On supratopologies, normalized families and Frankl conjecture

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Abstract

We introduce some generalized topological concepts to deal with union-closed families, and show that one can reduce the proof of Frankl conjecture to some families of so-called supratopological spaces. We prove some results on the structure of normalized families, presenting a new way of reducing such a family to a smaller one using dual families. Applying our reduction method, we prove a refinement of a conjecture originally proposed by Poonen. Finally, we show that Frankl Conjecture holds for the class of families obtained from successively applying the reduction process to a power set.

1 Introduction

Let $\mathcal{F}$ be a finite family of sets. In this context, by "family of sets" we mean a set consisting of sets. We say that $\mathcal{F}$ is *union-closed* if, for any $F, G \in \mathcal{F}$, we have that $F \cup G \in \mathcal{F}$. We define the *universe* of a family $\mathcal{F}$ as the union of all its member-sets, and denote it by $U(\mathcal{F})$. Of course, $\mathcal{F} \subseteq \mathcal{P}(U(\mathcal{F}))$, and if $\mathcal{F}$ is union-closed, then $U(\mathcal{F}) \in \mathcal{F}$.

Any bijection between two finite sets, $U \to V$, induces a bijection $\mathcal{P}(U) \to \mathcal{P}(V)$, that, of course, preserves unions, and therefore union-closed families, as well as all the properties about them that are pertinent in this paper. Thus, we will only be interested in families

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modulo the equivalence relation induced by bijections on the respective universes. We will then call two families *isomorphic* if such a bijection exists, and, as usual, we will say that a family is "such and such" *up to bijection*, meaning up to a bijection between their universes.

For $S \subseteq U(\mathcal{F})$, we define $\mathcal{F}_S = \{F \in \mathcal{F} : F \cap S \neq \varnothing\}$ and $\mathcal{F}_{\tilde{S}} = \{F \in \mathcal{F} : F \cap S = \varnothing\}$. For $a \in U(\mathcal{F})$, we denote $\mathcal{F}_{\{a\}}$ by $\mathcal{F}_a$, and similarly $\mathcal{F}_{\widetilde{\{a\}}}$ by $\mathcal{F}_{\tilde{a}}$.

The so called *Frankl conjecture*, also named the *union-closed sets conjecture* has been attracting the curiosity of many for a long time (see [5]), mostly due to its apparently simple statement.

Conjecture 1.1 (Frankl conjecture)**.** If a finite family of sets $\mathcal{F} \neq \varnothing, \{\varnothing\}$ is union-closed, then there is an element of its universe that belongs to at least half the sets of the family, i.e.

$$\exists\, a \in U(\mathcal{F}) : |\mathcal{F}_a| \geq \frac{|\mathcal{F}|}{2}.$$

The origin of Frankl conjecture is not completely clear. According to [3] it was well known by the mid-1970s as a "folklore conjecture", but it is usually attributed to Péter Frankl, as he stated it in terms of intersection-closed set families in 1979. The first such attribution seems to have being done by Dwight Duffus, in 1985, in [9].

The problem has been studied from several viewpoints and some interesting formulations have been obtained. For example, the conjecture admits a lattice-theoretical version, which has been proved firstly for modular lattices [1] by Abe and Nakano, and later for lower semimodular lattices by Reinhold [21]; it also admits a graph-theoretical version which is trivially true for non-bipartite graphs and it was proved to hold for the classes of chordal bipartite graphs, subcubic bipartite graphs, bipartite series-parallel graphs and bipartitioned circular interval graphs in [4]; and there is also a very interesting, yet seemingly unfruitful formulation, known as *the Salzborn formulation*, described in [25]. Compression techniques have been attempted, yielding some partial results. They were introduced in this context by Reimer in [20], and later further explored in [22] and [3]. The concept of Frankl Complete families, or FC-families, introduced by Sarvate and Renaud [23], and later formalized by Poonen [19], has also been studied by some ([17] and [24], for example). A more direct approach on the properties a hypothetical counterexample of minimal size was taken by Lo Faro in [10] and [11], with some interesting results.

A very thorough survey on the topic by Bruhn and Schaudt, [5], is suggested to the interested reader, as well as the Master thesis of the first author [7].

Recently, a breakthrough by Gilmer [12] provided the first known constant lower bound on the frequency of the most frequent element in union-closed families. This was later improved, and the current best bound is of 0.38234 [6, 26].

In this paper, we will focus on the study of *normalized* union-closed families. These constitute an interesting subclass of union-closed families, that are, in some sense, the smallest possible separating families for a given universe. They are particularly relevant because the above mentioned Salzborn formulation of Frankl conjecture only refers to normalized union-closed families, as opposed to Frankl conjecture that concerns all union-closed families of sets.

We will study normalized families, proving some of their properties, and detailing their construction. The main result of this paper is a reduction technique that can be applied to any normalized family to produce a smaller one. Denoting by $\mathcal{F} \ominus S$ the set $\{F \setminus S : F \in \mathcal{F}\}$, where $\mathcal{F}$ is a family of sets and $S \subseteq U(\mathcal{F})$, our result can be stated as:

Theorem 4.11. Let $\mathcal{N}$ be a n-normalized family and let M be any minimal non-empty set of $\mathcal{N}$. Then the family $\mathcal{N}' = (\mathcal{N} \setminus \{M\}) \ominus \{a_{\mathcal{N}}\}$ is $(n-1)$-normalized, for some $a_{\mathcal{N}} \in \mathcal{N}$.

Poonen, in [19] made the following refinement of Frankl conjecture:

Conjecture 5.1. Let $\mathcal{F}$ be a union-closed family of sets. Unless $\mathcal{F}$ is a power set, it contains an element that appears in strictly more than half of the sets.

We use the reduction introduced by Theorem 4.11 to prove that this statement can be weakened in the following way.

Conjecture 5.2. Let $\mathcal{F}$ be a union-closed family such that the most frequent element belongs to exactly half the sets in $\mathcal{F}$. Then $\mathcal{F}$ must be a power set.

Note that this conjecture only concerns families *sharply* satisfying the conclusion of Frankl conjecture, apparently making no claim on the original conjecture. However, we show that:

Theorem 5.3. Conjectures 5.1 and 5.2 are equivalent.

Naturally, the non-trivial part is proving that Conjecture 5.2 implies Frankl conjecture.

Finally, using the reduction of the statement of Theorem 4.11, we introduce a reduction technique for arbitrary families, and use it to prove that Frankl conjecture holds for a certain class of families obtained from successively reducing a power set, families that we call descendents of the original fmily.

Theorem 5.5. If a family is a descendent of a power set, then it satisfies Frankl Conjecture.

The paper is organized as follows. In Section 2, we present some preliminary notions and structural results on union-closed families and on Frankl conjecture, and show that the statement of Frankl conjecture can be generalized in a natural way: if true, then one can guarantee the existence of a subset of size k being in at least $\frac{1}{2^k}$ of the sets of a union-closed family. In Section 3, we show the relevance of some generalized topological concepts, namely supratopologies, for the study of union-closed families, introducing some separation axioms and prove that Frankl conjecture can be reduced to families satisfying some of those axioms. In Section 4, we start by presenting the already known relation between union-closed and normalized families in detail, combining the ideas from [25] and [14]. We then prove Theorem 4.11, establishing a natural way to reduce a normalized family to a smaller one, which looks non-trivial when seen in the original, non-normalized, context. Finally, we present some properties of normalized families obtained through this reduction and some of its connections to Frankl conjecture, including the weakening of Poonen conjecture, in Section 5, and we prove that all families descending from power sets satisfy Frankl conjecture.

2 Preliminaries and an equivalent formulation

Let $\mathcal{F}$ be a family of sets. As usual, a set $F \in \mathcal{F}$ is said to be *irreducible* in $\mathcal{F}$ if, for all $G, H \in \mathcal{F}$, we have $F = G \cup H \implies G = F$ or $H = F$. The set of all irreducible non-empty sets in a family $\mathcal{F}$ is denoted by $J(\mathcal{F})$. Given a family $\mathcal{G} \subseteq \mathcal{P}(U)$, we denote by $\langle \mathcal{G} \rangle$ the smallest union-closed family in $\mathcal{P}(U)$ that contains $\mathcal{G}$, which is the family whose elements are all possible finite unions of elements of $\mathcal{G}$, the empty set included, as it is the union of an empty family. When $\mathcal{F} = \langle \mathcal{G} \rangle$, we say that $\mathcal{G}$ is a generating family for $\mathcal{F}$. Of course, $\mathcal{F}$ is generated by its irreducible sets.

The family $\mathcal{F}$ is said to *separating* if, for every two distinct elements $a, b \in U(\mathcal{F})$, there is a set $O \in \mathcal{F}$ such that $|O \cap \{a, b\}| = 1$, i.e. $\mathcal{F}_a \neq \mathcal{F}_b$. A family is called *normalized* if it is a separating union-closed family such that $\varnothing \in \mathcal{F}$ and $|\mathcal{F}| = |U(\mathcal{F})| + 1$. We will call it *$n$-normalized* to mean that $|U(\mathcal{F})| = n$. Setting $[n] = \{1, 2, \ldots, n\}$, a simple example of an n-normalized family is given by the *n-staircase family* $\{\varnothing, [1], \ldots, [n]\}$. Another example is the family $\{\varnothing\} \cup \{[n] \setminus \{a\} : a \in \{2, 3, \ldots, n\}\} \cup \{[n]\}$.

We write $U_{\tilde{a}}$ instead of $U(\mathcal{F}_{\tilde{a}})$ when the family involved is clear, which is thus the set of all elements belonging to some set of $\mathcal{F}$ that does not contain a. The following results show that normalized families are the smallest possible families that are both separating and union-closed.

Lemma 2.1. A family $\mathcal{F}$ is separating if and only if $U_{\tilde{a}} \neq U_{\tilde{b}}$, for all $a \neq b \in U(\mathcal{F})$.

Proof. Suppose $\mathcal{F}$ is separating and let $a \neq b \in U(\mathcal{F})$. Without loss of generality, there is a set $O \in \mathcal{F}$ such that $a \in O$ and $b \notin O$. Then, $a \in O \subseteq U_{\tilde{b}}$ and $a \notin U_{\tilde{a}}$. Hence, $U_{\tilde{a}} \neq U_{\tilde{b}}$.

Conversely, if $\mathcal{F}$ is not separating, there are $a \neq b \in U(\mathcal{F})$ such that $\mathcal{F}_a = \mathcal{F}_b$. Thus, $\mathcal{F}_{\tilde{a}} = \mathcal{F}_{\tilde{b}}$, and therefore $U_{\tilde{a}} = U_{\tilde{b}}$. □

Proposition 2.2. A separating union-closed family of sets $\mathcal{F}$ with an universe with n elements has at least n sets. If, moreover, $\mathcal{F}$ is normalized, then there is $a \in U(\mathcal{F})$ such that $U_{\tilde{a}} = \varnothing$, i.e. a belongs to every non-empty set in $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is union-closed, then $U(\mathcal{F}) \in \mathcal{F}$ and $U_{\tilde{a}} \in \mathcal{F}$, for all $a \in U(\mathcal{F})$ such that $\mathcal{F}_a \neq \mathcal{F}$. Also, from the fact that $\mathcal{F}$ is separating, it follows that there can only be at most one $a \in U(\mathcal{F})$ such that $\mathcal{F}_a = \mathcal{F}$ and so, by the previous proposition, we have that $\mathcal{F}$ has at least n sets, and at least $n + 1$ in case there is no element belonging to all sets.

When $\mathcal{F}$ is normalized, since we require that $\varnothing \in \mathcal{F}$, if there was no element belonging to every non-empty set in $\mathcal{F}$, then there would be at least $n+2$ elements in $\mathcal{F}$, by the argument in the last paragraph. □

We now present a generalization of the concept of separation in union-closed families, which we call *independence*: we can think of it as a form of a weak separation between elements and sets. Independent families will have a relevant role later in this paper.

Definition 2.3. A family $\mathcal{F}$ of sets is called *independent* if, for all $a \in U(\mathcal{F})$ and for all $S \subseteq U(\mathcal{F}) \setminus \{a\}$, one of the following conditions holds:

- there is a set $O \in \mathcal{F}_{\tilde{a}}$ such that $O \cap S \neq \varnothing$;
- there is a set $O \in \mathcal{F}_a$ such that $O \cap S = \varnothing$.

We say that a family is *dependent* if it is not independent.

It is easy to see by the definition, that independent families are in particular separating, by just taking $|S| = 1$. We now present a characterization of independence that will become useful in Section 4.

Lemma 2.4. A family $\mathcal{F}$ of sets is dependent if and only if there exist $a \in U(\mathcal{F})$ and $S \subseteq U(\mathcal{F}) \setminus \{a\}$ such that $\mathcal{F}_a = \bigcup\limits_{b \in S} \mathcal{F}_b$.

Proof. Let $\mathcal{F}$ be a dependent family of sets. Then there exist $a \in U(\mathcal{F})$ and $S \subseteq U(\mathcal{F}) \setminus \{a\}$ such that, given any set $O \in \mathcal{F}$, we have that $a \in O$ if and only if $O \cap S \neq \varnothing$. So, for $O \in \mathcal{F}_a$, we have that $O \cap S \neq \varnothing$, thus we may take some element $b \in O \cap S$ to conclude that $O \in \mathcal{F}_b$. Therefore, $\mathcal{F}_a \subseteq \bigcup_{b \in S} \mathcal{F}_b$. Now, if $O \in \mathcal{F}_b$ for some $b \in S$, that means that $b \in O \cap S$, and so, in particular, $O \cap S \neq \varnothing$, from which it follows that $a \in O$. This shows that $\bigcup_{b \in S} \mathcal{F}_b \subseteq \mathcal{F}_a$.

Conversely, suppose there exists $a \in U(\mathcal{F})$ and $S \subseteq U(\mathcal{F})\backslash\{a\}$ such that $\mathcal{F}_a = \bigcup_{b\in S} \mathcal{F}_b$. Let $O \in \mathcal{F}$. If $a \in O$, then there exists $b \in S$ such that $O \in \mathcal{F}_b$, from which it follows that $O \cap S \neq \emptyset$; if $a \notin O$, then, for all $b \in S$, $O \notin \mathcal{F}_b$, and so $O \cap S = \emptyset$. Hence, $\mathcal{F}$ is dependent. □

Proposition 2.5. To prove Frankl conjecture, it suffices to show it holds for independent families.

Proof. Let $\mathcal{F}$ be a dependent union-closed family of sets. Then, by the previous proposition, there are $a \in U(\mathcal{F})$ and $S \subseteq U(\mathcal{F}) \setminus \{a\}$ such that $\mathcal{F}_a = \bigcup\limits_{b\in S} \mathcal{F}_b$. If we consider the family $\mathcal{F}' = \mathcal{F} \ominus \{a\}$, we have that $|\mathcal{F}'| = |\mathcal{F}|$, because if that was not the case, then there would be sets $O, O \cup \{a\} \in \mathcal{F}$ such that $a \notin O$. But then, $O \cup \{a\} \in \mathcal{F}_a$ implies that there is some $b \in S$ belonging to O, and so, $O \in \mathcal{F}_b \subseteq \mathcal{F}_a$, a contradiction. Hence, if Frankl conjecture holds for $\mathcal{F}'$, then it holds for $\mathcal{F}$. The family $\mathcal{F}'$ might not be independent, but if that is the case we continue this process, which will eventually stop in an independent family, as the cardinality of the universe decreases in each iteration. □

As mentioned in the introduction, some different formulations of the union-closed sets conjecture arose in different branches of mathematics. Out of those different formulations, one that seems very surprising and also, so far, surprisingly unfruitful is the Salzborn formulation that only refers to normalized families, a very restrict subclass of union-closed families.

Conjecture 2.6 (Salzborn formulation)**.** If a finite family of sets $\mathcal{F}$ is normalized, then there is an irreducible set of size at least $\frac{|\mathcal{F}|}{2}$, i.e.

$$\exists\, I \in J(\mathcal{F}) : |I| \geq \frac{|\mathcal{F}|}{2}.$$

Y. Jiang proposed, in [13] (the link is no longer available), the following generalization of Frankl conjecture:

Conjecture 2.7. Let $\mathcal{F}$ be a union-closed family of sets such that $n = |U(\mathcal{F})|$. Then, for any positive integer $k \leq n$, there exists at least one set $S \subseteq U(\mathcal{F})$ of size k that is contained in at least $2^{-k}|\mathcal{F}|$ of the sets in $\mathcal{F}$.

This conjecture is, in principle, not easier to prove, but it might be useful in finding eventual counterexamples to the problem. It turns out that it is, in fact, equivalent to Frankl conjecture. The equivalence is not hard to see, but since we have not found it stated in the literature, we include it here.

Proposition 2.8. Let $\mathcal{F}$ be a union-closed family of sets with $n = |U(\mathcal{F})|$. If the union-closed sets conjecture holds, then for any positive integer $k \leq n$ there are sets $S_k \subseteq U(\mathcal{F})$ such that $|S_k| = k$, $S_k \subseteq S_{k+1}$, and such that S_k is contained in at least $2^{-k}|\mathcal{F}|$ of the sets in $\mathcal{F}$. In particular, in that case, Conjecture 2.7 also holds.

Proof. We proceed by induction on k. The case $k = 1$ is trivial, since in this case Conjecture 2.7 reduces to the union-closed sets conjecture. Now, assume that we have some set $S_k \subseteq U(\mathcal{F})$ such that S_k is contained in at least $2^{-k}|\mathcal{F}|$ of the sets in $\mathcal{F}$ and consider the family $\mathcal{G} = \{F \in \mathcal{F} : S_k \subseteq F\}$. We know that $|\mathcal{G}| \geq 2^{-k}|\mathcal{F}|$, and it is clear that $\mathcal{G}$ is union-closed. Now, take the family $\mathcal{G} \ominus S_k = \{G \setminus S_k : G \in \mathcal{G}\}$. This new family is still union-closed: if $A, B \in \mathcal{G} \ominus S_k$, then $(A \cup S_k) \cup (B \cup S_k) = (A \cup B) \cup S_k \in \mathcal{G}$, and $A \cup B \in \mathcal{G} \ominus S_k$. Also, clearly, $|\mathcal{G} \ominus S_k| = |\mathcal{G}|$. Since we assume the union-closed sets conjecture is valid, we know there exists an element $x \in U(\mathcal{G} \ominus S_k) \subseteq U(\mathcal{F}) \setminus S_k$ such that x is in at least $\frac{|\mathcal{G} \ominus S_k|}{2} = \frac{|\mathcal{G}|}{2} \geq 2^{-(k+1)}|\mathcal{F}|$ sets. Now just take $S_{k+1} = S_k \cup \{x\}$. We have $|S_{k+1}| = k + 1$, and S_{k+1} is contained in at least $2^{-(k+1)}|\mathcal{F}|$ of the sets in $\mathcal{F}$. □

3 Generalized topologies and Frankl conjecture

A set X together with a union-closed family $\mathcal{F} \subseteq \mathcal{P}(X)$ was named a *supratopological space* when $X \in \mathcal{F}$ in [16]; a *generalized topological space* if $\varnothing \in \mathcal{F}$ in [8]; and a *strong generalized topological space*, e.g. in [18], when $\varnothing, X \in \mathcal{F}$. Many topological notions can readily be extended to this more general setting. For example, and this will be relevant later, calling the elements of $\mathcal{F}$ *open sets*, and its complements *closed sets*, the *interior* of a subset $A \subseteq X$ is, as usual, the biggest open set contained in A, i.e. the set $A^\circ = \bigcup\{O \in \mathcal{F} : O \subseteq A\}$, which is open, as $\mathcal{F}$ is union-closed. Note that, trivially, $A^{\circ\circ} = A^\circ$. Similarly, the *closure* of A is the smallest closed set containing A, which is $\bar{A} = \bigcap\{C \text{ closed} : A \subseteq C\} = (X \setminus A)^\circ$.

In this setting, the union-closed sets conjeture has the following reformulation.

Conjecture 1.1 (Frankl conjecture, topological reformulation)**.** If $(X, \mathcal{F})$ is a finite supratopological space, then there is a point that belongs to at least half the open sets.

Note that, in this context, $\mathcal{F}_x$ is the set of all neighborhoods of x, and one can also state Frankl conjecture as: there is a point whose neighborhoods consist of, at least, half of all open sets.

There are several separation axioms that are quite pertinent for our purposes, specially axioms between T_0 and T_1, that we will shortly recall from [2], to which we join the axiom that we introduced above, in Definition 2.3, and we separate as an axiom a condition in the definition of T_{DD} from [2]. Those axioms originated in the context of topological spaces, but they carry over to supratopological spaces without changes. However, there are relations among them that no longer hold, as we will point out. In particular, some do not remain between T_0 and T_1.

To state the axioms, it is convenient to denote the closure of $\{x\}$ by $\bar{x}$, for any given point $x \in X$ of a supratopological space, and we will call the *shadow*[1] of x to the set $\dot{x} = \bar{x} \setminus \{x\}$. It is easy to see that, with our notations, $\bar{x} = \{y \in X : \mathcal{F}_y \subseteq \mathcal{F}_x\} = X \setminus U_{\tilde{x}}$. Also, following [2], given two subsets, A and B, of a supratopological space X, we say that A is *weakly separated* from B, which we denote by $A \longmapsto B$, if there is an open set O of X such that $A \subseteq O$ and $B \cap O = \varnothing$. When dealing with a singular set, we will often write x for $\{x\}$. It is easy to see that $\bar{x} = \{y \in X : y \not\longmapsto x\}$.

In [2], the set $\hat{x} = \{y \in X : x \not\longmapsto y\}$ is called the *kernel* of x, and $\tilde{x} = \hat{x} \setminus \{x\}$ the *shell* of x. It is easy to see that $\hat{x} = \{y \in X : \mathcal{F}_x \subseteq \mathcal{F}_y\} = \{y \in X : x \notin U_{\tilde{y}}\}$, the set of all elements y that *dominate* x, in the language of [5].

We prefer the name "supratopological" because it directly suggests that the family of open sets is union-closed, but, **from now on**, to simplify matters and because it really does not make much difference, we assume that a supratopological space always contains the empty set.

Definition 3.1 (Separation Axioms)**.** A supratopological space $(X, \mathcal{F})$ is

- T_0 if, for any $x \neq y \in X$, either $x \longmapsto y$ oy $y \longmapsto x$, which is equivalent to $\mathcal{F}_x \neq \mathcal{F}_y$.
- $T_{\mathcal{I}}$ if $\mathcal{F}$ is independent, as specified above in Definition 2.3. Of course, $T_{\mathcal{I}} \subseteq T_0$.
- T_{UD} if $\dot{x}$ is a union of disjoint closed sets, for all $x \in X$. This is equivalent to require that, for all $x \in X$, $U_{\tilde{x}} \cup \{x\}$ is a intersection of open sets whose pairwise union is X.
- T_{D} if $\dot{x}$ is closed, for all $x \in X$. This is equivalent to demand that, for all $x \in X$, there is $O \in \mathcal{F}$ such that $O \cup \{x\} \in \mathcal{F}$, which turns out to be equivalent to $U_{\tilde{x}} \cup \{x\} \in \mathcal{F}$. Clearly, $T_{\mathrm{D}} \subseteq T_{\mathrm{UD}}$.
- T_{iD} if, for all $x \neq y \in X$, $\dot{x} \cap \dot{y} = \varnothing$, which is equivalent to $\{x, y\} \cup U_{\tilde{x}} \cup U_{\tilde{y}} = X$.
- $T_{\mathrm{DD}} = T_{\mathrm{D}} \cap T_{\mathrm{iD}}$.
- T_{F} if, for all $x \in X$ and finite $S \subseteq X \setminus \{x\}$, either $x \longmapsto S$ or $S \longmapsto x$.

[1] In [2] this set is denoted by $[x]'$ and called the derived set of x.

- T_{FF} if, for any pair of finite disjoint sets $S_1, S_2 \subseteq X$, either $S_1 \longmapsto S_2$ or $S_2 \longmapsto S_1$. Of course, $T_{\text{FF}} \subseteq T_{\text{F}}$.

- T_{Y} if, for all $x \neq y \in X$, one has $|\bar{x} \cap \bar{y}| \leq 1$. This is equivalent to either $U_{\tilde{x}} \cup U_{\tilde{y}} = X$, or $U_{\tilde{x}} \cup U_{\tilde{y}} = X \setminus \{z\}$ for some $z \in X$, for each pair $x \neq y$.

- T_{YS} if, for all $x \neq y \in X$, one has $\bar{x} \cap \bar{y} \in \{\varnothing, \{x\}, \{y\}\}$. This is equivalent to $U_{\tilde{x}} \cup U_{\tilde{y}} \in \{X, X \setminus \{x\}, X \setminus \{y\}\}$. It is clear that $T_{\text{YS}} \subseteq T_{\text{Y}} \cap T_{\text{iD}}$.

- T_1 if, for all $x \neq y \in X$, one has $\mathcal{F}_x \setminus \mathcal{F}_y \neq \varnothing$. This is equivalent to $\{x\}$ is closed, for all $x \in X$, or $U_{\tilde{x}} = X \setminus \{x\}$. It immediately follows that $T_1 \subseteq T_{\text{YS}}$.

Examples:

- The *indiscrete supratopology* on X, given by $\mathcal{F} = \{\varnothing, X\}$, is not T_0 when $|X| \geq 2$, but it is T_{iD} when $|X| = 2$, and thus $T_{\text{iD}} \nsubseteq T_0$.

- Let us denote by $\binom{X}{k}$ the set of all subsets of the set X with k elements, and by $\binom{X}{\geq k}$ the set of all such subsets with at least k elements, which is a union-closed family. Note that $\langle \binom{X}{k} \rangle = \binom{X}{\geq k} \cup \{\varnothing\}$. The supratopological space given by $\left([4], \langle \binom{[4]}{2} \rangle\right)$ is a finite non-discrete T_1 space, something that cannot exist in the topological case.

- The n-staircase family $\{\varnothing, [1], \ldots, [n]\}$ is T_{D} but not T_1.

- The space $X = [4]$ with the supratopology given by $\mathcal{F} = \langle \{1,2\}, \{3,4\} \rangle$ is T_{iD} but not T_{UD}.

- For $n \geq 3$, the family $\{\varnothing\} \cup \{[n] \setminus \{a\} : a \in \{2, 3, \ldots, n\}\} \cup \{[n]\}$ is T_{UD} but not T_{D}.

- The space $X = [5]$ with the supratopology given by

$$\mathcal{F} = \{\varnothing, \{1,2,3\}, \{1,4,5\}, \{1,2,3,4,5\}\}$$

  is T_{UD} but not T_0.

- The space $X = [3]$ with the supratopology given by $\mathcal{F} = \{\varnothing, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ is T_{YS} but not $T_{\mathcal{I}}$.

- The space $X = [4]$ with the supratopology given by the family

$$\mathcal{F} = \{\varnothing, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$$

  is T_{DD} but not T_{F}.

Remarks:

- The notion of T_0 corresponds exactly what it is called "separating" in the union-closed literature, as we did in the previous section. One can now give a topological proof of Lemma 2.1: when $(X, \mathcal{F})$ is T_0, the map $X \to \mathcal{F}$ given by $x \mapsto X \setminus \bar{x}$ is injective.
- If X is not T_0, then there exist $x \neq y \in X$ such that $\bar{x} = \bar{y}$, and hence $\{x, y\} \subseteq \bar{x} \cap \bar{y}$. This shows that $T_{\mathrm{Y}} \subseteq T_0$.
- We will see below that $T_{\mathrm{D}} \subseteq T_{\mathcal{I}} \subseteq T_0$.
- $T_{\mathrm{DD}} = T_{\mathrm{iD}} \cap T_{\mathrm{D}}$, by definition, and $T_{\mathrm{DD}} = T_{\mathrm{iD}} \cap T_{\mathcal{I}}$, by Proposition 3.6 below.
- $T_1 \nsubseteq T_{\mathrm{FF}}$: the space $X = [4]$ with $\mathcal{F} = \langle \binom{[4]}{3} \rangle$ is T_1 but not T_{FF}.
- $T_{\mathrm{iD}} \cap T_0 \subseteq T_{\mathrm{YS}}$.
- For topological spaces, one has the following relations (see [2]):

$$T_1 \subsetneq T_{\mathrm{DD}} \subsetneq T_{\mathrm{D}} \subsetneq T_{\mathrm{UD}} \subsetneq T_0;$$

$$T_1 \subsetneq T_{\mathrm{FF}} \subsetneq T_{\mathrm{Y}} \subsetneq T_{\mathrm{F}} \subsetneq T_{\mathrm{UD}};$$

$$T_1 \subsetneq T_{\mathrm{DD}} \subsetneq T_{\mathrm{YS}} \subsetneq T_{\mathrm{Y}}.$$

  Some of these do not hold for supratopological spaces, as they depend on the fact that the intersection of open sets is still open, which is not required to hold in a supratopological space. One can see that in this case the relations are as depicted in Figure 1, where all inclusions are straightforward by the definitions involved, except:

  - (1) holds when $\varnothing \notin \mathcal{F}$, which we assumed to be the case in supratopological spaces.
  - (2) and (3) follow easily from the fact that in a T_{F} space all points are either open or closed, which can be proved in an analogous way as Proposition 3.7 below.
  - (4) is the content of Proposition 3.4 below.

  We leave to the reader the verification that the examples given above show that there are no extra line segments in the diagram, as well as to verify that all inclusions are indeed strict.

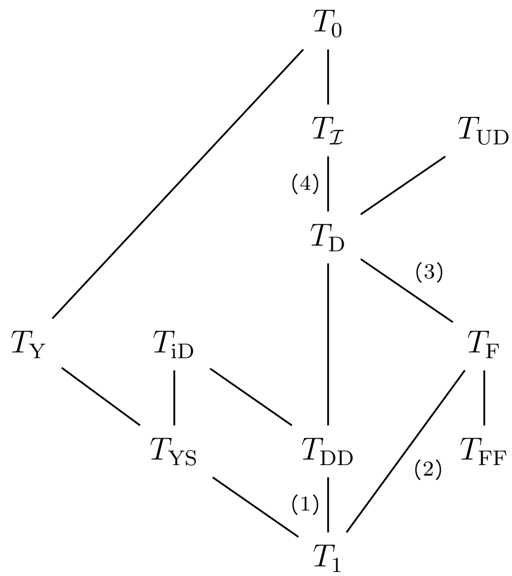

Figure 1: Hasse diagram for the separation axioms in supratopological spaces

It is well known that it suffices to verify Frankl conjecture for T_0 families. In fact, it is easy to see that one can suppose the union-closed family to have, for every $a \in U(\mathcal{F})$, an *a-problematic* set, which is a set $O \in \mathcal{F}$ such that $O \cup \{a\} \in \mathcal{F}$. The paper by Lo Faro [10] can be used as a reference for this result. This means:

Proposition 3.2. To prove Frankl conjecture, it suffices to do it for T_{D} supratopological spaces.

We can go even further and assume $\mathcal{F}$ has at least 3 a-problematic sets for each $a \in U(\mathcal{F})$ (cf. [7, Corollary 3.1.11]):

Theorem 3.3. It suffices to prove Frankl Conjecture for families $\mathcal{F}$ having at least three a-problematic sets for each $a \in U(\mathcal{F})$.

Proof. Suppose that Frankl Conjecture does not hold, and let $\mathcal{F}$ be a minimal counterexample to the conjecture with respect to the number of sets and with minimal universe size among the counterexamples with $|\mathcal{F}|$ sets. Let $a \in U(\mathcal{F})$ and

$$\Pi_a = \{O \in \mathcal{F}_{\tilde{a}} \mid O \cup \{a\} \in \mathcal{F}\}$$

be the subfamily of a-problematic sets of $\mathcal{F}$. By minimality of $\mathcal{F}$, we know that Frankl Conjecture holds for $\mathcal{F}' = \mathcal{F} \ominus \{a\}$, and so, $|\mathcal{F}'| = |\mathcal{F}| - |\Pi_a| < |\mathcal{F}|$ and $\Pi_a \neq \varnothing$. This implies that $U_{\tilde{a}} \in \Pi_a$, as, taking $O \in \Pi_a$, we have that $U_{\tilde{a}} \cup \{a\} = U_{\tilde{a}} \cup (O \cup \{a\}) \in \mathcal{F}$. Let $b \in U(\mathcal{F}')$ be an element belonging to at least $\frac{|\mathcal{F}'|}{2}$ sets of $\mathcal{F}'$, i.e., such that

$$|\mathcal{F}'_b| \geq \frac{|\mathcal{F}| - |\Pi_a|}{2}. \tag{1}$$

As the element b belongs to at least half the sets in $\mathcal{F}'$ and $\mathcal{F}_{\tilde{a}} \subseteq \mathcal{F}'$ is a subfamily such that $|\mathcal{F}_{\tilde{a}}| \geq \frac{|\mathcal{F}'|}{2}$, b must belong to $U_{\tilde{a}} \in \Pi_a$. In particular, $|(\Pi_a)_b| \geq 1$.

Since $\mathcal{F}$ is a counterexample to Frankl Conjecture, we have that

$$|\mathcal{F}'_b| + |(\Pi_a)_b| = |\mathcal{F}_b| < \frac{|\mathcal{F}|}{2}. \tag{2}$$

Combining (1) and (2), we get that

$$\frac{|\mathcal{F}| - |\Pi_a|}{2} + |(\Pi_a)_b| < \frac{|\mathcal{F}|}{2},$$

and so $|\Pi_a| > 2|(\Pi_a)_b| \geq 2$. □

Notice that the existence of an a-problematic set for each $a \in U(\mathcal{F})$ implies separation, since for $a, b \in U(\mathcal{F})$, taking $O, O \cup \{a\} \in \mathcal{F}$, if $b \in O$, then O separates a from b; if not then $O \cup \{a\}$ does. That is, $T_{\mathrm{D}} \subseteq T_0$. The concept of independence lies somewhere between T_0 and T_{D}, as we will now see, and which provides an alternative way to obtain the result in Proposition 2.5.

Proposition 3.4. Every T_{D} union-closed family is independent, i.e. $T_{\mathrm{D}} \subseteq T_{\mathcal{I}}$.

Proof. Suppose that $\mathcal{F}$ is dependent. In view of Lemma 2.4, there are $a \in U(\mathcal{F})$ and $S \subseteq U(\mathcal{F}) \setminus \{a\}$ such that $\mathcal{F}_a = \bigcup_{b \in S} \mathcal{F}_b$. This means that $\mathcal{F}$ has no a-problematic sets, since, if there was an a-problematic set O, then there would be some $b \in S$ belonging to $O \cup \{a\}$, which means that $b \in O$ and that contradicts the fact that $\mathcal{F}_b \subseteq \mathcal{F}_a$. Therefore, $\mathcal{F}$ is not a T_{D} family. □

It is proved in [11, Corollary 2] that, if $\mathcal{F}$ is a minimal counterexample to Frankl Conjecture, then, for all $z \in U(\mathcal{F})$, $1 \leq |\hat{z}| \leq 2$, i.e. $|\check{z}| \leq 1$. This implies that $\mathcal{F}$ satisfies the axiom T_{iD}: given $x, y \in U(\mathcal{F})$, we have that for every element $z \in U(\mathcal{F}) \setminus \{x, y\}$, it cannot happen that both x and y belong to $\hat{z}$, thus $z \in U_{\tilde{x}} \cup U_{\tilde{y}}$, and so $\{x, y\} \cup U_{\tilde{x}} \cup U_{\tilde{y}} = U(\mathcal{F})$. This shows:

Proposition 3.5. It suffices to prove Frankl conjecture for T_{iD} families

As noted above, the condition T_{iD} is not sufficient for a family to be T_{DD}. However, the following holds.

Proposition 3.6. Let X be a set endowed with a supratopology $\mathcal{F}$. If $(X, \mathcal{F})$ satisfies T_{iD} and $T_{\mathcal{I}}$, then $(X, \mathcal{F})$ is T_{DD}.

Proof. Let $(X, \mathcal{F})$ be $T_{\mathcal{I}}$ supratopological space satisfying T_{iD}, and $x \in X$. We want to prove that $U_{\tilde{x}} \cup \{x\} \in \mathcal{F}$. If $U_{\tilde{x}} = X \setminus \{x\}$, we are done since $X \in \mathcal{F}$. If not, then $\dot{x} \neq \varnothing$. Notice that, if $y \in \dot{x}$, then any open set containing y must also contain x, since $y \notin U_{\tilde{x}}$. That is, $\bigcup_{y \in \dot{x}} \mathcal{F}_y \subseteq \mathcal{F}_x$. Since $(X, \mathcal{F})$ is $T_{\mathcal{I}}$, the previous inclusion has to be strict, and hence there must exist some $S \in \mathcal{F}_x$ disjoint from $\dot{x} = X \setminus (U_{\tilde{x}} \cup \{x\})$, which implies that $S \subseteq U_{\tilde{x}} \cup \{x\}$. But then $U_{\tilde{x}} \cup \{x\} = S \cup U_{\tilde{x}} \in \mathcal{F}$. □

We finish this section by showing that the separation axiom T_{FF} is strong enough to imply Frankl conjecture.

Lemma 3.7. Let $(X, \mathcal{F})$ be a supratopological space. Then $(X, \mathcal{F})$ is T_{FF} if and only if every $S \subseteq X$ is either open or closed.

Proof. Assume that $(X, \mathcal{F})$ is T_{FF} and let $S \subseteq X$. Then, there must be $O \in F$ such that $S \subseteq O$ and $(X \setminus S) \cap O = \varnothing$, and so $S = O$, in which case S is open; or $X \setminus S \subseteq O$ and $S \cap O = \varnothing$, and so $O = X \setminus S$, in which case S is closed.

Now, assume that every $S \subseteq X$ is either open or closed, and let $S_1, S_2 \subseteq X$ be disjoint. If S_1 is open, then taking $O = S_1$ in the definition of T_{FF}, we get that $S_1 \subseteq O$ and $O \cap S_2 = \varnothing$. If, on the other hand, S_1 is closed, then taking $O = X \setminus S_1$, we obtain that $S_1 \cap O = \varnothing$ and $S_2 \subseteq O$, so $(X, \mathcal{F})$ is T_{FF}. □

Proposition 3.8. Let $\mathcal{F}$ be a finite T_{FF} union-closed family of sets. Then Frankl Conjecture holds for $\mathcal{F}$.

Proof. It follows from Lemma 3.7 that, for a T_{FF} union-closed family $\mathcal{F} \subseteq \mathcal{P}([n])$, we have that $|\mathcal{F}| \geq 2^{n-1}$, and so Frankl Conjecture holds by [15]. □

4 Dual and normalized families

The notion of *dual family* was introduced in [14], and a similar notion was used in [25] to prove the equivalence between the usual formulation of Frankl conjecture and the Salzborn formulation. The main difference between those two notions is that, in [14], the sets of the notion presented in [25] are replaced by their indexes, and the empty set is not included.

Also, the definition in [14] uses a minimal generating set, while the definition in [25] uses any generating set. We use the notion introduced in [14], with some small variations. We will describe the construction of the dual of a given family, illustrate it with examples, and give some of its basic properties. We then give some structural results on normalized families, and highlight their relation with Frankl conjecture.

Consider an indexed subfamily $\mathcal{H} = \{H_1, \dots, H_s\}$ of distinct non-empty sets in $\mathcal{P}([m])$ with $U(\mathcal{H}) = [m]$. For each $j \in [m]$, set

$$\mathcal{H}^{\iota_j} = \{i \in [s] : j \in H_i\} \in \mathcal{P}([s]),$$

i.e., $\mathcal{H}^{\iota_j}$ is the set of indices of sets in $\mathcal{H}$ to which j belongs, and put

$$\mathcal{H}^{\iota} = \{\mathcal{H}^{\iota_1}, \dots, \mathcal{H}^{\iota_m}\}.$$

Moreover, for any $A \subseteq [m]$, set

$$\mathcal{H}^{\iota_A} = \{i \in [s] : A \cap H_i \neq \varnothing\} = \bigcup_{j \in A} \mathcal{H}^{\iota_j}.$$

Note that $\mathcal{H}^{\iota_{A \cup B}} = \mathcal{H}^{\iota_A} \cup \mathcal{H}^{\iota_B}$. Also, note that

$$\mathcal{H}^{\iota\iota} = \mathcal{H}, \text{ since } j \in \mathcal{H}^{\iota\iota_i} \iff i \in \mathcal{H}^{\iota_j} \iff j \in H_i. \tag{3}$$

The choice of indices for the sets in $\mathcal{H}$ is irrelevant, as a different choice just induces a permutation of elements on the subsets of $\mathcal{P}([m])$, and one simply obtains an isomorphic family.

Now, given any subset $\mathcal{L}$ of $\mathcal{P}([m])$, we define $\mathcal{L}^*$ as the union-closed family generated by $\mathcal{H}^{\iota}$, where $\mathcal{H} = \mathcal{L} \setminus \{\varnothing\}$ is indexed in some way. From what was noted above, $\mathcal{L}^* = \{\mathcal{H}^{\iota_A} : A \subseteq [m]\}$. The special cases $\mathcal{L} = \mathcal{F}$ and $\mathcal{L} = J(\mathcal{F})$ will be particularly relevant. The family $\mathcal{F}^*$ is called the *dual family of* $\mathcal{F}$.

Remark 4.1. Note that $\mathcal{F}^{\iota_A}$ is the set consisting of the indices of the sets in $\mathcal{F}_A$, and thus, in particular, $|\mathcal{F}^{\iota_A}| = |\mathcal{F}_A|$.

It is clear from the definitions that $|U(\mathcal{L}^*)| = |\mathcal{L}| - \varepsilon_{\mathcal{L}}$, , where

$$\varepsilon_{\mathcal{L}} = \begin{cases} 0, & \text{if } \varnothing \notin \mathcal{L}, \\ 1, & \text{otherwise.} \end{cases}$$

and that $\mathcal{L}^*$ is separating.

The next proposition gives a procedure to build examples of normalized families, and, in fact, as we will see below, this procedure yields all normalized families. The result is equivalent to one contained in [25, Lemma 2.2], but the context is a bit different, and so we provide a complete proof which underlies its topological nature.

Proposition 4.2. Let $\mathcal{F}$ be a union-closed family, and $\mathcal{G}$ be a generating subfamily. Then $|\mathcal{G}^*| = |\mathcal{F}| + 1 - \varepsilon_{\mathcal{F}}$. In particular, we have that $\mathcal{F}^*$ is a normalized family.

Proof. Set $U = U(\mathcal{F}) = [m]$, and $\mathcal{G} \setminus \{\varnothing\} = \{H_1, \ldots, H_s\}$. As pointed out above, $\mathcal{G}^* = \{\mathcal{H}^{\iota A} : A \subseteq [m]\}$, and therefore $\mathcal{G}^* = \{\mathcal{H}^{\iota U \setminus A} : A \subseteq [m]\}$. Now, $\mathcal{H}^{\iota U\setminus A} = \mathcal{H}^{\iota U \setminus B}$ is equivalent to saying that

$$\forall i \in [s] \quad (U \setminus A) \cap H_i \neq \varnothing \iff (U \setminus B) \cap H_i \neq \varnothing,$$

which, of course, is the same as

$$\forall i \in [s] \quad (U \setminus A) \cap H_i = \varnothing \iff (U \setminus B) \cap H_i = \varnothing,$$

or

$$\forall i \in [s] \quad H_i \subseteq A \iff H_i \subseteq B.$$

But this is equivalent to $A^\circ = B^\circ$, since $\mathcal{G}$ is a generating family for $\mathcal{F}$. It follows from this that $\mathcal{G}^* = \{\mathcal{H}^{\iota U\setminus A^\circ} : A \subseteq [m]\} = \{\mathcal{H}^{\iota U \setminus O} : O \in \mathcal{F} \cup \{\varnothing\}\}$, and that the elements of this last set are distinct, which proves first the claim. The second one is now very easy to establish. □

Example 4.3. Suppose we want to construct a 6-normalized family. To do so, we need a union-closed family with 7 sets (or only 6, if the the empty set is excluded). Take, for example:

$$\mathcal{F} = \mathcal{P}([3]) \setminus \{\{1\}\} = \{\varnothing, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

One has $\mathcal{F}^{\iota 1} = \{3,4,6\}$, $\mathcal{F}^{\iota 2} = \{1,3,5,6\}$ and $\mathcal{F}^{\iota 3} = \{2,4,5,6\}$. Now, we simply build

$$\begin{aligned}\mathcal{F}^* &= \langle \{3,4,6\}, \{1,3,5,6\}, \{2,4,5,6\} \rangle \\ &= \{\varnothing, \{3,4,6\}, \{1,3,5,6\}, \{2,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\},\end{aligned}$$

which is 6-normalized.

The following technical lemma gives us two properties that will be useful later on.

Lemma 4.4. Let $\mathcal{F}$ be a union-closed family and $\mathcal{G}$ be a generating subfamily. We have the following:

1. if $\mathcal{F}$ is independent, then $J(\mathcal{G}^*) = \mathcal{G}^\iota$, $|J(\mathcal{G}^*)| = |U(\mathcal{F})|$ and $J(\mathcal{F}^*)^* = \mathcal{F}$.
2. if $\mathcal{F}$ is normalized, then $\mathcal{G}^\iota$ is union-closed.

Proof. Let $\mathcal{F}$ be an independent union-closed family and put $U(\mathcal{F}) = [m]$. Clearly, in $\mathcal{G}^*$ only the sets in $\mathcal{G}^\iota$ may be irreducible. Let $\mathcal{G} \setminus \{\varnothing\} = \{H_1, \dots, H_s\}$. If $\mathcal{H}^{\iota_c} = \mathcal{H}^{\iota_a} \cup \mathcal{H}^{\iota_b}$, for some $a, b, c \in [m]$, then, since $\mathcal{G}$ is generating, it would follow that $\mathcal{F}_c = \mathcal{F}_a \cup \mathcal{F}_b$, which contradicts independence by Proposition 2.4. This shows that $J(\mathcal{G}^*) = \mathcal{G}^\iota$.

Also, since $\mathcal{F}$ is independent (in particular, separating), the sets $\mathcal{H}^{\iota_j}$ are all distinct, and so we have that $|J(\mathcal{G}^*)| = |\mathcal{G}^\iota| = m$. Now, from what was proven in the last paragraph, we know that $J(\mathcal{F}^*) = (\mathcal{F} \setminus \{\varnothing\})^\iota$ and so $J(\mathcal{F}^*)^* = ((\mathcal{F} \setminus \{\varnothing\})^\iota)^* = \langle (\mathcal{F} \setminus \{\varnothing\})^{\iota\iota} \rangle = \mathcal{F}$. This completes the proof of the first claim.

Finally, let $\mathcal{F}$ be an n-normalized family. Then, by the previous lemma, $|\mathcal{G}^*| = n + 1 = |U(\mathcal{F})| + 1 = |\mathcal{G}^\iota| + 1$, since $\mathcal{F}$ is separating. This means that $\mathcal{G}^* = \mathcal{G}^\iota \cup \{\varnothing\}$, and thus $\mathcal{G}^\iota$ is union-closed. □

We can use this lemma to show that there is, up to bijection, only one independent normalized family, while by definition all normalized families are separating. This implies that the concept of independence is stronger than the concept of separation, i.e. $T_0 \not\subseteq T_{\mathcal{I}}$.

Proposition 4.5. The only independent n-normalized family is the *staircase* family $\mathcal{N} = \{\varnothing, [1], \dots, [n]\}$, up to bijection.

Proof. It is easy to see that the staircase family is independent for every $n \in \mathbb{N}$. Now let $\mathcal{N}$ be an independent n-normalized family of sets. By the previous lemma, it follows that $J(\mathcal{N}^*)$ is union-closed and all its sets are, of course, irreducible. Let $X, Y \in J(\mathcal{N}^*)$. Then, $X \cup Y \in J(\mathcal{N}^*)$ is an irreducible set, and so either $Y \subseteq X$ or $X \subseteq Y$. Therefore $J(\mathcal{N}^*)$ is a chain. Now, the previous lemma also implies that $\mathcal{N} = J(\mathcal{N}^*)^*$. But it is easily seen that the dual of a chain is still a chain, and that there is only one separating chain of sets with a given universe, up to bijection. □

The next two propositions show that any n-normalized family can be obtained as the dual of an indenpendent family.

Proposition 4.6. Let $\mathcal{N}$ be a normalized family. Then $\mathcal{N} = J(\mathcal{N})^{**}$.

Proof. Let $\mathcal{N}$ be an n-normalized family and set $\mathcal{H} = J(\mathcal{N}) = \{H_1, \dots, H_s\}$. We have $J(\mathcal{N})^* = \langle \mathcal{H}^{\iota_1}, \dots, \mathcal{H}^{\iota_n} \rangle$. From Proposition 4.2 it follows that $|J(\mathcal{N})^*| = |\mathcal{N}| = n+1$, since here $\mathcal{N}$ contains the empty set. Hence $J(\mathcal{N})^* = \{\varnothing, \mathcal{H}^{\iota_1}, \dots, \mathcal{H}^{\iota_n}\}$. But then

$$J(\mathcal{N})^{**} = \langle \mathcal{H}^{\iota\iota_1}, \dots, \mathcal{H}^{\iota\iota_n} \rangle = \langle H_1, \dots, H_n \rangle = \mathcal{N},$$

by (3). This proves the claim. □

Proposition 4.7. If $\mathcal{N}$ be a normalized family of sets, then $J(\mathcal{N})^*$ is independent. It follows that any normalized family is the dual of an independent family.

Proof. In view of the previous Proposition, is enough to show that $\mathcal{L} = J(\mathcal{N})^*$ is independent. Assume this is false, so that, from Lemma 2.4, and using the notations in the previous proof, there would exist $a \in [s]$ and $S \subseteq [s] \setminus \{a\}$ such that

$$\mathcal{L}_a = \bigcup_{b \in S} \mathcal{L}_b. \tag{4}$$

But $\mathcal{H}^{\iota_j} \in \mathcal{L}_a \iff a \in \mathcal{H}^{\iota_j} \iff j \in H_a$, and so one sees that (4) is equivalent to $H_a = \bigcup_{b \in S} H_b$, which contradicts the fact that the H_i are irreducible. ☐

Example 4.8. Let $\mathcal{N}$ be the following 7-normalized family:

$$\mathcal{N} = \{\varnothing, \{1,4,6,7\}, \{2,5,6,7\}, \{3,4,5,6\}, [7] \setminus \{3\}, [7] \setminus \{2\}, [7] \setminus \{1\}, [7]\}.$$

We have that $J(\mathcal{N}) = \{\{1,4,6,7\}, \{2,5,6,7\}, \{3,4,5,6\}\}$. Let us compute $J(\mathcal{N})^{**}$ and see that it coincides with $\mathcal{N}$, as Proposition 4.6 claims. To start with,

$$\begin{aligned} J(\mathcal{N})^* &= \langle \{\{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,2\}\} \rangle \\ &= \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{1,2\}\}. \end{aligned}$$

It follows that $J(\mathcal{N})^{**} = \langle \varnothing, \{1,4,6,7\}, \{2,5,6,7\}, \{3,4,5,6\} \rangle = \mathcal{N}$.

It is now very easy to show that the original form of the union-closed sets conjecture and its Salzborn formulation are equivalent. We state this result in a from that explicitly exhibits the families involved in the equivalence.

Theorem 4.9 (Salzborn Formulation, [25])**.** If $\mathcal{N}$ is a normalized family and the independent family $J(\mathcal{N})^*$ satisfies the union-closed sets conjecture, then $\mathcal{N}$ satisfies the Salzborn formulation of the conjecture. If $\mathcal{F}$ is an independent union-closed family such that $\mathcal{F}^*$ satisfies the Salzborn formulation of the conjecture, then $\mathcal{F} = J(\mathcal{F}^*)^*$ satisfies the union-closed sets conjecture.

Proof. Let $\mathcal{N}$ be an n-normalized family, and $J(\mathcal{N}) = \{I_1, \dots, I_s\}$. We saw in the proof of Proposition 4.6 that $J(\mathcal{N})^* = J(\mathcal{N})^\iota \cup \{\varnothing\}$. The hypothesis that this set satisfies the union-closed sets conjecture entails that there is an element $a \in U(J(\mathcal{N})^*)$ in at least half the sets of $J(\mathcal{N})^*$. Now, by the definition of $J(\mathcal{N})^\iota$, we have that $a \in J(\mathcal{N})^{\iota_j} \iff j \in I_a$. Therefore, if we have a in at least $\frac{n}{2}$ sets of $J(\mathcal{N})^*$, we have $|I_a| \geq \frac{n}{2}$, and so $\mathcal{N}$ satisfies the Salzborn formulation of the union-closed sets conjecture.

To prove the second statement, let $\mathcal{F}$ be an independent union-closed family. We know by Lemma 4.4 that $\mathcal{F} = J(\mathcal{F}^*)^*$. By Proposition 4.2, $\mathcal{F}^*$ is normalized, and so, by Lemma 4.4, $J(\mathcal{F}^*)^\iota$ is union-closed. Therefore, every set in $\mathcal{F} = J(\mathcal{F}^*)^*$ is of the form $J(\mathcal{F}^*)^{\iota_j}$. Now, if $\mathcal{F}^*$ satisfies the Salzborn condition, then, there exists $I \in J(\mathcal{F}^*)$ with $|I| \geq \frac{1}{2}|\mathcal{F}^*|$, and thus $|I| \geq \frac{1}{2}|\mathcal{F}|$, by Proposition 4.2. Let i be the index of I in $J(\mathcal{F}^*)$ used when one constructs $J(\mathcal{F}^*)^* = \mathcal{F}$. Then, since $k \in I \iff i \in J(\mathcal{F}^*)^{\iota_k}$, it follows that i belongs to at least half of the sets in $\mathcal{F}$. □

We saw in Proposition 2.2 that, in a normalized family, there is an element in all of its non-empty sets, which must be unique since the family is, by definition, separating.

Definition 4.10. Given a normalized family $\mathcal{N}$, we will denote by $a_{\mathcal{N}}$ the unique element belonging to all of its non-empty sets.

We are now ready to present one of the main results of this paper, which introduces a reduction process for normalized families.

Theorem 4.11. Let $\mathcal{N}$ be a n-normalized family and let M be any minimal non-empty set of $\mathcal{N}$. Then the family $\mathcal{N}' = (\mathcal{N} \setminus \{M\}) \ominus \{a_{\mathcal{N}}\}$ is $(n-1)$-normalized.

Proof. Note that if $\{a_{\mathcal{N}}\} \in \mathcal{N}$, then $M = \{a_{\mathcal{N}}\}$, and $\mathcal{N}' = \mathcal{N} \ominus \{a_{\mathcal{N}}\}$. We claim that the family $\mathcal{N}'$ is $(n-1)$-normalized. It is clear that $\mathcal{N}'$ is union closed, $\varnothing \in \mathcal{N}'$, and $|\mathcal{N}'| = |\mathcal{N}| - 1$. It remains to show that $|U(\mathcal{N}')| = |U(\mathcal{N})| - 1$ and that $\mathcal{N}'$ is separating.

In case $\{a_{\mathcal{N}}\} \in \mathcal{N}$, it is clear that $|U(\mathcal{N}')| = |U(\mathcal{N})| - 1$. Also, for $x, y \in U(\mathcal{N}')$, if $N \in \mathcal{N}$ is such that $x \in N$, $y \notin N$, then for $N' = N \setminus \{a_{\mathcal{N}}\} \in \mathcal{N}'$ it is still true that $x \in N'$, $y \notin N'$.

Let us now deal with the case $\{a_{\mathcal{N}}\} \notin \mathcal{N}$, which implies $|M| \geq 2$. Clearly $|U(\mathcal{N}')| \leq n-1$. Now, $|U(\mathcal{N}')| < n-1$ would imply the existence of an element in M not in any other set of $\mathcal{N} \setminus \{M\}$, thus $M = U(\mathcal{N})$, and $\mathcal{N} = \{\varnothing, M\}$, but then $|M| = 1$, a contradiction. Therefore, $|U(\mathcal{N}')| = n-1$.

Finally, let $x, y \in U(\mathcal{N}') = U(\mathcal{N}) \setminus \{a_{\mathcal{N}}\}$. Since $\mathcal{N}$ is separating, there exists $N \in \mathcal{N}$ such that $|N \cap \{x, y\}| = 1$. It is, of course, still true that $N' = N \setminus \{a_{\mathcal{N}}\}$ separates x from y. If $N' \neq M \setminus \{a_{\mathcal{N}}\}$, we are done. It remains to consider the case where M is the only set that separates x from y in $\mathcal{N}$. In this case, there cannot be any set $\varnothing \neq L \in \mathcal{N}$ such that $\{x, y\} \nsubseteq L$, otherwise $L \cup M$ would also separate x and y, and by the minimality of M, $M \cup L \neq M$. So, except for M, all other sets of $\mathcal{N}$ must contain $\{x, y\}$. Without loss of generality, we may assume that $x \in M$. It follows that x belongs to all sets of $\mathcal{N}$, contradicting the uniqueness of the element $a_{\mathcal{N}}$. □

Corollary 4.12. For every normalized family $\mathcal{N}$ there are distinct elements $a_i \in U(\mathcal{N})$ with frequency at least i, for all $1 \le i \le n$.

Proof. Follows directly from the Proposition 2.2 by applying Theorem 4.11 repeatedly. □

Notice that in the reduction from $\mathcal{N}$ to $\mathcal{N}'$ we remove a minimal set. To preserve closure under union we can only remove irreducible elements of the family. While it may be possible to weaken the condition of minimality, it can not be replaced with irreducibility, in general, as the next example shows.

Example 4.13. Take the normalized family

$$\mathcal{N} = \{\varnothing, \{5,7\}, \{3,6,7\}, \{3,5,6,7\}, \{2,4,5,6,7\}, [7] \setminus \{2\}, [7] \setminus \{1\}, [7]\}.$$

Then using the irreducible set $M = [7] \setminus \{2\}$ one gets

$$\mathcal{N}' = \{\varnothing, \{5\}, \{3,6\}, \{3,5,6\}, \{2,4,5,6\}, [6] \setminus \{1\}, [6]\}$$

which is not separating, as no set separates 2 from 4.

Given an n-normalized family $\mathcal{N}$, we can decompose it as $\mathcal{N} = \mathcal{S}_{\lambda_0} \cup \mathcal{S}_{\lambda_1} \cup \ldots \cup \mathcal{S}_{\lambda_s}$, with $\lambda_0 < \lambda_1 < \cdots < \lambda_s$, where $\mathcal{S}_\ell$ is the set of all subsets of $\mathcal{N}$ with ℓ elements. We set $k_i = |\mathcal{S}_{\lambda_i}|$. Since $\varnothing \in \mathcal{N}$, one has that $\lambda_0 = 0$ and $k_0 = 1$. Note that $|\mathcal{S}_1| \le 1$. We have the following corollary by applying Proposition 2.2 to all possible reductions.

Corollary 4.14. Let $\mathcal{N}$ be an n-normalized family. Then, using the notations such defined, there are k_i elements with frequency at least $n - \sum_{j=0}^{i-1} k_j$, for all $0 \le i \le s$.

Proof. Let M_1 and M_2 be two minimal sets of $\mathcal{N}$, and set $\mathcal{N}_i' = (\mathcal{N} \setminus \{M_i\}) \ominus \{a_{\mathcal{N}}\}$, for $i = 1,2$. If $a_{\mathcal{N}_1'} = a_{\mathcal{N}_2'}$, then this element, which is distinct from $a_{\mathcal{N}}$, would be in all sets of $\mathcal{N}$, contradicting the uniqueness of $a_{\mathcal{N}}$. The claim follows easily from this, by reducing $\mathcal{N}$ by successively removing sets of minimal length. □

Using the reduction process introduced in Theorem 4.11 for normalized families, one can introduce a reduction process for arbitrary union-closed familes as follows. Given such a family $\mathcal{F}$, we will call $\mathcal{F}_\downarrow = J((\mathcal{F}^*)')^*$ a *child* of $\mathcal{F}$. It depends on the minimal set chosen to be removed from $\mathcal{F}^*$ to form $(\mathcal{F}^*)'$, and different choices may lead to non-isomorphic families, but we do not include it in the notation, which would become a bit too heavy. However, please keep in mind that there may several distinct children of a family, and $\mathcal{F}_\downarrow$ just denotes one of them. Naturally, $\mathcal{F}$ is called a parent of $\mathcal{F}_\downarrow$. Children of the same parent are referred as *siblings*, and a family obtained by successive reductions from a given family of sets is called a *descendent* of that family.

Note that it follows from the results seen above that $|\mathcal{F}_{\downarrow}| = |\mathcal{F}| - 1$, when $\mathcal{F}$ contains the empty set, which will assume from now on. As usual, we define, $\mathcal{F}_{\downarrow\downarrow} = (\mathcal{F}_{\downarrow})_{\downarrow}$, etc., and $\mathcal{F}_{\downarrow k} = (\mathcal{F}_{\downarrow(k-1)})_{\downarrow}$, which again may depend on various choices of minimal sets, not conveyed by the notation. One has $|\mathcal{F}_{\downarrow k}| = |\mathcal{F}| - k$. The families $\mathcal{F}_{\downarrow k}$, for $k \in \mathbb{N}$, are the *descendents* of the family $\mathcal{F}$. It follows from Proposition 4.7 that, for any family, all its descendents are independent.

The relation between the family $\mathcal{F}$ and a family $\mathcal{F}_{\downarrow}$ seems rather misterious, in the sense that they may have different universes, and even when the universe is the same the relation between them does not seem to be evident.

Example 4.15. Consider again the union-closed family $\mathcal{F}$ of Example 4.3,

$$\mathcal{F} = \{\varnothing, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

There is only one minimal set in $\mathcal{F}^*$, namely $\{3,4,6\}$. It yields:

$$(\mathcal{F}^*)' = \{\varnothing, \{1,3,5\}, \{2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\{1,2,3,4,5\}\}.$$

Then $J((\mathcal{F}^*)') = \{\{1,3,5\}, \{2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}$, and thus

$$\mathcal{F}_{\downarrow} = J((\mathcal{F}^*)')^* = \{\varnothing, \{1,3\}, \{2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\},$$

which looks quite different from the original family.

We can picture the two parallel reduction processes in the following commutative diagram (the left dashed up arrow commutes with the others when $\mathcal{F}$ is an independent family):

$$\begin{array}{ccccccccc}
 & & J((\mathcal{F}^*)')^* & & & & & & \\
 & & \| & & & & & & \\
\mathcal{F} & \xrightarrow{\ \square_{\downarrow}\ } & \mathcal{F}_{\downarrow} & \longrightarrow & \mathcal{F}_{\downarrow\downarrow} & \dashrightarrow & \mathcal{F}_{\downarrow k} & \dashrightarrow & \\
{\scriptstyle \uparrow}\!\!\downarrow {\scriptstyle \square^*} & & {\scriptstyle \square^* \circ J}\ \uparrow\!\downarrow & & \uparrow\!\downarrow & & \uparrow\!\downarrow & & \\
\mathcal{F}^* & \xrightarrow{\ \square'\ } & (\mathcal{F}^*)' & \longrightarrow & (\mathcal{F}^*)'' & \dashrightarrow & (\mathcal{F}^*)^{(k)} & \dashrightarrow & \\
 & & \| & & & & & & \\
 & & (\mathcal{F}_{\downarrow})^* & & & & & &
\end{array} \tag{5}$$

where the down arrows in the middle are given by the "taking the dual" operator, $\square^*$; the right arrows on the second row are given by the reduction operator $\square'$; the right arrows on the first row by the reduction operator $\square_\downarrow = \square^* \circ J \circ \square' \circ \square^*$; and the up arrows in the middle by the $\square^* \circ J$ operator. Note that, by Theorem 4.11 and Proposition 4.6, we have that $(\mathcal{F}_{\downarrow k})^* = (\mathcal{F}^*)^{(k)}$ (i.e. the upper arrow followed by the down arrow is the identity operator, Id, or $\square^* \circ \square^* \circ J = Id$, for normalized families). From Lemma 4.4 it follows that, for independent families, $\square^* \circ J \circ \square^* = Id$, i.e. the down arrow followed by the up arrow is the identity, for independent families.

Remark 4.16. If $\mathcal{F}_{\downarrow k}$ satisfies Frankl conjecture and has an odd number of sets, then $\mathcal{F}_{\downarrow (k+1)}$ also satisfies the conjecture. If $\mathcal{F}_{\downarrow k}$ as an even number of sets and *strictly* satisfies Frankl conjecture, meaning that there is an element in strictly more than half the sets, then $\mathcal{F}_{\downarrow (k+1)}$ also strictly satisfies the conjecture.

Remark 4.17. Given a normalized family $\mathcal{N}$, the irreducibles of $\mathcal{N}' = (\mathcal{N} \setminus \{M\}) \ominus \{a_{\mathcal{N}}\}$ that do not belong to set $J(\mathcal{N}) \ominus \{a_{\mathcal{N}}\}$ have cardinality bigger that $|M|$.

5 Refinement of a conjecture of Poonen and descendents of power sets

It is easy to see that, for any given n-normalized family $\mathcal{N}$, there is always at least one family $\mathcal{M}$ such that $\mathcal{N} = \mathcal{M}'$, namely the family $\mathcal{M} = \{N \cup \{n+1\} : N \in \mathcal{N}\} \cup \{\varnothing\}$. Note that $\mathcal{M}$ is also normalized. Now, if $\mathcal{F}$ is an independent family, then take $\mathcal{M}$ such that $\mathcal{M}' = \mathcal{F}^*$, and $\mathcal{T} = J(\mathcal{M})^*$, which is an independent family as seen in the proof of Proposition 4.7. Then $\mathcal{T}^* = \mathcal{M}$, by Proposition 4.6, and thus $\mathcal{T}_\downarrow = J((\mathcal{T}^*)')^* = J(M')^* = J(\mathcal{F}^*)^* = \mathcal{F}$, by Lemma 4.4. This shows that, given any independent family $\mathcal{F}$ there is always a family $\mathcal{T}$ such that $\mathcal{F} = \mathcal{T}_\downarrow$. We will refer to this family $\mathcal{T}$ as *the trivial parent* of $\mathcal{F}$. In other words, the operator $\square'$ is surjective on normalized families, while the operator $\square_\downarrow$ is surjective on independent families. It is clear that if an idependent family satisfies the Frankl conjecture, so does its trivial parent.

Poonen conjectured in [19] that every union-closed family is either a power set or has an element in strictly more than half the sets.

Conjecture 5.1. Let $\mathcal{F}$ be a union-closed family of sets. Unless $\mathcal{F}$ is a power set, it contains an element that appears in strictly more than half of the sets

We propose the following seemingly weaker version of Poonen conjecture.

Conjecture 5.2. Let $\mathcal{F}$ be a union-closed family such that the most frequent element belongs to exactly half the sets in $\mathcal{F}$. Then $\mathcal{F}$ must be a power set.

We can now prove that these two conjectures are in fact equivalent. The advantage of this new statement is that it only concerns families *sharply* satisfying Frankl conjecture, apparently making no claim on the original conjecture: naturally, the non-trivial part is proving that Conjecture 5.2 implies Frankl conjecture.

Theorem 5.3. Conjectures 5.1 and 5.2 are equivalent.

Proof. Clearly Conjecture 5.1 implies Conjecture 5.2. We will prove that Conjecture 5.2 implies Conjecture 1.1. This is enough since Conjecture 5.2 together with Conjecture 1.1 implies Conjecture 5.1. Assume that Conjecture 5.2 holds while Frankl conjecture does not. By Theorem 4.9 there would be a family $\mathcal{F}$ such that $\mathcal{F}^*$ is an n-normalized counterexample to the Salzborn formulation of the conjecture. Put $\lambda = \max\{|I| : I \in J(\mathcal{F})\}$, and choose a set $I \in J(\mathcal{F})$ such that $|I| = \lambda$. Obviously, $|I| < \frac{n+1}{2}$. If we consider $n - 2\lambda + 1$ successive trivial parents of $\mathcal{F}$ we obtain a family $\mathcal{T}$ such that $T = I \cup \{n+1, \dots, 2n-2\lambda+1\}$ is a maximal element of $J(\mathcal{T})$ and $\frac{|T|}{|\mathcal{T}|} = \frac{n-\lambda+1}{2n-2\lambda+2} = \frac{1}{2}$. By Proposition 4.6, $\mathcal{T} = J(\mathcal{T})^{**}$. But, as seen in Proposition 4.7, $J(\mathcal{T})^*$ is an independent family, and therefore, by Lemma 4.4 applied to $\mathcal{F} = \mathcal{G} = J(\mathcal{T})^*$, we have that $T \in J(\mathcal{T}) = J(J(\mathcal{T}^*)^*) = (J(\mathcal{T})^*)^{\iota}$. Hence, the largest set $(J(\mathcal{T})^*)^{\iota_j}$, for $j \in U(J(\mathcal{T})^*)$, is T with cardinal $n - \lambda + 1$. Hence, the most frequent element in $J(\mathcal{T})^*$ has frequency $n - \lambda + 1$. Using Proposition 4.2, we see that $|J(\mathcal{T})^*| = |(J(\mathcal{T})^*)^*| = |\mathcal{T}| = 2n - 2\lambda + 2$. By the hypothesis, we deduce that $J(\mathcal{T})^*$ is a power set, but that is absurd since, by construction, $\mathcal{T} = (J(\mathcal{T})^*)^*$ has a singleton. ☐

We now prove Frankl conjecture for families that are descendents of power set families. To do so, we start with a technical lemma.

Lemma 5.4. Let $n \geq 6$ be an integer and $2 \leq k \leq \frac{n}{2}$. Then

$$2^{k+1} - 1 \leq \sum_{s=0}^{k-1} \binom{n}{s}.$$

Proof. For $k = 2$ we have that $2^{k+1} - 1 = 7$ and $\sum_{i=0}^{k-1} \binom{n}{i} = n + 1 \geq 7$. For $k = 3$, we have that $2^{k+1} - 1 = 15$ and $\sum_{s=0}^{k-1} \binom{n}{s} = \frac{n^2+n+2}{2}$, which is greater than 15 if $n \geq 6$.

Now assume that $k \geq 4$. Since the sum of each row of Pascal's triangle is half of the sum of the next row, and the triangle is symmetrical, we have that

$$2^{k+1} - 1 \leq 2^{k+1} = \sum_{s=0}^{k+1} \binom{k+1}{s} \leq \sum_{s=0}^{\lfloor \frac{k+2}{2} \rfloor} \binom{k+2}{s} \leq \sum_{s=0}^{k-1} \binom{k+2}{s} \leq \sum_{s=0}^{k-1} \binom{n}{s}.$$

☐

Theorem 5.5. If a family $\mathcal{F}$ is a descendent of a power set, then it satisfies Frankl Conjecture.

Proof. We may assume that $n \geq 6$ by [11, Theorem 10].

Suppose $\mathcal{F} = \mathcal{P}([n])$, for some n, and build its correspondent normalized family $\mathcal{N} = \mathcal{F}^*$. Then we have $|\mathcal{N}| = 2^n$, $|U(\mathcal{N})| = 2^n - 1$, $|J(\mathcal{N})| = n$, and $|I| = 2^{n-1}$ for every $I \in J(\mathcal{N})$. In fact, $\mathcal{N}$ has $\binom{n}{k}$ distinct sets of size $2^n - 2^k$, for every $k = 0, 1, \ldots, n$, which correspond to the sets containing the indices of the sets in $\mathcal{F}_F$, for $F \in \mathcal{F}$ with $|F| = k$. Indeed, it is easy to see that $\mathcal{F}_F \neq \mathcal{F}_G$ whenever $F \neq G$, and that for every $F \in \mathcal{F}$ with $|F| = k$ one has

$$|\mathcal{F}_F| = 2^{n-k} \sum_{i=1}^{k} \binom{k}{i} = 2^{n-k}(2^k - 1) = 2^n - 2^{n-k}.$$

We will prove that the normalized families obtained in successive reductions satisfy the Salzborn formulation of the conjecture. This suffices in view of Propositions 4.6 and 4.7, and Theorem 4.9.

Consider the subfamilies $\mathcal{N}_k \subseteq \mathcal{N}$ defined by $\mathcal{N}_k = \{N \in \mathcal{N} : |N| = 2^n - 2^{n-k}\}$. With this notation, we have that $J(\mathcal{N}) = \mathcal{N}_1$. Let $\mathcal{N}_k^{(i)}$ the subfamily obtained by the sets descending from the sets in $\mathcal{N}_k$ after i $\square'$ reductions. The sets in $\mathcal{N}_k^{(i)}$ have $2^n - 2^{n-k} - i$ elements, and, clearly, while doing the sucessive reductions, when the last set descending from a set in $\mathcal{N}_k$ is removed, then all sets descending from a set in $\mathcal{N}_{k+1}$ are irreducible, as they are then minimal.

We apply the reduction process by removing all sets descending from $\mathcal{N}_1$, then all sets descending from $\mathcal{N}_2$, and so on.

Put $J(\mathcal{N}) = \{\varnothing, I_1, \ldots, I_n\}$, where I_i is a set containing the indexes of sets in $\mathcal{F}$ having the element $i \in U(\mathcal{F})$. Assume, without loss of generality, that the removed minimal set is I_1. When we do so, $n - 1$ sets of $\mathcal{N}_2'$, with size $2^n - 2^{n-2} - 1$, become irreducible, namely the sets coming from the sets $I_1 \cup I_i$, for $2 \leq i \leq n$. This happens because $\{1, i\} \in \mathcal{F}$, which implies that the only irreducible sets of $\mathcal{N}$ containing the index of $\{1, i\}$ are I_i and I_1, and hence $I_1 \cup I_i$ cannot be written as union of sets not involving I_1. At this instance, we thus have irreducible sets of size $2^n - 2^{n-2} - 1$. When we do the ℓ-th reduction, we must have irreducible sets with at least $2^n - 2^{n-2} - \ell$ elements, because every set that is then irreducible belongs to $\mathcal{N}_k^{(\ell)}$ for some $k \geq 2$. Since

$$2^n - 2^{n-2} - \ell \geq \frac{2^n - \ell}{2} \iff \ell \leq 2^{n-1},$$

we have the conjecture verified up until $\mathcal{N}^{(2^{n-1})}$.

After doing 2^{n-1} reductions, our smallest irreducible elements are sets descending from $\mathcal{N}_{\frac{n+1}{2}}$, if n is odd (since the 2^{n-1} removed sets are precisely all sets from $\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_{\frac{n-1}{2}}$), or sets descending from $\mathcal{N}_{\frac{n}{2}}$, if n is even (since the 2^{n-1} removed sets are precisely all sets from $\mathcal{N}_1 \cup \cdots \cup \mathcal{N}_{\frac{n}{2}-1}$ together with half the sets from $\mathcal{N}_{\frac{n}{2}}$). Hence, it suffices to prove that, when removing sets coming from $\mathcal{N}_k$ with $k \geq \frac{n}{2}$, we always have irreducible sets of size greater than half the respective universe.

Let $k \geq \frac{n}{2}$. When we remove the i-th set coming from $\mathcal{N}_k$ (this corresponds to the $\left(\sum_{r=1}^{k-1}\binom{n}{r} + i\right)$-th reduction in total), the size of the remaining elements, if any, coming from $\mathcal{N}_k$ is $2^n - 2^{n-k} - \sum_{r=1}^{k-1}\binom{n}{r} - i$. If there are no remaining sets coming from $\mathcal{N}_k$, then the sets coming from $\mathcal{N}_{k+1}$ are now irreducible, and larger. The total number of sets in $\mathcal{N}$ is $2^n - \sum_{r=1}^{k-1}\binom{n}{r} - i$. We have that

$$2^n - 2^{n-k} - \sum_{r=1}^{k-1}\binom{n}{r} - i \geq \frac{2^n - \sum_{r=1}^{k-1}\binom{n}{r} - i}{2} \iff 2^{n-1} - 2^{n-k} - \frac{\sum_{r=1}^{k-1}\binom{n}{r}}{2} \geq \frac{i}{2}$$
$$\iff i \leq 2^n - 2^{n-k+1} - \sum_{r=1}^{k-1}\binom{n}{r}. \tag{6}$$

We know that $i \leq \binom{n}{k}$, since that is total number of sets in $\mathcal{N}_k$, so it suffices to prove that

$$\binom{n}{k} \leq 2^n - 2^{n-k+1} - \sum_{r=1}^{k-1}\binom{n}{r}.$$

We have that

$$\begin{aligned}\binom{n}{k} \leq 2^n - 2^{n-k+1} - \sum_{r=1}^{k-1}\binom{n}{r} &\iff 2^{n-k+1} \leq 2^n - \sum_{r=1}^{k}\binom{n}{r} \\ &\iff 2^{n-k+1} - 1 \leq 2^n - \sum_{r=0}^{k}\binom{n}{r} \\ &\iff 2^{n-k+1} - 1 \leq \sum_{r=k+1}^{n}\binom{n}{r} \\ &\iff 2^{n-k+1} - 1 \leq \sum_{r=0}^{n-k-1}\binom{n}{r}.\end{aligned}$$

If $k \leq n-2$, this follows from Lemma 5.4. The only case missing is the case $k = n-1$. Replacing k by $n-1$ in (6), we get that $i \leq n-2$, so we can remove the first $n-2$ sets of $\mathcal{N}_{n-1}$. Since $\mathcal{N}_{n-1}$ has n sets, when one of the two last sets is removed, then the universe (of the set in $\mathcal{N}_n$) becomes irreducible and the conjecture is satisfied. □

6 Future work

There are some things that we believe might be interesting to do using the construction of families using the reductions introduced in this paper. The first big question would be classifying the families which descend from power sets. It would also be interesting to uncover some relations between relatives. For example, could it be proved that if a descendent from a certain family satisfies Frankl Conjecture, then all its siblings do? If all children from a certain parent satisfy the conjecture, then so does the parent? If one parent satisfies the conjecture, then does every parent satisfy it too? If every parent of a famliy satisfies the conjecture, then so does that family?

Also, beyond the results presented in this paper, is it possible to further reduce the space of families for which it suffices to prove the conjecture? In particular, can it be reduced the T_1 families? Or to some class of supratopological spaces smaller than T_{D} or T_{iD} spaces, e.g. T_{DD} spaces? Since we have proven that the conjecture holds for T_{FF} spaces, and it is enough to prove it for T_{D} spaces, what can it be said about T_{F} spaces?

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