

# SOME CONNECTIONS BETWEEN THE ARITHMETIC AND THE GEOMETRY OF LIPSCHITZ INTEGERS

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ABSTRACT. Some relationships between the arithmetic and the geometry of Lipschitz and Hurwitz integers are presented. In particular, it is shown that the (ternary) vector product of a Lipschitz integer  $\alpha$  with two other Lipschitz integers, both orthogonal to  $\alpha$ , is a left and also a right multiple of  $\alpha$ , and that the vector product of two left multiples of  $\alpha$  with any other Lipschitz integer is still a left multiple of  $\alpha$ . We also provide new arithmetical proofs for some old results of Gordon Pall, and raise a geometric problem on the location of some integral quaternions that is related to the factorization of some integers.

## 1. INTRODUCTION

The arithmetical properties of integral quaternions have been studied since Lipschitz used them in 1886, in a paper on the real automorphs of the form corresponding to the sum of three squares [Lip86]. Ten years later, Hurwitz showed [Hur96] that the integral quaternions are contained in a slightly bigger ring that is arithmetically more interesting, the ring of the now called Hurwitz integers, and expanded their study in the 1919 monograph [Hur19]. In 1940, Gordon Pall published an interesting paper on quaternion arithmetic [Pal40], some results of which motivated the present work. The arithmetic of quaternions continues to be investigated in more recent papers, namely [CP12, AA<sup>+</sup>13, CK15, FG<sup>+</sup>16], and recently provided the tools to solve an open conjecture on a refinement of Lagrange's four squares theorem [Mac21a, Mac21b].

After describing, in the next section, the genesis behind the research that led to the results contained in this paper, in section 3 we summarize some of the basic facts on quaternions, we recall the unique factorization theorem for Hurwitz integers, and a result of Gordon Pall that offers some additional information on factorizations, for which we provide a new, entirely arithmetic proof. In section 4 we collect some results on relations between some

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arithmetical properties of integral quaternions and orthogonality (in  $\mathbb{R}^4$ ). In particular, we prove a simple result, Theorem 4.4, that yields some, albeit remote and tantalizing, hope of an integer factorization method that uses quaternions. Finally, in section 5, we prove some divisibility results having to do with the (triple) vector product. Namely, we show that the vector product of an integral quaternion with two other that are orthogonal to it is both a left and a right multiple of that quaternion, and that the vector product of two left (resp. right) multiples of an integral quaternion  $\alpha$  with any other integral quaternion is also a left (right) multiple of  $\alpha$ . We end the paper with a question naturally raised by this last result.

## 2. MOTIVATION

Frénicle de Bessy seems to have been the first to notice that one can obtain a factorization of an integer  $n$  from two different decompositions of  $n$  as a sum of two squares ([Dic92], vol. I, cap. XIV, p. 360). This amounts to the fact that a decomposition  $n = a^2 + b^2$  gives a factorization  $n = (a + bi)(a - bi)$  in  $\mathbb{Z}[i]$ , and if one has another decomposition  $n = c^2 + d^2$ , then, using the Euclidean algorithm in  $\mathbb{Z}[i]$ , one can compute the greatest common divisor of  $a + bi$  and  $c + di$ , whose norm yields a non-trivial factor of  $n$ .

As Bachet de Méziriac conjectured and Lagrange proved, every number is a sum of four squares ([Dic92], vol. II, cap. VIII, p. 275). Now, while there is no known fast algorithm to decompose a number as a sum of two squares, there is a very efficient probabilistic algorithm, due to Rabin and Shallit [RS86], to express a number as a sum of four squares. Such a decomposition of an integer  $n$  yields a factorization  $n = \alpha\bar{\alpha}$  in the ring of Hurwitz integers. Since this ring is both a left and a right Euclidean domain, it is natural to wonder if two distinct decompositions of a number as a sum of four squares could yield a factorization of that number in a manner analogous to what happens in  $\mathbb{Z}[i]$ .

However, if one has two essentially distinct factorizations of  $n$ ,  $n = \alpha\bar{\alpha} = \beta\bar{\beta}$ , it is not always the case that  $\alpha$  and  $\beta$  will have a non-trivial left or right greatest common divisor. In fact, for a number that is a product of two odd primes,  $n = pq$ , only a small (for  $p$  and  $q$  big) fraction, precisely  $\frac{p+q+2}{(p+1)(q+1)}$ , of all possible pairs  $\alpha, \beta$  will have a (left or right) greatest common divisor whose norm is neither 1 nor  $n$ . But, in [Pal40], Gordon Pall proves a series of interesting results (namely, Theorems 6 and 7), which imply, in particular, that given two quaternions  $\alpha$  and  $\beta$  with integral coprime

coordinates, if they are orthogonal and have the same norm, then they either have the same right divisors, or the same left divisors, or both. This suggests looking for orthogonal decompositions of an integer as a sum of four squares, i.e. orthogonal integral quaternions whose norm is that integer. If one could find some efficient way to find such decompositions, one would hope to get an interesting factorization algorithm.

It was this line of thought that made us study ways of constructing quaternions that are orthogonal to a given quaternion, namely using the ternary vector product, and that led to the discovery of the main results here presented, namely Theorems 5.3 and 5.5. Although these results are negative for the purposes mentioned above, we believe that they are interesting in their own way, showing some intimate connections between geometry and arithmetic in the realm of quaternions.

### 3. QUATERNIONS, LIPSCHITZ AND HURWITZ INTEGERS

We start by recalling that the quaternion ring  $\mathbb{H}$  is the division ring consisting of the additive group  $\mathbb{R}^4$  endowed with the only multiplication (so one gets a ring structure) determined by choosing  $\mathbf{e}_1 = 1$ , the multiplicative unit, and by the relations:

$$\mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_4^2 = \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_4 = -\mathbf{e}_1 = -1,$$

where  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$  is the canonical basis of  $\mathbb{R}^4$ . Usually, in this context, one denotes the elements of this basis by  $1, i, j, k$ , respectively. One can easily check that, then:

$$\begin{aligned} (3.1) \quad (u_0 + u_1 i + u_2 j + u_3 k)(v_0 + v_1 i + v_2 j + v_3 k) &= \\ &= (u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3) + (u_0 v_1 + u_1 v_0 + u_2 v_3 - u_3 v_2) i \\ &+ (u_0 v_2 - u_1 v_3 + u_2 v_0 + u_3 v_1) j + (u_0 v_3 + u_1 v_2 - u_2 v_1 + u_3 v_0) k. \end{aligned}$$

Given a quaternion  $u = a + bi + cj + dk$ , its *conjugate* is defined by  $\bar{u} = a - bi - cj - dk$ , and its *norm* is  $N(u) = u\bar{u}$ . We set  $\Re(u) = a$ , the *real* part of  $u$ , and  $\Im(u) = bi + cj + dk$ , the *imaginary* or *vector* part of  $u$ .

The quaternions with integral coordinates are called *Lipschitz integers*, and they form a subring of  $\mathbb{H}$  that we will denote by  $\mathcal{L}$ . This is almost a left (and right) Euclidean ring for the norm, in the sense that for any  $\alpha, \beta \in \mathcal{L}$  one can find  $q, r \in \mathcal{L}$  such that  $\alpha = \beta q + r$  and  $N(r) \leq N(\beta)$ , but a strict inequality cannot always be guaranteed (and the same for right division). However, one needs only to slightly enlarge  $\mathcal{L}$  by adding the quaternions whose coordinates are all halves of odd numbers to obtain a (left and right) Euclidean ring. This yields the set  $\mathcal{H} = \mathcal{L} \cup (\omega + \mathcal{L})$ , with  $\omega = \frac{1}{2}(1+i+j+k)$ ,

whose elements are called *Hurwitz integers*. One can easily show that any Hurwitz integer has both a left and a right associate which is a Lipschitz integer (see [Voi22, Lemma 11.2.9]).

The euclidianity of  $\mathcal{H}$  implies that every left and every right ideal of  $\mathcal{H}$  is principal, and from this a sort of unique factorization into *primes*, Hurwitz integers whose norm is a rational prime, can be deduced for *primitive* Hurwitz integers, i.e. those not divisible by a rational prime.

**Theorem 3.1** (Unique Factorization Theorem). *Corresponding to each factorization of the norm  $n$  of a primitive Hurwitzian integer  $\alpha$  into a product  $p_1 p_2 \cdots p_{k-1} p_k$  of rational primes, there is a factorization*

$$\alpha = \pi_1 \pi_2 \cdots \pi_{k-1} \pi_k$$

*of  $\alpha$  into a product of Hurwitzian primes that is said to be modelled on that factorization of  $n$ , that is, with  $N(\pi_i) = p_i$ .*

*Moreover, if  $\alpha = \pi_1 \pi_2 \cdots \pi_{k-1} \pi_k$  is any one factorization modelled on  $p_1 p_2 \cdots p_{k-1} p_k$ , then all the others have the form*

$$\alpha = \pi_1 \varepsilon_1 \cdot \varepsilon_1^{-1} \pi_2 \varepsilon_2 \cdot \varepsilon_2^{-1} \pi_3 \varepsilon_3 \cdot \cdots \cdot \varepsilon_{k-2}^{-1} \pi_{k-1} \varepsilon_{k-1} \cdot \varepsilon_{k-1}^{-1} \pi_k,$$

*where  $\varepsilon_1, \dots, \varepsilon_{k-1} \in \mathcal{H}^*$ , i.e. the factorization on a given model is unique up to unit-migration.*

This result is essentially contained in [Lip86] (p. 434), where Lipschitz proves that integral quaternions have that same sort of unique factorization up to factors of norm 2. For a modern proof see Theorem 2, p. 57 in [CS03].

Given  $m \in \mathbb{N}$ , a quaternion  $\alpha = a + bi + cj + dk \in \mathcal{L}$  is said to be *primitive modulo  $m$*  if  $\gcd(a, b, c, d, m) = 1$ . In [Pal40, Theorem 1], Pall proves the following result, using some classical results about quadratic forms, and of which we give here a completely arithmetical proof.

**Theorem 3.2.** *If  $\alpha \in \mathcal{L}$  is primitive modulo  $m$ , where  $m$  is odd and positive with  $m \mid N(\alpha)$ , then  $\alpha$  has a unique, up to left associates, right divisor of norm  $m$ , in  $\mathcal{L}$ . One has an analogous result for left divisors.*

*Proof.* Since every left ideal of  $\mathcal{H}$  is principal, and every Hurwitz integer has a Lipschitz associate, there exists  $\delta \in \mathcal{L}$  such that  $\mathcal{H}\alpha + \mathcal{H}m = \mathcal{H}\delta$ . In particular,  $\delta = \beta\alpha + \gamma m$ , for some  $\beta, \gamma \in \mathcal{H}$ . But then  $N(\delta) = N(\alpha)N(\beta) + 2\Re(\beta\alpha\bar{\gamma})m + N(\gamma)m^2$ , and thus  $m \mid N(\delta)$ . Let  $t \in \mathbb{N}$  be the respective quotient, so that  $N(\delta) = mt$ , and let  $\sigma, \tau \in \mathcal{H}$  be such that  $\alpha = \sigma\delta$  and  $m = \tau\delta$ . Then  $m^2 = N(\tau)N(\delta) = N(\tau)mt$  shows that  $\tau\delta = m = \tau\bar{\tau}t$ , and thus  $\delta = \bar{\tau}t$ . The fact that  $\alpha$  is primitive modulo  $m$  now entails  $t = 1$ , showing the existence of a right divisor of  $\alpha$  with norm  $m$  in  $\mathcal{H}$ . It remains to

show that  $\sigma \in \mathcal{L}$ . Since  $m$  is odd, there are  $x, y \in \mathbb{Z}$  such that  $2x + my = 1$ . But then  $\sigma = 2\sigma x + \alpha \bar{\delta} y \in \mathcal{L}$ .

To prove uniqueness, up to left associates, assume that  $\alpha = \xi\mu$  for some  $\xi, \mu \in \mathcal{L}$ , with  $N(\mu) = m$ . Then  $\delta = \beta\alpha + \gamma m = (\beta\xi + \gamma\bar{\mu})\mu$ , showing that  $\mu$  is a right divisor of  $\delta$ . Since they have the same norm, one has  $\epsilon := \beta\xi + \gamma\bar{\mu} \in \mathcal{H}^*$ . Using again the fact that  $m$  is odd, one shows that  $\epsilon \in \mathcal{L}^*$  by noticing that  $\epsilon = 2\epsilon x + \epsilon my = (2\epsilon)x + (\epsilon\mu)\bar{\mu}y = (2\epsilon)x + \delta\bar{\mu}y$ .  $\square$

**Remarks:**

- This last result does not hold for  $m$  even, as the following example shows:  $1 + i + j + k = (1 + i)(1 + j) = (1 + k)(1 + i)$ , while  $1 + j$  and  $1 + i$  are not left associates in  $\mathcal{L}$ . But it is very easy to see that the result does unconditionally hold in  $\mathcal{H}$ .
- The map  $\mathcal{H} \rightarrow \mathcal{H}$  given by  $\alpha \mapsto \bar{\alpha}$  is an anti-automorphism, and so any divisibility result on the left also holds on the right.

Notice that while Theorem 3.1 relates factorizations modelled on the same prime decomposition of the norm, Theorem 3.2 gives information about factorizations of a primitive quaternion modelled on different prime decomposition of its norm. For example, if  $\alpha = \pi_1\pi_2\pi_3$  is a factorization of a primitive quaternion  $\alpha$  corresponding to  $N(\alpha) = p_1p_2p_3$ , and  $\alpha = \pi'_2\pi'_1\pi'_3$  is a factorization corresponding to  $N(\alpha) = p_2p_1p_3$ , then it follows from the last theorem that  $\pi_1\pi_2$  and  $\pi'_2\pi'_1$  are right associates, and therefore  $\pi_3$  and  $\pi'_3$  are left associates. It turns out that the version of Theorem 3.2 for Hurwitz integers implies Theorem 3.1, as it is fairly easy to see.

#### 4. ORTHOGONALITY AND ARITHMETIC

From the expression (3.1) above, that gives the product of two generic quaternions,  $u$  and  $v$ , it immediately follows that, for the inner product  $u \cdot v$  (as vectors of  $\mathbb{R}^4$ ), one has:

$$(4.1) \quad u \cdot v = \Re(u\bar{v}) = \frac{1}{2}(u\bar{v} + v\bar{u}).$$

Then, for all  $u, v, \alpha \in \mathbb{H}$ ,

$$(4.2) \quad u\alpha \cdot v = \Re((u\alpha)\bar{v}) = \Re(u(\alpha\bar{v})) = u \cdot \overline{\alpha v} = u \cdot v\bar{\alpha},$$

and using the obvious fact that  $u \cdot v = \bar{u} \cdot \bar{v}$ , one also has:

$$(4.3) \quad \alpha u \cdot v = \bar{u}\bar{\alpha} \cdot \bar{v} = \bar{u} \cdot \bar{v}\alpha = u \cdot \bar{\alpha}v.$$

In what follows, we will use the notation  $u \perp v$  to mean that the quaternions  $u$  and  $v$  are orthogonal, i.e.  $u \cdot v = 0$ . In [Pal40], Pall shows that there

are interesting connections between arithmetic properties of Lipschitz integers and orthogonality. We here exhibit some others, and provide a simpler arithmetical proof for a particular case of a result of Pall.

**Proposition 4.1.** *For any  $u, v, w \in \mathbb{H}$ , one has*

$$(uv) \cdot (uw) = N(u) (v \cdot w).$$

*In particular, if  $\alpha, \beta \in \mathcal{L}$  have a common left divisor  $\tau$ , then  $N(\tau) \mid \alpha \cdot \beta$ . One has analogous results for right common divisors.*

*Proof.* This is an immediate consequence of (4.3).  $\square$

**Corollary 4.2.** *Let  $\epsilon, \delta \in \{1, i, j, k\}$  with  $\epsilon \neq \delta$ . Then, for any  $\alpha \in \mathbb{H}$ ,  $\alpha\epsilon \perp \alpha\delta$  and  $\epsilon\alpha \perp \delta\alpha$ .*

*Proof.* This is an immediate consequence of the previous proposition, and the fact that  $\epsilon \perp \delta$ .  $\square$

It follows from Theorem 6 in [Pal40] that two non-associate Hurwitzian primes that have the same norm cannot be orthogonal. We show here that this can be directly deduced from the unique factorization theorem.

**Theorem 4.3.** *If  $\alpha, \beta \in \mathcal{H}$  are primes with the same norm, and  $\alpha \perp \beta$ , then each one is a left, as well as a right associate of the other.*

*Proof.* Let  $p = N(\alpha) = N(\beta)$ . From  $\alpha \perp \beta$  one gets that  $\alpha\bar{\beta} = -\beta\bar{\alpha}$ . Now, if the quaternion  $\gamma = \alpha\bar{\beta}$  is not primitive, then  $m \mid \gamma$  for some  $m \in \mathbb{N}$  with  $m > 1$ . But then, from  $m^2 \mid N(\gamma) = p^2$ , it follows that  $m = p$ . But then  $\alpha\bar{\beta} = p\varepsilon = \varepsilon p$ , for some unit  $\varepsilon$ . Since  $p = \beta\bar{\beta}$ , one gets  $\alpha = \varepsilon\beta$ . From  $\beta\bar{\alpha} = -\alpha\bar{\beta} = -p\varepsilon$ , one gets  $\beta = -\varepsilon\alpha$  (and in this case one sees that  $\varepsilon^2 = -1$ , and therefore  $\varepsilon = \pm i, \pm j, \pm k$ ).

If  $\gamma$  is primitive, then  $\alpha\bar{\beta}$  and  $-\beta\bar{\alpha}$  are two factorizations of  $\gamma$  modelled on  $N(\gamma) = pp$ , and the unique factorization theorem implies that  $\alpha$  and  $\beta$  are right associates.

Finally note that  $\alpha \perp \beta \Rightarrow \bar{\alpha} \perp \bar{\beta}$ , which allows to deduce the left version of the result from its right version, and vice-versa.  $\square$

With non-primes one can obtain examples that are a little more interesting. For instance, from the previous corollary, it follows that if  $\pi$  and  $\rho$  are any two quaternions, then  $\pi i \rho$  and  $\pi \rho$  have the same norm and are orthogonal. The question of what exactly is the left greatest common divisor of these two quaternions, leads to:

**Proposition 4.4.** *Let  $\gamma = z + wj \in \mathcal{L}$ , with  $z, w \in \mathbb{Z}[i]$ , be an odd quaternion (i.e.  $\gamma$  has an odd norm). Then:*

$$i\gamma\mathcal{H} + \gamma\mathcal{H} = 1 \quad \iff \quad (z, w) = 1 \quad (\text{in } \mathbb{Z}[i]).$$

*Proof.* Set  $I = i\gamma\mathcal{H} + \gamma\mathcal{H}$ . It is clear that if  $\delta \mid z$  and  $\delta \mid w$ , with  $\delta \in \mathbb{Z}[i]$ , then  $\delta$  is a left divisor of  $i\gamma$ , since of course  $\delta \mid \gamma$  and it commutes with  $i$ . Therefore,  $(z, w) = (\delta)$  implies that  $I \subseteq \delta\mathcal{H}$ .

On the other hand, since  $i\gamma = zi + wk$ ,  $\gamma i = zi - wk$ , one has:

$$2zi = i\gamma + \gamma i \in I$$

and

$$2wk = i\gamma - \gamma i \in I.$$

Hence:

$$2z, 2w \in I.$$

Now,  $(2, N(\gamma)) = 1$  implies that there are  $x, y \in \mathbb{Z}$  such that  $2x + \gamma\bar{\gamma}y = 1$ . In particular, there is  $x \in \mathbb{Z}$  with  $2x \equiv 1 \pmod{I}$ . From this one concludes that  $z, w \in I$ , and so, if these are coprime, it follows that  $I = 1$ .  $\square$

Note that from an algorithm to compute  $\pi i\rho$  from the quaternion  $\pi\rho$  one would get a factorization algorithm for some integers, namely semi-primes. In fact, suppose we have a semi-prime number  $n = pq$  with  $p$  and  $q$  to be determined. Using an algorithm like the one in [RS86], one can find  $\alpha \in \mathcal{L}$  such that  $N(\alpha) = n$ , and one has  $\alpha = \pi\rho$ , for some primes  $\pi, \rho \in \mathcal{H}$ . If one could determine  $\pi i\rho$  from  $\alpha$ , then using the Euclidean algorithm, one would get  $\pi$  and  $\rho$ , since the previous result easily implies that  $\pi i\rho\mathcal{H} + \pi\rho\mathcal{H} = \pi\mathcal{H}$  (and, in an entirely analogous way,  $\mathcal{H}\pi i\rho + \mathcal{H}\pi\rho = \mathcal{H}\rho$ ). This would yield  $p$  and  $q$ . In order to get an interesting factoring algorithm along these lines it would, of course, be enough to find a method of determining a reasonable sized neighborhood in the orthogonal space to  $\alpha$  where  $\pi i\rho$  would be located. Given that integer factorization seems to be a very hard problem, it is to be expected that the relation between the coordinates of  $\pi\rho$  and the ones of  $\pi i\rho$  will be rather subtle.

We leave here just one example that illustrates the seeming lack of relation between the coordinates of  $\pi\rho$  and  $\pi i\rho$ , as well as the natural variations:

$$\begin{array}{l|l} \pi = 1 + i + 3j + 6k & (N(\pi)=47) & \rho = 1 + 2i + 5j + 7k & (N(\rho)=79) \\ \pi\rho = -58 - 6i + 13j + 12k & & \rho\pi = -77 + 4i + 10j + 14k & \\ \pi i\rho = -12 + 56 - 12j - 17k & & \rho i\pi = 6 + 56i - 10j - 21k & \\ \pi j\rho = -3 - 10i + 30j - 52k & & \rho j\pi = -13 - 12i + 30j - 50k & \\ \pi k\rho = -14 - 21i - 50j - 24k & & \rho k\pi = -12 - 17i - 52j - 54k & \end{array}$$

We end this section by showing that one can easily get a  $\mathbb{Z}$ -basis for the  $\mathbb{Z}$ -module of the integral quaternions that are orthogonal to a given primitive integral quaternion.

**Proposition 4.5.** *Let  $\alpha = a+bi+cj+dk \in \mathcal{L}$  be a primitive quaternion. Let  $g_1, g_2 \in \mathbb{Z}$  be such that  $g_1\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$ ,  $g_2\mathbb{Z} = c\mathbb{Z} + d\mathbb{Z}$ , and  $x_0, y_0, z_0, t_0 \in \mathbb{Z}$  be such that:  $ax_0 + by_0 = g_1$ ,  $cz_0 + dt_0 = g_2$ . In case  $g_1 = 0$ , we choose  $x_0 = 0$  and  $y_0 = 1$ , and similarly if  $g_2 = 0$ . We make the convention that  $\frac{0}{0} + \frac{0}{0}i = 1$ . Then the  $\mathbb{Z}$ -module  $\alpha^\perp \cap \mathcal{L}$  is generated by the quaternions:*

$$g_2(x_0 + y_0i) - g_1(z_0 + t_0i)j, \quad \frac{b}{g_1} - \frac{a}{g_1}i, \quad \left( \frac{d}{g_2} - \frac{c}{g_2}i \right) j$$

*Proof.* Suppose we are given  $\alpha = a + bi + cj + dk \in \mathcal{L}$ , primitive. We want to find all vectors in  $\mathcal{L} \cap \alpha^\perp$ . Let  $g_1, g_2$  be as in the statement above, and assume first that  $g_1g_2 \neq 0$ . Let  $x_0, y_0, z_0, t_0 \in \mathbb{Z}$  be such that:

$$(4.4) \quad ax_0 + by_0 = g_1$$

$$(4.5) \quad cz_0 + dt_0 = g_2$$

Now, any quaternion  $\gamma = x + yi + zj + tk \in \mathcal{L}$  such that

$$ax + by + cz + dt = 0.$$

must satisfy

$$ax + by = r_1g_1$$

$$cz + dt = r_2g_2,$$

for some  $r_1, r_2 \in \mathbb{Z}$  with  $r_1g_1 + r_2g_2 = 0$ . Because  $\alpha$  is primitive,  $g_1\mathbb{Z} + g_2\mathbb{Z} = 1$ , and therefore there exists  $r \in \mathbb{Z}$  such that  $r_1 = rg_2$  and  $r_2 = -rg_1$ . It then follows from the well known characterization of the solutions of linear Diophantine equations that:

$$(4.6) \quad \begin{aligned} x &= r_1x_0 + \frac{b}{g_1}s = rg_2x_0 + \frac{b}{g_1}s \\ y &= r_1y_0 - \frac{a}{g_1}s = rg_2y_0 - \frac{a}{g_1}s \\ z &= r_2z_0 + \frac{d}{g_2}u = -rg_1z_0 + \frac{d}{g_2}u \\ t &= r_2t_0 - \frac{c}{g_2}u = -rg_1t_0 - \frac{c}{g_2}u, \end{aligned}$$

for some  $s, u \in \mathbb{Z}$ .

It is easy to check that the result holds in the two cases in which  $g_1g_2 = 0$ , if one uses the conventions formulated in the statement of the lemma.  $\square$

## 5. THE VECTOR PRODUCT IN $\mathbb{H}$ AND THE ARITHMETIC OF $\mathcal{L}$

We show in this section that some triple vector products of some quaternions involving a given quaternion  $\alpha$ , being orthogonal to  $\alpha$ , are nevertheless multiples of  $\alpha$ . The results are really of an algebraic nature, in the sense that they follow from some polynomial identities, which one can (implicitly) verify using, for instance, SageMath [Sage21]. Albeit the Sage verification



being a proof of the following two theorems, it not a very enlightning one. We provide proofs that we believe to be interesting in their own right.

We start by recalling the notion of vector product in  $\mathbb{R}^n$ .

**Definition 5.1.** For  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1} \in \mathbb{R}^n$ , define their vector product by

$$\times(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}) = \mathbf{u}_1 \times \mathbf{u}_2 \times \dots \times \mathbf{u}_{n-1} := \begin{vmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{vmatrix}$$

(with the obvious meaning, using Laplace expansion on the last row), where  $\mathbf{e}_i$  is the  $i$ -th vector of the canonical basis of  $\mathbb{R}^n$ , and  $u_{i,j}$  is the  $j$ -th coordinate of the vector  $\mathbf{u}_i$ , on that same basis.

It immediately follows from this definition that, for any vectors  $\mathbf{u}_i, \mathbf{v} \in \mathbb{R}^n$ , where  $i = 1, \dots, n-1$ , one has:

$$(5.1) \quad (\mathbf{u}_1 \times \mathbf{u}_2 \times \dots \times \mathbf{u}_{n-1}) \cdot \mathbf{v} = \begin{vmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \\ v_1 & v_2 & \cdots & v_n \end{vmatrix}$$

$$(5.2) \quad (\mathbf{u}_1 \times \mathbf{u}_2 \times \dots \times \mathbf{u}_{n-1}) \perp \mathbf{u}_i, \text{ for all } i = 1, \dots, n-1.$$

**Proposition 5.2.** For any vectors  $\mathbf{u}_i, \mathbf{v}_j \in \mathbb{R}^n$ , with  $i, j = 1, \dots, n-1$ , one has:

$$(\mathbf{u}_1 \times \mathbf{u}_2 \times \dots \times \mathbf{u}_{n-1}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{n-1}) = \det(\mathbf{u}_i \cdot \mathbf{v}_j).$$

*Proof.* Since the two maps from  $(\mathbb{R}^n)^{n-1}$  to  $\mathbb{R}$  given by

$$(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}) \rightarrow (\mathbf{u}_1 \times \mathbf{u}_2 \times \dots \times \mathbf{u}_{n-1}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{n-1})$$

and by

$$(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}) \rightarrow \det(\mathbf{u}_i \cdot \mathbf{v}_j)$$

are both multilinear, it is enough to check the validity of the claimed result for  $\mathbf{u}_i, \mathbf{v}_j \in \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ . And because the mentioned maps are also alternate, it is enough to check that:

$$(\mathbf{e}_{\sigma(1)} \times \mathbf{e}_{\sigma(2)} \times \dots \times \mathbf{e}_{\sigma(n-1)}) \cdot (\mathbf{e}_{\tau(1)} \times \mathbf{e}_{\tau(2)} \times \dots \times \mathbf{e}_{\tau(n-1)}) = \det(\mathbf{e}_{\sigma(i)} \cdot \mathbf{e}_{\tau(j)}),$$

for all  $\sigma, \tau \in \mathcal{S}_n$ , where  $\mathcal{S}_n$  denotes the symmetric group on  $\{1, \dots, n\}$ .

Now,  $\mathbf{e}_{\sigma(1)} \times \dots \times \mathbf{e}_{\sigma(n-1)} = \sum_{\tau \in \mathcal{S}_n} \text{sgn}(\tau) e_{\sigma(1)\tau(1)} \cdots e_{\sigma(n-1)\tau(n-1)} \mathbf{e}_{\tau(n)}$ , where, as in the notation used in Definition 5.1,  $e_{ij}$  denotes the  $j$ -th coordinate of  $\mathbf{e}_i$ . The only non-zero term of this sum is the one where  $\tau = \sigma$ . It follows that:

$$\mathbf{e}_{\sigma(1)} \times \mathbf{e}_{\sigma(2)} \times \dots \times \mathbf{e}_{\sigma(n-1)} = \text{sgn}(\sigma) \mathbf{e}_{\sigma(n)}.$$

Therefore:

$$(\mathbf{e}_{\sigma(1)} \times \mathbf{e}_{\sigma(2)} \times \dots \times \mathbf{e}_{\sigma(n-1)}) \cdot (\mathbf{e}_{\tau(1)} \times \mathbf{e}_{\tau(2)} \times \dots \times \mathbf{e}_{\tau(n-1)}) = \text{sgn}(\sigma\tau) \delta_{\sigma(n), \tau(n)}.$$

On the other hand,  $\det(\mathbf{e}_{\sigma(i)} \cdot \mathbf{e}_{\tau(j)}) = \sum_{\gamma \in \mathcal{S}_{n-1}} \text{sgn}(\gamma) (\mathbf{e}_{\sigma(1)} \cdot \mathbf{e}_{\tau(\gamma(1))}) \cdots (\mathbf{e}_{\sigma(n-1)} \cdot \mathbf{e}_{\tau(\gamma(n-1))})$ , which is non-zero only for  $\gamma = \tau^{-1} \sigma|_{\{1, \dots, n-1\}}$ , and this happens only when  $\sigma(n) = \tau(n)$ . It easily follows that:

$$\det(\mathbf{e}_{\sigma(i)} \cdot \mathbf{e}_{\tau(j)}) = \text{sgn}(\tau^{-1} \sigma) e_{\sigma(n)\tau(n)},$$

proving what we wanted to show.  $\square$

From this proposition, one sees that, for any  $\alpha, \beta, \gamma \in \mathbb{H}$ ,

$$N(\alpha \times \beta \times \gamma) = (\alpha \times \beta \times \gamma) \cdot (\alpha \times \beta \times \gamma) = \begin{vmatrix} N(\alpha) & \alpha \cdot \beta & \alpha \cdot \gamma \\ \alpha \cdot \beta & N(\beta) & \beta \cdot \gamma \\ \alpha \cdot \gamma & \beta \cdot \gamma & N(\gamma) \end{vmatrix},$$

from which one easily gets

$$\begin{aligned} N(\alpha \times \beta \times \gamma) &= N(\alpha\beta\gamma) - N(\alpha)(\beta \cdot \gamma)^2 - N(\beta)(\alpha \cdot \gamma)^2 - \\ &\quad - N(\gamma)(\alpha \cdot \beta)^2 + 2(\alpha \cdot \beta)(\alpha \cdot \gamma)(\beta \cdot \gamma). \end{aligned}$$

In particular, if  $\beta \perp \alpha$  and  $\gamma \perp \alpha$ , then  $N(\alpha) \mid N(\alpha \times \beta \times \gamma)$ . It follows from Theorem 3.2 that, in this case,  $\alpha \times \beta \times \gamma$  has both a left and a right divisor with the same norm as  $\alpha$ . We will show that, in both cases, it turns out that  $\alpha$  is that divisor.

**Theorem 5.3.** *Given  $\alpha \in \mathcal{L}$ , and  $\beta, \gamma \in \mathcal{L}$  such that  $\beta \perp \alpha$  and  $\gamma \perp \alpha$ , one has*

$$\alpha \times \beta \times \gamma \in \alpha \mathcal{L} \cap \mathcal{L} \alpha.$$

*Proof.* Let  $\alpha = a + bi + cj + dk \in \mathcal{L} \setminus \{0\}$ , and assume that  $d \neq 0$  (if not, the following argument still works *mutatis mutandis*). Then an  $\mathbb{Q}$ -basis for  $\alpha^\perp$  is given by  $\beta_1 = d - ak, \beta_2 = di - bk, \beta_3 = dj - ck$ . Now, simple computations yield:

$$\begin{aligned} \alpha \times \beta_1 \times \beta_2 &= -\alpha \Im(j\alpha) d = -d \Im(\alpha j) \alpha, \\ \alpha \times \beta_1 \times \beta_3 &= \alpha \Im(i\alpha) d = -d \Im(\alpha i) \alpha, \\ \alpha \times \beta_2 \times \beta_3 &= \alpha \Im(\alpha) d = d \Im(\alpha) \alpha, \end{aligned}$$

which, by the multilinearity of the vector product, proves the claim.  $\square$

Using corollary 4.2, one sees that, for example,  $\alpha \times \alpha i \times \alpha j \in \alpha \mathcal{L} \cap \mathcal{L} \alpha$ . While doing some computational experiments, we noticed that, for example,  $\alpha \times \alpha i \times \beta \in \alpha \mathcal{L}$ , for all  $\beta \in \mathcal{L}$ . This eventually led to the discovery of the next results that connect the vector product with the multiplication of quaternions.

**Theorem 5.4.** *For any  $\alpha, \beta, \gamma, \delta \in \mathbb{H}$ , one has:*

$$\begin{aligned} \alpha \beta \times \alpha \gamma \times \alpha \delta &= N(\alpha) \alpha (\beta \times \gamma \times \delta), \\ \beta \alpha \times \gamma \alpha \times \delta \alpha &= N(\alpha) (\beta \times \gamma \times \delta) \alpha. \end{aligned}$$

*Proof.* We will show the first equality, the proof of the second being entirely analogous.

Recall (see [Voi22, Remark 3.3.8]) that the determinant of the left regular representation  $\mathbb{H} \rightarrow \mathbb{H}$  given by  $x \mapsto \alpha x$  is equal to  $N(\alpha)^2$ , and note

that by multilinearity and alternatingness it is enough to show the claimed equality for  $\beta, \gamma, \delta \in \{1, i, j, k\}$ . In order to do that, let  $\varepsilon_i, i = 1, 2, 3, 4$ , be distinct units such that  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, i, j, k\}$ , and set  $\varepsilon_4 = \varepsilon_1 \times \varepsilon_2 \times \varepsilon_3$ . Then  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$  is an  $\mathbb{R}$  basis for  $\mathbb{H}$ , and  $N(\alpha)^2$  is the value of the determinant whose rows are the coordinates of  $\alpha\varepsilon_i, i = 1, 2, 3, 4$ , which is equal to  $(\alpha\varepsilon_1 \times \alpha\varepsilon_2 \times \alpha\varepsilon_3) \cdot \alpha\varepsilon_4$ . Using (4.2), one then has  $N(\alpha)^2 = \bar{\alpha}(\alpha\varepsilon_1 \times \alpha\varepsilon_2 \times \alpha\varepsilon_3) \cdot \varepsilon_4$ . Since, of course,  $\bar{\alpha}(\alpha\varepsilon_1 \times \alpha\varepsilon_2 \times \alpha\varepsilon_3) \cdot \varepsilon_i = 0$  for  $i = 1, 2, 3$ , one gets that

$$\bar{\alpha}(\alpha\varepsilon_1 \times \alpha\varepsilon_2 \times \alpha\varepsilon_3) = N(\alpha)^2 \varepsilon_4 = N(\alpha)^2 \varepsilon_1 \times \varepsilon_2 \times \varepsilon_3,$$

from which the result follows.  $\square$

The main result of this section is now easy to prove.

**Theorem 5.5.** *Given  $\alpha, \beta, \gamma, \delta \in \mathcal{L}$ , one has*

$$\alpha\beta \times \alpha\gamma \times \delta \in \alpha\mathcal{L} \quad \text{and} \quad \beta\alpha \times \gamma\alpha \times \delta \in \mathcal{L}\alpha.$$

*Proof.* Again, we will just show the first claim, the proof of the second being entirely analogous.

Using the previous result, one has:

$$\begin{aligned} \alpha\beta \times \alpha\gamma \times \delta &= \alpha\beta \times \alpha\gamma \times \alpha\alpha^{-1}\delta \\ &= N(\alpha)^2 \alpha (\beta \times \gamma \times \alpha^{-1}\delta) \\ &= \alpha N(\alpha) (\beta \times \gamma \times \bar{\alpha}\delta), \end{aligned}$$

which finishes the proof, given explicitly the respective right quotient.  $\square$

Observe that, in particular, this result entails that  $N(\alpha) \mid N(\alpha\beta \times \alpha\gamma \times \delta)$ , and hence Theorem 3.2 implies, when  $N(\alpha)$  is odd, that  $\alpha\beta \times \alpha\gamma \times \delta$ , if primitive, has a unique, up to left associates, divisor of norm equal to the norm of  $\alpha$ . However, it is not true that  $\alpha\beta \times \alpha\gamma \times \delta \in \mathcal{L}\alpha$ . For example, for  $\alpha = 1 + i + j + 2k, \beta = j, \gamma = i$  and  $\delta = 1 + i + j$ , one has:

$$\alpha\beta \times \alpha\gamma \times \delta \in \mathcal{L}\bar{\alpha}k.$$

Notice that when  $\nu = \alpha\beta \times \alpha\gamma \times \delta$  is such that  $\Re(\nu) = 0$ , which is the case when  $\delta \in \mathbb{R}$ , then the fact that  $\nu = \alpha\lambda$  obviously entails that  $\nu = -\bar{\nu} = -\bar{\lambda}\bar{\alpha}$ .

In many examples, the right divisor of  $\alpha\beta \times \alpha\gamma \times \delta$  is *comorphic* to  $\alpha$ , i.e. the absolute values of their coordinates are the same, up to order. However, for  $\alpha = 1 + i + 3j + 6k$ , a prime of norm 47, one has, taking  $\delta = 1 + 2i + 5j + 7k$  (a prime above 79):

$$\alpha \times \alpha i \times \delta = (1 - 2i - 2j - 2k)(2 + 3i + 5j + 3k),$$

a product of two primes, and where the second factor is a prime above 47 that is not comorphic to  $\alpha$ .

This raises the following problem:

**Question:** Given  $\beta, \gamma, \delta \in \mathcal{L}$  such that  $\alpha\beta \times \alpha\gamma \times \delta$  is primitive, can one describe the relation between  $\alpha$  and the unique, up to left associates, right divisor of the quaternion  $\alpha\beta \times \alpha\gamma \times \delta$  with norm equal  $N(\alpha)$ ? When it is comorphic to  $\alpha$ ?

Note that from the equality obtained in the proof of Theorem 5.5 it follows that, for  $\alpha, \beta, \gamma, \delta \in \mathcal{L}$ ,

$$(\alpha\beta \times \alpha\gamma \times \delta) \mathsf{N}(\delta) \in \mathcal{L}\bar{\alpha}\delta.$$

In particular, when  $\delta$  is a unity one gets that

$$\alpha\beta \times \alpha\gamma \times \delta \in \mathcal{L}\bar{\alpha}\delta,$$

which is comorphic to  $\alpha$ .

We finish by pointing out some identities that can easily be deduced from the results in this section, or directly from the definition of the triple vector product, and that are valid for all  $\alpha, \beta, \gamma, \delta \in \mathbb{H}$ :

- (1)  $\mathsf{N}(u) = 1 \Rightarrow u(\alpha \times \beta \times \gamma) = u\alpha \times u\beta \times u\gamma$  (and the same on the right)
- (2)  $(\alpha \times \beta \times \gamma) \cdot \delta = -(\alpha \times \beta \times \delta) \cdot \gamma$
- (3)  $\alpha \times \beta \times 1 = \frac{1}{2}(\alpha\beta - \beta\alpha) = \mathfrak{I}(\alpha) \times \mathfrak{I}(\beta)$
- (4)  $\alpha\beta \times \alpha\gamma \times 1 = (\alpha \times \bar{\beta} \times \bar{\gamma})\bar{\alpha}$
- (5)  $\overline{\alpha \times \beta \times \gamma} = -\bar{\alpha} \times \bar{\beta} \times \bar{\gamma}.$

## 6. FINAL REMARKS

As pointed out in the introduction, the results presented here were obtained while musing on a possible extension to integral quaternions of the method of factoring an integer from two of its representations as a sum of two squares. Some results of G. Pall, contained in [Pal40], led us to look for integral quaternions that are orthogonal to a given one. To construct these, we turned our attention to the vector product, just to find out that this did not yield what we were looking for. Nevertheless, we obtained results that seem interesting in their own way, and that led to a question that seems worth pondering about.

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