Optimal insurance, consumption and investment decisions: a duality approach

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Abstract

We employ a duality approach and martingale techniques to characterize the solutions to a stochastic optimal control problem modeling the choices available to an economic agent seeking to maximize the expected utility derived from consumption, terminal wealth, and life insurance coverage, while facing both investment risks and mortality uncertainty. The agent dynamically allocates wealth between a financial market composed of one risk-free asset and multiple risky assets, and term life insurance premiums subject, respectively, to uncertainty associated with the market conditions and the agent uncertain lifespan. Our results provide insights into the trade-offs between consumption, wealth accumulation, and life insurance demand in the presence of financial and mortality risks.

- **Keywords**: Stochastic optimal control; Convex duality; Martingale methods; Consumption-investment problem; Life-insurance.
- **MSC2020 classification**: 49N90; 91G15; 91G30; 91G80; 93E20.

1 Introduction

We consider the problem faced by a wage earner who must continuously make decisions about three strategies: consumption, investment, and life insurance purchases during a random time interval $[0, \min\{\tau, T\}]$, where T is a fixed point in the future representing the wage earner's retirement date, and τ is a random variable representing the time of death. We assume that the wage earner receives income that ceases either at death or retirement, whichever occurs first. One of our main assumptions is that the wage earner's lifetime τ is a non-negative continuous random variable. Therefore, the wage earner needs to buy life insurance to protect their family against the possibility of premature death. The life insurance depends on an insurance premium payment rate $p(t)$, such that if the insured pays $p(t)\delta(t)$ and dies during the ensuing short interval of length $\delta(t)$, the insurance company will

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pay $p(t)/\eta(t)$ to the insured's estate, where $\eta(t)$ is an amount predetermined by the insurance company. Thus, this is akin to term life insurance with an infinitesimal term. We also assume that the wage earner seeks to maximize the expected utility derived from a consumption process with rate $c(t)$. In addition to consumption and the purchase of a life insurance policy, we assume the wage earner invests all of their savings in a complete financial market consisting of one money market and a fixed, finite number of risky securities with diffusive terms driven by a multi-dimensional Brownian motion. The financial market considered here has a general form, with all coefficients being progressively measurable stochastic processes, without any assumption concerning the Markov property.

The wage earner must find strategies that maximize the utility of: (i) their family's consumption for all $t \leq \min\{\tau, T\}$; (ii) their wealth at the retirement date T if they live that long; and (iii) the value of their estate in the event of premature death. Various quantitative models have been proposed to model and analyze problems that involve at least one of these three objectives. Yaari [23] considered the problem of optimal financial planning decisions for an individual with an uncertain lifetime, in what is generally regarded as the starting point for modern research on the demand for life insurance. Later, Samuelson [22] and Merton [15, 16], using methods of dynamic programming, emphasized optimal consumption and investment decisions without considering life insurance, with Merton focusing on the problem in continuous time. Richard [20] combined the previous approaches and analyzed a life-cycle life insurance and consumption-investment problem in a continuous-time model. Later, Pliska and Ye [19] generalized the previously considered models, combining more realistic features while considering a different boundary condition that led to somewhat different economic interpretations than those provided by Richard. Pliska and Ye's paper considered only one risky security and assumed that the market was complete. The main difference between Richard's paper and Pliska and Ye's paper is that, while Richard assumed the lifetime of the wage earner was limited by some fixed number, Pliska and Ye considered that the lifetime of the individual is a random variable, independent of the stochastic process associated with the underlying financial market, but allowed to take any positive value. In this setup, the fixed horizon T is seen as the time when the wage earner retires. Duarte *et* al. [6] extended Pliska and Ye's paper to a more general setup, allowing for an arbitrary finite number of risky securities and without assuming that the market was complete. We remark that all the references cited above rely on dynamic programming techniques to characterize (and, in some cases, compute) the optimal controls in feedback form. The use of the dynamic programming approach is possible due to the assumptions imposed on the underlying financial market. More precisely, the market coefficients are given by deterministic functions. This assumption ensures that the Markov property holds for the price processes of the financial market assets, thereby enabling the use of the dynamic programming principle.

Duality theory offers an alternative approach to dynamic programming by converting the original stochastic optimal control problem into a more manageable dual problem. In the primal formulation, the objective is to determine the optimal control policy and its corresponding value function. In contrast, the dual problem typically involves identifying a function − often a Lagrange multiplier − that satisfies specific optimality conditions. Martingale techniques are frequently employed in this context. Specifically, the dual problem can be interpreted as finding an optimal martingale measure or adjusting the probability measure governing the system's dynamics. In his seminal paper [1], Bismut introduces techniques from convex analysis and duality theory to analyze stochastic optimal control problems. In [2], he provides an overview of the fundamental concepts and establishes a framework that shows how duality can simplify the solution of such problems. Pliska [18] applies convex analysis techniques to study the problem of selecting a portfolio of securities in order to maximize the expected utility of wealth at a terminal planning horizon, in cases where security prices are semimartingales. In [9], Karatzas *et* al. use martingale techniques to analyze a general consumption and investment problem for an agent seeking to maximize the total expected discounted utility of both consumption and terminal wealth. Meanwhile, in [5], Cvitanić and Karatzas employ duality theory to solve portfolio optimization problems under constraints. The highly influential book by Karatzas and Shreve [11] also deserves mention, as it provides comprehensive discussions of duality techniques in stochastic optimal control and mathematical finance. In particular, in [11, Section 3], they address the problem of joint utility maximization of consumption and terminal wealth in a complete market, but without considering life insurance or any income beyond the individual's initial endowment. Further contributions to the field were made by Cox and Huang [3, 4], who used martingale techniques to characterize optimal consumption-portfolio policies in the presence of non-negativity constraints on agents' consumption and final wealth.

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In this work, we employ a duality approach to extend the results of [6, 19] to a framework that includes financial assets for which the Markov property does not necessarily hold, thus broadening the current literature on consumption-investment problems to incorporate the analysis of optimal life insurance purchases. The structure of the paper is as follows. In Section 2, we describe the financial market model, the life insurance market, the wage earner's wealth dynamics based on their decisions, and the utility functions and optimal control problem for the wage earner.

In Section 3, we solve the problem under consideration using martingale methods, starting with admissible strategies and their hedging, followed by an examination of the associated static optimization problem. In Section 4, we focus on the case of deterministic coefficients and apply the earlier results to derive solutions for power utility functions with constant relative risk aversion and logarithmic utility functions. The paper concludes with final remarks in Section 5.

2 Problem formulation

In this section, we define the setting under which the wage earner makes decisions regarding their consumption, investment, and life insurance purchases. We begin by defining the financial and insurance markets, followed by a brief description of the wage earner's wealth process, utility functions, and the optimization problem.

2.1 The financial market model

Let $T > 0$ be a fixed and finite planning horizon. We consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which a standard N-dimensional Brownian motion $W = \{W(t) = (W^{(1)}(t), \ldots, W^{(N)}(t))' : t \in [0,T]\}$ is defined. Here, and in what follows, we represent vectors as column-vectors and use the prime sign to denote transposition.

Let $\{\mathcal{F}(t), t \in [0,T]\}$ denote the P-augmentation of the filtration generated by the Brownian motion W, *i.e.*, the sigma-algebras $\sigma\{W(s), 0 \leq s \leq t\}$, for $t \geq 0$, augmented by their P-null subsets. This filtration naturally represents the flow of information available to any economic agent continuously observing the financial market. Namely, $\mathcal{F}(t)$ can be interpreted as the information available to an investor at time $t \in [0,T]$. We shall always define progressive measurability of stochastic processes with respect to this filtration.

The financial market we consider here is composed of a *risk-free asset* (or a *money market* such as a bond) and an arbitrary finite number of (risky) *stocks*. The price of one share of the *money market* and of a certain *stock* is denoted, respectively, by $(S_0(t))_{0 \leq t \leq T}$ and $(S_n(t))_{0 \leq t \leq T}$, $n = 1, \ldots, N$, and evolve according to the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}S_0(t) &= S_0(t)(r(t)\mathrm{d}t + \mathrm{d}A(t)) \\ \mathrm{d}S_n(t) &= S_n(t)\left(\mu_n(t)\mathrm{d}t + \mathrm{d}A(t) + \sum_{d=1}^{N} \sigma_{nd}\mathrm{d}W^{(d)}(t)\right), \quad n = 1, \ldots, N \ , \end{aligned}$$

where $r(\cdot)$ is a progressively measurable process, called the *risk-free rate process*, satisfying

$$\int_0^T |r(t)| \, \mathrm{d}t < \infty \qquad \text{a.s.} \ ,$$

the process $A(\cdot)$ is a progressively measurable, singularly continuous process with finite variation, $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_N(\cdot))'$ is a progressively measurable process, called the *mean rate of return process*, that satisfies

$$\int_0^T ||\mu(t)|| \, \mathrm{d}t < \infty \qquad \text{a.s.} \ ,$$

where $||\cdot||$ denoted the euclidean norm in R^N, and $\sigma(\cdot)$ is a $(N\times N)$-matrix-valued process, invertible for Lebesgue-almost-every $t\in[0,T]$ almost-surely, called the *volatility process*, that satisfies

$$\sum_{n=1}^{N}\sum_{d=1}^{N}\int_0^T \sigma_{nd}^2(t)\,\mathrm{d}t<\infty \qquad \text{a.s. .}$$

We also assume that a strictly positive, constant vector of initial stock prices $S(0)=(S_1(0),\ldots,S_N(0))'$ is given. For simplicity of exposition and without loss of generality, we will take the initial price of the risk-free asset to be unitary: $S_0(0)=1$.

Additionally, we will suppose that the risky assets are dividend-paying, *i.e.*, there is a progressively measurable process $\delta(\cdot)=(\delta_1(\cdot),\ldots,\delta_N(\cdot))'$, called the *dividend rate process*, for which

$$\int_0^T ||\delta(t)||\,\mathrm{d}t<\infty \qquad \text{a.s. .}$$

We define the *risk premium* process $\alpha(t)$ as

$$\alpha(t)=\mu(t)+\delta(t)-r(t)\underline{1}_N \ ,$$

where $\underline{1}_N$ denotes the N-dimensional column vector with all components equal to one, and we define the *market price of risk* process $\theta(t)$ as

$$\theta(t)=\sigma(t)^{-1}\alpha(t)=\sigma(t)^{-1}\left(\mu(t)+\delta(t)-r(t)\underline{1}_N\right) \ .$$

Moreover, we will assume that the following integrability conditions hold:

$$\int_0^T ||\theta(t)||^2\,\mathrm{d}t<\infty \qquad \text{a.s.}$$
$$\mathbb{E}\left[\exp\left\{-\int_0^T \theta'(t)\,\mathrm{d}W(t)-\frac{1}{2}\int_0^T ||\theta(t)||^2\,\mathrm{d}t\right\}\right]=1 \ ,$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation with respect to the probability measure P. These two conditions ensure that there are no arbitrage opportunities in the financial market under consideration, and the almost-sure invertibility of the volatility process $\sigma(\cdot)$ ensures market completeness (see, e.g., [11]).

We will make use of the process $Z_0(t)$ defined by

$$Z_0(t)=\exp\left\{-\int_0^t \theta'(s)\,\mathrm{d}W(s)-\frac{1}{2}\int_0^t ||\theta(s)||^2\,\mathrm{d}s\right\} \ .$$

We observe that Z_0 is a local martingale. Moreover, since Z_0 is bounded from below, it is a supermartingale. We define the *state price density* $H_0(t)$ as

$$H_0(t)=\frac{Z_0(t)}{S_0(t)}$$

and assume that it satisfies the integrability condition

$$\mathbb{E}\left[\int_0^T H_0(t)\,\mathrm{d}t+H_0(T)\right]<\infty \ .$$

A sufficient condition for the condition above to hold is that the risk-free asset S_0 is almost-surely bounded away from zero.

Remark 1. *In most situations, it is assumed that the market is* standard *in the sense that the local martingale process $Z_0(t)$ is indeed a martingale, which holds for instance if the* Novikov condition *is satisfied:*

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T ||\theta(t)||^2\,\mathrm{d}t\right\}\right] < \infty\ .$$

This allows for the definition of a P-equivalent standard martingale measure *on $\mathcal{F}(T)$, to be denoted P_0 from this point onward, which is given by*

$$P_0(A) = \mathbb{E}\left[Z_0(T)I_A\right]\ ,\quad A \in \mathcal{F}(T)\ ,$$

where I_A denotes the indicator function of the set A. The standard martingale measure P_0 incorporates information about the so called market fundamentals as given by the market price of risk process $\theta(\cdot)$. Furthermore, by Girsanov's Theorem, the process

$$W_0(t) = W(t) + \int_0^t \theta(s)\,\mathrm{d}s \tag{2.1}$$

is a Brownian motion under the measure P_0 relative to the filtration $\{\mathcal{F}(t)\}$ of the original Brownian motion $W(t)$ (see, for instance, [10, 17]). In the sequel, we will only make use of the local martingale property of $Z_0(\cdot)$.

2.2 The life insurance market model

We suppose that the wage earner is alive at time $t = 0$ and that their lifetime is a nonnegative continuous random variable τ defined on the probability space $(\Omega, \mathcal{F}, P)$. Furthermore, we assume that the random variable τ is independent of the filtration $\{\mathcal{F}(t)\}$ and has a distribution function $F : [0,\infty) \to [0,1]$ with a bounded and continuous density function $f : [0,\infty) \to \mathbb{R}^+$ so that

$$F(t) = P(\tau \le t) = \int_0^t f(s)\,\mathrm{d}s\ .$$

We define the *survival function* $\overline{F} : [0,\infty) \to [0,1]$ as the wage earner survival probability to at least time t:

$$\overline{F}(t) = P(\tau > t) = 1 - F(t)\ ,$$

and the *force of mortality* $\lambda : [0,\infty) \to \mathbb{R}^+$ as the instantaneous rate of mortality of the wage earner measured on an annualized basis:

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(t < \tau \le t + \delta t | \tau > t)}{\delta t}\ .$$

We note that the force of mortality λ is a continuous and deterministic function of time t, which can be written in terms of the survival function $\overline{F}$ and the corresponding probability density function f as

$$\lambda(t) = \frac{f(t)}{\overline{F}(t)}\ . \tag{2.2}$$

Due to the uncertainty concerning their lifetime, the wage earner buys a life insurance policy to protect their family against the eventuality of premature death. The life insurance policy is available continuously and the wage earner buys it by paying a *life insurance premium* at a rate $p(t)$ dollars per year. The insurance contract is like term insurance with an infinitesimally small (instantaneous) term. If the wage earner dies at time $\tau < T$ while buying insurance at rate $p(t)$, the insurance company pays an amount $p(\tau)/\eta(\tau)$ to their estate, where $\eta(\cdot)$ is a positive, progressively measurable, almost-surely uniformly bounded process, called the *insurance premium-payout ratio*, which is fixed by the insurance company. The insurance contract ends when the wage earner dies or achieves retirement age at time T, whichever happens first. Therefore, the wage earner's total legacy or bequest to their estate in the event of premature death at time $\tau < T$ is given by

$$Z(\tau) = X(\tau) + \frac{p(\tau)}{\eta(\tau)} \ , \tag{2.3}$$

where $X(t)$ denotes the wage earner's wealth at time t.

2.3 The wealth process

We will now introduce the wage earner wealth process given their income and choices concerning consumption, investment and life-insurance purchase. We assume that the wage earner is endowed with some *initial wealth* x, and receives *income* at a rate $i(t)$ dollars per year continuously over the period $[0, \min\{T, \tau\}]$, *i.e.* the income will be terminated either by death or retirement date, whichever happens first. We assume that $i(\cdot)$ is a non-negative, progressively measurable process, almost-surely uniformly bounded. Under such assumption, we have that

$$\int_0^T i(t)\,\mathrm{d}t < \infty \qquad \text{a.s. .}$$

The *consumption process* $(c(t))_{0\leq t\leq T}$ is a progressively measurable non-negative process satisfying

$$\int_0^T c(t)\,\mathrm{d}t < \infty \qquad \text{a.s.}$$

and the *life insurance premium payment rate* $(p(t))_{0\leq t\leq T}$ is a predictable process satisfying

$$\int_0^T |p(t)|\,\mathrm{d}t < \infty \qquad \text{a.s. .}$$

Similarly, the *portfolio process* $(\pi_0(t), \pi_1(t), \ldots, \pi_N(t))'_{0\leq t\leq T}$ is a progressively measurable process with $\pi_0(t)$ representing the *monetary amount* invested in the money market at time t, while the components $\pi_n(t)$, with $n = 1, \ldots, N$, represent the *monetary amount* invested in the risky asset $n \in \{1, \ldots, N\}$ at time t. We will adhere to a common abuse of language and refer to the process $\pi(t) = (\pi_1(t), \ldots, \pi_N(t))'_{0\leq t\leq T}$ reflecting the investment in the risky assets also as the portfolio process. Indeed, under the assumption that the wage earner invests the full amount of their savings in the financial market (money market *plus* risky assets), the amount invested in the money market can always be determined from the $\pi(t)$ via the financing condition

$$X(t) = \pi_0(t) + \pi'(t)\underline{1}_N \ , \tag{2.4}$$

where $X(t)$ represents the wealth of the wage earner at time t. Finally, we will assume that the following integrability conditions are satisfied:

$$\int_0^T |\pi'(t)\alpha(t)|\,\mathrm{d}t < \infty \qquad \text{a.s.}$$
$$\int_0^T ||\pi'(t)\sigma(t)||^2\,\mathrm{d}t < \infty \qquad \text{a.s.}\ . \tag{2.5}$$

We conclude this section by observing that *wealth process* $X(t)$ satisfies the SDE

$$\begin{aligned}\mathrm{d}X(t) &= (i(t) - c(t) - p(t))\mathrm{d}t + \frac{\pi_0(t)}{S_0(t)}\mathrm{d}S_0(t) \\ &\quad + \sum_{n=1}^{N} \frac{\pi_n(t)}{S_n(t)}\left(\mathrm{d}S_n(t) + \delta_n(t)\mathrm{d}t\right)\ , \qquad t \geq 0\ .\end{aligned}$$

Moreover, we note that $\pi_n(t)/S_n(t)$ gives the number of shares of security n that the wage earner holds at time t, and that the differential term in the summation above represents the dynamics of the *yield per share* of the risky assets. Using the financing condition (2.4), and the equations governing the asset prices, the *discounted wealth* can be written in integral form as

$$\begin{aligned}\frac{X(t)}{S_0(t)} &= x + \int_0^t \frac{i(u) - c(u) - p(u) + \pi'(u)\alpha(u)}{S_0(u)}\,\mathrm{d}u \\ &\quad + \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)\,\mathrm{d}W(u)\ .\end{aligned}$$

Furthermore, resorting to the Brownian motion $W_0(\cdot)$ under the martingale measure P_0 (as defined in (2.1)), the discounted wealth process may be written more concisely as

$$\frac{X(t)}{S_0(t)} = x + \int_0^t \frac{i(u) - c(u) - p(u)}{S_0(u)}\,\mathrm{d}u + \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)\,\mathrm{d}W_0(u)\ . \tag{2.6}$$

2.4 Utility functions and preference structures

We will impose only rather general conditions on the utility functions describing the wage earner preferences. Namely, we will say that a *utility function* is a concave, nondecreasing, upper semi-continuous function $U : \mathbb{R} \to [-\infty, \infty)$ for which there exists a nonnegative constant

$$\bar{x} = \inf\{x \in \mathbb{R} : U(x) > -\infty\} \in \mathbb{R}$$

such that U' is continuous, positive and strictly decreasing on $(\bar{x}, \infty)$, and it holds that

$$\lim_{x\to\infty} U'(x) = 0\ .$$

Let

$$U'(\bar{x}^+) = \lim_{x\downarrow\bar{x}} U'(x) > 0$$

and observe that the derivative U' has a continuous and strictly decreasing inverse $I : (0, U'(\bar{x}^+)) \to (\bar{x}, \infty)$, which can be continuously extended to the whole half-line $(0, \infty]$ by setting $I(y) = \bar{x}$ for every $y \in [U'(\bar{x}^+), \infty]$.

We will now briefly recall the notion of *convex dual* (see [21, 8] for more results on convexity and convex duality). The *convex dual* (also referred to as *convex conjugate*) of the function U is defined as

$$\widetilde{U}(y) = \sup_{x \in \mathbb{R}} \{U(x) - xy\} \ , \quad y \in \mathbb{R} \ .$$

We can deduce that, for $y > 0$, the convex dual $\widetilde{U}$ is given by

$$\widetilde{U}(y) = U(I(y)) - yI(y) \ . \tag{2.7}$$

In what follows, we will consider triplets (U_1, U_2, U_3), where $U_1, U_2 : [0,T] \times \mathbb{R} \to [-\infty, \infty)$ and $U_3 : \mathbb{R} \to [-\infty, \infty)$, are such that

i) for each fixed $t \in [0,T]$, $U_1(t,\cdot)$ and $U_2(t,\cdot)$ are utility functions (with respect to their second variable) and each of the following two functions,

$$\bar{c}(t) = \inf\{c \in \mathbb{R} : U_1(t,c) > -\infty\} \tag{2.8}$$

and

$$\bar{Z}(t) = \inf\{z \in \mathbb{R} : U_2(t,z) > -\infty\} \ , \tag{2.9}$$

is continuous on $[0,T]$ and takes values on $[0,\infty)$.

ii) the functions U_1 and U_1', where the prime denotes differentiation with respect to the second component, are both continuous on the set

$$D_1 = \{(t,c) \in [0,T] \times (0,\infty) : \ c > \bar{c}(t)\} \ ,$$

and the functions U_2 and U_2', where the prime once more denotes differentiation with respect to the second component, are continuous on the set

$$D_2 = \{(t,z) \in [0,T] \times (0,\infty) : \ z > \bar{Z}(t)\} \ .$$

iii) U_3 is a utility function.

The functions $\bar{c}$ and $\bar{Z}$ defined in (2.8) and (2.9) are called, respectively, *subsistence consumption* and *subsistence legacy in the case of premature death*, and the level of wealth

$$\bar{x} = \inf\{x \in \mathbb{R} : U_3(x) > -\infty\}$$

is called the *subsistence terminal wealth*.

Since for each fixed $t \in [0,T]$ we have that $U_i(t,\cdot)$, $i = 1,2$, is a utility function with respect to its second variable, we are able to define its convex dual, denoted $\widetilde{U}_i(t,\cdot)$, $i = 1,2$, as already detailed above. The corresponding inverse marginal utility, *i.e.* the inverse of the derivative of U_i with respect to its second variable (which, recall, is continuous and strictly decreasing), is denoted $I_i(t,\cdot)$, $i = 1,2$, so that relation (2.7) becomes

$$\widetilde{U}_i(t,y) = U_i(I_i(t,y)) - yI_i(t,y) \ , \quad \text{for } y > 0 \ .$$

We also define the analogous functions for $U_3(\cdot)$, employing similar notation. Finally, we extend each of the functions $I_1(t,\cdot), I_2(t,\cdot)$ and $I_3(\cdot)$ to the whole half-line $(0,\infty]$ as discussed before when describing the general case. Thus, we have that

$$\begin{aligned} I_1(t,c) &= \bar{c}(t) \quad \text{for all } c \in [U_1'(t,\bar{c}(t)^+), \infty] \\ I_2(t,z) &= \bar{Z}(t) \quad \text{for all } z \in [U_2'(t,\bar{Z}(t)^+), \infty] \\ I_3(x) &= \bar{x} \quad \text{for all } x \in [U_3'(\overline{x}^+), \infty] \ . \end{aligned}$$

Finally, we observe that the functions I_1 and I_2 are jointly continuous on $[0,T] \times (0,\infty]$.

2.5 Admissible strategies and the optimal control problem

We will describe the set of admissible strategies before introducing the optimal control problem under consideration herein. We start by defining the wage earner *human capital* at time t, denoted $b(t)$ and given by

$$b(t) = \frac{1}{H_0(t)}\mathbb{E}_t\left[\int_t^T \frac{H_0(s)i(s)}{\exp(\int_t^s \eta(u)\,\mathrm{d}u)}\,\mathrm{d}s\right] ,$$

where $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\mathcal{F}(t)]$ denotes expectation conditional on the sigma-algebra $\mathcal{F}(t)$, *i.e.*, conditional on the information available up to time t. Defining the process

$$D(t) = \exp\left(\int_0^t \eta(u)\,\mathrm{d}u\right) ,$$

and observing that $D(t) > 1$ for all $t > 0$, we are able to introduce the *modified state price density* $\overline{H}_0(t)$ as

$$\overline{H}_0(t) = \frac{H_0(t)}{D(t)} = \frac{Z_0(t)}{S_0(t)D(t)} ,$$

and to rewrite the wage earner human capital $b(t)$ as

$$b(t) = \frac{1}{\overline{H}_0(t)}\mathbb{E}_t\left[\int_t^T \overline{H}_0(s)i(s)\,\mathrm{d}s\right] . \tag{2.10}$$

Our terminology for the human capital function b adheres to that introduced by Richard in [20] for the analogue case when all of the processes $i(\cdot), S_0(\cdot)$ and $\eta(\cdot)$ are deterministic. It can be interpreted as the present value (valued at time t) of the wage earner's future income from time t through to the finite time horizon T, using for discount factor the modified state price density $\overline{H}_0$, *i.e.*, thus accounting for both the money market and the market fundamentals via its dependence on the local martingale $Z_0(\cdot)$, as well as for the cost of life insurance via its dependence on the insurance premium-payout rate $\eta(\cdot)$. Richard interprets this as a certainty equivalent for the wage earner's future income, that is assumed to be sure if they are alive at the time horizon.

Given an initial wealth x, we define the set $\mathcal{A}(x)$ of admissible strategies as follows. If $x + b(0) < 0$, we set $\mathcal{A}(x) = \emptyset$. For $x + b(0) \geq 0$, $\mathcal{A}(x)$ is the set of triples (c, π, p) representing the wage earner choices regarding consumption, investment and life insurance purchase defined in Section 2.3, under the constraint that $X(t)+b(t) \geq 0$ and $Z(t) \geq 0$, almost-surely. The quantity $X(t) + b(t)$ may be regarded as the potential wealth (present wealth plus present value of future income) of the wage earner at time t. We should note that, in our case, the wealth $X(t)$ can be negative, provided it is bounded from below by $-b(t)$ so that the wage earner potential wealth is nonnegative. Moreover, if $X(t) + b(t) = 0$ the wage earner is in a situation of bankruptcy, and it can be shown by using the optional sampling theorem that this state is absorbing, *i.e.*, their future wealth, consumption and legacy will all be zero after the time when bankruptcy occurs, and consequently the premium process and portfolio will all be zero after bankruptcy. This is analogous to the common bankruptcy absorbing condition $X(t) = 0$ in the case where there is no income. Finally, the condition $Z(t) \geq 0$ may be interpreted as the wage earner choosing not to leave any debts to their family in the event of premature death.

We will now shift our focus to the statement of our central object of study, namely, the optimal control problem describing the wage earner choice of admissible strategies maximizing the joint expected utility of:

1. their family consumption rate over the random time interval $[0, \min\{\tau, T\}]$;
2. their wealth at retirement date T, if they survive up to that age;
3. the value of their estate, in the event of premature death.

The wage earner's problem can then be formulated as follows: for a given initial wealth x, find admissible strategies $(c, \pi, p) \in \mathcal{A}(x)$ that maximize the expected utility

$$V(x) = \sup_{(c,\pi,p)\in\mathcal{A}_1(x)} \mathbb{E}\Bigg[\int_0^{\min\{\tau,T\}} U_1(s, c(s))\,\mathrm{d}s + U_2(\tau, Z(\tau))I_{\{\tau\leq T\}} + U_3(X(T))I_{\{\tau>T\}}\Bigg], \quad (2.11)$$

where

$$\mathcal{A}_1(x) = \Bigg\{(c,\pi,p) \in \mathcal{A}(x) :\ \mathbb{E}\Bigg[\int_0^{\tau\wedge T} \min\{0, U_1(s, c(s))\}\,\mathrm{d}s + \min\{0, U_2(\tau, Z(\tau))\}I_{\{\tau\leq T\}} + \min\{0, U_3(X(T))\}I_{\{\tau>T\}}\Bigg] > -\infty\Bigg\}.$$

In the case where $\mathcal{A}_1(x) = \emptyset$, we define $V(x) = -\infty$.

The following lemma allows us to restate the above problem as an equivalent one with a fixed planning horizon (see [7, Lemma 2.3] for a proof).

Lemma 1. *Suppose that the random variable τ is independent of the filtration $\{\mathcal{F}(t)\}$. We have that*

$$V(x) = \sup_{(c,\pi,p)\in\mathcal{A}_1(x)} \mathbb{E}\Bigg[\int_0^T \overline{F}(s)U_1(s, c(s)) + f(s)U_2(s, Z(s))\,\mathrm{d}s + \overline{F}(T)U_3(X(T))\Bigg],$$

where $\overline{F}(t)$ and $f(t)$ are, respectively, the survival function of τ and the corresponding force of mortality.

Remark 2. *Given the assumptions on the utility functions, for $(c, \pi, p) \in \mathcal{A}_1(x)$ we have that for Lebesgue-almost-every $t \in [0, T]$, almost-surely: $c(t) \geq \overline{c}(t)$, $Z(t) \geq \overline{Z}(t)$, $X(T) \geq \overline{x}$, otherwise (2.11) would be $-\infty$. This further justifies the use of the terminology* subsistence consumption, legacy and terminal wealth.

3 Optimal strategies: martingale method

In this section, we discuss certain characteristics of admissible strategies, including budget constraints. Employing the martingale method approach [11, 12], we solve first the *representation step* of the martingale method, which consists of finding portfolio and life-insurance rules corresponding to given terminal wealth and legacy processes, which we then optimize.

3.1 Admissibility

In this subsection, we address the *representation step* of consumption, bequest, and terminal wealth processes by admissible strategies. We derive a *budget constraint* for admissible strategies and show that, if one starts with consumption, bequest and terminal wealth processes satisfying such budget constraint, it is possible to replicate these processes behavior by an adequate choice of an admissible portfolio and life insurance policy purchase.

Using relation (2.3), we obtain from (2.6) that

$$\frac{X(t)}{S_0(t)} = x + \int_0^t \frac{i(u) - c(t) - \eta(u)Z(u) + \eta(u)X(u)}{S_0(u)}\,\mathrm{d}u + \int_0^t \frac{1}{S_0(u)}\pi'(u)\sigma(u)\,\mathrm{d}W_0(u)\ .$$

Applying Itô's lemma to the product of $X(t)/S_0(t)$ with $1/D(t)$ gives

$$\frac{X(t)}{D(t)S_0(t)} + \int_0^t \frac{c(u) + \eta(u)Z(u)}{D(u)S_0(u)}\,\mathrm{d}u = x + \int_0^t \frac{i(u)}{D(u)S_0(u)}\,\mathrm{d}u + \int_0^t \frac{1}{D(u)S_0(u)}\pi'(u)\sigma(u)\,\mathrm{d}W_0(u)\ .$$

Applying Itô's lemma once more, now to the product of the process process above with $Z_0(t)$, and recalling (2.1), allows us to write the discounted wealth process (with the discounting accomplished by the state price density $\overline{H}_0(\cdot)$) as follows:

$$\overline{H}_0(t)X(t) + \int_0^t \overline{H}_0(u)\left(c(u) + \eta(u)Z(u)\right)\,\mathrm{d}u = x + \int_0^t \overline{H}_(u)i(u)\,\mathrm{d}u + \int_0^t \overline{H}_0(u)\left(\pi'(u)\sigma(u) - X(u)\theta'(u)\right)\,\mathrm{d}W(u)\ .$$

Finally, adding $\overline{H}_0(t)b(t)$ to both sides of the identity above and recalling (2.10), we obtain

$$\overline{H}_0(t)\left(X(t) + b(t)\right) + \int_0^t \overline{H}_0(u)\left(c(u) + \eta(u)Z(u)\right)\,\mathrm{d}u = x + \mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] + \int_0^t \overline{H}_0(u)\left(\pi'(u)\sigma(u) - X(u)\theta'(u)\right)\,\mathrm{d}W(u)\ . \tag{3.1}$$

Since the left-hand side of (3.1) is non-negative by admissibility of (c, π, p), the right-hand side of (3.1) is not just a local martingale but, by Fatou's lemma, a super-martingale too. Thus, setting $t = T$ and recalling that $b(T) = 0$, we obtain the following *budget constraint* for every $(c, \pi, p) \in \mathcal{A}(x)$:

$$\mathbb{E}\left[\overline{H}_0(T)X(T) + \int_0^T \overline{H}_0(u)(c(u) + \eta(u)Z(u))\,\mathrm{d}u\right] \leq x + b(0)\ . \qquad (3.2)$$

Hence, we may conclude that admissible strategies impose the budget constraint (3.2).

The representation problem may be regarded as a converse to the previous statement. More precisely, if we start with a consumption process, a non-negative random variable representing the target terminal wealth, and a non-negative stochastic process representing the target legacy in the event of premature death, all of which such that the budget constraint (3.2) is satisfied, then there is a portfolio process and a life insurance policy which are admissible and that lead to the target terminal wealth and legacy in the event of premature death. This is the content of the next result.

Theorem 1. *Let $x + b(0) \geq 0$ be given, let $c(\cdot)$ be a consumption process, ξ a non-negative $\mathcal{F}(T)$-measurable random variable, and $\phi(\cdot)$ a non-negative, progressively measurable process satisfying the budget constraint*

$$\mathbb{E}\left[\int_0^T \overline{H}_0(u)\left(c(u) + \eta(u)\phi(u)\right)\,\mathrm{d}u + \overline{H}_0(T)\xi\right] = x + b(0) \geq 0\ .$$

Then there exist a portfolio process $\pi(\cdot)$ and a life insurance premium process $p(\cdot)$ such that $(c, \pi, p) \in \mathcal{A}(x)$, $X(T) = \xi$ and $Z(t) = \phi(t)$ for all $t \in [0, T]$.

Proof. Define the process

$$J(t) = \int_0^t \overline{H}_0(u)(c(u) + \eta(u)\phi(u) - i(u))\,\mathrm{d}u$$

and the martingale

$$M(t) = \mathbb{E}_t\left[J(T) + \overline{H}_0(T)\xi\right]\ .$$

Observe that $M(0) = x$. Using the martingale representation theorem, there exists a progressively measurable $\mathbb{R}^N$-valued process $\psi(\cdot)$ with finite $L_2[0, T]$-norm almost-surely, that is

$$||\psi||_2^2 = \int_0^T ||\psi(u)||^2\,\mathrm{d}u < \infty \qquad \text{a.s.}\ ,$$

and such that $M(\cdot)$ may be written as

$$M(t) = x + \int_0^t \psi'(u)\,\mathrm{d}W(u)\ .$$

Define a process $X(\cdot)$ by

$$\begin{aligned}
\overline{H}_0(t)X(t) &= M(t) - J(t) \\
&= x + \int_0^t \psi'(u)\,\mathrm{d}W(u) - \int_0^t \overline{H}_0(u)(c(u) + \eta(u)\phi(u) - i(u))\,\mathrm{d}u \\
&= \mathbb{E}_t\left[\int_t^T \overline{H}_0(u)(c(u) + \eta(u)\phi(u) - i(u))\,\mathrm{d}u + \bar{H}_0(T)\xi\right]\ . \qquad (3.3)
\end{aligned}$$

Using (2.10), we rewrite (3.3) as

$$\overline{H}_0(t)\,(X(t)+b(t)) + \int_0^t \overline{H}_0(u)(c(u)+\eta(u)\phi(u))\,\mathrm{d}u$$
$$= x + \mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] + \int_0^t \psi'(u)\,\mathrm{d}W(u)\ . \quad (3.4)$$

Recalling identity (3.1), suggests defining a portfolio process as

$$\pi(t) \quad = \quad \frac{1}{\overline{H}_0(t)}(\sigma'(t))^{-1}\psi(t) + X(t)(\sigma'(t))^{-1}\theta(t)\ , \quad (3.5)$$

and a premium process by

$$p(t) = \eta(t)(\phi(t) - X(t))\ . \quad (3.6)$$

Substituting (3.5) and (3.6) into the wealth process (3.1), and comparing with (3.4), we obtain that $X(T) = \xi$ and $Z(t) = \phi(t) \geq 0$, $t \in [0,T]$ almost-surely. Moreover, using (3.3), we have

$$\overline{H}_0(t)(X(t)+b(t)) = \mathbb{E}_t\left[\int_t^T \overline{H}_0(u)(c(u)+\eta(u)\phi(u))\,\mathrm{d}u + \overline{H}_0(T)\xi\right]\ ,$$

from which we arrive at $X(t)+b(t) \geq 0$ for all $t \in [0,T]$. Thus, we conclude that the triple $(c,\pi,p) \in \mathcal{A}(x)$ is admissible and leads to the target terminal wealth ξ and target legacy $\phi(\cdot)$ in the event of premature death.

It remains to check that the portfolio process and the premium process defined by (3.5) and (3.6) satisfy the integrability conditions (2.5). Observe that $M(\cdot)$ has continuous paths and that

$$||M|| = \max_{0\leq t\leq T}|M(t)| < \infty \qquad \text{a.s.}\ .$$

Analogously, it holds that $||J|| < \infty$ and $||S_0|| < \infty$. Since $\eta(\cdot)$ is uniformly bounded, we have $||\eta|| < \infty$ and also $||D|| < \infty$. In addition, observe that $\kappa_1 = ||1/Z_0|| < \infty$ almost-surely, and that $||\theta||_2 < \infty$ almost-surely. This implies that

$$||1/\overline{H}_0|| = \kappa_1\ ||S_0||\ ||D|| < \infty \qquad \text{a.s.}\ .$$

Using Hölder's inequality, we obtain that

$$\int_0^T |\pi'(t)(\mu(t)+\delta(t)-r(t)\underline{1}_N|\,\mathrm{d}t$$
$$= \int_0^T \frac{1}{\overline{H}_0(t)}|\psi'(t)\theta(t) + ||\theta(t)||^2(M(t)-J(t))|\,\mathrm{d}t$$
$$\leq ||1/\overline{H}_0||\left(||\psi||_2||\theta||_2 + ||\theta||_2^2(||M||+||J||)\right) < \infty\ ,$$

and so the portfolio process defined by (3.5) satisfies the first integrability condition of (2.5). We also have that

$$\int_0^T ||\sigma'(t)\pi(t)||^2\,\mathrm{d}t$$
$$= \int_0^T \frac{1}{\overline{H}_0(t)^2}\Big|\Big|\psi(t)+\theta(t)(M(t)-J(t))\Big|\Big|^2\,\mathrm{d}t$$
$$\leq ||1/\overline{H}_0||^2\Big(||\psi||_2^2 + ||\theta||_2^2(||M||+||J||)^2 + 2(||M||+||J||)||\psi||_2||\theta||_2\Big) < \infty\ ,$$

and so the portfolio process defined by (3.5) satisfies the second integrability condition of (2.5).

For the premium process defined by (3.6), we have that

$$\begin{aligned}
&\int_0^T |p(t)|\,\mathrm{d}t \\
&\quad \leq \int_0^T \eta(t)\left(\phi(t) + \frac{1}{\overline{H}_0(t)}|M(t)-J(t)|\right)\mathrm{d}t \\
&\quad \leq \int_0^T \frac{1}{\overline{H}_0(t)}\eta(t)\overline{H}_0(t)\phi(t)\,\mathrm{d}t + ||1/\overline{H}_0||\;||\eta||\;T(||M||+||J||) \\
&\quad \leq ||1/\overline{H}_0||\left(\int_0^T \overline{H}_0(t)\eta(t)\phi(t)\,\mathrm{d}t + ||\eta||\;T(||M||+||J||)\right) < \infty\ ,
\end{aligned}$$

thus completing the proof that all required integrability conditions are satisfied. □

3.2 Utility maximization

We now address the second step of the martingale method. This step involves solving a static optimization problem by a Lagrange multiplier argument, taking advantage of the utility preference structure described in Section 2.4, and making use of tools from convex duality through the convex dual or convex conjugate transform. This approach will allow us to find the optimal consumption rate, the optimal legacy in the case of premature death, and the optimal terminal wealth, leading to the optimal strategies for consumption, investment, and life insurance purchase, to be found as described in the previous subsection by means of the solution to the representation step given in Theorem 1.

Before proceeding, we define the auxiliary function

$$\begin{aligned}
\mathcal{X}(y) = \mathbb{E}\bigg[\int_0^T \overline{H}_0(t)I_1\left(t,\frac{y\overline{H}_0(t)}{\overline{F}(t)}\right) + \eta(t)\overline{H}_0(t)I_2\left(t,\frac{y\overline{H}_0(t)\eta(t)}{f(t)}\right)\mathrm{d}t \\
+\overline{H}_0(T)I_3\left(\frac{y\overline{H}_0(T)}{\overline{F}(T)}\right)\bigg]\ .
\end{aligned} \tag{3.7}$$

In what follows, we will assume that the auxiliary function $\mathcal{X}(y)$ defined above is such that $\mathcal{X}(y) < \infty$ for all $y > 0$.

The auxiliary function $\mathcal{X}(\cdot)$ depends heavily on the utility function structure described in Section 2.4. Its main properties, which we will use in the sequel, are stated in the next lemma.

Lemma 2. *The following hold:*

i) The function $\mathcal{X}(\cdot)$ is continuous and non-increasing on $(0,\infty)$.

ii) As $y \to 0^+$, we have that

$$\mathcal{X}(0^+) := \lim_{y\to 0^+}\mathcal{X}(y) = \infty\ .$$

iii) As $y \to \infty$, we have that $\mathcal{X}(\infty) := \lim_{y\to\infty} \mathcal{X}(y)$ satisfies

$$\mathcal{X}(\infty) = \mathbb{E}\left[\int_0^T \overline{H}_0(t)(\overline{c}(t) + \eta(t)\overline{Z}(t))\,\mathrm{d}t + \overline{H}_0(T)\overline{x}\right] < \infty\ . \tag{3.8}$$

iv) The function $\mathcal{X}(\cdot)$ is strictly decreasing on $(0, r)$, where r is given by

$$r = \sup\{y > 0 : \mathcal{X}(y) > \mathcal{X}(\infty)\} > 0\ .$$

Furthermore, when restricted to $(0, r)$ the function $\mathcal{X}(\cdot)$ has a strictly decreasing inverse, which we will henceforth denote as $\mathcal{Y} : (\mathcal{X}(\infty), \infty) \to (0, r)$.

Proof. Recalling that the functions $I_1(t, \cdot)$, $I_2(t, \cdot)$ and $I_3(\cdot)$ are continuous and non-increasing, we see that $\mathcal{X}(\cdot)$ is non-increasing. Appealing to the monotone convergence theorem, we obtain that $\mathcal{X}(\cdot)$ is right-continuous and, additionally, that assertion ii) holds as all of the functions $I_1(t, \cdot)$, $I_2(t, \cdot)$ and $I_3(\cdot)$ approach ∞ when the (non-temporal) variables approach zero. Assertion i) is a consequence of the dominated convergence theorem, after observing that finiteness of $\mathcal{X}(\cdot)$ implies left continuity of $\mathcal{X}(\cdot)$.

The dominated convergence theorem also implies assertion iii) once we observe that

$$\lim_{c\to\infty} I_1(t, c) = \overline{c}(t)\ , \quad \lim_{Z\to\infty} I_2(t, Z) = \overline{Z}(t) \quad \text{and} \quad \lim_{x\to\infty} I_3(x) = \overline{x}\ .$$

Let r be as given in item iv) of the statement above and let $\lambda \otimes P$ denote the product measure of the Lebesgue measure on $[0, T]$ and the probability measure P on $(\Omega, \mathcal{F})$. Take $y \in (0, r)$ arbitrary. From the definition of r, we obtain immediately that $\mathcal{X}(y) > \mathcal{X}(\infty)$. Moreover, comparing (3.7) with (3.8) and recalling the considerations at the end of Section 2.4, we conclude that either

$$\frac{y\overline{H}_0(t, \omega)}{\overline{F}(t)} < U_1'\left(t, \overline{c}(t)^+\right) \tag{3.9}$$

for all (t, ω) in a set of positive $\lambda \otimes P$ measure, or

$$\frac{y\overline{H}_0(t, \omega)\eta(t, \omega)}{f(t)} < U_2'\left(t, \overline{Z}(t)^+\right) \tag{3.10}$$

for all (t, ω) in a set of positive $\lambda \otimes P$ measure, or even

$$\frac{y\overline{H}_0(T, \omega)}{\overline{F}(T)} < U_3'\left(\overline{x}^+\right) \tag{3.11}$$

for all ω in a set of positive P measure. Since $I_1(t, \cdot)$ is strictly decreasing on $(0, U_1'(t, \overline{c}(t)^+))$, $I_2(t, \cdot)$ is strictly decreasing on $(0, U_2'(t, \overline{Z}(t)^+))$, and $I_3(\cdot)$ is strictly decreasing on $(0, U_3'(\overline{x}^+))$, any of the inequalities (3.9)–(3.11) is enough to guarantee that $\mathcal{X}(y - \epsilon) > \mathcal{X}(y)$ for all $\epsilon \in (0, y)$. Since the argument above applies to any $y \in (0, r)$, we conclude that $\mathcal{X}(\cdot)$ is strictly decreasing on $(0, r)$, thus completing the proof of assertion iv). □ □

Observe that if $(c,\pi,p)\in\mathcal{A}_1(x)$ we have that

$$\mathbb{E}\left[\int_0^T \overline{H}_0(t)(c(t)+\eta(t)Z(t))\,\mathrm{d}t+\overline{H}_0(T)X(T)\right]\geq\mathcal{X}(\infty)$$

and that the left-hand side of the inequality above satisfies the budget constraint (3.2). At this point, we separate the analysis into three cases, depending on whether $x+b(0)$ is less than, equal to, or greater than $\mathcal{X}(\infty)$.

In the first case, if $x+b(0)<\mathcal{X}(\infty)$, we must have that $\mathcal{A}_1(x)=\emptyset$ and, as consequence, $V(x)=-\infty$.

As for the second case, if $x+b(0)=\mathcal{X}(\infty)$, any triple $(c,\pi,p)\in\mathcal{A}_1(x)$ must satisfy

$$c(t)=\bar{c}(t)\ ,\quad Z(t)=\bar{Z}(t)\ ,\quad\text{and}\quad X(T)=\bar{x}\ ,$$

for Lebesgue almost-every $t\in[0,T]$ almost-surely. By Theorem 1, we can find both a portfolio process $\overline{\pi}(\cdot)$ and a life insurance premium process $\overline{p}(\cdot)$ corresponding to the given legacy and terminal wealth. Hence, we obtain that when $x+b(0)=\mathcal{X}(\infty)$, the expected utility is given by

$$\int_0^T \overline{F}(t)U_1(t,\overline{c}(t))+f(t)U_2(t,\overline{Z}(t))\,\mathrm{d}t+U_3(\overline{x})\ .$$

The third and last possible case occurs when $x+b(0)>\mathcal{X}(\infty)$. To proceed with our analysis, we need to restate our problem in an equivalent form. Namely, we want to find $(c,\pi,p)\in\mathcal{A}_1(x)$ for the equivalent objective function given in Lemma 1 subject to the *budget constraint* (3.2). We use a Lagrange multiplier argument to find the optimal strategies, *i.e.* the optimal consumption, legacy, and terminal wealth. Let $y>0$ be a Lagrange multiplier and consider the unconstrained maximization of

$$\begin{aligned}\mathbb{E}&\left[\int_0^T \overline{F}(t)U_1(t,c(t))+f(t)U_2(t,Z(t))\,\mathrm{d}t+\overline{F}(T)U_3(X(T))\right]\\&+y\left(x+b(0)-\mathbb{E}\left[\overline{H}_0(T)X(T)+\int_0^T \overline{H}_0(t)(c(t)+\eta(t)Z(t))\,\mathrm{d}t\right]\right)\end{aligned}\tag{3.12}$$

After rearranging (3.12) and considering the convex duals of the utility functions, we obtain that

$$
\begin{aligned}
(x+b(0))y + \mathbb{E}\left[\int_0^T \overline{F}(t)U_1(t,c(t)) - y\overline{H}_0(t)c(t)\,\mathrm{d}t\right] \\
+\mathbb{E}\left[\int_0^T f(t)U_2(t,Z(t)) - y\overline{H}_0(t)\eta(t)Z(t)\,\mathrm{d}t\right] \\
+\mathbb{E}\Big[\overline{F}(T)U_3(X(T)) - y\overline{H}_0(T)X(T)\Big] \\
\leq (x+b(0))y + \mathbb{E}\left[\int_0^T \overline{F}(t)\widetilde{U}_1\left(t, \frac{y\overline{H}_0(t)}{\overline{F}(t)}\right)\mathrm{d}t\right] \\
+\mathbb{E}\left[\int_0^T f(t)\widetilde{U}_2\left(t, \frac{y\overline{H}_0(t)\eta(t)}{f(t)}\right)\mathrm{d}t\right] \\
+\mathbb{E}\left[\overline{F}(T)\widetilde{U}_3\left(\frac{y\overline{H}_0(T)}{\overline{F}(T)}\right)\right] ,
\end{aligned}
$$

with equality if and only if

$$
\begin{aligned}
c(t) &= I_1\left(t, \frac{y\overline{H}_0(t)}{\overline{F}(t)}\right) \\
Z(t) &= I_2\left(t, \frac{y\overline{H}_0(t)\eta(t)}{f(t)}\right) \\
X(T) &= I_3\left(\frac{y\overline{H}_0(T)}{\overline{F}(T)}\right) .
\end{aligned}
$$

In order to satisfy the budget constraint (3.2) with equality we must have $\mathcal{X}(y) = x+b(0)$. Since we are assuming that $x+b(0) > \mathcal{X}(\infty)$ in this third case, by Lemma 2 there is a unique solution to the equation $\mathcal{X}(y) = x+b(0)$, given by $y = \mathcal{Y}(x+b(0))$. We conclude that the candidates for optimality given by the Lagrange multiplier method are:

$$
c^*(t) = I_1\left(t, \frac{\mathcal{Y}(x+b(0))\overline{H}_0(t)}{\overline{F}(t)}\right) \tag{3.13}
$$

$$
\Psi^*(t) = I_2\left(t, \frac{\mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)}{f(t)}\right) \tag{3.14}
$$

$$
\xi^* = I_3\left(\frac{\mathcal{Y}(x+b(0))\overline{H}_0(T)}{\overline{F}(T)}\right) . \tag{3.15}
$$

The next theorem guarantees optimality (3.13)–(3.15).

Theorem 2. *Suppose the previous assumptions hold, let $x+b(0) \in (\mathcal{X}(\infty), \infty)$ be given, and let $c^*(\cdot)$, $\Psi^*(\cdot)$, and $\xi^*(\cdot)$ be as in (3.13)–(3.15). Then there exists an optimal triple $(c^*, \pi^*, p^*) \in \mathcal{A}_1(x)$ such that $X^*(T) = \xi^*$ and $Z^*(t) = \Psi^*(t)$:*

$$
V(x) = \mathbb{E}\left[\int_0^T \overline{F}(t)U_1(t,c^*(t)) + f(t)U_2(t,Z^*(t))\,\mathrm{d}t + \overline{F}(T)U_3(X^*(T))\right] .
$$

Proof. The processes defined by (3.13)–(3.15) satisfy the conditions of Theorem 1, *i.e.*, each is non-negative and, by the definition of the auxiliary function (3.7), we have that

$$\mathbb{E}\left[\int_0^T \overline{H}_0(t)\left(c^*(t)+\eta(t)\Psi^*(t)\right)\,\mathrm{d}t+\overline{H}_0(T)\xi^*\right]=\mathcal{X}(\mathcal{Y}(x+b(0))=x+b(0)\ .$$

Thus, the target consumption, legacy and terminal wealth satisfy the budget constraint with equality. As a consequence, we obtain that there is $(c^*,\pi^*,p^*)\in\mathcal{A}(x)$ such that $X^*(T)=\xi^*$ and $Z^*(t)=\Psi^*(t)$.

We will now prove that $(c^*,\pi^*,p^*)\in\mathcal{A}_1(x)$. We choose

$$\hat{c}>\max\left\{\overline{x},\max_{0\le t\le T}\overline{c}(t)\right\}$$

and

$$\hat{z}>\max\left\{\overline{x},\max_{0\le t\le T}\overline{Z}(t)\right\}\ ,$$

yielding

$$\int_0^T |U_1(t,\hat{c}+\hat{z})|+|U_1(t,\hat{c}+\hat{z})|\,\mathrm{d}t+|U_3(\hat{c}+\hat{z})|<\infty\ .$$

Using convex duality, we obtain that the following three inequalities hold for all $t\in[0,T]$:

$$\begin{aligned}\overline{F}(t)U_1(t,c^*(t))&-\mathcal{Y}(x+b(0))\overline{H}_0(t)c^*(t)\\&=\overline{F}(t)\widetilde{U}_1\left(t,\frac{\mathcal{Y}(x+b(0))\overline{H}_0(t)}{\overline{F}(t)}\right)\\&\ge\overline{F}(t)\left(U_1(t,\hat{c}+\hat{z})-\frac{\mathcal{Y}(x+b(0))\overline{H}_0(t)(\hat{c}+\hat{z})}{\overline{F}(t)}\right)\ ,\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}f(t)U_2(t,Z^*(t))&-\mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)Z^*(t)\\&=f(t)\widetilde{U}_2\left(t,\frac{\mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)}{f(t)}\right)\\&\ge f(t)\left(U_2(t,\hat{c}+\hat{z})-\frac{\mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)(\hat{c}+\hat{z})}{f(t)}\right)\ ,\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}\overline{F}(T)U_3(X^*(T))&-\mathcal{Y}(x+b(0))\overline{H}_0(T)X^*(T)\\&=\overline{F}(T)\widetilde{U}_3\left(\frac{\mathcal{Y}(x+b(0))\overline{H}_0(T)}{\overline{F}(T)}\right)\\&\ge\overline{F}(T)\left(U_3(\hat{c}+\hat{z})-\frac{\mathcal{Y}(x+b(0))\overline{H}_0(T)(\hat{c}+\hat{z})}{\overline{F}(T)}\right)\ .\end{aligned}\tag{3.18}$$

Using inequalities (3.16)–(3.18), we obtain

$$\mathbb{E}\Bigg[\int_0^T \min\{0,\overline{F}(t)U_1(t,c^*(t))\} + \min\{0, f(t)U_2(t,Z^*(t))\}\,\mathrm{d}t$$
$$+\min\{0,\overline{F}(T)U_3(X^*(T))\}\Bigg]$$
$$\geq \int_0^T \min\{0,\overline{F}(t)U_1(t,\hat{c}+\hat{z})\} + \min\{0, f(t)U_2(t,\hat{c}+\hat{z})\}\,\mathrm{d}t$$
$$+\min\{0,\overline{F}(T)U_3(\hat{c}+\hat{z})\}$$
$$-\mathcal{Y}(x+b(0))(\hat{c}+\hat{z})\mathbb{E}\left[\int_0^T \overline{H}_0(t)(1+\eta(t))\,\mathrm{d}t + \overline{H}_0(T)\right] > -\infty\ ,$$

thus concluding the proof that $(c^*,\pi^*,p^*)\in\mathcal{A}_1(x)$.

We now prove optimality $(c^*,\pi^*,p^*)\in\mathcal{A}_1(x)$. Let $(c',\pi',p')\in\mathcal{A}_1(x)$, and denote by Z' and $X'(T)$, respectively, the legacy and terminal wealth associated with (c',π',p'). Using convex duality, we obtain that each of the following three inequalities holds for Lebesgue almost-every $t\in[0,T]$:

$$\overline{F}(t)U_1(t,c^*(t)) - \mathcal{Y}(x+b(0))\overline{H}_0(t)c^*(t)$$
$$\geq \overline{F}(t)U_1(t,c'(t)) - \mathcal{Y}(x+b(0))\overline{H}_0(t)c'(t)\ , \quad (3.19)$$

and

$$f(t)U_2(t,Z^*(t)) - \mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)Z^*(t)$$
$$\geq f(t)U_2(t,Z'(t)) - \mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)Z'(t)\ , \quad (3.20)$$

and

$$\overline{F}(T)U_3(X^*(T)) - \mathcal{Y}(x+b(0))\overline{H}_0(T)X^*(T)$$
$$\geq \overline{F}(T)U_3(X'(T)) - \mathcal{Y}(x+b(0))\overline{H}_0(T)X'(T)\ . \quad (3.21)$$

Using inequalities (3.19)–(3.21), we obtain that

$$\mathbb{E}\left[\int_0^T \overline{F}(t)U_1(t,c^*(t)) + f(t)U_2(t,Z^*(t))\,\mathrm{d}t + \overline{F}(T)U_3(X^*(T))\right]$$
$$\geq \mathbb{E}\left[\int_0^T \overline{F}(t)U_1(t,c'(t)) + f(t)U_2(t,Z'(t))\,\mathrm{d}t + \overline{F}(T)U_3(X'(T))\right]$$
$$+\mathcal{Y}(x+b(0))\mathbb{E}\left[\int_0^T \overline{H}_0(t)\left(c^*(t)+\eta(t)Z^*(t)\right)\,\mathrm{d}t + \overline{H}_0(T)X^*(T)\right]$$
$$-\mathcal{Y}(x+b(0))\mathbb{E}\left[\int_0^T \overline{H}_0(t)\left(c'(t)+\eta(t)Z'(t)\right)\,\mathrm{d}t + \overline{H}_0(T)X'(T)\right]$$
$$\geq \mathbb{E}\left[\int_0^T \overline{F}(t)U_1(t,c'(t)) + f(t)U_2(t,Z'(t))\,\mathrm{d}t + \overline{F}(T)U_3(X'(T))\right]\ ,$$

where in the last inequality we have used the budget constraint (3.2) applied to both the triple (c', π', p') and the triple (c^*, π^*, p^*), the latter of which satisfies the constraint with equality. We conclude that (c^*, π^*, p^*) is optimal and that the value function has the form given in the statement. □

Remark 3. *Assuming that $V(x) < \infty$, from the proof of the preceding theorem we also obtain that $c^*(\cdot)$, $Z^*(\cdot)$ and $\xi^*(\cdot)$ are unique almost-everywhere relative to the product measure of Lebesgue in $[0,T]$ and P. This implies that both the optimal portfolio $\pi^*(\cdot)$ and the optimal life insurance premium $p^*(\cdot)$ are unique almost-everywhere.*

Combining the optimality results of Theorem 2 and the representation step of Theorem 1, we arrive at the following result.

Theorem 3. *Under the assumptions of Theorem 2:*

i) The optimal wealth process is given by

$$X^*(t) = \frac{1}{\overline{H}_0(t)}\mathbb{E}_t\left[\int_t^T \overline{H}_0(u)(c^*(u) + \eta(u)Z^*(u) - i(u))\,\mathrm{d}u + \overline{H}_0(T)\xi^*\right] \quad (3.22)$$

ii) The optimal portfolio is given by

$$\pi(t) = \frac{1}{\overline{H}_0(t)}(\sigma'(t))^{-1}\psi(t) + (\sigma'(t))^{-1}X^*(t)\theta(t)\ , \quad (3.23)$$

where $\psi(\cdot)$ is such that

$$M(t) = x + \int_0^t \psi'(u)\,\mathrm{d}W(u)$$

and $M(\cdot)$ is the martingale defined by

$$M(t) = \mathbb{E}_t\left[\int_0^T \overline{H}_0(u)(c^*(u) + \eta(u)Z^*(u) - i(u))\,\mathrm{d}u + \overline{H}_0(T)\xi^*\right]\ . \quad (3.24)$$

iii) The optimal life insurance premium is given by

$$p^*(t) = \eta(t)(Z^*(t) - X^*(t))\ . \quad (3.25)$$

The optimal wealth process $X^*(\cdot)$ given by (3.22) may be written alternatively as the full wealth process $X^*(\cdot) + b(\cdot)$, representing present plus future wealth:

$$X^*(t) + b(t) = \frac{1}{\overline{H}_0(t)}\mathbb{E}_t\left[\int_t^T \overline{H}_0(u)(c^*(u) + \eta(u)Z^*(u))\,\mathrm{d}u + \overline{H}_0(T)\xi^*\right]\ .$$

4 Solutions for constant relative risk aversion utilities

In this section, we will assume that $A(\cdot)$, $r(\cdot)$, $\theta(\cdot)$, $\eta(\cdot)$ and $i(\cdot)$ are continuous and deterministic functions.

4.1 Constant relative risk aversion power utilities

We will now derive the optimal strategies for the case where the wage earner's utility functions for consumption, legacy, and terminal wealth are all discounted *constant relative risk aversion* (CRRA) power utility functions, given by

$$\begin{aligned} U_1(t,c) &= \mathrm{e}^{-\rho t}\frac{c^\gamma}{\gamma} \\ U_2(t,Z) &= \mathrm{e}^{-\rho t}\frac{Z^\gamma}{\gamma} \\ U_3(X) &= \mathrm{e}^{-\rho T}\frac{X^\gamma}{\gamma}\ , \end{aligned} \tag{4.1}$$

where the risk aversion parameter γ is such that $\gamma < 1$ and $\gamma \neq 0$, and the time discount rate ρ is non-negative.

Differentiating the utility functions in (4.1) with respect to their second variable (sole variable in the case of U_3) and inverting, we obtain

$$I_1(t,y) = I_2(t,y) = \mathrm{e}^{\rho t/(\gamma-1)} y^{1/(\gamma-1)} \quad \text{and} \quad I_3(y) = \mathrm{e}^{\rho T/(\gamma-1)} y^{1/(\gamma-1)}\ .$$

Using these functions in the auxiliary function (3.7), we obtain the relation

$$\mathcal{X}(y) = \mathcal{X}(1) y^{1/(\gamma-1)}\ ,$$

which we can then invert to get

$$\mathcal{Y}(x) = \left(\frac{x}{\mathcal{X}(1)}\right)^{\gamma-1}\ .$$

Recalling (3.13)–(3.15) and using (2.2), we obtain that

$$c^*(t) = \mathrm{e}^{\rho t/(\gamma-1)}\left(\frac{x+b(0)}{\mathcal{X}(1)}\right)\left(\frac{\overline{H}_0(t)}{\overline{F}(t)}\right)^{1/(\gamma-1)} \tag{4.2}$$

$$Z^*(t) = \mathrm{e}^{\rho t/(\gamma-1)}\left(\frac{x+b(0)}{\mathcal{X}(1)}\right)\left(\frac{\eta(t)}{\lambda(t)}\frac{\overline{H}_0(t)}{\overline{F}(t)}\right)^{1/(\gamma-1)} \tag{4.3}$$

$$\xi^* = \mathrm{e}^{\rho T/(\gamma-1)}\left(\frac{x+b(0)}{\mathcal{X}(1)}\right)\left(\frac{\overline{H}_0(T)}{\overline{F}(T)}\right)^{1/(\gamma-1)}\ . \tag{4.4}$$

Substituting (4.2)–(4.4) into the optimal wealth process (3.22), we obtain the full wealth process

$$X^*(t) + b(t) = \frac{x+b(0)}{\mathcal{X}(1)\overline{H}_0(t)}\mathbb{E}_t\left[\int_t^T \mathrm{e}^{\rho u/(\gamma-1)}\frac{\overline{H}_0(u)^{\gamma/(\gamma-1)}}{\overline{F}(u)^{1/(\gamma-1)}}K(u)\,\mathrm{d}u + \mathrm{e}^{\rho T/(\gamma-1)}\frac{\overline{H}_0(T)^{\gamma/(\gamma-1)}}{\overline{F}(T)^{1/(\gamma-1)}}\right]\ , \tag{4.5}$$

where the function $K(\cdot)$ is given by

$$K(u) = 1 + \frac{\eta(u)^{\gamma/(\gamma-1)}}{\lambda(u)^{1/(\gamma-1)}} \ .$$

We will now rewrite the integrand term in (4.5) in a more suitable way. Define a martingale $\Lambda(\cdot)$ as

$$\Lambda(t) = \exp\left\{\frac{\gamma}{1-\gamma}\int_0^t \theta'(u)\,\mathrm{d}W(u) - \frac{\gamma^2}{2(1-\gamma)^2}\int_0^t ||\theta(u)||^2\,\mathrm{d}u\right\} \ ,$$

which may be regarded as the stochastic exponential of the process $-\frac{\gamma}{1-\gamma}\theta(t)$. Define also the deterministic function $m(\cdot)$ as being

$$m(t) = \exp\left(\widetilde{m}(t)\right) \ ,$$

where

$$\begin{aligned}\widetilde{m}(t) &= -\frac{1}{1-\gamma}\left(\rho t + \int_0^t \lambda(u)\,\mathrm{d}u\right) + \frac{\gamma}{1-\gamma}\left(A(t) + \int_0^t r(u) + \eta(u)\,\mathrm{d}u\right) \\ &\quad + \int_0^t \frac{\gamma}{2(1-\gamma)^2}||\theta(u)||^2\,\mathrm{d}u \ .\end{aligned}$$

Then, for all $t \in [0,T]$, we have that

$$\mathrm{e}^{\rho t/(\gamma-1)}\frac{\overline{H}_0(t)^{\gamma/(\gamma-1)}}{\overline{F}(t)^{1/(\gamma-1)}} = m(t)\Lambda(t) \ . \tag{4.6}$$

Combining (4.6) with (4.5), and recalling that $\Lambda(\cdot)$ is a martingale, we rewrite the full wealth process as

$$\begin{aligned}X^*(t) + b(t) &= \frac{x+b(0)}{\mathcal{X}(1)\overline{H}_0(t)}\mathbb{E}_t\left[\int_t^T K(u)m(u)\Lambda(u)\,\mathrm{d}u + m(T)\Lambda(T)\right] \\ &= \frac{x+b(0)}{\mathcal{X}(1)\overline{H}_0(t)}\Lambda(t)\left(\int_t^T K(u)m(u)\,\mathrm{d}u + m(T)\right) \ . \end{aligned} \tag{4.7}$$

Solving (4.7) for the quantity $(x+b(0))/\mathcal{X}(1)$, substituting in (4.2) and (4.3), and using (4.6), we obtain the optimal consumption and legacy in feedback form on the present full wealth level:

$$c^*(t) = \frac{1}{e(t)}(X^*(t) + b(t)) \tag{4.8}$$

$$Z^*(t) = d(t)(X^*(t) + b(t)) \ , \tag{4.9}$$

where

$$e(t) = \frac{1}{m(t)}\left(\int_t^T K(u)m(u)\,\mathrm{d}u + m(T)\right)$$

and

$$d(t) = \left(\frac{\eta(t)}{\lambda(t)}\right)^{1/(\gamma-1)}\frac{1}{e(t)} \ .$$

Finally, from (4.9) and (3.25), we obtain that the optimal life insurance premium is given by

$$\begin{aligned} p^*(t) &= \eta(t)(Z^*(t) - X^*(t)) \\ &= \eta(t)((d(t)-1)X^*(t) + d(t)b(t)) \ . \end{aligned} \tag{4.10}$$

To obtain an explicit formula for the optimal portfolio, we first have find the differential of the martingale $M(\cdot)$ given by (3.24). Substituting (4.2), (4.3) and (4.4) into (3.24), we get

$$\begin{aligned} M(t) = \frac{x+b(0)}{\mathcal{X}(1)}\mathbb{E}_t &\left[\int_0^T K(u)m(u)\Lambda(u)\,\mathrm{d}u + m(T)\Lambda(T)\right] \\ &-\mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] \\ = \frac{x+b(0)}{\mathcal{X}(1)} &\left(\int_0^t K(u)m(u)\Lambda(u)\,\mathrm{d}u + \Lambda(t)e(t)m(t)\right) \\ &-\mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] \ . \end{aligned}$$

Computing the differential of $M(\cdot)$, we obtain

$$\begin{aligned} \mathrm{d}M(t) &= \frac{x+b(0)}{\mathcal{X}(1)}e(t)m(t)\mathrm{d}\Lambda(t) - \mathrm{d}\mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] \\ &= \overline{H}_0(t)(X^*(t)+b(t))\frac{\gamma}{1-\gamma}\theta'(t)\mathrm{d}W(t) + b(t)\overline{H}_0(t)\theta'(t)\mathrm{d}W(t) \ . \end{aligned}$$

Combining the stochastic integral representation of the martingale $M(\cdot)$ with (3.23), we arrive at

$$\begin{aligned} \pi(t) &= \frac{1}{1-\gamma}(\sigma'(t))^{-1}\theta(t)(X^*(t)+b(t)) \\ &= \frac{1}{1-\gamma}(\sigma(t)\sigma'(t))^{-1}\alpha(t)(X^*(t)+b(t)) \ . \end{aligned} \tag{4.11}$$

These results coincide with the ones obtained in [6], using dynamic programming and the Hamilton-Jacobi-Bellman equation, in the case where the financial market is assumed to be complete, there is no dividend payment, and the singularly continuous component of the money market is zero. For the economic interpretations of these results we refer the reader to that paper.

4.2 Logarithmic utilities

We will now derive the optimal strategies for the case where the utility functions are discounted logarithmic utilities, given by

$$\begin{aligned} U_1(t,c) &= \mathrm{e}^{-\rho t}\log c \\ U_2(t,Z) &= \mathrm{e}^{-\rho t}\log Z \\ U_3(X) &= \mathrm{e}^{-\rho T}\log X \ , \end{aligned} \tag{4.12}$$

where the time discount rate ρ is non-negative. These utilities may be regarded as the limit case of the constant relative risk aversion utility functions (4.1) of the previous subsection when the risk aversion coefficient γ approaches zero.

Differentiating the utility functions in (4.12) with respect to their second variable (sole variable in the case of U_3) and inverting, we obtain

$$I_1(t,y) = I_2(t,y) = \frac{\mathrm{e}^{-\rho t}}{y} \quad \text{and} \quad I_3(y) = \frac{\mathrm{e}^{-\rho T}}{y} \ .$$

Using these functions in the auxiliary function (3.7), we obtain the relation $\mathcal{X}(y) = \mathcal{X}(1)/y$, which we can invert to get

$$\mathcal{Y}(x) = \frac{\mathcal{X}(1)}{x} \ ,$$

where $\mathcal{X}(1)$ is given by

$$\mathcal{X}(1) = \int_0^T \mathrm{e}^{-\rho t}\overline{F}(t)(1+\lambda(t))\,\mathrm{d}t + \mathrm{e}^{-\rho T}\overline{F}(T) \ .$$

Recalling (3.13)–(3.15), we obtain that

$$c^*(t) = \mathrm{e}^{-\rho t}\frac{\overline{F}(t)}{\mathcal{Y}(x+b(0))\overline{H}_0(t)} \tag{4.13}$$

$$Z^*(t) = \mathrm{e}^{-\rho t}\frac{\lambda(t)\overline{F}(t)}{\mathcal{Y}(x+b(0))\overline{H}_0(t)\eta(t)} \tag{4.14}$$

$$\xi^* = \mathrm{e}^{-\rho T}\frac{\overline{F}(T)}{\mathcal{Y}(x+b(0))\overline{H}_0(T)} \ . \tag{4.15}$$

Substituting (4.13)–(4.15) into the optimal wealth process (3.22), we obtain that the full wealth process may be written as

$$X^*(t) + b(t) = \frac{a(t)}{\mathcal{Y}(x+b(0))\overline{H}_0(t)} \ , \tag{4.16}$$

where $a(t)$ is given by

$$a(t) = \int_t^T \mathrm{e}^{-\rho u}\overline{F}(u)(1+\lambda(u))\,\mathrm{d}u + \mathrm{e}^{-\rho T}\overline{F}(T) \ .$$

Combining (4.16) with (4.13) and (4.14), we obtain the optimal consumption and optimal legacy in feedback form on the present full wealth level:

$$c^*(t) = g_c(t)(X^*(t)+b(t)) \qquad \text{and} \qquad Z^*(t) = g_Z(t)(X^*(t)+b(t)) \ , \tag{4.17}$$

where $g_c(\cdot)$ and $g_Z(\cdot)$ are given by

$$g_c(t) = \mathrm{e}^{-\rho t}\frac{\overline{F}(t)}{a(t)} \qquad \text{and} \qquad g_Z(t) = \mathrm{e}^{-\rho t}\frac{\lambda(t)}{\eta(t)}\frac{\overline{F}(t)}{a(t)} \ .$$

Combining (4.17) with (3.25), we get

$$\begin{aligned} p^*(t) &= \eta(t)(Z^*(t) - X^*(t)) \\ &= \eta(t)\left((g_Z(t)-1)X^*(t) + g_Z(t)b(t)\right) \ . \end{aligned}$$

To obtain an explicit formula for the optimal portfolio, we recall that $a(0) = \mathcal{X}(1)$ and rewrite the martingale $M(\cdot)$ in (3.24) in the form

$$\begin{aligned} M(t) &= \frac{1}{\mathcal{Y}(x+b(0))}a(0) - \mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] \\ &= x + b(0) - \mathbb{E}_t\left[\int_0^T \overline{H}_0(u)i(u)\,\mathrm{d}u\right] \ . \end{aligned}$$

It is then possible to check that $M(\cdot)$ admits the stochastic representation

$$\mathrm{d}M(t) = b(t)\overline{H}_0(t)\theta'(t)\mathrm{d}W(t) \ ,$$

which, combined with (3.23), results in the following explicit representation for the optimal portfolio

$$\begin{aligned} \pi(t) &= (\sigma'(t))^{-1}\theta(t)(X^*(t)+b(t)) \\ &= (\sigma(t)\sigma'(t))^{-1}\alpha(t)(X^*(t)+b(t)) \ . \end{aligned} \tag{4.18}$$

It is also possible to check that the optimal solutions for logarithmic utilities given in (4.17) and (4.18) are equal to the optimal solutions for constant relative risk aversion power utilities given by (4.8), (4.10) and (4.11) when the risk aversion coefficient γ is set equal to zero, which we couldn't guarantee *a priori* since constant relative risk aversion power utilities are undefined when γ equals zero.

5 Conclusions

In this paper, we have employed the duality approach of convex analysis and martingale techniques to address a stochastic optimal control problem faced by an agent making decisions about family consumption, financial investments, and life insurance purchases. By formulating the problem within a general framework that accounts for both the financial market and life insurance options, we have extended previous work on consumption-investment problems to include the important dimension of life insurance, which has significant practical relevance for risk-averse individuals planning for uncertain lifetimes.

The use of duality techniques, combined with martingale methods, enabled us to transform the original dynamic optimization problem into a static one, thus simplifying the analysis while allowing us to maintain the generality of the model under consideration. One of the key advantages of the duality approach is its flexibility in handling non-standard financial market settings, such as those with incomplete or non-Markovian asset prices. As a special case of our analysis, we derived optimal strategies for consumption and portfolio selection, as well as life insurance purchase, for the case of both power and logarithmic utilities.

Future research could further explore extensions to more complex market settings, such as those with stochastic interest rates, labor income, or more general mortality rates. Additionally, investigating the impact of additional constraints on consumption and portfolio strategies, such as liquidity restrictions or tax considerations, would enhance the practical applicability of the model.

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