Separation of solutions of Caputo fractional differential equations

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Abstract

A simpler proof of known results regarding the separation of solutions of a fractional initial value problem is presented. This proof does not rely on a Lipschitz type condition.

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1 Preamble

The problem under consideration in this work was firstly stated and discussed in [3, Conjecture 1.2] and it was completely solved in [2, Theorem 3.5]. Specifically, it was aimed to show that any two continuous solutions, say x and y, of the initial value problem (here f is a continuous function of its arguments),

$$^{C}D^{\alpha}_{0+}z(t) = f(t, z(t)),\ z(0) = z_0, \quad t \in [0, T]\ (T > 0),\ 0 < \alpha < 1, \tag{1}$$

with $x(0) \neq y(0)$ do not intersect in the common domain of definition of x and y. Other contributions to this problem may be found in [1, Remark 3.13], as well as some applications of it to some fractional variational problems in [4].

In the following section we give a proof of this result which is far more simpler than the one provided in [2, Theorem 3.5]. It is based on a non-canonical representation for the Caputo fractional derivative given by [5, Theorem 5.2].

2 Main result and some observations

We will state and prove the aforementioned result:

Theorem 1. *Suppose that* $0 < \alpha < 1$ *and* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Let* $x, y \in C[0,T]$ *be solutions of* (1) *with* $x(0) \neq y(0)$*. Then,* $x(t) \neq y(t)$ *on* $[0,T]$.

Proof. Suppose without loss of generality that $x(0) < y(0)$. Assume that $t^\star \in (0,T]$ is the first point where $x(t^\star) = y(t^\star)$. By the continuity of x and y, we have that $x(t) < y(t)$ for $t \in [0, t^\star)$. Now, by the linearity of the Caputo fractional derivative, we know that

$$^{C}D^{\alpha}_{0+} z(t^\star) = f(t^\star, x(t^\star)) - f(t^\star, y(t^\star)) = 0,$$

where $z(t) = x(t) - y(t)$. However, since f is continuous (hence $^{C}D^{\alpha}_{0+} z$ is continuous on $[0, t^\star]$) we obtain, upon using [5, Theorem 5.2],

$$^{C}D^{\alpha}_{0+} z(t^\star) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{z(t^\star) - z(0)}{(t^\star)^\alpha} + \alpha \int_0^{t^\star} (t^\star - s)^{-\alpha-1} (z(t^\star) - z(s)) ds \right) > 0,$$

which is absurd. The proof is done. □

In the works [2] and [3], an additional hypoethesis is included in Theorem 1, namely, that f satisfies the Lipschitz condition,

$$|f(t,x) - f(t,y)| \leq L(t)|x - y|, \quad t \in [0,T],\ x, y \in \mathbb{R},\ L \in (C[0,T], \mathbb{R}^+). \tag{2}$$

While (2) is a sufficient condition to prove existence and uniqueness of solution to (1), we have shown in Theorem 1 that this condition is not actually needed to show the separation of two solutions under different initial conditions. We emphasyze that in [2, 3] the authors used (2) to prove the *separation theorem*, i.e., to prove the conclusion of Theorem 1.

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References

[1] L. Bourdin and R. A. C. Ferreira, Legendre's necessary condition for fractional Bolza functionals with mixed initial/final constraints, J. Optim. Theory Appl. **190** (2021), no. 2, 672–708.

[2] N. D. Cong and H. T. Tuan, Generation of nonlocal fractional dynamical systems by fractional differential equations, J. Integral Equations Appl. **29** (2017), no. 4, 585–608.

[3] K. Diethelm, On the separation of solutions of fractional differential equations, Fract. Calc. Appl. Anal. **11** (2008), no. 3, 259–268.

[4] R. A. C. Ferreira, Sign of the solutions of linear fractional differential equations and some applications, Vietnam J. Math. **51** (2023), no. 2, 451–461.

[5] G. Vainikko, Which functions are fractionally differentiable?, Z. Anal. Anwend. **35** (2016), no. 4, 465–487.