second and third relations in (2.6) correspond to condition (1.1) of Definition 1.1. The first defining relation in (2.6) corresponds to condition (1.2). The endomorphisms and derivations of Proposition 1.2 are given by $\sigma_x(t) = pt$, $\delta_x(t) = 0$ and $\sigma_y(t) = p^{-1}t$, $\delta_y(t) = 0$. Therefore $\mathcal{H}_{p,q} = \sigma(K[t])\langle x, y\rangle$ is a bijective skew PBW extension of endomorphism type. Also (see [6, Proposition 3.4]) $\mathcal{H}_{p,q}$ is the iterated Ore extension

$$K[t][x; \sigma_x][y; \sigma_y, \delta_y],$$

where $\sigma_x(t) = pt$, $\sigma_y(t) = p^{-1}t$, $\sigma_y(x) = qx$, $\delta_y(t) = 0$ and $\delta_y(x) = t$. □

Remark 2.3. In [10] the first defining relation in (2.6) from $\mathcal{H}_{p,q}$ is taken to be $yx - q^2xy = t^2$ (see [6]). In contexts, such as with the following properties, where it is relevant to work with a graded algebra, we will work with this homogeneous relation instead.

Let $A = R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ be an iterated Ore extension. Then A is called a *graded iterated Ore extension* if $x_1, \dots, x_n$ have degree 1 in A,

$$\sigma_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \to R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \tag{2.8}$$

is a graded algebra automorphism and

$$\delta_i : R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}](-1) \to R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \tag{2.9}$$

is a graded σ_i-derivation, $2 \leq i \leq n$, where (-1) in (2.9) denotes a shift.

The class of graded iterated Ore extensions is strictly contained in the class of graded skew PBW extensions [24, Remark 2.11].

Corollary 2.4. *Let $\mathcal{H}_{p,q}$ be as in Remark 2.3. Then $\mathcal{H}_{p,q}$ is a connected graded skew PBW extension of $R = K[t]$ and a graded iterated Ore extension.*

Proof. By the comment above, it suffices to show that $\mathcal{H}_{p,q}$ is a connected graded iterated Ore extension. As the homogeneous component of degree 0 of $\mathcal{H}_{p,q}$ is K, then $\mathcal{H}_{p,q}$ is connected. Now, by Proposition 2.2, we have that $\mathcal{H}_{p,q}$ is an iterated Ore extension. It is clear that $\sigma_x(t) = pt$, $\sigma_y(t) = q^{-1}t$, $\sigma_y(x) = q^2x$ are graded algebra automorphisms and $\delta_y(x) = t^2$ is a graded σ_y-derivation of degree 1. So $\mathcal{H}_{p,q}$ is indeed a graded iterated Ore extension. □

A graded algebra is right (left) noetherian if and only if it is graded right (left) noetherian [14, Proposition 1.4], which means that every graded right (left) ideal is finitely generated.

Corollary 2.5. *Two-parameter quantum Heisenberg algebras $\mathcal{H}_{p,q}$ are left and right noetherian. If $\mathcal{H}_{p,q}$ is as in Remark 2.3, then it is graded left and right noetherian.*

Proof. By Proposition 2.2 $\mathcal{H}_{p,q}$ is a bijective skew PBW extension of $K[t]$. Since $K[t]$ is left and right noetherian, then so is $\mathcal{H}_{p,q}$, by [5, Theorem 3.1.5]. If $\mathcal{H}_{p,q}$ is as in Remark 2.3 then $\mathcal{H}_{p,q}$ is graded and by [14, Proposition 1.4] it is graded left and right noetherian. □

Theorem 2.6. *Let $\mathcal{H}_{p,q}$ be a two-parameter quantum Heisenberg algebra, as in Remark 2.3. Then:*

(i) *$\mathcal{H}_{p,q}$ is Koszul.*

(ii) *$\mathcal{H}_{p,q}$ is Artin-Schelter regular.*

(iii) *$\mathcal{H}_{p,q}$ is graded skew Calabi-Yau.*

(iv) *The Ore extensions $\mathcal{H}_{p,q}[z;\nu]$ and $\mathcal{H}_{p,q}[z^{\pm};\nu]$ are graded Calabi-Yau, where ν is the Nakayama automorphism of $\mathcal{H}_{p,q}$.*

Proof. By Corollary 2.4, $\mathcal{H}_{p,q}$ is a graded skew PBW extension of $K[t]$. Note that $R = K[t]$ is finitely presented, connected, Auslander-regular, Koszul, Artin-Scelter regular and graded skew Calabi-Yau.

(i) By [25, Theorem 5] we have that $\mathcal{H}_{p,q}$ is a Koszul algebra.

(ii) By [24, Proposition 3.5 (iii)] $\mathcal{H}_{p,q}$ is Artin-Schelter regular.

(iii) By [24, Theorem 4.5 (ii)] $\mathcal{H}_{p,q}$ is skew Calabi-Yau.

(iv) Let ν be a Nakayama automorphism of $\mathcal{H}_{p,q}$ and z be a new variable. Note that $\mathcal{H}_{p,q}$ is noetherian, by Corollary 2.5, connected, by [25, Remark 2.10 (ii)], and by the previous items it is Koszul and graded skew Calabi-Yau. Then $\text{hdet}(\nu) = 1$ [21, Theorem 6.3], where hdet is the homological determinant. As, in addition, $\mathcal{H}_{p,q}$ is Artin-Schelter regular, then by [21, Proposition 7.3] we have that $\mathcal{H}_{p,q}[z;\nu]$ and $\mathcal{H}_{p,q}[z^{\pm};\nu]$ are graded Calabi-Yau.

□

Gaddis in [6, Proposition 3.2] had already shown that $\mathcal{H}_{p,q}$ is Artin-Schelter regular, using the fact that t is a normal regular element. Here we use the fact that $\mathcal{H}_{p,q}$ is a graded skew PBW extensions. Koszul, Artin-Schelter regular and skew Calabi-Yau properties can also be shown using the fact that $K[t]$ is Koszul, Artin-Schelter regular and skew Calabi-Yau, and $\mathcal{H}_{p,q}$ is a graded iterated Ore extension of $K[t]$: A graded iterated Ore extension of R is Koszul if and only if R is Koszul [25, Proposition 3.1]; the graded skew Calabi-Yau property of a connected graded algebra is preserved by graded Ore extensions [15] and the same occurs for the Artin-Schelter regular property, since a connected graded algebra is skew Calabi-Yau if and only if it is Artin-Schelter regular [21, Lemma 1.2]. Note that the Calabi-Yau property is not preserved by Ore extensions. For example, the Jordan plane $A = K\langle x,y\rangle/\langle yx - xy - x^2\rangle = K[x][y;\sigma,\delta]$, where $\sigma(x) = x$ and $\delta(x) = x^2$, is a graded skew PBW extension of a Calabi-Yau algebra $K[x]$, but A is not Calabi-Yau, since the Nakayama automorphism ν of the Jordan plane is given by $\nu(x) = x$ and $\nu(y) = 2x + y$, which is not inner.

3 Generalized Heisenberg algebras

Generalized Heisenberg algebras were formally introduced in [4] and later Lü and Zhao in [19] introduced a more general definition for this type of algebras.

Definition 3.1. Given a polynomial $f \in K[t]$, the *generalized Heisenberg* algebra associated to f, denoted $\mathcal{H}(f)$, is the algebra with generators x, y and t, with defining relations:

$$tx = xf, \quad yt = fy, \quad yx - xy = f - t. \tag{3.1}$$

In the present paper we only study generalized Heisenberg algebras $\mathcal{H}(f)$ according to the previous definition, which differs from [4].

Theorem 3.2. *A two-parameter quantum Heisenberg algebra $\mathcal{H}_{p,q}$ is isomorphic to a generalized Heisenberg algebra if and only if either $p = 1$ or $q = 1$, but not both. In particular, the enveloping algebra of the Heisenberg Lie algebra $\mathcal{H}_{1,1}$ is not a generalized Heisenberg algebra.*

Proof. Suppose that $\mathcal{H}(f)$ is isomorphic to $\mathcal{H}_{p,q}$, for some $f \in K[t]$ and $p, q \in K^*$. By Corollary 2.5, $\mathcal{H}_{p,q}$ is noetherian, so the same holds for $\mathcal{H}(f)$. By [16, Proposition 2.1], it follows that $\deg f = 1$. So set $f(t) = \lambda t + \mu$, with $\lambda, \mu \in K$ and $\lambda \neq 0$.

Case 1: $\lambda = 1$. If $\mu = 0$ then $f(t) = t$ and $\mathcal{H}(f)$ is the commutative polynomial ring $K[t, x, y]$. Thus, $\mathcal{H}_{p,q}$ is commutative, whence $0 = [y, x] = (q - 1)xy + t$. But Proposition 2.2 implies that $\{t^i x^j y^k \mid i, j, k \in \mathbb{N}\}$ is a K-basis of $\mathcal{H}_{p,q}$, which contradicts the previous relation. Therefore, $\mathcal{H}_{p,q}$ is never commutative and $\mu \neq 0$.

Suppose that I is an ideal of $\mathcal{H}(t + \mu)$ such that $\mathcal{H}(t + \mu)/I$ is commutative. Then $1 + I = \mu^{-1}[y, x] + I = \mu^{-1}[y + I, x + I] = 0 + I$, so $1 \in I$. It follows that $\mathcal{H}(t + \mu)$ has no nonzero commutative epimorphic images. On the other hand, there is an algebra homomorphism $\phi : \mathcal{H}_{p,q} \longrightarrow K[z]$ such that $\phi(x) = \phi(t) = 0$ and $\phi(y) = z$, and ϕ is surjective. So $\mathcal{H}_{p,q}/\ker\phi \simeq K[z]$ has nonzero commutative epimorphic images, which contradicts the hypothesis that $\mathcal{H}(f) \simeq \mathcal{H}_{p,q}$.

Case 2: $\lambda \neq 1$. Then there is an isomorphism $\psi : \mathcal{H}(\lambda t) \longrightarrow \mathcal{H}(\lambda t + \mu)$ defined on the generators by $\psi(t) = t + \frac{\mu}{\lambda - 1}$, $\psi(x) = x$, $\psi(y) = y$. Indeed, computing in $\mathcal{H}(\lambda t + \mu)$,

$$\begin{aligned}
\psi(tx - \lambda xt) &= \left(t + \frac{\mu}{\lambda - 1}\right) x - \lambda x \left(t + \frac{\mu}{\lambda - 1}\right) \\
&= tx - \lambda xt + (\lambda - 1)^{-1}(\mu - \lambda\mu)x \\
&= \mu x - \mu x = 0; \\
\psi(yx - xy - (\lambda - 1)t) &= yx - xy - (\lambda - 1)\left(t + \frac{\mu}{\lambda - 1}\right) \\
&= (\lambda - 1)t + \mu - (\lambda - 1)t - \mu = 0;
\end{aligned}$$

and similarly $\psi(yt - \lambda ty) = 0$. The inverse isomorphism sends $x \in \mathcal{H}(\lambda t + \mu)$ to $x \in \mathcal{H}(\lambda t)$, $y \in \mathcal{H}(\lambda t + \mu)$ to $y \in \mathcal{H}(\lambda t)$ and $t \in \mathcal{H}(\lambda t + \mu)$ to $t - \frac{\mu}{\lambda - 1} \in \mathcal{H}(\lambda t)$. Thus, without loss of generality, we can assume that $\mu = 0$. So the defining relations of $\mathcal{H}(f)$ are

$$tx = \lambda xt, \quad yt = \lambda ty, \quad yx - xy = (\lambda - 1)t.$$

Let I be the minimal ideal of $\mathcal{H}(\lambda t)$ such that $\mathcal{H}(\lambda t)/I$ is commutative. Then, since $\lambda \neq 1$, I is generated by the normal element t: $I = t\mathcal{H}(\lambda t) = \mathcal{H}(\lambda t)t$.

Subcase 2A: $p = 1$. Then $t \in \mathcal{H}_{1,q}$ is central. The minimal ideal J such that $\mathcal{H}_{1,q}/J$ is commutative is generated by $[y, x] = (q-1)xy + t$.

Suppose, by contradiction, that $q = 1$. Then $[y, x] = t$ is central and $J = t\mathcal{H}_{1,q}$. If $\phi : \mathcal{H}_{1,q} \longrightarrow \mathcal{H}(\lambda t)$ is an isomorphism, then $\phi(J) = I$. As the only units in $\mathcal{H}(f)$ are the nonzero scalars, this forces $\phi(t) = \gamma t$, for some $\gamma \in K^*$. But this is a contradiction as $t \in \mathcal{H}_{1,q}$ is central yet $t \in \mathcal{H}(\lambda t)$ is not, because $\lambda \neq 1$. This forces $q \neq 1$, as needed.

Subcase 2B: $p \neq 1$. Suppose, by contradiction, that $q \neq 1$.

The space of maximal ideals of codimension 1 of $\mathcal{H}(\lambda t)$ (equivalently, as a set, the 1-dimensional representations of $\mathcal{H}(\lambda t)$) is given by

$$\{(x-\alpha, y-\beta, t-\gamma) \mid \alpha, \beta, \gamma \in K,\ \alpha\gamma = \beta\gamma = \gamma = 0\} = \{(x-\alpha, y-\beta, t) \mid \alpha, \beta \in K\} \simeq \mathbb{A}^2,$$

an irreducible algebraic variety of dimension 2.

On the other hand, assuming that $p, q \neq 1$, the space of maximal ideals of codimension 1 of $\mathcal{H}_{p,q}$ is given by

$$\begin{aligned} \{(x-\alpha, y-\beta, t-\gamma) \mid \alpha, \beta, \gamma \in K,\ \alpha\gamma = \beta\gamma = 0,\ (1-q)\alpha\beta = \gamma\} = \\ = \{(x-\alpha, y-\beta, t) \mid \alpha, \beta \in K,\ \alpha\beta = 0\}, \end{aligned}$$

a reducible algebraic variety of dimension 1.

This contradiction shows that if $p \neq 1$ then $q = 1$, as stated. This concludes the direct implication.

For the converse implication, and to avoid confusion between the generators of each one of these algebras, we denote the defining generators of $\mathcal{H}(f)$ by x, y, t and the defining generators of $\mathcal{H}_{p,q}$ by X, Y, T.

We assume first that $p = 1$ and $q \neq 1$. So we have

$$YX - qXY = T, \quad XT = TX, \quad YT = TY.$$

We claim that there is an isomorphism $\phi : \mathcal{H}_{1,q} \longrightarrow \mathcal{H}(qt)$ such that $\phi(X) = \frac{x}{q-1}$, $\phi(Y) = y$ and $\phi(T) = t - xy$. Indeed we have

$$\begin{aligned} \phi(YX - qXY - T) &= (q-1)^{-1}(yx - qxy) - t + xy \\ &= (q-1)^{-1}((1-q)xy + (q-1)t) - t + xy = 0; \\ \phi(XT - TX) &= (q-1)^{-1}(x(t-xy) - (t-xy)x) \\ &= (q-1)^{-1}x((1-q)t - xy + yx) = 0; \end{aligned}$$

and similarly $\phi(YT - TY) = 0$. The inverse map takes x to $(q-1)X$, y to Y and t to $T + (q-1)XY$.

Now, if $p \neq 1$ and $q = 1$, we use Proposition 2.1 and the previous result to obtain

$$\mathcal{H}_{p,1} \simeq \mathcal{H}_{1,p} \simeq \mathcal{H}(pt).$$

□

Corollary 3.3. *Up to isomorphism, the classes of generalized Heisenberg algebras $\mathcal{H}(f)$ and quantum Heisenberg algebras $\mathcal{H}_q$ are disjoint.*

Proof. The quantum Heisenberg algebra $\mathcal{H}_q$ is the two-parameter quantum Heisenberg algebra $\mathcal{H}_{q,q}$, and the result follows from Theorem 3.2. □

Note that $\mathcal{H}(f) = K[t, x, y]$ if and only if $f = t$ (see [19]).

Theorem 3.4. *A generalized Heisenberg algebra $\mathcal{H}(f)$ is a bijective skew PBW extension of endomorphism type of $R = K[t]$ if and only if $\deg f = 1$.*

Proof. Suppose that $\mathcal{H}(f)$ is a bijective skew PBW extension of endomorphism type of $R = K[t]$. Since $K[t]$ is noetherian, then so is $\mathcal{H}(f)$. By [16, Proposition 2.4], $\deg f = 1$.

Conversely, suppose that $f(t) = qt + k$ with $k, q \in K$ and $q \neq 0$. Then, by [16], $\mathcal{H}(f)$ is the iterated Ore extension $K[t][x; \sigma_x][y; \sigma_y, \delta_y]$, where $\sigma_x(t) = q^{-1}t - q^{-1}k$, $\sigma_y(x) = x$, $\sigma_y(t) = qt + k$ (which are bijective on R) and $\delta_y(x) = (q-1)t + k$. It follows that $\mathcal{H}(f)$ is a left free R-module, with basis the set of standard monomials

$$\{x^{\alpha_1}y^{\alpha_2} \mid \alpha_1, \alpha_2 \in \mathbb{N}\}.$$

Then $xt = (q^{-1}t - q^{-1}k)x$, $yt = (qt+k)y$, and $yx - xy = (q-1)t + k$. Thus the conditions (1.1) and (1.2) of the definition of a skew PBW extension are satisfied for $R = K[x]$. □

Corollary 3.5. *A generalized Heisenberg algebra $\mathcal{H}(f)$ is a skew PBW extension of K if and only if f has degree 1.*

Proof. If $\deg f = 1$ we have that $f = qt + k$ with $k, q \in K$ and $q \neq 0$. From (3.1) we have that $xt - q^{-1}tx = -q^{-1}kx$, $yt - qty = ky$, $\quad yx - xy = (q-1)t + k$. Thus the condition (1.2) of Definition of skew PBW extension is satisfied for $R = K$ and the variables t, x, y.

If $\deg f > 1$ then, for every $\lambda \in K^*$, $yx - \lambda xy = (1-\lambda)xy + (f - t) \notin K + Kt + Kx + Ky$, which contradicts the relation (1.2) of Definition 1.1.

Finally, if $\deg f \leq 0$, say $f(t) = k \in K$, then for every $\lambda \in K^*$, $xt - \lambda tx = xt - k\lambda x \notin K + Kt + Kx + Ky$, again contradicting relation (1.2) of Definition 1.1. □

Corollary 3.6. *Let $\mathcal{H}(f)$ be a generalized Heisenberg algebra. Then $\mathcal{H}(f)$ is left or right noetherian if and only if $\mathcal{H}(f)$ is a skew PBW extension of $K[t]$ or of K.*

Proof. By [16, Proposition 2.4] we have that $\mathcal{H}(f)$ is left (right) noetherian if and only if $\deg f = 1$. Thus, $\mathcal{H}(f)$ is left (right) noetherian if and only if it is a skew PBW extension of $K[t]$ or K. □

Proposition 3.7. *Let $0 \neq f \in K[t]$. Then a generalized Heisenberg algebra $\mathcal{H}(f)$ is a skew PBW extension of $R = K\langle x, t\rangle / \langle tx - xf\rangle$ in the variable y.*

Proof. Since $\{x^{\alpha_1}t^{\alpha_2}y^{\alpha_3} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}\}$ is a basis of $\mathcal{H}(f)$ then $H(f)$ is a left free R-module with basis the set of standard monomials $\{y^\alpha \mid \alpha \in \mathbb{N}\}$. Since $yt - fy = 0 \in R$, with $f \neq 0$, and $yx - xy = f - t \in R$, then condition (1.1) of Definition 1.1 is satisfied. Therefore $\mathcal{H}(f) = \sigma(R)\langle y\rangle$. □

Note that a generalized Heisenberg algebra $\mathcal{H}(f)$ is a graded algebra with x, y and t in degree 1 if and only if $f = t$. In this case $\mathcal{H}(f) = K[t, x, y]$. Thus, $\mathcal{H}(f)$ is graded (skew) Calabi-Yau or Koszul or Artin-Schelter regular if and only if $f = t$.

4 Quantum generalized Heisenberg algebras

Lopes and Razavinia in [17] introduced the quantum generalized Heisenberg algebras which depend on a parameter q and two polynomials $f, g \in K[t]$. The class of quantum generalized Heisenberg algebras includes generalized down-up algebras, the Heisenberg algebra $\mathcal{H}$, quantum Heisenberg algebras $\mathcal{H}(q)$, two-parameter quantum Heisenberg algebras $\mathcal{H}_{p,q}$, the deformations of the enveloping algebra of $\mathfrak{sl}_2$ introduced by Jing and Zhang in [12], and many others.

Definition 4.1 ([17], Definition 1.1)**.** Let K be an arbitrary field and fix $q \in K$ and $f, g \in K[t]$. The *quantum generalized Heisenberg algebra*, denoted by $\mathcal{H}_q(f, g)$, is the algebra generated by x, y and t, with defining relations

$$tx = xf, \quad yt = fy, \quad yx - qxy = g. \tag{4.1}$$

Remark 4.2. (i) Any generalized Heisenberg algebra $\mathcal{H}(f)$ is a quantum generalized Heisenberg algebra $\mathcal{H}_q(f, g)$, by setting $q = 1$ and $g = f - t$, i.e. $\mathcal{H}(f) = \mathcal{H}_1(f, f - t)$.

(ii) A Heisenberg algebra $\mathcal{H}$ is a quantum generalized Heisenberg $\mathcal{H}_q(f, g)$, setting $f = g = t$ and $q = 1$.

(iii) A quantum Heisenberg algebra $\mathcal{H}_q$ is a quantum generalized Heisenberg $\mathcal{H}_q(f, g)$, setting $f = q^{-1}t$ and $g = t$.

(iv) A two-parameter quantum Heisenberg algebra $\mathcal{H}_{p,q}$ is a quantum generalized Heisenberg algebra $\mathcal{H}_q(f, g)$, by setting $f = p^{-1}t$, $g = t$ and $q = p$.

Cassidy and Shelton in [3] introduced the *generalized down-up algebra* $L(g, p_1, p_2, p_3)$ as the unital associative algebra generated by d, u and t with defining relations

$$dt = p_1 td - p_3 d, \quad tu = p_1 ut - p_3 u, \quad du - p_2 ud + g = 0, \tag{4.2}$$

where $p_1, p_2, p_3 \in K$ and $g \in K[t]$. The generalized down-up algebra $L(g, p_1, p_2, p_3)$ is isomorphic to the quantum generalized Heisenberg algebra $\mathcal{H}_{p_2}(p_1 t - p_3, -g)$ and conversely, any quantum generalized Heisenberg algebra $\mathcal{H}_q(f, g)$ such that $f = at + b$, with $a, b \in K$, is a generalized down-up algebra of the form $L(-g, a, q, -b)$ (see [17, Proposition 1.3]).

Proposition 4.3. *The quantum generalized Heisenberg algebra* $\mathcal{H}_q(f, g)$ *is a bijective skew PBW extension of* $K[t]$ *if and only if* $\deg f = 1$ *and* $q \neq 0$.

Proof. The proof is analogous to the proof of Theorem 3.4, since $g \in K[t]$. $\square$

Proposition 4.4. *A quantum generalized Heisenberg algebra $\mathcal{H}_q(f,g)$ is a skew PBW extension of K if and only if $q \neq 0$, $\deg f = 1$ and $\deg g \leq 1$.*

Proof. Let $f = pt + e$ and $g = ct + d$ with $e, c, d \in K$, $p, q \in K^*$. Let's see that $\mathcal{H}_q(f,g)$ is a skew PBW extension of K in the variables x, t, y. Using [17, Lemma 2.1] we have that $\mathcal{H}_q(f,g)$ is a left free K-module, with basis the set $\{x^i t^j y^k \mid i,j,k \in \mathbb{N}\}$. As $\mathcal{H}_q(f,g)$ is a K-algebra, condition (1.1) of Definition 1.1 is immediate. From (4.1) we have that $xt - p^{-1}tx = -p^{-1}ex$, $yt - pty = ey$ and $yx - qxy = ct + d$, which correspond to the conditions (1.2) of Definition 1.1 for the variables x, t, y. Then $\mathcal{H}_q(f,g)$ is a skew PBW extension of K in the variables t, x, y. Note that, in addition, $\mathcal{H}_q(f,g)$ is bijective.

Suppose now that $\mathcal{H}_q(f,g)$ is a skew PBW extension of K. If $q = 0$ then from (4.1) and (1.1) we have that $yx = g = rxy + s$ for some $r \in K^*$ and $s \in K + Kt + Kx + Ky$, which is a contradiction by [17, Lemma 2.1], so $q \neq 0$.

From (1.2) there is some $d_{t,y} \in K^*$ such that $yt - d_{t,y}ty \in K + Kt + Kx + Ky$. But by (4.1) we have $yt - d_{t,y}ty = (f(t) - d_{t,y}t)y$, which is in $K + Kt + Kx + Ky$ if and only if $f(t) - d_{t,y}t \in K$, again by [17, Lemma 2.1]. Thence, $\deg f = 1$.

Similarly, from (1.2) there is some $d_{x,y} \in K^*$ such that $yx - d_{x,y}xy \in K + Kt + Kx + Ky$. By (4.1), $yx - d_{x,y}xy = (q - d_{x,y})xy + g(t)$ and [17, Lemma 2.1] implies that $d_{x,y} = q$ and $\deg g \leq 1$. $\square$

The first-named author and Razavinia in [18, Lemma 2.1] proved that

$$\mathcal{H}_q(f,g) \cong \mathcal{H}_q(f(t-\alpha) + \alpha, g(t-\alpha)),$$

for any $\alpha \in K$. Let $f_1 = pt$, $f = pt + k$ with $1 \neq p \in K^*$. Then

$$\begin{aligned} f_1(t - k(1-p)^{-1}) + k(1-p)^{-1} &= p(t - k(1-p)^{-1}) + k(1-p)^{-1} \\ &= pt - pk(1-p)^{-1} + k(1-p)^{-1} = pt + k = f. \end{aligned}$$

Then by [18, Lemma 2.1] we have that $\mathcal{H}_q(f_1, g_1) \cong \mathcal{H}_q(f,g)$, i.e.,

$$\mathcal{H}_q(pt, g(t)) \cong \mathcal{H}_q(pt + k, g(t - k(1-p)^{-1})),$$

for $g(t) \in K[t]$ and $1 \neq p \in K^*$.

From Proposition 4.3 and relations (4.1) the following holds.

Corollary 4.5. *Let $\mathcal{H}_q(f,g)$ be a quantum generalized Heisenberg algebra with $f = pt$ and $p, q \in K^*$.*

(i) *If $g = 0$ then $\mathcal{H}_q(f,g) = K\langle t,x,y\rangle / \langle xt - p^{-1}tx, yt - pty, yx - qxy\rangle$ is a quantum polynomial algebra and a graded skew PBW extension of K or $K[t]$.*

(ii) *If g is a homogeneous polynomial of degree 2, then $\mathcal{H}_q(f,g) = K\langle t,x,y\rangle / \langle xt - p^{-1}tx, yt - pty, yx - qxy - g\rangle$ is a connected graded skew PBW extension of $K[t]$.*

Gómez and the second-named author, in [9, Theorem 3.2], proved that if $A = \sigma(R)\langle x, y\rangle$ is a connected graded skew PBW extension of an algebra R then A is a connected graded double Ore extension of R. James Zhang and Jun Zhang in [32, Theorem 0.2] proved that if A is a connected graded double Ore extension of an Artin-Schelter regular algebra R, then A is Artin-Schelter regular and $\operatorname{gldim} A = \operatorname{gldim} A + 2$.

Theorem 4.6. *Let $\mathcal{H}_q(f, g)$ be quantum generalized Heisenberg algebra with $f = pt$, $p, q \in K^*$ and g a homogeneous polynomial of degree 2 then $\mathcal{H}_q(f, g)$ is Koszul and Artin-Schelter regular of dimension 3.*

Proof. By Corollary 4.5, $\mathcal{H}_q(f, g)$ is a connected graded skew PBW extension of $K[t]$. Therefore, by [9, Theorem 3.2], $\mathcal{H}_q(f, g)$ is a connected graded double Ore extension of $K[t]$. Note that $K[t]$ is a finitely presented Koszul algebra. Then, by [25, Theorem 5.5], $\mathcal{H}_q(f, g)$ is a Koszul Algebra. Note also that $K[t]$ is an Artin-Schelter regular algebra of global dimension 1. By [32, Theorem 0.2] we have that $\mathcal{H}_q(f, g)$ is Artin-Schelter regular of dimension 3. □

Note that $\mathcal{H}_q(f, g)$, with q, f and g as above, is of type S_1' in the classification of regular algebras of global dimension 3 generated in degree 1, given by Artin and Schelter in [1, Theorem 10].

Let V be a 3-dimensional vector space, $\deg(V) = 1$ and TV the tensor algebra. Fix a basis $\{t, x, y\}$ for V. The *cyclic partial derivative* with respect to x of a word Φ in the letters t, x, y is $\partial_x(\Phi) := \sum_{\Phi=uxv} vu$ where the sum is taken over all such factorizations. We extend ∂_x to TV by linearity. We define ∂_y and ∂_t in a similar way. The *Jacobian algebra* $J(\Phi)$ associated to $\Phi \in TV$ is the quotient algebra of TV by the ideal generated by the cyclic partial derivatives, i.e., $J(\Phi) := TV/\langle \partial_x(\Phi), \partial_y(\Phi), \partial_t(\Phi)\rangle$. The linear span, $R_\Phi := \operatorname{span}\{\partial_x(\Phi), \partial_y(\Phi), \partial_t(\Phi)\}$, does not depend on the choice of basis for V (see [20]). Given $\Phi_3 \in V^{\otimes 3}$ (in this case Φ_3 is called a *homogeneous potential* of degree 3), Mori and Smith [20, Theorem 1.3] proved that $J(\Phi)$ is graded 3-Calabi-Yau if and only if it is a 3-dimensional Artin-Schelter regular algebra.

From this point on we assume that the field K is algebraically closed of characteristic not 2 or 3.

Theorem 4.7. *Let $\mathcal{H}_q(f, g)$ be quantum generalized Heisenberg algebra with $f = pt$, $p \in K^*$, $q = p^{-1}$ and let g be a homogeneous polynomial of degree 2. Then $\mathcal{H}_q(f, g)$ is a graded Calabi-Yau algebra of dimension 3.*

Proof. Suppose that $f = pt$, $p \in K^*$, $q = p^{-1}$ and $g = ct^2$ is a homogeneous polynomial of degree 2. Let $\Phi_3 = xyt + pytx - ptxy + yxt - xty - ptyx + 3^{-1}pct^3$. Then $\partial_x(\Phi_3) = yt - pty$, $\partial_y(\Phi_3) = tx - pxt$ and $\partial_t(\Phi_3) = xy - pyx + pct^2 = -p(yx - p^{-1}xy - ct^2)$. Then the Jacobian algebra associated to Φ_3 is

$$J(\Phi_3) = K\langle t, x, y\rangle/\langle yt - pty, tx - pxt, yx - qxy - ct^2\rangle = \mathcal{H}_q(f, g).$$

Then the quantum generalized Heisenberg algebra $\mathcal{H}_q(f,g)$ is a Jacobian algebra associated to a homogeneous potential Φ_3. By Theorem 4.6, $\mathcal{H}_q(f,g)$ is Artin-Schelter regular of dimension 3. Then by [20, Theorem 1.3] we have that $\mathcal{H}_q(f,g)$ is graded Calabi-Yau of dimension 3. □

Berger and Taillefer [2, Theorem 3.1] proved that if the *Jacobian algebra* $A = J(\Phi_{N+1})$ is a graded Calabi-Yau algebra of dimension 3, where Φ_{N+1} is a homogeneous potential of degree $N+1$, and $\Phi = \Phi_{N+1} + \Phi' = \Phi_{N+1} + \Phi_N + \cdots + \Phi_1 + \Phi_0$ is a potential with $\deg \Phi_j = j$ for each $0 \leq j \leq N+1$, then $A' := J(\Phi)$ is a PBW deformation of A. They also proved that if $A = J(\Phi_{N+1})$ is a graded Calabi-Yau algebra of dimension 3 and $A' := J(\Phi)$ is a PBW deformation of A associated to a potential $\Phi = \Phi_{N+1} + \Phi'$ with $\deg \Phi' \leq N$, then A' is Calabi-Yau of dimension 3 [2, Theorem 3.6].

Theorem 4.8. *Let $\mathcal{H}_q(f',g')$ be quantum generalized Heisenberg algebra for $f',g' \in K[t]$, with $f' = pt+k$, $g' = ct^2+dt+e$ with $c,d,e,k \in K$, $p \in K^*$ and $q = p^{-1}$. Then $\mathcal{H}_q(f',g')$ is a Calabi-Yau algebra of dimension 3.*

Proof. By Theorem 4.7 and its proof we have that for $f = pt$, $g = ct^2$ and $q = p^{-1}$, the quantum generalized Heisenberg algebra $\mathcal{H}_q(f,g)$ is a graded Calabi-Yau algebra of dimension 3, which is a Jacobian algebra associated to a homogeneous potential $\Phi_3 = xyt + pytx - ptxy + yxt - xty - ptyx + 3^{-1}pct^3$. Let $\Phi' = kxy + 2^{-1}dt^2 + et$ and let $\Phi = \Phi_3 - \Phi' = xyt + pytx - ptxy + yxt - xty - ptyx + 3^{-1}pct^3 - kxy - 2^{-1}dt^2 - et$ be a potentials. Then $\partial_x(\Phi) = yt - pty - ky = yt - f'y$, $\partial_y(\Phi) = tx - pxt - kx$ and $\partial_t(\Phi) = yx - qxy - ct^2 - dt - e$. Then the Jacobian algebra associated to a potential Φ is

$$J(\Phi) = K\langle t,x,y\rangle / \langle yt - f'y, tx - xf', yx - qxy - g'\rangle = \mathcal{H}_q(f',g').$$

By [2, Theorem 3.1], $\mathcal{H}_q(f',g')$ is a PBW deformation of $\mathcal{H}_q(f,g)$. Since $\mathcal{H}_q(f,g)$ is a graded Calabi-Yau algebra of dimension 3 and $\deg \Phi' = 2 \leq \deg(\Phi_3)$ then by [2, Theorem 3.6] we have that $\mathcal{H}_q(f',g')$ is Calabi-Yau algebra of dimension 3. □

Corollary 4.9. *The quantum Heisenberg algebra $\mathcal{H}_q$ is a Calabi-Yau algebra.*

Proof. Note that $\mathcal{H}_q = \mathcal{H}_q(f',g')$ for $f' = q^{-1}t$ and $g' = t$. Then by Theorem 4.8 we have that $\mathcal{H}_q$ is Calabi-Yau. □

Corollary 4.10. *If $f = t + k$, with $k \in K$, then the generalized Heisenberg algebra $\mathcal{H}(f)$ is Calabi-Yau.*

Proof. If $f = t + k$ with $k \in K$ then a generalized Heisenberg algebra $\mathcal{H}(f)$ is the quantum generalized Heisenberg algebra $\mathcal{H}_1(f,k)$. Thus by Theorem 4.8 we have that $\mathcal{H}(f)$ is Calabi-Yau. □

Acknowledgement

The first-named author was partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020. He would like to thank the invitation and hospitality of the Universidad Pedagógica y Tecnológica de Colombia, in Tunja, and his present co-authors for the mathematical discussions held during a visit in 2024.

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