Estimates for the number of zeros of shifted combinations of completed Dirichlet series

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Abstract

In a previous paper [24], Yakubovich and the author of this article proved that certain shifted combinations of completed Dirichlet series have infinitely many zeros on the critical line. Here we provide some lower bounds for the number of critical zeros of a subclass of shifted combinations.

Contents

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1 Introduction and main results

Let $\eta(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$. A. Dixit, N. Robles, A. Roy and A. Zaharescu [8] proved the following theorem.

Theorem A: *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-zeros real numbers such that $\sum_{j=1}^{\infty}|c_j| < \infty$. Also, let $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers that attains its bounds. Then the function*

$$F(s) = \sum_{j=1}^{\infty} c_j\,\eta\left(s+i\lambda_j\right) := \sum_{j=1}^{\infty} c_j\,\pi^{-\frac{s+i\lambda_j}{2}}\Gamma\left(\frac{s+i\lambda_j}{2}\right)\zeta\left(s+i\lambda_j\right)$$

has infinitely many zeros on the critical line $Re(s)=\frac{1}{2}$.

See also the introduction of [8] for an excellent survey on results about zeros of certain shifts of the Riemann zeta function. Based on an integral representation of Jacobi's transformation formula due to Dixit [[6], p. 374, eq. (1.13)], A. Dixit, R. Kumar, B. Maji and A. Zaharescu [7] later generalized the aforementioned result and proved the more general theorem.

Theorem B: *Let $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ be as in Theorem A. Also, let $\mathscr{R}$ denote the region of the complex plane defined by $\mathscr{R} := \left\{z\in\mathbb{C}\,:\,|Re(z)|<\sqrt{\frac{\pi}{2}},\,|Im(z)|<\sqrt{\frac{\pi}{2}}\right\}$. Then, for any $z\in\mathscr{R}$, the function*

$$F_z(s) = \sum_{j=1}^{\infty} c_j\,\eta\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{1-(s+i\lambda_j)}{2};\,\frac{1}{2};\,\frac{z^2}{4}\right) + {}_1F_1\left(\frac{1-(\overline{s}-i\lambda_j)}{2};\,\frac{1}{2};\,\frac{\overline{z}^2}{4}\right)\right\} \tag{1.1}$$

has infinitely many zeros on the critical line $Re(s)=\frac{1}{2}$.

The proof of Theorem B employed a variant of Hardy's method of studying the moments of the real function $\eta\left(\frac{1}{2}+it\right)$ [[14], [31], Chapter X], as well as the elegant transformation formula

$$2x^{1/4}\psi(x,z) - x^{-1/4}e^{-z^2/4} = 2e^{-z^2/4}x^{-1/4}\,\psi\left(\frac{1}{x},\,iz\right) - x^{1/4}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right){}_1F_1\left(\frac{1}{4}+\frac{it}{2};\,\frac{1}{2};\,-\frac{z^2}{4}\right)x^{-\frac{it}{2}}\,dt, \tag{1.2}$$

where

$$\psi(x,z) := \sum_{n=1}^{\infty} e^{-\pi n^2 x}\cos\left(\sqrt{\pi x}\,n\,z\right),\quad \text{Re}(x)>0,\quad z\in\mathbb{C}. \tag{1.3}$$

The first equality in (1.2) is, of course, due to Jacobi but the integral representation appears for the first time in [[6], p. 374, eq. (1.13)].

Together with Yakubovich [24], the author of this paper extended Theorem B to a class of Dirichlet series satisfying Hecke's functional equation. A direct generalization of Theorem B concerns Dirichlet series attached to positive powers of the $\theta-$function,

$$\theta(x) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},\quad \text{Re}(x)>0, \tag{1.4}$$

defined as follows: for any $\alpha > 0$, $r_{\alpha}(n)$ is the arithmetical function [17] described by the expansion

$$\theta^{\alpha}(x) - 1 := \sum_{n=1}^{\infty} r_{\alpha}(n)\, e^{-\pi n x}, \quad \text{Re}(x) > 0. \tag{1.5}$$

For $\text{Re}(s)$ sufficiently large, one may consider the Dirichlet series

$$\zeta_{\alpha}(s) := \sum_{n=1}^{\infty} \frac{r_{\alpha}(n)}{n^{s}}, \tag{1.6}$$

and motivate its study through the transformation formula for $\theta(x)$. When $\alpha = k \in \mathbb{N}$, it is effortless to see that $r_{\alpha}(n)$ reduces to the arithmetical function counting the number of representations of n as a sum of k squares. Also, when $\alpha = 1$, $\zeta_1(s)$ reduces to $2\zeta(2s)$. Like $\zeta_k(s)$ and $\zeta(2s)$, $\zeta_{\alpha}(s)$ satisfies Hecke's functional equation

$$\eta_{\alpha}(s) := \pi^{-s}\Gamma(s)\,\zeta_{\alpha}(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_{\alpha}\left(\frac{\alpha}{2}-s\right) := \eta_{\alpha}\left(\frac{\alpha}{2}-s\right), \tag{1.7}$$

from which it is possible to conclude that $\eta_{\alpha}(s)$ is real on the critical line $\text{Re}(s) = \frac{\alpha}{4}$. The extension of Theorem B to the class of zeta functions $\zeta_{\alpha}(s)$ is described by the following result (cf. [[24], p. 6, Theorem 1.1]).

Theorem C: *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_j |c_j| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of real numbers attaining its bounds. Then, for any z satisfying the condition*

$$z \in \mathscr{D}_{\alpha} := \left\{ z \in \mathbb{C} \, : \, |Re(z)| < \sqrt{\frac{\pi\alpha}{2}}, \, |Im(z)| < \sqrt{\frac{\pi\alpha}{2}} \right\}, \tag{1.8}$$

the function

$$F_{z,\alpha}(s) := \sum_{j=1}^{\infty} c_j\, \pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_{\alpha}(s+i\lambda_j)\,\left\{ {}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right) + {}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\, \frac{\alpha}{2};\, \frac{\overline{z}^2}{4}\right)\right\} \tag{1.9}$$

has infinitely many zeros on the critical line $Re(s) = \frac{\alpha}{4}$.

Note that Theorem B is a particular case of Theorem C (obtained when $\alpha = 1$). One of the main ingredients in the proof of Theorem C is a generalization of Dixit's integral formula (1.2) obtained in [[24], p. 30, eq. (2.60)]. This identity takes the form

$$\begin{aligned} x^{\frac{\alpha}{4}}\psi_{\alpha}(x,z) - x^{-\alpha/4}\,e^{-\frac{z^2}{4}} &= e^{-\frac{z^2}{4}}\,x^{-\alpha/4}\,\psi_{\alpha}\left(\frac{1}{x},\, iz\right) - x^{\alpha/4} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_{\alpha}\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\,x^{-it}\,dt, \end{aligned} \tag{1.10}$$

where $\eta_{\alpha}(s)$ is the completed Dirichlet series (1.7) and $\psi_{\alpha}(x,z)$ is the generalization of Jacobi's $\psi-$function,

$$\psi_{\alpha}(x,z) = 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi x}\,z\right)^{1-\frac{\alpha}{2}}\,\sum_{n=1}^{\infty} r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,x}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\, n\, x}\,z), \quad \text{Re}(x) > 0, \;\; z \in \mathbb{C}. \tag{1.11}$$

The transformation formula (1.10) can be also taken in a general setting, with $\zeta_{\alpha}(s)$ being replaced by any Dirichlet series satisfying Hecke's functional equation. For example, let $f(\tau)$ be a holomorphic cusp form with weight $k \geq 12$ for the full modular group whose Fourier expansion is given by

$$f(\tau) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n\tau}, \quad \text{Im}(\tau) > 0. \tag{1.12}$$

If we construct the Dirichlet series associated to $f(\tau)$,

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s},\quad \text{Re}(s)>\frac{k+1}{2}, \tag{1.13}$$

we know that $L(s,f)$ can be analytically continued to an entire function obeying Hecke's functional equation

$$\eta_f(s):=(2\pi)^{-s}\Gamma(s)\,L(s,f)=(-1)^{k/2}\,(2\pi)^{-(k-s)}\,\Gamma(k-s)\,L(k-s,f):=(-1)^{k/2}\eta_f\,(k-s)\,. \tag{1.14}$$

Analogously to $\psi_{\alpha}(x,z)$, which generalizes Jacobi's $\psi-$function, we can construct a generalization of the cusp form $f(\tau)$ in the form[1]

$$\psi_f(x,z):=(k-1)!\,\left(\sqrt{\frac{\pi x}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n\,x}\,z\right),\ \text{Re}(x)>0,\ z\in\mathbb{C}. \tag{1.15}$$

Just like $\psi_{\alpha}(x,z)$, $\psi_f(x,z)$ obeys to the following transformation

$$x^{\frac{k}{2}}\psi_f(x,z)=(-1)^{k/2}\,e^{-\frac{z^2}{4}}x^{-k/2}\,\psi_f\left(\frac{1}{x},\,iz\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}(2\pi)^{-\frac{k}{2}-it}\,\Gamma\left(\frac{k}{2}+it\right)\,L\left(\frac{k}{2}+it,f\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^2}{4}\right)\,x^{-it}\,dt$$

$$:=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_f\left(\frac{k}{2}+it\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^2}{4}\right)\,x^{-it}dt, \tag{1.16}$$

where $\eta_f(s)$ is the completed Dirichlet series (1.14). Using (1.16), we have been able to establish the following Theorem [[24], Theorem 1.4].

Theorem D: *Let $f(\tau)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_f(n)$. Consider the Dirichlet series,*

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s},\quad Re(s)>\frac{k+1}{2}. \tag{1.17}$$

If $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$, $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers[2] *and z satisfies the condition*

$$z\in\mathscr{D}:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<2\sqrt{\pi},\ |Im(z)|<2\sqrt{\pi}\right\}, \tag{1.18}$$

then the function

$$G_{z,f}\,(s):=\sum_{j=1}^{\infty}c_j\,(2\pi)^{-s-i\lambda_j}\,\Gamma(s+i\lambda_j)\,L\,(s+i\lambda_j,f)\,\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{\overline{z}^2}{4}\right)\right\} \tag{1.19}$$

has infinitely many zeros at the critical line $Re(s)=\frac{k}{2}$.

Our main goal in this paper is to prove quantitative analogues of Theorems C and D above for a subclass of shifted combinations. This is, when we impose some further restrictions on $(\lambda_j)_{j\in\mathbb{N}}$, we aim to find lower bounds for the number of critical zeros of the functions $F_{z,\alpha}(s)$ and $G_{z,f}(s)$. We need to introduce the following assumptions:

[1] Note that, when $z=0$, $\psi_f(x,0)=f(ix)$

[2] In the statement of Theorem 1.4. of [24] we require that the sequence $(\lambda_j)_{j\in\mathbb{N}}$ attains its bounds. However, in view of Remark 5.4 of [24], this condition is not necessary when we are working with combinations involving entire Dirichlet series.

1. **Assumpion 1:** If the shift $s \to s + i\lambda_j$ is introduced, then the symmetric shift $s \to s - i\lambda_j$ needs to be introduced as well and having the same weight c_j. Concerning Theorem C, for example, this means that if the term
$$c_j\, \eta_\alpha\left(s + i\lambda_j\right) \mathrm{Re}\left\{ {}_1F_1\left(\frac{\alpha}{2} - s - i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right)\right\}$$
belongs to the combination (1.9), then the term
$$c_j\, \eta_\alpha\left(s - i\lambda_j\right) \mathrm{Re}\left\{ {}_1F_1\left(\frac{\alpha}{2} - s + i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right)\right\}$$
also belongs to it. This assumption essentially says that we are enlarging the sequences $\left(\lambda_j\right)_{j\in\mathbb{N}}$ in (1.9) and (1.19) in such a way that each element has a symmetric pair contained in it. Therefore, we can write it as a sequence over $\mathbb{Z}\setminus\{0\}$ in the form $\left(\lambda_j\right)_{j\in\mathbb{Z}\setminus\{0\}} = \left(\lambda_j\right)_{j\in\mathbb{N}} \cup \left(-\lambda_j\right)_{j\in\mathbb{N}}$, once we take the convention that $\lambda_{-j} := -\lambda_j$. The same can be done to $(c_j)_{j\in\mathbb{N}}$ under the convention $c_{-j} := c_j$.

2. **Assumption 2:** In each one of the cases (1.9) and (1.19), the parameter z in the hypergeometric functions will be a real number belonging to an interval contained in the regions (1.8) and (1.18).

Under these assumptions, we can write the shifted combinations (1.9) and (1.19) in the symmetric forms
$$\tilde{F}_{z,\alpha}(s) := \sum_{j\neq 0} c_j\, \pi^{-(s+i\lambda_j)}\Gamma\left(s + i\lambda_j\right)\, \zeta_\alpha\left(s + i\lambda_j\right)\, \left\{ {}_1F_1\left(\frac{\alpha}{2} - s - i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right) + {}_1F_1\left(\frac{\alpha}{2} - \overline{s} + i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right)\right\} \tag{1.20}$$
and
$$\tilde{G}_{z,f}(s) := \sum_{j\neq 0} c_j\, \left(2\pi\right)^{-s-i\lambda_j}\, \Gamma(s + i\lambda_j)\, L\left(s + i\lambda_j, f\right)\, \left\{ {}_1F_1\left(k - s - i\lambda_j;\, k;\, \frac{z^2}{4}\right) + {}_1F_1\left(k - \overline{s} + i\lambda_j;\, k;\, \frac{z^2}{4}\right)\right\}. \tag{1.21}$$
In both cases, the sum is taken over $\mathbb{Z}\setminus\{0\}$ and we are under the conventions (see Assumption 1) $c_{-j} := c_j$ and $\lambda_{-j} := -\lambda_j$.

Note also that $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4} + it\right)$ is a real-valued and even function of $t \in \mathbb{R}$. These properties come immediately from the functional equation (1.7) and assumptions 1 and 2. In the same lines one can check that, when the coefficients $a_f(n)$ are real, $i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2} + it\right)$ is a real function of t. Furthermore, it is an even function if $k \equiv 0 \mod 4$ and it is odd if $k \equiv 2 \mod 4$. Although the additional assumptions given above restrict some of the general aspects of Theorems C and D, they prove to be very helpful in deriving some estimates for the number of zeros of the functions (1.20) and (1.21). We begin with a theorem about the number of zeros of $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4} + it\right)$.

Theorem 1.1. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_{j=1}^{\infty} |c_j| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of real numbers attaining its bounds. Suppose that $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ can be extended to $\mathbb{Z}\setminus\{0\}$ in the form $c_{-j} = c_j$ and $\lambda_{-j} = -\lambda_j$. Assume also that $z \in \mathbb{R}$ satisfies the condition*
$$z \in \left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}, \frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]. \tag{1.22}$$

Moreover, let $N_{\alpha,z}(T)$ *be the number of zeros written in the form* $s=\frac{\alpha}{4}+it,\ 0\leq t\leq T$*, of the function*

$$\tilde{F}_{z,\alpha}(s):=\sum_{j\neq 0}c_j\,\pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_{\alpha}\left(s+i\lambda_j\right)\,\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\}. \tag{1.23}$$

Then there exists some $c>0$ *such that*

$$\liminf_{T\to\infty}\frac{N_{\alpha,z}(T)}{\sqrt{T}/\log(T)}\geq c. \tag{1.24}$$

The quantitative estimate (1.24) is somewhat general and it seems very difficult to improve on the power of T using the same method. In the final section of this paper, we make some conjectures regarding a possible improvement. When the shifted combination is trivial and reduces to the term $\lambda_1=0$, our estimate (1.24) extends, for all $\alpha>0$, a result of the author and Yakubovich [[25], Corollary 3.5] (cf. Remark 1.1 below).

Our proof of Theorem 1.1 employs a method developed by Fekete [10], which was essentially inspired by a lemma of Fejér [9]. See [[31], p. 259] for a clear explanation of Fekete's method, as well as an interesting paper by Berlowitz which uses the same idea [[3], p. 206, Lemma 2]. However, in order to apply Fejér's lemma, it will be crucial in our argument to use the fact that $\zeta_{\alpha}(s)$ has a simple pole located at $s=\frac{\alpha}{2}$.

Since $L(s,f)$ is an entire function of s, a quantitative analogue of Theorem D will have to use a different idea. In any case, we have been able to establish the following result.

Theorem 1.2. *Let* $f(\tau)$ *be a cusp form of weight* k *for the full modular group with real Fourier coefficients* $a_f(n)$ *and consider the Dirichlet series,* $L(s,f)$*, attached to it. Let* $(c_j)_{j\in\mathbb{N}}$ *be a sequence of non-zero real numbers such that* $\sum_{j=1}^{\infty}|c_j|<\infty$ *and let* $(\lambda_j)_{j\in\mathbb{N}}$ *be a bounded sequence of distinct real numbers. Suppose that* $(c_j)_{j\in\mathbb{N}}$ *and* $(\lambda_j)_{j\in\mathbb{N}}$ *can be extended to* $\mathbb{Z}\setminus\{0\}$ *in the form* $c_{-j}=c_j$ *and* $\lambda_{-j}=-\lambda_j$*. Assume also that* z *is a real number satisfying the condition*

$$z\in\left[-\frac{\sqrt{\pi}}{3},\frac{\sqrt{\pi}}{3}\right]. \tag{1.25}$$

If $N_{f,z}(T)$ *denotes the number of zeros written in the form* $s=\frac{k}{2}+it,\ 0\leq t\leq T$*, of the function*

$$\tilde{G}_{z,f}(s):=\sum_{j\neq 0}c_j\,\left(2\pi\right)^{-s-i\lambda_j}\,\Gamma(s+i\lambda_j)\,L\left(s+i\lambda_j,f\right)\,\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{z^2}{4}\right)\right\}, \tag{1.26}$$

then there exists some $d>0$ *such that*

$$\limsup_{T\to\infty}\frac{N_{f,z}(T)}{\sqrt{T}}\geq d. \tag{1.27}$$

Our proof of Theorem 1.2 uses a variant of a method due to de la Vallée Poussin [5]. The author of this paper has recently used this method to establish a quantitative estimate for the number of critical zeros of $L-$functions attached to half-integral weight cusp forms [23].

This paper is organized as follows. In the next section we give the necessary technical lemmas to establish Theorem 1.1. The most important of these, Lemma 2.2, is obtained via a generalization of the theta transformation formula obtained in [24]. Section 3 is devoted to a proof of Theorem 1.1. Next, we follow a similar structure in sections 4 and 5. Finally, we end this paper with some conjectures concerning further extensions of the main results here presented. Before moving on, we introduce a couple of remarks which describe some interesting particular cases of our Theorems 1.1 and 1.2, as well as other theorems that can be proved via the same methods.

Remark 1.1. The intervals (1.22) and (1.25) are respectively contained in the regions (1.8) and (1.18) of Theorems C and D. As we shall see below, it is possible to enlarge the length of the interval (1.22) by a more careful choice of the parameter λ in the proof of Lemma 2.2 and a more precise inequality for $|I_{\nu}(x)|$. See Remark 2.1 below.

Remark 1.2. When $z=0$, we deduce from Theorem 1.1 that the arbitrary shifted combination given by

$$\sum_{j\neq 0} c_j\,\pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_{\alpha}\left(s+i\lambda_j\right)$$

has $\gg T^{1/2}/\log(T)$ zeros of the form $s=\frac{\alpha}{4}+it$, $0\leq t\leq T$. Since $\zeta_1(s):=2\zeta(2s)$, this shows that the function

$$F(s):=\sum_{j\neq 0} c_j\,\pi^{-\frac{1}{2}(s+i\lambda_j)}\Gamma\left(\frac{s+i\lambda_j}{2}\right)\,\zeta\left(s+i\lambda_j\right) \tag{1.28}$$

has $\gg T^{1/2}/\log(T)$ on the critical line $\text{Re}(s)=\frac{1}{2}$. Also, when $z\neq 0$ and $\alpha=1$, Theorem 1.1 yields an extension of Theorem B above. The generality of this result, with respect to the shifted combination and the explicit estimate for the number of critical zeros, seems to be unnoticed in the literature. We should remark that Selberg's outstanding result about the positive proportion of combinations of $L-$functions with degree one [26] does not seem to cover the case (1.28), because we are considering shifted combinations of $\eta(s)$.

Remark 1.3. Using an inductive method involving generalized Epstein zeta functions [25], Yakubovich and the author of this paper proved that, when $\alpha>4$, there always exist $T_0(\alpha)$ and $c:=c(\alpha)>0$ such that, for any $T\geq T_0(\alpha)$, there is a zero $\rho=\frac{\alpha}{4}+i\gamma$ of $\zeta_{\alpha}(s)$ with $\gamma\in[T,T+cT^{1/2}\log(T)]$. See [[25], Corollary 3.5] for details. In particular, if $N_{\alpha}(T):=\#\left\{0\leq t\leq T\,:\,\zeta_{\alpha}\left(\frac{\alpha}{4}+it\right)=0\right\}$, then

$$\liminf_{T\to\infty}\frac{N_{\alpha}(T)}{\sqrt{T}/\log(T)}\geq c'>0,\quad \alpha>4. \tag{1.29}$$

Our Theorem 1.1 above generalizes the estimate (1.29) to shifted combinations of $\zeta_{\alpha}(s)$. Moreover, when $z=0$ and $c_1=1$, $\lambda_1=0$, $c_2=c_3=...=0$, it actually extends (1.29) also to the range $0<\alpha\leq 4$. As remarked in [25], when α is any integer greater than 3, it is known that $N_{\alpha}(0,T)\asymp T$ [28] and so, in this case, (1.29) does not say anything new. However, for non-integral α, this result seems to be novel.

Remark 1.4. Analogues of our Theorem 1.1 can be proved when $\zeta_{\alpha}(s)$ is replaced by any of the meromorphic Dirichlet series mentioned in [24]. For example, it is still valid if we replace $\zeta_{\alpha}(s)$ by an Epstein zeta function $\zeta(s,Q)$ attached to an integral quadratic form $Q(m,n)=Am^2+Bmn+Cn^2$ such that $\sqrt{4AC-B^2}\equiv 2 \mod 4$. Of course, the admissible region for z in this case would have to depend on the discriminant of Q.

Remark 1.5. A lower bound for d appearing in (1.27) can be explicitly calculated. A simple numerical computation (see the proof of Lemma 4.2 below) gives the value $d=\frac{1}{36\pi}$ as admissible for d.

Remark 1.6. Since the method of proof of Theorem 1.2 works well for any entire Dirichlet series, there is an analogue of this result for some Dirichlet $L-$functions. See [[24], Theorem 1.3] for details. Furthermore, we can use this method to prove (1.27) for a general combination involving shifts of $\zeta_\alpha(s)$. Since $s\left(s-\frac{\alpha}{2}\right)\eta_\alpha(s)$ is an entire function of s, then our proof of Theorem 1.2 can be applied to study the number of zeros of the function

$$\sum_{j\neq 0} c_j\,(s+i\lambda_j)\left(s+i\lambda_j-\frac{\alpha}{2}\right)\eta_\alpha\,(s+i\lambda_j)\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\}, \quad (1.30)$$

whenever z satisfies (1.22). If $\tilde{N}_{\alpha,z}(T)$ denotes the number of zeros of the above combination that can be written in the form $s=\frac{\alpha}{4}+it$, $0\leq t\leq T$, then the following estimate takes place

$$\limsup_{T\to\infty}\frac{\tilde{N}_{\alpha,z}(T)}{\sqrt{T}}\geq\tilde{c}, \quad (1.31)$$

for some $\tilde{c}>0$. In particular, when we reduce the combination (1.30) to the case where $\lambda_1=0$ and $c_j=0$ for any $|j|\geq 2$, we see that, besides (1.29), we also have $N_\alpha(T)=\Omega\left(T^{\frac{1}{2}}\right)$.

Remark 1.7. Using the properties of the slash operator, it is possible to establish Theorem 1.2 when $f(z)$ is a cusp form of weight k on a congruence subgroup $\Gamma_0(N)$ with N being a perfect square. We have to put the additional condition that $\left(f|W_N\right)(z)=\pm f(z)$, where W_N is the Fricke involution. The same method also works for half-integral weight cusp forms in $\Gamma_0(4N)$, N being a perfect square.

2 Lemmas for the proof of Theorem 1.1

We start by recalling Stirling's formula for the Gamma function,

$$\Gamma(\sigma+it)=(2\pi)^{\frac{1}{2}}\,t^{\sigma+it-\frac{1}{2}}\,e^{-\frac{\pi t}{2}-it+\frac{i\pi}{2}(\sigma-\frac{1}{2})}\left(1+\frac{1}{12(\sigma+it)}+O\left(\frac{1}{t^2}\right)\right),\quad t\to\infty, \quad (2.1)$$

valid whenever $-\infty<\sigma_1\leq\sigma\leq\sigma_2<\infty$. A similar formula can be written for $t<0$ as t tends to $-\infty$ by using the fact that $\Gamma(\overline{s})=\overline{\Gamma(s)}$. Of course, a direct consequence of this exact version is

$$|\Gamma(\sigma+it)|=(2\pi)^{\frac{1}{2}}\,|t|^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right),\quad |t|\to\infty. \quad (2.2)$$

Let us now recall some basic facts about the Dirichet series $\zeta_\alpha(s)$. It is well-known that the theta function $\vartheta_3(\tau):=\sum_{n\in\mathbb{Z}}e^{\pi i n^2\tau}$ is a modular form of weight $\frac{1}{2}$ with a multiplier system with respect to the theta group Γ_θ (see [17], p. 15-16 for details). Therefore, for any $\alpha>0$, $\vartheta_3^\alpha(\tau)$ is a modular form of weight $\alpha/2$ with a multiplier system on the same group. By definition, $r_\alpha(n)$ are the Fourier coefficients of the expansion of $\vartheta_3(\tau)$ at the cusp $i\infty$, this is (cf. [24] and [17] for a clearer explanation of this expansion)

$$\vartheta_3^\alpha(ix):=\theta^\alpha(x)=1+\sum_{n=1}^{\infty}r_\alpha(n)\,e^{-\pi n\,x},\quad x>0. \quad (2.3)$$

The order of growth of $r_\alpha(n)$ as $n\to\infty$ is determined by classical estimates due to Petersson and Lehner [17,18]. These estimates show that

$$r_\alpha(n)\ll_\alpha\begin{cases}n^{\alpha/2-1} & \alpha>4\\ n^{\alpha/2-1}\log(n) & \alpha=4\\ n^{\alpha/4} & 0<\alpha<4.\end{cases} \quad (2.4)$$

Thus, we can rigorously define the Dirichlet series (1.6) in the form

$$\zeta_\alpha(s) := \sum_{n=1}^{\infty} \frac{r_\alpha(n)}{n^s}, \quad \text{Re}(s) > \sigma_\alpha := \begin{cases} \frac{\alpha}{2} & \alpha \geq 4 \\ 1 + \frac{\alpha}{4} & 0 < \alpha < 4 \end{cases}. \tag{2.5}$$

Mimicking Riemann's paper, one can easily show that $\zeta_\alpha(s)$ can be analytically continued to the entire complex plane as a meromorphic function with a simple pole located at $s = \frac{\alpha}{2}$, whose residue is $\text{Res}_{s=\alpha/2}\zeta_\alpha(s) = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}$. Moreover, it satisfies Hecke's functional equation

$$\eta_\alpha(s) := \pi^{-s}\Gamma(s)\,\zeta_\alpha(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2} - s\right)\,\zeta_\alpha\left(\frac{\alpha}{2} - s\right) := \eta_\alpha\left(\frac{\alpha}{2} - s\right). \tag{2.6}$$

It also follows from the Phragmén-Lindelöf principle that $\zeta_\alpha(s)$ obeys to the convex estimate

$$\zeta_\alpha\left(\sigma + it\right) \ll_\alpha |t|^{\sigma_\alpha - \sigma + \delta}, \quad \frac{\alpha}{2} - \sigma_\alpha - \delta < \sigma < \sigma_\alpha + \delta, \tag{2.7}$$

for any $\delta > 0$ and σ_α defined by (2.5).

In order to estimate the series given in (1.20), we need an asymptotic formula for the confluent hypergeometric function valid when $|s| \to \infty$. Following the reasoning in [[6], p. 379], recall that the Whittaker function $M_{\lambda,\mu}(z)$ has the asymptotic formula, [[19], p. 341, eq. (13.21.1)]

$$M_{\lambda,\mu}(z) = \frac{z^{1/4}}{\sqrt{\pi}}\lambda^{-\mu-\frac{1}{4}}\Gamma\left(2\mu+1\right)\,\cos\left(2\sqrt{\lambda z} - \frac{\pi}{4} - \mu\pi\right) + O\left(|\lambda|^{-\mu-\frac{3}{4}}\right) \tag{2.8}$$

as $|\lambda| \to \infty$ and z such that $|\arg(\lambda z)| < 2\pi$. Furthermore, since

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}\,{}_1F_1\left(\mu - \lambda + \frac{1}{2};\, 2\mu+1;\, z\right), \tag{2.9}$$

we see that, as $|t| \to \infty$ and fixed $z \in \mathbb{R}$, the substitutions in (2.8) and (2.9) give the bound

$$\left|{}_1F_1\left(\frac{\alpha}{4} - it;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right)\right| = \left(\frac{|z|}{2}\right)^{-\frac{\alpha}{2}} e^{\frac{z^2}{8}}\left\{\Gamma\left(\frac{\alpha}{2}\right)\sqrt{\frac{|z|}{2\pi}}\,|t|^{\frac{1-\alpha}{4}}\,\exp\left(\sqrt{\frac{|t|}{2}}\,|z|\right) + O\left(|t|^{-\frac{\alpha}{4}-\frac{1}{4}}\right)\right\}, \quad |t| \to \infty. \tag{2.10}$$

This estimate is more than enough to justify most of the steps in this paper. By Stirling's formula and (2.7), one can see that

$$\left|\eta_\alpha\left(\frac{\alpha}{4} + it\right)\right| \ll_\alpha |t|^{A(\alpha)}\,e^{-\frac{\pi}{2}|t|}, \quad |t| \to \infty, \tag{2.11}$$

where $A(\alpha) = \sigma_\alpha - \frac{1}{2}$, σ_α being given by (2.5). When combined with (2.10), the convex estimate (2.11) yields the bound

$$\left|\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4} + it\right)\right| \leq \sum_{j\neq 0}\left|c_j\,\eta_\alpha\left(\frac{\alpha}{4} + i\left(t + \lambda_j\right)\right)\text{Re}\left({}_1F_1\left(\frac{\alpha}{4} - i\left(t+\lambda_j\right);\, \frac{\alpha}{2};\, \frac{z^2}{4}\right)\right)\right| \ll_{\alpha,z} C_\lambda \sum_{j=1}^{\infty}|c_j|\,|t|^{B(\alpha)}\,e^{-\frac{\pi}{2}|t| + |z|\sqrt{|t|}}, \tag{2.12}$$

where we have used the fact that $\left(\lambda_j\right)_{j\in\mathbb{N}}$ is a bounded sequence and $\left(c_j\right)_{j\in\mathbb{N}} \in \ell^1$. The term C_λ only stands for a positive constant which depends on the bounds of the sequence $\left(\lambda_j\right)_{j\in\mathbb{N}}$.

Next, let us note that an explicit way of writing the first equality in (1.2) is

$$1 + 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi x}\,z\right)^{1-\frac{\alpha}{2}}\,\sum_{n=1}^{\infty} r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,x}\,J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n\,x}\,z\right)$$

$$= \frac{e^{-\frac{z^2}{4}}}{x^{\alpha/2}}\left\{1 + 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\frac{\pi}{x}}\,z\right)^{1-\frac{\alpha}{2}}\,\sum_{n=1}^{\infty} r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{\alpha}{2}-1}\left(\sqrt{\frac{\pi n}{x}}\,z\right)\right\}, \tag{2.13}$$

whenever $\text{Re}(x) > 0$ and $z \in \mathbb{C}$. Since the summation formula (2.13) transforms a generalized $\psi-$function involving the Bessel functions of the first kind, it will be useful in our next argument to have a bound for the modified Bessel function, $I_\nu(z)$, $\nu > -1$. In the previous paper [[24], p. 42] we have used the famous Hankel expansion for $I_\nu(z)$, $|z| \to \infty$. However, for our alternative argument we need a bound for $|I_\nu(z)|$ that is somewhat uniform in ν and z.

The following simple bound will be useful: first, let us note that, if $\nu \geq 0$ and $k \in \mathbb{N}_0$,

$$\Gamma(k+\nu+1) = \Gamma(\nu+1)(\nu+1)...(\nu+k) \geq k!\,\Gamma(\nu+1),$$

which gives, after the use of the power series of $I_\nu(z)$,

$$|I_\nu(z)| \leq \left(\frac{|z|}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{|z|^{2k}}{2^{2k}k!\,\Gamma(k+\nu+1)} \leq \frac{(|z|/2)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{|z|^{2k}}{2^{2k}(k!)^2} \leq \frac{(|z|/2)^\nu}{\Gamma(\nu+1)} \left\{\sum_{k=0}^{\infty} \frac{|z|^k}{2^k k!}\right\}^2 = \left(\frac{|z|}{2}\right)^\nu \frac{e^{|z|}}{\Gamma(\nu+1)}, \quad \nu \geq 0. \tag{2.14}$$

If, on the other hand, we want to extend (2.14) to $-1 < \nu < 0$, we begin to note that, for $k \geq 1$,

$$\Gamma(k+\nu+1) = \Gamma(\nu+2)\,(\nu+2)...(\nu+k) \geq (k-1)!\,\Gamma(\nu+2).$$

Hence, if $-1 < \nu < 0$,

$$|I_\nu(z)| \leq \left(\frac{|z|}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{|z|^{2k}}{2^{2k}k!\,\Gamma(k+\nu+1)} \leq \frac{(|z|/2)^\nu}{\Gamma(\nu+1)} + \frac{(|z|/2)^{\nu+2}}{\Gamma(\nu+2)} \sum_{k=0}^{\infty} \frac{|z|^{2k}}{2^{2k}(k!)^2} < \left(\frac{|z|}{2}\right)^\nu \frac{e^{|z|}}{\Gamma(\nu+2)} \left(1 + \frac{|z|^2}{4}\right). \tag{2.15}$$

To proceed, let us recall some functions and facts already mentioned at the introduction. From this point on, we will let $\alpha > 0$ and $r_\alpha(n)$ be defined as the coefficients of the $q-$expansion of $\theta^\alpha(x)$. For $\text{Re}(x) > 0$ and $z \in \mathbb{C}$, we let $\psi_\alpha(x, z)$ denote the analogue of Jacobi's $\psi-$function,

$$\psi_\alpha(x, z) = 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right) \left(\sqrt{\pi x}\, z\right)^{1-\frac{\alpha}{2}} \sum_{n=1}^{\infty} r_\alpha(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\pi n x}\, J_{\frac{\alpha}{2}-1}(\sqrt{\pi\, n\, x}\, z). \tag{2.16}$$

As stated above, the transformation formula for $\psi_\alpha(x, z)$, (2.13), will play a major role in this paper. We shall also need some auxiliary summation formulas given in [[24], pp. 37-40]. Analogously to $r_\alpha(n)$, we can consider positive powers of the theta function

$$\vartheta_2(\tau) = 2\sum_{n=0}^{\infty} e^{\pi i \tau \left(n+\frac{1}{2}\right)^2}, \quad \text{Im}(\tau) > 0, \tag{2.17}$$

from the Fourier expansion at the cusp $\tau = i\infty$. We can create a new arithmetical function $\tilde{r}_\alpha(m)$ as the coefficient coming from the expansion (see [[24], p.36, eq. (3.5)] for details)

$$\begin{aligned}
\vartheta_2^\alpha(ix) := \theta_2^\alpha(x) &= \left(2\sum_{n=0}^{\infty} e^{-\pi x\left(n+\frac{1}{2}\right)^2}\right)^\alpha = \left(2\,e^{-\frac{\pi x}{4}} + 2\sum_{n=1}^{\infty} e^{-\pi x(n+\frac{1}{2})^2}\right)^\alpha \\
&= 2^\alpha e^{-\frac{\pi\alpha x}{4}} \left(1 + \sum_{n=1}^{\infty} e^{-\pi x(n^2+n)}\right)^\alpha = 2^\alpha\, e^{-\frac{\pi\alpha x}{4}} \sum_{j=0}^{\infty} \binom{\alpha}{j} \left(\sum_{n=1}^{\infty} e^{-\pi x(n^2+n)}\right)^j \\
&:= \sum_{m=0}^{\infty} \tilde{r}_\alpha(m)\, e^{-\pi\left(m+\frac{\alpha}{4}\right)x}, \qquad \text{Re}(x) > 0.
\end{aligned} \tag{2.18}$$

Analogously to $r_\alpha(n)$, the coefficients $\tilde{r}_\alpha(n)$ grow polynomially with n (cf.[[24], Lemma 3.1]). Note also that $\tilde{r}_\alpha(0) := 2^\alpha$ by the construction (2.18). Using Jacobi's transformation formula for $\vartheta_2(\tau)$, one can derive a summation formula connecting $\tilde{r}_\alpha(n)$ with $r_\alpha(n)$. The next lemma, given in [[24], p. 40, Lemma 3.3], establishes this correspondence.

Lemma 2.1 ([24], Lemma 3.3)**.** *Let $r_\alpha(n)$ be the coefficients of the series expansion of $\theta^\alpha(x)-1$, (1.5), and $\tilde{r}_\alpha(n)$ be defined by (2.18). Then, for Re$(x)>0$ and $y\in\mathbb{C}$, the following identity holds*

$$\sum_{n=1}^{\infty}(-1)^n\, r_\alpha(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\pi n x}\, J_{\frac{\alpha}{2}-1}(y\sqrt{\pi\, n}) = -\frac{y^{\frac{\alpha}{2}-1}\pi^{\frac{\alpha}{4}-\frac{1}{2}}}{2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)}$$
$$+\frac{e^{-\frac{y^2}{4x}}}{x}\sum_{n=0}^{\infty}\tilde{r}_\alpha(n)\,\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\frac{\pi}{x}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\, y}{x}\right). \tag{2.19}$$

The previous lemma was necessary in [24] to prove that, for z under the condition (1.8) and any $m\in\mathbb{N}_0$,

$$\frac{d^m}{d\omega^m}\left(1+\psi_\alpha\left(e^{2i\omega},z\right)\right)\to 0, \quad \text{as} \quad \omega\to\frac{\pi}{4}^{-}. \tag{2.20}$$

This curious transformation formula will now play an essential role in giving a more precise version of (2.20). This is done in the next lemma, where we shall estimate uniformly the derivatives of the function $\psi_\alpha\left(ie^{-2iu},z\right)$, $0<u<\frac{\pi}{4}$.

Lemma 2.2. *Let $\psi_\alpha(x,z)$ be the generalized Jacobi's $\psi-$function defined by (1.11) and assume that $z\in\mathbb{R}$ satisfies the condition*

$$-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\leq z\leq\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}. \tag{2.21}$$

Then there are two positive constants A and C (depending only on α) such that, for any $0<u<\frac{\pi}{4}$,

$$\left|\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right| < C\,\frac{2^{7k}k!}{u^{\frac{\alpha}{2}+k}}e^{-\frac{A}{u}}, \quad \alpha>2 \tag{2.22}$$

and

$$\left|\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right| < C\,\frac{2^{7k}k!}{u^{\frac{\alpha}{2}+k+2}}e^{-\frac{A}{u}}, \quad 0\leq\alpha\leq 2. \tag{2.23}$$

Proof. Let us fix $0<u_0<\frac{\pi}{4}$. We shall prove (2.22) only, as the bound (2.23) can be similarly deduced. At the end of the proof we just outline the difference in getting (2.23). The derivative of the function $\psi_\alpha\left(ie^{-2iu},z\right)$ at the point u_0 is taken by integrating along a circle with center u_0 and having radius λu_0, $0<\lambda<1$. Throughout this proof, the parameter λ will be arbitrary up to the point where the condition (2.21) enters in the argument. For any $w\in D_{\lambda u_0}(u_0):=\text{int}(C_{\lambda u_0}(u_0))$, it is simple to check that Re$\left(ie^{-2iw}\right)>0$, so that $\psi_\alpha\left(ie^{-2iw},z\right)$ is analytic inside the circle $C_{\lambda u_0}(u_0)$ [[24], Corollary 2.2]. Hence, by Cauchy's formula,

$$\left[\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right]_{u=u_0} = \frac{k!}{2\pi i}\int_{C_{\lambda u_0}(u_0)}\frac{1+\psi_\alpha\left(ie^{-2iw},z\right)}{(w-u_0)^{k+1}}\,dw. \tag{2.24}$$

Now, let us bound trivially the integral above: to do it, it will be crucial to employ the transformation formula (2.19). Indeed, since $ie^{-2iw}=i+2e^{-iw}\sin(w)$ and, by definition (1.11),

$$1+\psi_\alpha(ie^{-2iw},z) := 1+2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi}\,e^{i\frac{\pi}{4}}e^{-iw}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}(-1)^n r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-2\pi n e^{-iw}\sin(w)}\,J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n}\,e^{i\left(\frac{\pi}{4}-w\right)}\,z\right),$$

an application of (2.19) with $x = 2e^{-iw}\sin(w)$ and $y = e^{i\left(\frac{\pi}{4}-w\right)}z$ gives

$$\begin{aligned}
1+\psi_\alpha(ie^{-2iw},z) &= 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi}\,e^{i\left(\frac{\pi}{4}-w\right)}z\right)^{1-\frac{\alpha}{2}} \\
&\times \frac{e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\sum_{n=0}^{\infty}\tilde{r}_\alpha(n)e^{-\frac{\pi i}{2}\left(n+\frac{\alpha}{4}\right)}\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}}e^{-\frac{\pi}{2\tan(w)}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,e^{\frac{i\pi}{4}}z}{2\sin(w)}\right). \qquad (2.25)
\end{aligned}$$

We shall bound $|1+\psi_\alpha\left(ie^{-2iw},z\right)|$ by estimating the second expression on the right-hand side of (2.25). We will track each factor independently. Clearly, for any $w \in C_{\lambda u_0}(u_0)$,

$$\left|2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi}\,e^{i\frac{\pi}{4}}e^{-iw}z\right)^{1-\frac{\alpha}{2}}\right| \leq 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{1}{2}-\frac{\alpha}{4}}e^{\frac{\pi}{4}\left|1-\frac{\alpha}{2}\right|}|z|^{1-\frac{\alpha}{2}},$$

where we have noted that $\lambda u_0 \leq \frac{\pi}{4}$. Next,

$$\begin{aligned}
\left|\frac{e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\right| &= \frac{e^{-\frac{z^2}{8}-\mathrm{Im}(w)}\exp\left(\frac{z^2}{8}\mathrm{Im}\left(\frac{1}{\tan(w)}\right)\right)}{2\sqrt{\sin^2\left(\mathrm{Re}(w)\right)+\sinh^2\left(\mathrm{Im}(w)\right)}} \\
&= \frac{e^{-\frac{z^2}{8}-\mathrm{Im}(w)}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right)}{2\sqrt{\sin^2\left(\mathrm{Re}(w)\right)+\sinh^2\left(\mathrm{Im}(w)\right)}} \\
&\leq \frac{e^{-\frac{z^2}{8}+\lambda u_0}}{2\sin\left(\mathrm{Re}(w)\right)}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right),
\end{aligned}$$

where in the last step we just have used the fact that, for any $w \in C_{\lambda u_0}(u_0)$, $|\mathrm{Im}(w)| \leq \lambda u_0$. Moreover, we have used the explicit expression

$$\frac{1}{\tan(w)} = \frac{\sin(2\mathrm{Re}(w)) - i\,\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}. \qquad (2.26)$$

Using now the elementary Jordan inequality $\sin(x) > \frac{2x}{\pi}$, $0 < x < \frac{\pi}{2}$, and the fact that $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0 < \frac{\pi}{2}$, we obtain

$$\begin{aligned}
\left|\frac{e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\right| &< \frac{\pi e^{-\frac{z^2}{8}+\lambda u_0}}{4\mathrm{Re}(w)}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right) \\
&\leq \frac{\pi e^{-\frac{z^2}{8}+\lambda u_0}}{4(1-\lambda)u_0}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right), \quad w \in C_{\lambda u_0}(u_0). \qquad (2.27)
\end{aligned}$$

We now address each term of the series on the right-hand side of (2.25). Since we first aim at proving (2.22), we assume here that $\alpha > 2$, so that $\frac{\alpha}{2}-1 > 0$. Using the first uniform inequality for $I_\nu(z)$, (2.14), we have that the term involving the Bessel function admits the bound

$$\begin{aligned}
\left|I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,e^{\frac{i\pi}{4}}z}{2\sin(w)}\right)\right| &\leq \frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)|\sin(w)|^{\frac{\alpha}{2}-1}}\exp\left(\frac{\sqrt{\pi\left(n+\frac{\alpha}{4}\right)}\,|z|}{2\,|\sin(w)|}\right) \\
&\leq \frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)\sin^{\frac{\alpha}{2}-1}\left(\mathrm{Re}(w)\right)}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right) \\
&\leq \frac{\pi^{\frac{3\alpha}{4}-\frac{3}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\frac{3\alpha}{2}-3}\Gamma\left(\frac{\alpha}{2}\right)(1-\lambda)^{\frac{\alpha}{2}-1}u_0^{\frac{\alpha}{2}-1}}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right). \qquad (2.28)
\end{aligned}$$

In the second inequality above we have used the fact that $|\sin(w)| \geq |\sin(\mathrm{Re}(w))|$ and in the third inequality we have once more invoked Jordan's inequality $\sin(x) > \frac{2x}{\pi}$, $0 < x < \frac{\pi}{2}$, together with the fact that $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0 < \frac{\pi}{2}$. Returning to (2.25) and recalling once more (2.26), from (2.27) and (2.28) we deduce

$$\left|1+\psi_\alpha(ie^{-2iw},z)\right| < d_\alpha \frac{\pi^{\frac{\alpha}{2}} e^{-\frac{z^2}{8}+\lambda u_0}}{2^\alpha (1-\lambda)^{\frac{\alpha}{2}} u_0^{\frac{\alpha}{2}}} \sum_{n=0}^{\infty} |\tilde{r}_\alpha(n)| \exp\left\{-\frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))} P_w\left(\sqrt{n+\frac{\alpha}{4}}\right)\right\}, \tag{2.29}$$

where d_α is some constant depending on α and $P_w(X)$ is the real-valued polynomial

$$P_w(X) := X^2 - \sqrt{\frac{2}{\pi}} \frac{|z|}{\sin(2\mathrm{Re}(w))} \sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\, X + \frac{z^2}{4\pi}\frac{\sinh(2\mathrm{Im}(w))}{\sin(2\mathrm{Re}(w))}. \tag{2.30}$$

The inequality (2.29) is valid for any $w \in C_{\lambda u_0}(u_0)$. Taking out the first term of the series on (2.29) and using the fact that $\tilde{r}_\alpha(0) := 2^\alpha$, we get from (2.29)

$$\begin{aligned}\left|1+\psi_\alpha(ie^{-2iw},z)\right| < \frac{d_\alpha\, \pi^{\frac{\alpha}{2}} e^{-\frac{z^2}{8}+\frac{u_0}{2}}}{(1-\lambda)^{\frac{\alpha}{2}} u_0^{\frac{\alpha}{2}}} \exp\left[-\frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))} P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right] \\ \times \sum_{n=0}^{\infty}\left|\frac{\tilde{r}_\alpha(n)}{\tilde{r}_\alpha(0)}\right| \exp\left[-Q_w(n)\right].\end{aligned} \tag{2.31}$$

The exponent showing up in the infinite series is explicitly

$$Q_w(n) := \frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\left[P_w\left(\sqrt{n+\frac{\alpha}{4}}\right) - P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right],$$

where $P_w(X)$ is the polynomial given by (2.30). We will show below that there is a choice of λ such that, for any z satisfying (2.21), $Q_w(n) > 0$, $\forall n \in \mathbb{N}$. Since $|\tilde{r}_\alpha(n)| < A_\alpha\, n^{\frac{\alpha}{2}}$ (see [[24], p. 38]), the infinite series that appears on the right-hand side of (2.31) is convergent. The remaining part of our proof will be to find a uniform bound for it, with a constant depending on λ and α. We want to find an upper bound for

$$\sum_{n=0}^{\infty}\left|\frac{\tilde{r}_\alpha(n)}{\tilde{r}_\alpha(0)}\right| \exp\left[-\frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\left\{n - \sqrt{\frac{2}{\pi}}|z|\frac{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}{\sin(2\mathrm{Re}(w))}\left(\sqrt{n+\frac{\alpha}{4}}-\frac{\sqrt{\alpha}}{2}\right)\right\}\right].$$

Since $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0$, we know from Jordan's inequality

$$\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w)) \geq 1 - \cos(2(1-\lambda)u_0) = \int_0^{2(1-\lambda)u_0} \sin(t)\, dt > \frac{2}{\pi}\int_0^{2(1-\lambda)u_0} t\, dt = \frac{4(1-\lambda)^2}{\pi} u_0^2,$$

which implies

$$\frac{1}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}} < \frac{\sqrt{\pi}}{2(1-\lambda)u_0}. \tag{2.32}$$

On the other hand, we can find the upper bound

$$\begin{aligned}\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w)) &= \int_0^{2\mathrm{Im}(w)} \sinh(t)\, dt + \int_0^{2\mathrm{Re}(w)} \sin(t)\, dt \leq \int_0^{2\mathrm{Im}(w)} t\, e^{\frac{t^2}{6}}\, dt + \int_0^{2\mathrm{Re}(w)} t\, dt \\ &\leq 2\mathrm{Im}(w)^2 e^{\frac{2}{3}\mathrm{Im}(w)^2} + 2\mathrm{Re}(w)^2 < 4e^{\frac{2\lambda^2}{3}u_0^2}(1+\lambda)^2 u_0^2,\end{aligned} \tag{2.33}$$

where we have used the fact that $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0$, $-\lambda u_0 \leq \mathrm{Im}(w) \leq \lambda u_0$. We have also used the known inequality

$$\sinh(x) \leq x\, e^{\frac{x^2}{6}}, \quad x > 0. \tag{2.34}$$

Since $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0 < \frac{\pi}{2}$, another application of Jordan's inequality gives

$$\sin\left(2\mathrm{Re}(w)\right) > \begin{cases} \frac{4}{\pi}\mathrm{Re}(w), & 0 < \mathrm{Re}(w) < \frac{\pi}{4} \\ 2 - \frac{4}{\pi}\mathrm{Re}(w), & \frac{\pi}{4} \leq \mathrm{Re}(w) < \frac{\pi}{2} \end{cases} \geq \frac{4}{\pi}\left(1-\lambda\right)u_0. \tag{2.35}$$

Thus, combining (2.33) with (2.35) and recalling that $0 < u_0 < \frac{\pi}{4}$, we get the inequality

$$\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w))} > \frac{(1-\lambda)}{\pi e^{\frac{2\lambda^2}{3}u_0^2}\left(1+\lambda\right)^2 u_0} > \frac{(1-\lambda)}{\pi e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2 u_0}. \tag{2.36}$$

Thus, the infinite series on the right-hand side of (2.31) admits the bound

$$\sum_{n=0}^{\infty}\left|\frac{\tilde{r}_{\alpha}(n)}{\tilde{r}_{\alpha}(0)}\right| \exp\left[-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right) - \cos\left(2\mathrm{Re}(w)\right)}\left\{n - \frac{\sqrt{\frac{2}{\pi}}\,|z|\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)}{\sin(2\mathrm{Re}(w))}\sqrt{\cosh\left(2\mathrm{Im}(w)\right) - \cos\left(2\mathrm{Re}(w)\right)}\right\}\right]$$
$$< \sum_{n=0}^{\infty}\left|\frac{\tilde{r}_{\alpha}(n)}{\tilde{r}_{\alpha}(0)}\right| \exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2} - \frac{\pi\,|z|}{2\sqrt{2}(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right].$$

Using the condition (2.21), one sees that the exponent in the previous expression is bounded by

$$\exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2} - \frac{\pi\,|z|}{2\sqrt{2}(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right]$$
$$\leq \exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2} - \frac{\pi^{\frac{3}{2}}\sqrt{\alpha}}{24(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right]$$
$$\leq \exp\left[-\frac{n}{u_0}\left\{\frac{1-\lambda}{2e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2} - \frac{\pi^{\frac{3}{2}}}{24(1-\lambda)}\right\}\right],$$

where in the last inequality we have used the mean value theorem for the function $f(x) = \sqrt{x+\frac{\alpha}{4}}$. If we select the value $\lambda = 0.01$, we have that

$$\frac{1-\lambda}{2e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2} - \frac{\pi^{\frac{3}{2}}}{24(1-\lambda)} > \frac{1}{4},$$

and so, since $0 < u_0 < \frac{\pi}{4}$,

$$\sum_{n=0}^{\infty}\left|\frac{\tilde{r}_{\alpha}(n)}{\tilde{r}_{\alpha}(0)}\right| \exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2} - \frac{\pi\,|z|}{2\sqrt{2}(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right]$$
$$\leq \sum_{n=0}^{\infty}\left|\frac{\tilde{r}_{\alpha}(n)}{\tilde{r}_{\alpha}(0)}\right| \exp\left[-\frac{n}{4u_0}\right] < \frac{A_{\alpha}}{2^{\alpha}}\sum_{n=0}^{\infty} n^{\frac{\alpha}{2}} \exp\left[-\frac{n}{\pi}\right] \leq M_{\alpha}. \tag{2.37}$$

Combining (2.37) with (2.31) and using the fact that $0 < u_0 < \frac{\pi}{4}$, the following bound holds (note that we are now selecting λ as equal to 0.01)

$$\left|1+\psi_{\alpha}(ie^{-2iw}, z)\right| < \frac{d_{\alpha}M_{\alpha}e^{\frac{\pi}{8}}\,\pi^{\frac{\alpha}{2}}\,e^{-\frac{z^2}{8}}}{(2u_0)^{\frac{\alpha}{2}}}\exp\left[-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right) - \cos\left(2\mathrm{Re}(w)\right)}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right], \tag{2.38}$$

which is almost (2.22) with $k=0$. To conclude the proof in this direction, we just need to bound uniformly the term

$$\exp\left[-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right],\quad w\in C_{\lambda u_0}(u_0)$$

under the hypothesis on z (2.21) and the choice $\lambda=0.01$. This can be done by appealing to (2.34), (2.32) and (2.36), which give

$$\exp\left[-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right]<\exp\left[-\frac{\alpha}{8}\frac{(1-\lambda)}{e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2u_0}+\sqrt{\frac{\alpha}{2}}\frac{\pi\,|z|}{4(1-\lambda)u_0}+\frac{\pi z^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{16(1-\lambda)^2u_0}\right]$$

$$\leq\exp\left[-\frac{\alpha}{u_0}\left\{\frac{1-\lambda}{8e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2}-\frac{\pi^{\frac{3}{2}}}{48(1-\lambda)}-\frac{\pi^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{1152(1-\lambda)^2}\right\}\right].\tag{2.39}$$

However, for $\lambda=0.01$, a numerical confirmation gives

$$\frac{1-\lambda}{8e^{\frac{\pi^2\lambda^2}{24}}\left(1+\lambda\right)^2}-\frac{\pi^{\frac{3}{2}}}{48(1-\lambda)}-\frac{\pi^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{1152(1-\lambda)^2}>0.004,$$

and so, returning to (2.38), we get from the previous inequality that

$$\left|1+\psi_\alpha(ie^{-2iw},z)\right|<\frac{d_\alpha M_\alpha e^{\frac{\pi}{8}}\,\pi^{\frac{\alpha}{2}}e^{-\frac{z^2}{8}}}{(2u_0)^{\frac{\alpha}{2}}}\exp\left[-\frac{0.04\,\alpha}{u_0}\right]=\frac{C}{u_0^{\alpha/2}}\,e^{-\frac{A}{u_0}}\tag{2.40}$$

where C and $A:=0.04\,\alpha$ only depend on α. This proves (2.22) for $k=0$. For the remaining cases, let us invoke Cauchy's integral formula (2.24) and use (2.40) to get

$$\left|\left[\frac{d^k}{du^k}\psi_\alpha\left(ie^{-2iu},z\right)\right]_{u=u_0}\right|\leq\frac{k!}{2\pi\left(\lambda u_0\right)^{k+1}}\int_{C_{\lambda u_0}(u_0)}\left|1+\psi_\alpha\left(ie^{-2iw},z\right)\right||dw|<C\,\frac{k!}{u_0^{\frac{\alpha}{2}+k}\lambda^k}e^{-\frac{A}{u_0}}.\tag{2.41}$$

Finally, under the choice $\lambda=0.01$, we know $\frac{1}{\lambda}<2^7$, and so we derive our estimate (2.22) in its final form.

To get the second formula (2.23), we use the bound (2.15) for $I_\nu(z)$, $-1<\nu<0$, and we invoke the same kind of inequalities as in (2.28) to obtain

$$\left|I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,e^{\frac{i\pi}{4}}z}{2\sin(w)}\right)\right|<\frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}+1\right)|\sin(w)|^{\frac{\alpha}{2}-1}}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right)\left(1+\frac{\pi^3(n+\frac{\alpha}{4})|z|^2}{64(1-\lambda)^2u_0^2}\right)$$

$$\leq\mathcal{A}\,\frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}+\frac{1}{2}}|z|^{\frac{\alpha}{2}+1}}{2^{\frac{3\alpha}{2}-3}\Gamma\left(\frac{\alpha}{2}+1\right)(1-\lambda)^{\frac{\alpha}{2}+1}u_0^{\frac{\alpha}{2}+1}}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right),\tag{2.42}$$

for some absolutely large constant $\mathcal{A}$. There is no formal difference between (2.42) and (2.28), the only exception being in the power of $1/u_0$. Thus, the same argument carries through with an extra factor of $1/u_0^2$. This gives (2.23) and finishes the proof. □

Remark 2.1. Of course, the condition (1.22) may be improved if we take a more careful choice of the parameter λ in the previous proof. Moreover, when $\nu>-\frac{1}{2}$, the uniform bounds (2.14) and (2.15) can be replaced by the better inequality,

$$|I_\nu(z)|\leq\left(\frac{|z|}{2}\right)^\nu\frac{e^{|\mathrm{Re}(z)|}}{\Gamma(\nu+1)},\quad\nu>-\frac{1}{2},\tag{2.43}$$

which can be obtained via the integral representation [[19], p. 252, eq. (10.32.2)]

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^{1} \left(1-t^2\right)^{\nu-\frac{1}{2}} e^{zt}\, dt, \quad \nu > -\frac{1}{2}.$$

Using (2.43) instead of (2.14) in the proof above, one may enlarge the interval $\left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}, \frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]$ in the condition (2.21) to $\left[-\frac{1}{4}\sqrt{\frac{\pi\alpha}{2}}, \frac{1}{4}\sqrt{\frac{\pi\alpha}{2}}\right]$ when $\alpha \geq 1$. Therefore, there are some cases where technical improvements lead to stronger results than those stated in our Theorems 1.1 and 1.2.

We now give another lemma that will be crucial in the proof of our result.

Lemma 2.3. *Let* $h : \mathbb{C} \longmapsto \mathbb{C}$ *be an analytic function. If* $z \in \left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}, \frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]$ *then, for every* $p \in \mathbb{N}_0$, *the following relation holds*

$$\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right\} = -2\sinh\left(\frac{z^2}{8}\right) h^{(2p)}(u) + \mathcal{A}_{\alpha,p}(u,z), \tag{2.44}$$

where, for any $0 < u < \frac{\pi}{4}$, $\mathcal{A}_{\alpha,p}(u,z)$ *satisfies the inequalities*

$$|\mathcal{A}_{\alpha,p}(u,z)| < D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\,\|h\|_{L^\infty(C_1(0))}, \quad \alpha > 2, \tag{2.45}$$

$$|\mathcal{A}_{\alpha,p}(u,z)| < D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2+2p}}\,\|h\|_{L^\infty(C_1(0))}, \quad 0 \leq \alpha \leq 2, \tag{2.46}$$

with A *and* D *depending only on* α.

Proof. First observe that

$$\begin{aligned}\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right\} &= -2\sinh\left(\frac{z^2}{8}\right) h^{(2p)}(u) + e^{\frac{z^2}{8}}\,\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(1+\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right\} \\ &:= -2\sinh\left(\frac{z^2}{8}\right) h^{(2p)}(u) + \mathcal{A}_{\alpha,p}(u,z). \end{aligned} \tag{2.47}$$

By the well-known Cauchy's estimates for the coefficients of analytic functions and (2.22), we obtain

$$\begin{aligned}\left|e^{\frac{z^2}{8}}\,\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(1+\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right\}\right| &\leq e^{\frac{\pi\alpha}{576}}\sum_{k=0}^{2p}\binom{2p}{k}\left|h^{(2p-k)}(u)\right|\left|\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right| \\ &\leq e^{\frac{\pi\alpha}{576}}\,\|h\|_{L^\infty(D(0,1))}\sum_{k=0}^{2p}\binom{2p}{k}(2p-k)!\left|\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right| \\ &< Ce^{\frac{\pi\alpha}{576}}\,\|h\|_{L^\infty(D(0,1))}\,\frac{(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}}}\sum_{k=1}^{2p}\frac{2^{7k}}{u^k} \\ &< C'\,\|h\|_{L^\infty(C_1(0))}\,\frac{(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\,\frac{(128)^{2p+1}-u^{2p+1}}{128-u} \\ &< C''\,\|h\|_{L^\infty(C_1(0))}\,\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}} \end{aligned}$$

where we have invoked the maximum modulus principle. This completes the proof of (2.45). The proof of (2.46) is analogous. □

To conclude this section, let us recall the useful notation introduced in [8]. From now on, it will be convenient to write the terms of the sequence $(\lambda_j)_{j\in\mathbb{N}}$ in polar coordinates

$$\frac{i\alpha}{2}-2\lambda_j:=r_j\,e^{i\theta_j},\;\;0<\theta_j<\pi. \tag{2.48}$$

According to our assumption 1, we need to extend this definition to negative indices: since $\lambda_{-j}=-\lambda_j$, then we have that $r_{-j}=r_j$ and $\theta_{-j}=\pi-\theta_j$.

Using these polar coordinates, we shall introduce a simple lemma which gives an integral representation for the moments of $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$. Its proof follows closely the steps given in [[24], pp. 44-45], so we shall omit some details.

Lemma 2.4. *Let $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$, $p\in\mathbb{N}_0$ and $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ be the function defined by (1.23). Then the following integral representation holds*

$$\int_0^{\infty}t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,\cosh\left(2\omega t\right)\,dt=-\frac{8\pi}{2^{2p}}\sum_{j=1}^{\infty}c_jr_j^{2p}\left[\cos\left(2p\theta_j\right)\,\cos\left(\frac{\omega\alpha}{2}\right)\cosh\left(2\omega\lambda_j\right)+\sin\left(2p\theta_j\right)\,\sin\left(\frac{\omega\alpha}{2}\right)\sinh\left(2\omega\lambda_j\right)\right]$$
$$+\frac{2\pi\,e^{z^2/8}}{2^{2p}}\,Re\left(\frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j\neq0}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)\right)\right\}\right), \tag{2.49}$$

where $\psi_{\alpha}(x,z)$ is the generalized Jacobi's $\psi-$function (1.11) and (r_j,θ_j) are the polar coordinates attached to $(\lambda_j)_{j\in\mathbb{N}}$ defined by (2.48).

Proof. We start by using the integral representation given at the introduction (1.10), replacing there x by $e^{2i\omega}$, $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$: we obtain the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_{\alpha}\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\,e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,\psi_{\alpha}\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\alpha}{2}}\,e^{-z^2/4}. \tag{2.50}$$

Using Kummer's identity [[1], p. 191, eq. (4.1.11)],

$${}_1F_1\left(a;\,c;\,x\right)=e^{x}\,{}_1F_1\left(c-a;\,c;\,-x\right), \tag{2.51}$$

an equivalent version of (2.50) reads

$$\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_{\alpha}\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\,e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,e^{z^2/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\alpha}{2}}\,e^{-z^2/8}. \tag{2.52}$$

In [[24], p. 43, eq. (3.42)], we have used (2.52) and the estimates (2.12) to obtain the following formula

$$\int_{-\infty}^{\infty}t^{2p}\,F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,e^{2\omega t}\,dt=-\frac{8\pi}{2^{2p}}\sum_{j=1}^{\infty}c_j\,r_j^{2p}e^{-2\omega\lambda_j}\,\cos\left(2p\,\theta_j+\frac{\omega\alpha}{2}\right)$$
$$+\frac{4\pi}{2^{2p}}\,\text{Re}\left(e^{z^2/8}\,\frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j=1}^{\infty}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)\right)\right\}\right),$$

where $0<\omega<\frac{\pi}{4}$, $z\in\mathbb{C}$, $\psi_\alpha(x,z)$ is the generalized theta function (1.11) and $F_{z,\alpha}(s)$ is the function given by (cf. (1.9) above)

$$F_{z,\alpha}(s):=\sum_{j=1}^{\infty}c_j\,\eta_\alpha\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{\overline{z}^2}{4}\right)\right\}.$$

According to assumption 1, our symmetric shifted combination, $\tilde{F}_{z,\alpha}(s)$, has the same expression as $F_{z,\alpha}(s)$ with an additional extension to the negative integers (see (1.20) above). Therefore, assuming also that $z\in\mathbb{R}$ (see assumption 2), we immediately find that

$$\begin{aligned}\int_{-\infty}^{\infty}t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,e^{2\omega t}\,dt&=-\frac{8\pi}{2^{2p}}\sum_{j\neq0}c_j\,r_j^{2p}e^{-2\omega\lambda_j}\,\cos\left(2p\,\theta_j+\frac{\omega\alpha}{2}\right)\\&+\frac{4\pi}{2^{2p}}e^{z^2/8}\,\text{Re}\left(\frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j\neq0}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}\right).\end{aligned}\tag{2.53}$$

But it is clear that the first infinite series admits the expression

$$\begin{aligned}\sum_{j\neq0}c_j\,r_j^{2p}e^{-2\omega\lambda_j}\,\cos\left(2p\,\theta_j+\frac{\omega\alpha}{2}\right)&=\sum_{j=1}^{\infty}c_jr_j^{2p}\left[e^{-2\omega\lambda_j}\cos\left(2p\,\theta_j+\frac{\omega\alpha}{2}\right)+e^{2\omega\lambda_j}\cos\left(2p\,\theta_j-\frac{\omega\alpha}{2}\right)\right]\\&=2\sum_{j=1}^{\infty}c_jr_j^{2p}\left[\cos\left(2p\theta_j\right)\,\cos\left(\frac{\omega\alpha}{2}\right)\cosh\left(2\omega\lambda_j\right)+\sin\left(2p\theta_j\right)\,\sin\left(\frac{\omega\alpha}{2}\right)\sinh\left(2\omega\lambda_j\right)\right],\end{aligned}\tag{2.54}$$

because $\lambda_{-j}=-\lambda_j$, $r_{-j}=r_j$ and $\theta_{-j}=\pi-\theta_j$ by hypothesis. Since $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ is a real and an even function of t, using the previous expression allows to rewrite (2.53) in the form

$$\begin{aligned}\int_{0}^{\infty}t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,\cosh\left(2\omega t\right)\,dt&=-\frac{8\pi}{2^{2p}}\sum_{j=1}^{\infty}c_jr_j^{2p}\left[\cos\left(2p\theta_j\right)\,\cos\left(\frac{\omega\alpha}{2}\right)\cosh\left(2\omega\lambda_j\right)+\sin\left(2p\theta_j\right)\,\sin\left(\frac{\omega\alpha}{2}\right)\sinh\left(2\omega\lambda_j\right)\right]\\&+\frac{2\pi\,e^{z^2/8}}{2^{2p}}\,\text{Re}\left(\frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j\neq0}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}\right),\end{aligned}$$

which completes the proof. □

3 Proof of Theorem 1.1

3.1 Outline of the Proof

At the core of our method is the following lemma, which is an immediate consequence of a Theorem of Fejér [9,10,31].

Lemma 3.1. *Let $(p_n)_{n\in\mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$ with $p_0:=0$. Then the number of variations of sign in the interval $[0,a]$ of a real continuous function $f(x)$ is not less than the number of variations of sign of the following sequence*

$$\mathcal{F}_{p_n}(a):=\begin{cases}f(0) & n=0\\ \int_0^a f(t)\,t^{p_n-1}dt & n\geq1.\end{cases}$$

Our proof of Theorem 1.1 is essentially divided in four parts: in the first of these, we employ lemmas 2.3 and 2.4 to rewrite (2.49) in a form which is useful to apply the zero counting method suggested by the previous lemma. By the conditions of our Theorem 1.1, the sequence $(\lambda_j)_{j\in\mathbb{N}}$ attains its bounds and is made of distinct elements. Hence, we know that exists some $M \in \mathbb{N}$ such that

$$|\lambda_M| = \max_{j\in\mathbb{N}}\{|\lambda_j|\}, \quad \lambda_j \neq \lambda_M \text{ for } j \neq M. \tag{3.1}$$

This implies that $r_j < r_M$ for $j \neq M$ by definition of the polar coordinates (2.48). During the proof of our result, we will assume without any loss of generality that $\lambda_M < 0$.[3] According to the representation (2.48), this assumption implies that $\theta_M \in \left(0, \frac{\pi}{2}\right)$.

Using the existence of λ_M given in (3.1), the first section of our proof is devoted to show that, for $0 < u < \frac{\pi}{4}$,

$$\int_0^{\infty} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2}-2u\right)t\right) dt = -\frac{8\pi c_M r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\mathcal{G}_p\left(\theta_M\right)$$
$$\times\left\{1+\frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\,\mathcal{G}_p\left(\theta_M\right)}+\tilde{E}\left(X,z\right)+\tilde{H}\left(X,z\right)\right\}+\frac{2\pi\, e^{z^2/8}}{2^{2p}}\,\mathrm{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right], \tag{3.2}$$

where $\mathcal{A}_{\alpha,p}(u,z)$ and the terms inside the braces will be specified later. Moreover, θ_M is the angular coordinate of $\frac{i\alpha}{2} - 2\lambda_M := r_M e^{i\theta_M}$ in the convention (2.48) and $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$ is the sequence given by

$$\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}} := \cos(2p\theta_M)\,\cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi\lambda_M}{2}\right)+\sin(2p\theta_M)\,\sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi\lambda_M}{2}\right). \tag{3.3}$$

In the second part we construct a sequence of integers $(q_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\mathcal{G}_{q_n}(\theta_M)$ and $\mathcal{G}_{q_{n+1}}(\theta_M)$ have always different signs. In the third part of our proof, we prove that the terms

$$\mathcal{A}_{\alpha,p}(u,z),\ \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\,\mathcal{G}_p\left(\theta_M\right)},\ \tilde{E}\left(X,z\right),\ \tilde{H}\left(X,z\right)$$

are very small if we pick u small enough and take p as a very large integer. Finally, in the fourth part of our proof we combine our collected data with Fejér's lemma 3.1 to reach the desired conclusion.

We should remark that, throughout this proof, we will only work with the assumption that $\alpha > 2$, so that we will apply bounds (2.22) and (2.45) respectively. Of course, in the range $0 \le \alpha \le 2$ the proof is entirely analogous but the auxiliary computations are slightly different.

3.2 A suitable integral representation

Let $0 < u < \frac{\pi}{4}$ and take $\omega = \frac{\pi}{4} - u$ on the integral representation (2.49). If $\mathcal{I}_{2p}(u)$ denotes the integral

$$\mathcal{I}_{2p}(u) := \int_0^{\infty} t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right) dt, \quad 0 < u < \frac{\pi}{4},$$

then, according to (2.49), $\mathcal{I}_{2p}(u)$ can be explicitly given by

$$-\frac{8\pi}{2^{2p}}\sum_{j=1}^{\infty} c_j r_j^{2p}\left[\cos\left(2p\theta_j\right)\,\cos\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\cosh\left(2\lambda_j\left(\frac{\pi}{4}-u\right)\right)+\sin\left(2p\theta_j\right)\,\sin\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\sinh\left(2\lambda_j\left(\frac{\pi}{4}-u\right)\right)\right]$$
$$+\frac{2\pi\, e^{z^2/8}}{2^{2p}}\,\mathrm{Re}\left(\frac{d^{2p}}{du^{2p}}\left\{e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)}\sum_{j\neq 0} c_j e^{-2\lambda_j\left(\frac{\pi}{4}-u\right)}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu},z\right)\right)\right\}\right) = \mathcal{I}_{2p}(u). \tag{3.4}$$

[3] There is no loss of generality because we can interchange λ_j with $-\lambda_j$ in the function $\tilde{F}_{z,\alpha}(s)$.

We shall rewrite (3.4) by appealing to Lemma 2.3. Consider the following function

$$h_\alpha(u) = e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\lambda_j\left(\frac{\pi}{4}-u\right)}. \tag{3.5}$$

Since $\sum |c_j| < \infty$, it is clear that $h_\alpha(u)$ is an analytic function of u. Therefore, by (2.47) and the notation (2.48),

$$\begin{aligned}\frac{d^{2p}}{du^{2p}}\left\{\left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right)\, e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\lambda_j\left(\frac{\pi}{4}-u\right)}\right\} &= -2\sinh\left(\frac{z^2}{8}\right) h_\alpha^{(2p)}(u) + \mathcal{A}_{\alpha,p}(u,z)\\ &= -2\sinh\left(\frac{z^2}{8}\right)\, e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j r_j^{2p} e^{2ip\theta_j}\, e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} + \mathcal{A}_{\alpha,p}(u,z).\end{aligned} \tag{3.6}$$

We will now find a suitable bound for $\mathcal{A}_{\alpha,p}(u,z)$, which will invoke Lemma 2.3. Since $(\lambda_j)_{j\in\mathbb{N}}$ is bounded by hypothesis, there exists some μ such that $|\lambda_j| < \mu$ for every $j \in \mathbb{N}$: hence

$$\|h_\alpha\|_{L^\infty(C_0(1))} \leq e^{\frac{\alpha}{2}} \sum_{j\neq 0} |c_j| e^{\left(2-\frac{\pi}{2}\right)\lambda_j} < e^{(4-\pi+\alpha)\mu} \sum_{j\neq 0} |c_j| \leq \mathcal{M}, \tag{3.7}$$

where $\mathcal{M}$ only depends on the sequence the value of the convergent series $\sum_{j\neq 0} |c_j| := 2\sum_{j=1}^{\infty} |c_j|$. Recalling (2.45), we arrive at the following bound

$$|\mathcal{A}_{\alpha,p}(u,z)| < D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\,\|h\|_{L^\infty(C_1(0))} \leq \mathscr{D}\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}, \tag{3.8}$$

for some $\mathscr{D}$ depending only on the sequence $(c_j)_{j\in\mathbb{N}}$ and α. Returning to the second term of (3.4) and using (3.6), we find that

$$\begin{aligned}&e^{z^2/8}\mathrm{Re}\left(\frac{d^{2p}}{du^{2p}}\left\{\left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right)\, e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\right\}\right)\\ &= e^{z^2/8}\,\mathrm{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right] - 2e^{\frac{z^2}{8}}\sinh\left(\frac{z^2}{8}\right) \sum_{j\neq 0} c_j r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\,\cos\left(2p\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right).\end{aligned} \tag{3.9}$$

We can rewrite the infinite series on the previous expression by repeating the same computations leading to (2.54): we see that

$$\begin{aligned}2\sum_{j=1}^{\infty} c_j r_j^{2p}\left\{\cos(2p\theta_j)\,\cos\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\cosh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right) + \sin(2p\theta_j)\,\sin\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\sinh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right)\right\}\\ = \sum_{j\neq 0} c_j r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\,\cos\left(2p\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right),\end{aligned} \tag{3.10}$$

and so, after combining (3.10) with (3.4), we derive the representation

$$\begin{aligned}\mathcal{I}_{2p}(u) &= \frac{2\pi\, e^{z^2/8}}{2^{2p}}\,\mathrm{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right] - \frac{8\pi}{2^{2p}}\left(1 + e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\\ &\times \sum_{j=1}^{\infty} c_j\, r_j^{2p}\left\{\cos(2p\theta_j)\,\cos\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\cosh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right) + \sin(2p\theta_j)\,\sin\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\sinh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right)\right\}.\end{aligned} \tag{3.11}$$

Of course, by (3.9) and (3.4), an equivalent way of writing (3.11) is returning to the sum over $j \in \mathbb{Z} \setminus \{0\}$ and get

$$\mathcal{I}_{2p}(u) = -\frac{4\pi}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) \sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right)$$
$$+ \frac{2\pi\, e^{z^2/8}}{2^{2p}} \operatorname{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right]. \tag{3.12}$$

We will now analyze the infinite series on the first term on the right side of (3.11) and approximate it by its value for $u = 0$. For this analysis it will be easier to manipulate the more compact expression (3.12). By the mean value theorem and the existence of an element λ_M satisfying (3.1), we know that

$$\left|\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \left\{\cos\left(2p\,\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right) - \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8}\right)\right\}\right| \leq \frac{\alpha}{2} e^{\frac{\pi}{2}|\lambda_M|} r_M^{2p} \sum_{j\neq 0} |c_j|\, u \leq C_\alpha\, r_M^{2p}\, u, \tag{3.13}$$

where, by the definition (2.48), $\frac{i\alpha}{2} - 2\lambda_M = r_M e^{i\theta_M}$. Thus, we can write the infinite series in (3.12) in the approximate form

$$\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right) = \sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8}\right) + \mathcal{B}_{\alpha,p}(u)$$
$$= 2\sum_{j=1}^{\infty} c_j\, r_j^{2p} \left\{\cos(2p\theta_j)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin(2p\theta_j)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_j\right)\right\} + \mathcal{B}_{\alpha,p}(u), \tag{3.14}$$

where, according to (3.13), $\mathcal{B}_{\alpha,p}(u)$ satisfies the estimate

$$|\mathcal{B}_{\alpha,p}(u)| \leq C_\alpha\, r_M^{2p}\, u. \tag{3.15}$$

Assume that $p \in \mathbb{N}$ is such that

$$\mathcal{G}_p(\theta_M) := \cos(2p\theta_M)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_M\right) + \sin(2p\theta_M)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_M\right) \neq 0. \tag{3.16}$$

Then, for every p satisfying (3.16), we can write (3.14) as

$$\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right) = 2\sum_{j=1}^{\infty} c_j r_j^{2p}\, \mathcal{G}_p(\theta_j) + \mathcal{B}_{\alpha,p}(u)$$
$$= 2c_M r_M^{2p}\, \mathcal{G}_p(\theta_M) \left\{1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\, \mathcal{G}_p(\theta_M)} + \tilde{E}(X,z) + \tilde{H}(X,z)\right\}, \tag{3.17}$$

where

$$\tilde{E}(X,z) := \sum_{j\neq M,\, j\leq X} \frac{c_j}{c_M} \left(\frac{r_j}{r_M}\right)^{2p} \frac{\cos(2p\theta_j)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin(2p\theta_j)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_j\right)}{\cos(2p\theta_M)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_M\right) + \sin(2p\theta_M)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_M\right)} \tag{3.18}$$

and

$$\tilde{H}(X,z) := \sum_{j\neq M,\, j> X} \frac{c_j}{c_M} \left(\frac{r_j}{r_M}\right)^{2p} \frac{\cos(2p\theta_j)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin(2p\theta_j)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_j\right)}{\cos(2p\theta_M)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_M\right) + \sin(2p\theta_M)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_M\right)}, \tag{3.19}$$

for some large parameter X that will be chosen later. Using (3.12), we arrive at the expression

$$\mathcal{I}_{2p}(u) = -\frac{8\pi c_M r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) \mathcal{G}_p(\theta_M) \left\{1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\, \mathcal{G}_p(\theta_M)} + \tilde{E}(X,z) + \tilde{H}(X,z)\right\}$$
$$+ \frac{2\pi\, e^{z^2/8}}{2^{2p}} \operatorname{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right], \tag{3.20}$$

which is exactly the one claimed at the introduction of our proof, (3.2).

In the remaining parts of our proof, the strategy will be to see that $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$ is the dominant term in (3.17) and in (3.20). This will be concluded after showing that $\mathcal{A}_{\alpha,p}(u,z)$, $\mathcal{B}_{\alpha,p}(u)$, $\tilde{E}(X,z)$ and $\tilde{H}(X,z)$ are very small for infinitely many values of p and for sufficiently small u. Since (3.3) will change its sign infinitely often for a suitable sequence of integers $(q_n)_{n\in\mathbb{N}}$, our proof will be concluded after this observation.

3.3 Studying the sign changes of $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$

We will now show that we can always construct a sequence $(q_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ for which $\mathcal{G}_{q_n}(\theta_M)$ and $\mathcal{G}_{q_{n+1}}(\theta_M)$ always have a different sign. To that end, we divide our construction in two cases: $\frac{\theta_M}{\pi} \notin \mathbb{Q}$ or $\frac{\theta_M}{\pi} \in \mathbb{Q}$.

1st case: $\theta_M/\pi \notin \mathbb{Q}$ Since we are assuming that $\lambda_M < 0$ (so that $\theta_M \in (0, \frac{\pi}{2})$), we may write the sequence (3.3) as follows

$$(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}} = \cos(2p\theta_M)\cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi|\lambda_M|}{2}\right) - \sin(2p\theta_M)\sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi|\lambda_M|}{2}\right).$$

We will study the sign changes of the previous sequence through the study of the function

$$G(\phi) := \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi|\lambda_M|}{2}\right)\cos(\phi) - \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi|\lambda_M|}{2}\right)\sin(\phi).$$

In order to study $G(\phi)$, let us assume that $8k < \alpha < 8k+4$, for some $k \in \mathbb{N}_0$. The remaining cases $8k+4 < \alpha < 8k+8$ and $\alpha \equiv 0 \mod 4$ are analogous and will be sketched at the end of the argument. Under the assumption $8k < \alpha < 8k+4$, we know that $\cos\left(\frac{\pi\alpha}{8}\right)$ and $\sin\left(\frac{\pi\alpha}{8}\right)$ must have the same sign and, without loss of generality, we assume it to be positive.

Let us define the sets

$$\mathcal{A}^+ := \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \pi, \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right)\right) \tag{3.21}$$

and

$$\mathcal{A}^- := \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right), \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \pi\right). \tag{3.22}$$

Clearly, if $\phi \in \mathcal{A}^+$ then $G(\phi) > 0$, while $G(\phi) < 0$ when $\phi \in \mathcal{A}^-$.We now consider the subintervals of $\mathcal{A}^+$ and $\mathcal{A}^-$ defined by

$$\mathcal{I}^+ := \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} - \theta_M; \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} + \theta_M\right), \tag{3.23}$$

$$\mathcal{I}^- := \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} - \theta_M; \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} + \theta_M\right). \tag{3.24}$$

Then $\mathcal{I}^+$ (resp. $\mathcal{I}^-$) is an arc in the center of $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) with length $2\theta_M$. We can therefore write the partition

$$\mathcal{A}^+ = \mathcal{A}_1^+ \cup \mathcal{I}^+ \cup \mathcal{A}_2^+, \tag{3.25}$$

where

$$\mathcal{A}_1^+ = \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \pi, \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} - \theta_M\right),$$

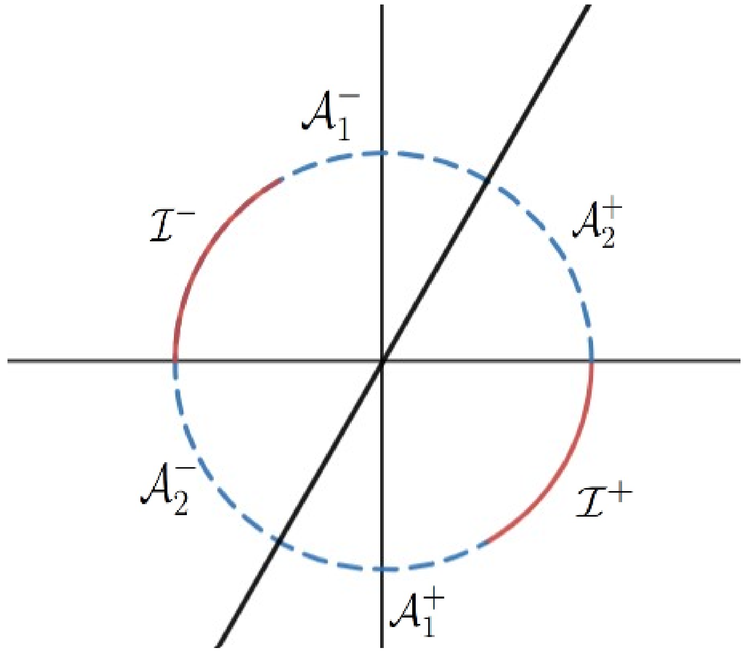

Figure 1: The partition of the unit circle into $\mathcal{A}^+$ and $\mathcal{A}^-$ and the representations of the subsets $\mathcal{A}_1^\pm$, $\mathcal{A}_2^\pm$ and $\mathcal{I}^\pm$. The line dividing the unit circle is defined by the equation $Y = \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) X$.

while

$$\mathcal{A}_2^+ = \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} + \theta_M, \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right)\right).$$

Analogously, we can find the partition of $\mathcal{A}^-$,

$$\mathcal{A}^- = \mathcal{A}_1^- \cup \mathcal{I}^- \cup \mathcal{A}_2^-,$$

where

$$\mathcal{A}_1^- = \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right), \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} - \theta_M\right)$$

and

$$\mathcal{A}_2^- = \left(\arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} + \theta_M, \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \pi\right).$$

Since $\theta_M/\pi \notin \mathbb{Q}$, by Kronecker's lemma the set of points $\{n\theta_M/\pi\}_{n\in\mathbb{N}}$ is dense in $(0,1)$. Thus, there exist two infinite sequences $(q_n^+)_{n\in\mathbb{N}}$ and $(q_n^-)_{n\in\mathbb{N}}$ such that

$$(q_n^+)_{n\in\mathbb{N}} := \left\{q \in \mathbb{N} \,:\, 2q\theta_M \in \mathcal{I}^+\right\}, \quad (q_n^-)_{n\in\mathbb{N}} := \left\{q \in \mathbb{N} \,:\, 2q\theta_M \in \mathcal{I}^-\right\}. \tag{3.26}$$

The main claim of this section establishes that $(q_n^+)_{n\in\mathbb{N}}$ and $(q_n^-)_{n\in\mathbb{N}}$ interlace.

Claim 3.1. *For every $n \in \mathbb{N}$, we have that*

$$q_n^+ < q_n^- < q_{n+1}^+ < q_{n+1}^- < q_{n+2}^+ < \dots .$$

Proof. First, let us observe that, if $a \in (q_n^+)_{n\in\mathbb{N}}$ then $a+1 \notin (q_n^+)_{n\in\mathbb{N}}$, because $2(a+1)\theta_M = 2a\theta_M + 2\theta_M$ and the length of $\mathcal{I}^+$ is, by construction, equal to $2\theta_M$. Therefore, either $2(a+1)\theta_M \in \mathcal{I}^-$ (i.e., $a+1 \in (q_n^-)_{n\in\mathbb{N}}$) or $2(a+1)\theta_M \in \mathcal{A}_2^+ \cup \mathcal{A}_1^-$. This actually happens because $2\theta_M < \pi$ and, since $2a\theta_M \in \mathcal{I}^+$ by hypothesis, then

$$2a\theta_M + 2\theta_M < \arctan\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} + \theta_M,$$

which assures that $2(a+1)\theta_M \notin \mathcal{A}_2^-$. Since $a \in (q_n^+)_{n\in\mathbb{N}}$ then $a = q_{m_0}^+$ for some m_0. If $a+1 \in (q_n^-)_{n\in\mathbb{N}}$, then we have that $q_{m_0}^+ < q_{m_0}^-$. On the other hand, if $2(a+1)\theta_M \in \mathcal{A}_2^+ \cup \mathcal{A}_1^-$ then there is some $j \geq 2$ such that $2(a+j)\theta_M \in \mathcal{I}^-$ because the length of $\mathcal{I}^-$ is equal to $2\theta_M$. Therefore, the translation $a+j$, $j \in \mathbb{N}_0$, cannot go from $q_{m_0}^+$ to $q_{m_0+1}^+$ without passing first by an element of the sequence $(q_n^-)_{n\in\mathbb{N}}$. This completes the proof of the claim. □

The interlacing property of q_n^+ and q_n^- shows that we can construct a new sequence

$$(q_n)_{n\in\mathbb{N}} := \left(q_n^+\right)_{n\in\mathbb{N}} \cup \left(q_n^-\right)_{n\in\mathbb{N}}$$

in such a way that $q_{2n-1} := q_n^+$ and $q_{2n} := q_n^-$. Moreover, we have that $\mathcal{G}_{q_m}(\theta_M)$ and $\mathcal{G}_{q_{m+1}}(\theta_M)$ have opposite signs.

Finally, we remark that the cases $8k+4 < \alpha < 8k+8$ and $\alpha \equiv 0 \mod 4$ are analogous. In the first case, $\cos\left(\frac{\pi\alpha}{8}\right)$ and $\sin\left(\frac{\pi\alpha}{8}\right)$ have opposite signs. However, a similar strategy as the one given above works and the only modification is replacing $\mathcal{A}_1^+, \mathcal{I}^+$ and $\mathcal{A}_2^+$ respectively by $\mathcal{A}_1^-, \mathcal{I}^-$ and $\mathcal{A}_2^-$.

On the other hand, if $\alpha = 8k$ or $\alpha = 8k+4$ for some $k \in \mathbb{N}_0$, then

$$\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}} = (-1)^k \cosh\left(\frac{\pi|\lambda_M|}{2}\right)\cos\left(2p\theta_M\right), \text{ if } \alpha = 8k \tag{3.27}$$

and

$$\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}} = (-1)^{k-1} \sinh\left(\frac{\pi|\lambda_M|}{2}\right)\sin\left(2p\theta_M\right), \text{ if } \alpha = 8k+4. \tag{3.28}$$

In any of these cases a similar construction of the sets $\mathcal{A}_1^{\pm}$, $\mathcal{A}_2^{\pm}$ and $\mathcal{I}^{\pm}$ can be made.

2nd case: $\theta_M/\pi \in \mathbb{Q}$ The conclusions given above in the case $\theta_M/\pi \in \mathbb{Q}$ are analogous but instead of arguing via Kronecker's lemma, the sequences (3.26) can be explicitly constructed in this case. Since $|\mathcal{I}^+| = |\mathcal{I}^-| = 2\theta_M$, we know that there exist q_1 and $q_2 > q_1$ such that $2q_1\theta_M \in \mathcal{I}^+$ and $2q_2\theta_M \in \mathcal{I}^-$. Since $\theta_M = \frac{a}{b}\pi$, $(a,b)=1$, then $2q_1\theta_M \equiv 2\left(q_1 + nb\right)\theta_M \mod 2\pi$ and $2q_2\theta_M \equiv 2\left(q_2+nb\right)\theta_M \mod 2\pi$. Hence, if we construct the sequence $(q_n)_{n\in\mathbb{N}}$ in such a way that[4]

$$q_{2n-1} := q_1 + nb, \quad q_{2n} := q_2 + nb, \tag{3.29}$$

then $\mathcal{G}_{q_n}(\theta_M)\cdot\mathcal{G}_{q_{n+1}}(\theta_M) < 0$, i.e., $\mathcal{G}_{q_n}(\theta_M)$ and $\mathcal{G}_{q_{n+1}}(\theta_M)$ have opposite signs.

[4]Note that the interlacing property established by Claim 3.1 is obviously valid in this case.

3.4 The dominance of $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$.

Having constructed a sequence $(q_n)_{n\in\mathbb{N}}$ for which $(\mathcal{G}_{q_n}(\theta_M))_{n\in\mathbb{N}}$ is always alternating its sign, we are now ready to proceed with the proof. Our next claim establishes that the sequence $\mathcal{G}_{q_n}(\theta_M)$ dominates, in some sense, the second term (involving $\mathcal{A}_{\alpha,p}(u,z)$) appearing in (3.20).

Claim 3.2. *There exists a sufficiently large η_0 such that, for any z satisfying (2.21), $\mathcal{A}_{\alpha,p}(\cdot,z)$ satisfies*

$$e^{z^2/8}\left|Re\left[\mathcal{A}_{\alpha,p}\left(\frac{1}{\eta_0\, p\log(p)},z\right)\right]\right| < |c_M|\, r_M^{2p}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)|\mathcal{G}_p\left(\theta_M\right)| \tag{3.30}$$

for any $p\in(q_n)_{n\in\mathbb{N}}$.

Proof. By (3.8), we have that

$$|\mathcal{A}_{\alpha,p}(u,z)|\leq \mathscr{D}\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}},\quad 0<u<\frac{\pi}{4},$$

so that the inequality (3.30) holds for any $p\in(q_n)_{n\in\mathbb{N}}$ if

$$\frac{\mathscr{D}2^{14p}(2p)!\,e^{z^2/8}}{|c_M|r_M^{2p}\,u^{\frac{\alpha}{2}+2p}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)|\mathcal{G}_p\left(\theta_M\right)|}<e^{A/u} \tag{3.31}$$

holds for any $p\in(q_n)_{n\in\mathbb{N}}$. By construction of the sets $\mathcal{I}^+$ and $\mathcal{I}^-$ (see (3.23) and (3.24) above), if $p\in(q_n)_{n\in\mathbb{N}}$ then

$$|\mathcal{G}_p\left(\theta_M\right)|=\left|\cos\left(2p\theta_M\right)\,\cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_M\right)+\sin\left(2p\theta_M\right)\,\sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_M\right)\right|\geq\epsilon_0, \tag{3.32}$$

for some $\epsilon_0>0$ (only depending on θ_M). Therefore, since $e^{z^2/8}<e^{\frac{\pi\alpha}{576}}$ (by our condition (1.22)), (3.31) is plainly satisfied if, for any $p\in(q_n)_{n\in\mathbb{N}}$,

$$\frac{\mathscr{D}\exp\left(\frac{\pi\alpha}{576}\right)}{\epsilon_0|c_M|}\,\frac{2^{14p}(2p)!}{r_M^{2p}u^{\frac{\alpha}{2}+2p}}<e^{A/u}. \tag{3.33}$$

Letting $u=\frac{1}{\eta_0\,p\log(p)}$ and taking the logarithm on both sides of (3.33), we see that (3.33) holds if

$$\frac{1}{Ap\log(p)}\log\left(\frac{\mathscr{D}\exp\left(\frac{\pi\alpha}{576}\right)}{\epsilon_0|c_M|}\right)+\frac{2\,\log\left(128/r_M\right)}{A\log(p)}+\frac{\log(2p)!}{Ap\log(p)}+\frac{1}{A\log(p)}\left(\log(\eta)+\log(p)+\log(\log(p))\right)\left(2+\frac{\alpha}{2p}\right)<\eta_0. \tag{3.34}$$

From Stirling's formula, we know that $\log(2p)!=O\left(p\log(p)\right)$, from which it follows that (3.34) holds for every $p\in(q_n)_{n\in\mathbb{N}}$ if η_0 is a sufficiently large number. This completes the proof of (3.30). □

We have proved that the second term of (3.20) is dominated by the term involving $\mathcal{G}_p\left(\theta_M\right)$ once p belongs to a suitable sequence of integers. Similar to what we have done in [[24], p. 47], a bound for the functions $\tilde{E}(X,z)$ and $\tilde{H}(X,z)$ also holds.

Claim 3.3. *There exists a sufficiently large N_0 such that, for any $p\in(q_n)_{n\geq N_0}$ and z satisfying (1.22), the following inequalities hold*

$$|\tilde{E}(X,z)|<\frac{1}{6},\quad|\tilde{H}\left(X,z\right)|<\frac{1}{6} \tag{3.35}$$

and, for any $\eta>0$,

$$\frac{\left|\mathcal{B}_{\alpha,p}\left(\frac{1}{\eta p\log(p)}\right)\right|}{2\,|c_M|r_M^{2p}\,|\mathcal{G}_p\left(\theta_M\right)|}<\frac{1}{6}, \tag{3.36}$$

with $\mathcal{B}_{\alpha,p}(u)$ is defined by (3.14).

Proof. Since $p \in (q_n)_{n \geq N_0}$, then (3.32) holds for some $\epsilon_0 \geq 0$. Thus, since $(c_j)_{j \in \mathbb{N}} \in \ell^1$, we can choose a sufficiently large $X \geq X_0$ such that

$$\begin{aligned} |\tilde{H}(X,z)| &\leq \frac{1}{\epsilon_0} \sum_{j \neq M,\, j > X} \frac{|c_j|}{|c_M|} \left(\frac{r_j}{r_M}\right)^{2p} \left|\cos(2p\theta_j)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin(2p\theta_j)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi}{2}\lambda_j\right)\right| \\ &\leq \frac{2e^{\frac{\pi}{2}\lambda_M}}{\epsilon_0 |c_M|} \sum_{j \neq M, j > X} |c_j| < \frac{1}{6}. \end{aligned}$$

On the other hand, we may bound $\tilde{E}(X,z)$ as follows

$$\left|\tilde{E}(X,z)\right| \leq \frac{2e^{\frac{\pi}{2}\lambda_M}}{\epsilon_0|c_M|} \mu_X^{2p} \sum_{j \neq M,\, j \leq X} |c_j|,$$

where

$$\mu_X = \max_{j \leq X} \left\{\frac{r_j}{r_M}\right\}.$$

By property (3.1), we know that $\mu_X < 1$ and so, for N_0 sufficiently large and $p \in (q_n)_{n \geq N_0}$,

$$|\tilde{E}(X,z)| < \frac{1}{6},$$

which proves (3.35). To get (3.36), we just need to use (3.15) with $u = \frac{1}{\eta p\, \log(p)}$,

$$\frac{\left|\mathcal{B}_{\alpha,p}\left(\frac{1}{\eta p \log(p)}\right)\right|}{2\,|c_M| r_M^{2p}\, |\mathcal{G}_p(\theta_M)|} \leq \frac{C_\alpha}{2\epsilon_0 |c_M| \eta p\, \log(p)} < \frac{1}{6},$$

whenever $p \geq q_{N_0}$ and N_0 is sufficiently large. This completes the proof of the claim.

□

3.5 Conclusion of the argument

Finally, we return to the integral representation (3.20),

$$\begin{aligned} \int_0^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4} + it\right) \cosh\left(\left(\frac{\pi}{2} - 2u\right) t\right) dt &= -\frac{8\pi c_M r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) \mathcal{G}_p(\theta_M) \\ \times \left\{1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2 c_M r_M^{2p}\, \mathcal{G}_p(\theta_M)} + \tilde{E}(X,z) + \tilde{H}(X,z)\right\} &+ \frac{2\pi\, e^{z^2/8}}{2^{2p}} \operatorname{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right], \end{aligned} \tag{3.37}$$

and we take there $p \in (q_n)_{N_0 \leq n \leq N}$, where $N \geq 2N_0 + 2$ and N_0 is a sufficiently large number for which (3.35) and (3.36) hold. Replacing u by $\frac{1}{\eta q_N \log(q_N)}$ in (3.37), it follows from our choices of N_0 and X in the previous claim that

$$1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2 c_M r_M^{2p}\, \mathcal{G}_p(\theta_M)} + \tilde{E}(X,z) + \tilde{H}(X,z) > \frac{1}{2}, \tag{3.38}$$

by (3.35) and (3.36). Moreover, according to Claim 3.2, namely (3.30), we also have the inequality

$$\frac{2\pi\, e^{z^2/8}}{2^{2p}} \left|\operatorname{Re}\left[\mathcal{A}_{\alpha,p}\left(\frac{1}{\eta\, p \log(p)}, z\right)\right]\right| < \frac{1}{4} \cdot \frac{8\pi |c_M|\, r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) |\mathcal{G}_p(\theta_M)|, \tag{3.39}$$

for any $\eta \geq \eta_0$ and η_0 sufficiently large (such that (3.34) holds). Combining (3.38) with (3.39) one concludes that, for any $p \in (q_n)_{N_0 \leq n \leq N}$ and η_0 chosen as in the proof of Claim 3.2, the sequence of moments

$$(M_p)_{p\in(q_n)_{N_0\leq n\leq N}} := \int_0^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2} - \frac{2}{\eta_0 q_N \log(q_N)}\right) t\right) dt \tag{3.40}$$

must have the same sign as the sequence defined by

$$(s_p)_{p\in(q_n)_{N_0\leq n\leq N}} := -\frac{8\pi c_M r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) \mathcal{G}_p(\theta_M). \tag{3.41}$$

Moreover, from (3.38) and (3.39), the modulus of M_p, (3.40), satisfies the inequality

$$\begin{aligned}\left|\int_0^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2} - \frac{2}{\eta_0 q_N \log(q_N)}\right) t\right) dt\right| &> \frac{1}{4}\cdot \frac{8\pi |c_M| r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) |\mathcal{G}_p(\theta_M)| \\ &= \frac{2\pi |c_M| r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) |\mathcal{G}_p(\theta_M)| . \end{aligned} \tag{3.42}$$

In order to be able to finally apply Fejér's lemma, we establish the following simple bound.

Claim 3.4. *There exists $\eta_1 > \eta_0$ such that, for any $\eta \geq \eta_1$ and z satisfying (1.22), the following inequality takes place*

$$\left|\int_{\eta^2 q_N^2 \log^2(q_N)}^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2} - \frac{2}{\eta_0 q_N \log(q_N)}\right) t\right) dt\right| < \frac{2\pi |c_M| r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) |\mathcal{G}_p(\theta_M)| , \tag{3.43}$$

whenever $p \in (q_n)_{N_0 \leq n \leq N}$.

Proof. Using (2.12), it is simple to check that, for a sufficiently large η_1 and $\eta \geq \eta_1$, that the left-hand side of (3.43) can be bounded as follows

$$\begin{aligned} &\int_{\eta^2 q_N^2 \log^2(q_N)}^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2} - \frac{2}{\eta_0 q_N \log(q_N)}\right) t\right) dt \\ &\leq e^{-\frac{\eta^2}{\eta_0} q_N \log(q_N)} \int_{\eta^2 q_N^2 \log^2(q_N)}^\infty t^{2q_N} \left|\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| \exp\left(\left(\frac{\pi}{2} - \frac{1}{\eta_0 q_N \log(q_N)}\right) t\right) dt \\ &\ll_{\alpha,z,\Lambda} e^{-\frac{\eta^2}{\eta_0} q_N \log(q_N)} \int_{\eta^2 q_N^2 \log^2(q_N)}^\infty t^{2q_N + B(\alpha)} \exp\left(-\frac{t}{\eta_0 q_N \log(q_N)} + |z|\sqrt{t}\right) dt \\ &\ll_{\alpha,z,\Lambda} e^{-\frac{\eta^2}{\eta_0} q_N \log(q_N)} \int_{\eta^2 q_N^2 \log^2(q_N)}^\infty t^{2q_N + B(\alpha)} \exp\left(-\frac{t}{2\eta_0 q_N \log(q_N)}\right) dt \\ &\ll_{\alpha,z,\Lambda} (2\eta_0 q_N \log(q_N))^{2q_N + B(\alpha)+1}\, e^{-\frac{\eta^2}{\eta_0} q_N \log(q_N)}\, (2q_N + [B(\alpha)] + 1)! . \end{aligned}$$

In the above inequalities, $\ll_{\alpha,z,\Lambda}$ stands for a constant depending on α, z and the pair of sequences $(c_j, \lambda_j)_{j\in\mathbb{Z}\setminus\{0\}}$. By Stirling's formula, we have that the latter expression is bounded by

$$\exp\left\{-\frac{\eta^2}{\eta_0} q_N \log(q_N) + (2q_N + B(\alpha) + 1)\log(2\eta_0 q_N \log(q_N)) + (2q_N + [B(\alpha)] + 1)\log(2q_N + [B(\alpha)] + 1) + O(q_N)\right\}$$

and so it is clear that, for $\eta \geq \eta_1 > \eta_0$ sufficiently large, the dominant term in the exponential will be $-\frac{\eta^2}{\eta_0} q_N \log(q_N)$ and this proves (3.43). □

By the previous claim and (3.42), we found that for each $p \in (q_n)_{N_0 \leq n \leq N}$, there exists some sufficiently large number $\eta_1 > \eta_0$ such that the sequence of moments

$$\left(\tilde{M}_p\right)_{p\in(q_n)_{N_0\leq n\leq N}} := \int_0^{\eta_1^2 q_N^2 \log^2(q_N)} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2} - \frac{2}{\eta_0 q_N \log(q_N)}\right) t\right) dt$$

must have the same sign as the sequence $(s_p)_{p\in(q_n)_{N_0\leq n\leq N}}$ defined by (3.41). According to Claim 3.1, s_ℓ and $s_{\ell+1}$ have always distinct signs, and so the sequence $(s_p)_{p\in(q_n)_{N_0\leq n\leq N}}$ has exactly $N - N_0 - 1$ sign changes. By Fejér's theorem, we then have that

$$N_{\alpha,z}\left(\eta_1^2\, q_N^2 \log^2(q_N)\right) \geq N - N_0 - 1 > \frac{N}{2}, \tag{3.44}$$

since $N \geq 2N_0 + 2$ by hypothesis. To conclude, we just need to connect q_N with N. We divide the final argument in two cases (recall that we are assuming that $0 < \theta_M < \frac{\pi}{2}$).

1. If $\theta_M/\pi \in \mathbb{Q}$, say $\theta_M = \pi\frac{a}{b}$, then the explicit construction (3.29) shows that, for $N \geq 3$, $N > \frac{q_N - q_2}{b} \geq \frac{q_N}{2b}$. Thus, (3.44) implies

$$N_{\alpha,z}\left(\eta_1^2\, q_N^2 \log^2(q_N)\right) > \frac{q_N}{4b},$$

proving (1.24).

2. If $\theta_M/\pi \notin \mathbb{Q}$ then, by Kronecker's lemma, the sequence $\left(\left\{n\, \frac{2\theta_M}{\pi}\right\}\right)_{n\in\mathbb{N}}$ is uniformly distributed on the interval $[0,1]$, which means that

$$\lim_{X\to\infty} \frac{\#\left\{1 \leq m \leq X \,:\, 2m\theta_M \in (\mathcal{I}^+ \cup \mathcal{I}^-)\right\}}{X} = \frac{|\mathcal{I}^+ \cup \mathcal{I}^-|}{2\pi} = \frac{2\theta_M}{\pi}. \tag{3.45}$$

Thus, taking $X = q_N$ in (3.45) we obtain, for $N \geq 2N_0 + 2$ sufficiently large,

$$N_{\alpha,z}\left(\eta_1^2\, q_N^2 \log^2(q_N)\right) > \frac{\theta_M}{\pi} q_N,$$

which proves (1.24). ■

4 Lemmas for the proof of Theorem 1.2

Analogously to (2.12), we can use the estimate (2.10) to prove that

$$\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right| \leq \sum_{j\neq 0}\left|c_j\, \eta_f\left(\frac{k}{2}+i\,(t+\lambda_j)\right) \mathrm{Re}\left({}_1F_1\left(\frac{k}{2}-i\,(t+\lambda_j);\, k;\, \frac{z^2}{4}\right)\right)\right| \ll_{k,z} C_\lambda \sum_{j=1}^{\infty} |c_j|\, |t|^{B(k)} e^{-\frac{\pi}{2}|t|+|z|\sqrt{|t|}}, \tag{4.1}$$

for some exponent $B(k)$ depending on the weight k (which comes from convex estimates for $L(s,f)$). The next lemma is an analogue of Lemma 2.1. For a proof, we refer to [[24], p. 34, eq. (2.78)].

Lemma 4.1. *Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and $a_f(n)$ be its Fourier coefficients. Then for $Re(x)>0$ and any $y\in\mathbb{C}$, the following transformation formula takes place*

$$\sum_{n=1}^{\infty}a_f(n)\,n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n}\,y\right)=\frac{(-1)^{k/2}\,e^{-\frac{y^2}{4x}}}{x}\sum_{n=1}^{\infty}a_f(n)\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{x}}\,I_{k-1}\left(\sqrt{2\pi n}\,\frac{y}{x}\right) \tag{4.2}$$

or, equivalently,

$$\psi_f(x,z)=(-1)^{k/2}\,e^{-\frac{z^2}{4}}\,x^{-k}\,\psi_f\left(\frac{1}{x},\,iz\right),\quad Re(x)>0,\,z\in\mathbb{C}, \tag{4.3}$$

where $\psi_f(x,z)$ is the analogue of Jacobi's $\psi-$function (1.15),

$$\psi_f(x,z):=(k-1)!\left(\sqrt{\frac{\pi x}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n\,x}\,z\right),\ Re(x)>0,\ z\in\mathbb{C}. \tag{4.4}$$

Lemma 4.2. *Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and $\psi_f(x,z)$ be the generalized Jacobi's $\psi-$function attached to it, (4.4). Assume that $z\in\mathbb{R}$ satisfies the condition*

$$-\frac{\sqrt{\pi}}{3}\leq z\leq\frac{\sqrt{\pi}}{3}. \tag{4.5}$$

Then there is an absolute constant A and a constant C depending only on k such that, for any $0<u<\frac{\pi}{4}$ and every $p\in\mathbb{N}_0$,

$$\left|\frac{d^p}{du^p}\psi_f\left(ie^{-2iu},z\right)\right|<C\,\frac{2^{7p}p!}{u^{k+p}}e^{-\frac{A}{u}}. \tag{4.6}$$

Proof. The proof is exactly the same as in Lemma 2.2, so we only give a brief sketch. Starting with Cauchy's formula,

$$\left[\frac{d^p}{du^p}\psi_f\left(ie^{-2iu},z\right)\right]_{u=u_0}=\frac{p!}{2\pi i}\int_{C_{\lambda u_0}(u_0)}\frac{\psi_f\left(ie^{-2iw},z\right)}{(w-u_0)^{p+1}}\,dw, \tag{4.7}$$

we need to find a suitable bound for $\psi_f\left(ie^{-2iw},z\right)$, $w\in C_{\lambda u_0}(u_0)$. This is done through Lemma 4.1 above: replacing there x by $2e^{-iw}\sin(w)$ and y by $e^{i\left(\frac{\pi}{4}-w\right)}\,z$, we obtain the transformation formula

$$\begin{aligned}\psi_f\left(ie^{-2iw},z\right)&=(k-1)!\left(\sqrt{\frac{\pi ie^{-2iw}}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)n^{\frac{1-k}{2}}\,e^{-2\pi n\cdot 2e^{-iw}\sin(w)}\,J_{k-1}\left(\sqrt{2\pi nie^{-2iw}}\,z\right)\\&=(k-1)!\left(\sqrt{\frac{\pi}{2}}e^{i\left(\frac{\pi}{4}-w\right)}z\right)^{1-k}\frac{(-1)^{k/2}\,e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\sum_{n=1}^{\infty}(-1)^n a_f(n)\,n^{\frac{1-k}{2}}\,e^{-\frac{\pi n}{\tan(w)}}\,I_{k-1}\left(\sqrt{\frac{\pi n}{2}}\frac{e^{i\frac{\pi}{4}}z}{\sin(w)}\right).\end{aligned}$$

From this point on, we can easily adapt the steps given in the proof of the bounds (2.29) and (2.31). Extracting the first term of the series in the above expression,[5] we obtain

$$\begin{aligned}\left|\psi_f\left(ie^{-2iw},z\right)\right|&\leq d_k\,\frac{\pi^k e^{-\frac{z^2}{8}+\frac{u_0}{2}}}{(2u_0)^k}\exp\left\{-\frac{\pi\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}P_w^{\star}(1)\right\}\\&\quad\times\sum_{n=1}^{\infty}\left|\frac{a_f(n)}{a_f(1)}\right|\exp\left[-\frac{\pi\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\left(P_w^{\star}(\sqrt{n})-P_w^{\star}(1)\right)\right],\end{aligned} \tag{4.8}$$

[5]without any loss of generality, we suppose that $a_f(1)\neq 0$. If c is the smallest integer such that $a_f(c)\neq 0$, then it would be possible to enlarge the interval (1.25) to $\left[-\frac{\sqrt{\pi c}}{3},\,\frac{\sqrt{\pi c}}{3}\right]$.

where $P_w^\star(X)$ is the real-valued polynomial

$$P_w^\star(X) := X^2 - \frac{|z|}{\sqrt{\pi}\sin(2\text{Re}(w))}\sqrt{\cosh(2\text{Im}(w)) - \cos(2\text{Re}(w))}\, X + \frac{z^2}{8\pi}\frac{\sinh(2\text{Im}(w))}{\sin(2\text{Re}(w))}.$$

Note that our expression (4.8) is completely analogous to (2.31). Repeating the steps in (2.37), we find that the series on the right-hand side of (4.8) is uniformly bounded for any $w \in C_{0.01u_0}(u_0)$ and z satisfying (4.5). The resulting constant, M_k, will only dependent on k. Analogously to (2.38) and (2.39),

$$\begin{aligned}
\left|\psi_f\left(ie^{-2iw}, z\right)\right| &\leq \frac{d_k M_k e^{\frac{\pi}{8}}\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left\{-\frac{\pi\sin(2\text{Re}(w))}{\cosh(2\text{Im}(w)) - \cos(2\text{Re}(w))}P_w^\star(1)\right\} \\
&< \frac{d_k M_k e^{\frac{\pi}{8}}\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left[-\frac{1}{u_0}\left\{\frac{1-\lambda}{e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2} - \frac{\pi|z|}{2(1-\lambda)} - \frac{\pi z^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{16(1-\lambda)^2}\right\}\right] \\
&< \frac{d_k M_k e^{\frac{\pi}{8}}\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left[-\frac{1}{u_0}\left\{\frac{1-\lambda}{e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2} - \frac{\pi^{\frac{3}{2}}}{6(1-\lambda)} - \frac{\pi^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{144(1-\lambda)^2}\right\}\right].
\end{aligned}$$

Thus, if we pick $\lambda = 0.01$, the term on the braces is greater than 0.03: this implies that

$$\left|\psi_f\left(ie^{-2iw}, z\right)\right| < \frac{d_k M_k e^{\frac{\pi}{8}}\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left[-\frac{0.03}{u_0}\right] = \frac{C}{u_0^k}e^{-\frac{A}{u_0}},$$

where C depends on k and A is an absolute constant, say $A = 0.03$. Using Cauchy's formula (4.7) and mimicking the steps in (2.41), we find that (4.6) must hold for every $p \in \mathbb{N}_0$. □

We finish this section with two lemmas analogous to Lemmas 2.3 and 2.4. Since their proofs are exactly the same, we shall omit them.

Lemma 4.3. *Let $g : \mathbb{C} \longmapsto \mathbb{C}$ be an analytic function. If $z \in \left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}, \frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]$ then, for arbitrary $p \in \mathbb{N}_0$ and $0 < u < \frac{\pi}{4}$, the following relation holds*

$$\left|\frac{d^{2p}}{du^{2p}}\left\{g(u)\,\psi_f\left(ie^{-2iu}, z\right)\right\}\right| < D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{k+2p}}\|g\|_{L^\infty(C_1(0))}, \tag{4.9}$$

where A is some absolute constant and D only depends on k.

Lemma 4.4. *Let $-\frac{\pi}{4} < \omega < \frac{\pi}{4}$, $p \in \mathbb{N}$ and $\tilde{G}_{z,f}\left(\frac{k}{2} + it\right)$ be the function defined by (1.23). Then the following integral representations hold*[6]

$$\int_0^\infty i^{-\frac{k}{2}}\,t^{2p}\tilde{G}_{z,f}\left(\frac{k}{2} + it\right)\cosh(2\omega t)\,dt = \frac{2\pi\,e^{\frac{z^2}{4}}}{2^{2p}}\,Re\left(i^{-\frac{k}{2}}\frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j\neq 0}c_j\,e^{i\omega k - 2\omega\lambda_j}\,\psi_f\left(e^{2i\omega}, z\right)\right\}\right), \quad k \equiv 0 \mod 4 \tag{4.10}$$

and

$$\int_0^\infty i^{-\frac{k}{2}}\,t^{2p+1}\tilde{G}_{z,f}\left(\frac{k}{2} + it\right)\cosh(2\omega t)\,dt = \frac{2\pi}{2^{2p}}\,e^{\frac{z^2}{4}}\,Re\left(i^{-\frac{k}{2}}\frac{d^{2p+1}}{d\omega^{2p+1}}\left\{\sum_{j\neq 0}c_j\,e^{i\omega k - 2\omega\lambda_j}\,\psi_f\left(e^{2i\omega}, z\right)\right\}\right), \quad k \equiv 2 \mod 4, \tag{4.11}$$

where $\psi_f(x, z)$ is the "cusp form analogue" of Jacobi's $\psi-$function (4.4).

[6]Note that the integral is written in terms of $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2} + it\right)$ because it defines a real function of $t \in \mathbb{R}$ which is even if $k \equiv 0 \mod 4$ or odd if $k \equiv 2 \mod 4$.

5 Proof of Theorem 1.2

In this section we follow closely the author's variant of de la Vallée Poussin's method [5, 23]. The details of the proof are given for the case where $k \equiv 0 \mod 4$ and a similar argument for $k \equiv 2 \mod 4$ will be devised at the end of the proof. Let $(\rho_n)_{n\in\mathbb{N}}$ be the sequence of zeros of odd order of $\tilde{G}_{z,f}(s)$ such that $\text{Re}(\rho_n) = \frac{k}{2}$. Then we can write $\rho_n := \frac{k}{2} + i\tau_n$, with $\tau_n > 0$ being an increasing sequence[7]. If we show that there is some $h > 0$ such that, for infinitely many values of n, $\tau_n < h\,n^2$, our Theorem 1.2 is proved. This is the case because if we choose the sequence $T_n := hn^2$, then we find that $N_{f,z}(T_n) \geq N_{f,z}(\tau_n) = n = \sqrt{\frac{T_n}{h}}$. This establishes that

$$\limsup_{T\to\infty} \frac{N_{f,z}(T)}{\sqrt{T}} > \frac{1}{\sqrt{h}}, \quad \text{or equivalently } N_{f,z}(T) = \Omega\left(T^{\frac{1}{2}}\right). \tag{5.1}$$

Hence, for the sake of contradiction, let us assume that there is some N_0 such that, for every $n \geq N_0$ and any $h > 0$, $\tau_n \geq h\,n^2$. We will now show that there exists some h large enough that this assumption is contradicted. Indeed, if we construct the real and entire function[8]

$$\varphi_{f,z}(y) = \prod_{\ell=1}^{\infty}\left(1 - \frac{y^2}{\tau_\ell^2}\right) = \sum_{\ell=0}^{\infty}(-1)^\ell a_{2\ell}\, y^{2\ell}, \tag{5.2}$$

we see that $a_0 = 1$ and, for $\ell \geq 1$,

$$a_{2\ell} = \sum_{r_1\geq 1}\sum_{r_2>r_1}\cdots\sum_{r_\ell>r_{\ell-1}} \frac{1}{\tau_{r_1}^2\cdot\ldots\cdot\tau_{r_\ell}^2} = \sum_{1\leq r_1<r_2<\ldots<r_\ell} \frac{1}{\tau_{r_1}^2\cdot\ldots\cdot\tau_{r_\ell}^2}, \tag{5.3}$$

where we are summing over $(r_1, ..., r_\ell) \in \mathbb{N}^\ell$ such that $r_1 < r_2 < ... < r_\ell$. Note that, in the k^{th} nested series in (5.3), the index r_k always satisfies $r_k \geq k$, due to the condition $r_k > r_{k-1} > ... > r_1 \geq 1$. From this point on, we just need to find a suitable bound for a_{2j}. Considering the nested sum above, we have two possibilities: if $1 \leq k \leq \ell$ and $r_k \geq N_0$, we know by the contradiction hypothesis that $\tau_{r_k}^{-2} \leq \frac{r_k^{-4}}{h^2}$, while if $1 \leq r_k \leq N_0 - 1$, we have that $\tau_{r_k}^{-2} \leq \frac{r_k^{-4}}{h^{\star 2}}$ where $h^\star := \min_{1\leq n\leq N_0-1}\left\{\frac{\tau_n}{n^2}\right\}$. But then if follows from the argument given in [[23], p. 16, eq. (4.6)] that for some $\mathscr{B}$ (only depending on h and N_0),

$$a_{2\ell} \leq \frac{\mathscr{B}}{h^{2\ell}} \sum_{1\leq r_1<r_2<\ldots<r_\ell} \frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4}. \tag{5.4}$$

Defining the new coefficient on the right-hand side of (5.4)

$$b_{2\ell} := \sum_{1\leq r_1<\ldots<r_\ell} \frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4}, \tag{5.5}$$

and following the steps given in [[23], p. 17], this sequence of numbers comes from Euler's infinite product representation of the function

$$\frac{\sinh(\pi\sqrt{y})\sin(\pi\sqrt{y})}{\pi^2 y} = \prod_{\ell=1}^{\infty}\left(1+\frac{y}{\ell^2}\right)\prod_{j=1}^{\infty}\left(1-\frac{y}{\ell^2}\right) = \prod_{j=1}^{\infty}\left(1-\frac{y^2}{\ell^4}\right) := 1 + \sum_{\ell=1}^{\infty}(-1)^\ell b_{2\ell}\, y^{2\ell}. \tag{5.6}$$

[7]Note that if $\tilde{G}_{z,f}\left(\frac{k}{2}\right) = 0$, we are excluding this real zero from the sequence.

[8]The infinite product can be written because, due to Theorem 1.4 of [24], $\tilde{G}_{z,f}(s)$ has infinitely many zeros at the critical line $\text{Re}(s) = \frac{k}{2}$.

Thus, we can get precise information about $b_{2\ell}$ (and, consequently, about $a_{2\ell}$) by interpreting them as the coefficients of the Taylor series for the function on the left-hand side of (5.6). A standard computation of these coefficients gives

$$b_{2\ell} := \sum_{1\leq r_1<\ldots<r_\ell} \frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4} = \frac{2^{2\ell+1}\pi^{4\ell}}{(4\ell+2)!} \implies a_{2\ell} \leq \mathscr{B}\,\frac{2^{2\ell+1}\pi^{4\ell}}{h^{2\ell}(4\ell+2)!}. \tag{5.7}$$

Our proof will be concluded by seeing that (5.7) contradicts (4.10) and the bounds found in Lemma 4.2 above. Recall that, under the assumption $k\equiv 0 \mod 4$, $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$ is an even and real function and $\varphi_{f,z}(t)$ has the same odd zeros as this function. Hence, by this construction, $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\,\varphi_{f,z}(t)$ must be a real and even function with constant sign for any $t\in\mathbb{R}$. Considering the continuous $Q_f:\left(0,\,\frac{\pi}{4}\right)\longmapsto\mathbb{R}$ defined by the integral

$$Q_f(u) := \int_0^\infty i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\varphi_{f,z}(t)\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt,\quad 0<u<\frac{\pi}{4},$$

we then have that $|Q_f(u)|$ will be positive decreasing. Our proof will now show that this cannot be true if the bound (5.7) takes place.

If we use the power series (5.2) for $\varphi_{f,z}(t)$, we see that $Q_f(u)$ can be written as an infinite series of the form

$$\begin{aligned} Q_f(u) &= \int_0^\infty i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\,\varphi_{f,z}(t)\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt \\ &= \sum_{j=0}^\infty(-1)^\ell a_{2\ell}\int_0^\infty i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\,t^{2\ell}\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt. \end{aligned} \tag{5.8}$$

Note that the interchange of the orders of summation and integral in (5.8) comes from Fubini's theorem and the bounds (5.7): indeed

$$\begin{aligned} \int_0^\infty\sum_{\ell=0}^\infty|a_{2\ell}|t^{2\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt &\leq \mathscr{B}\int_0^\infty\sum_{j=0}^\infty\frac{2^{2\ell+1}\pi^{4\ell}}{h^{2\ell}(4\ell+2)!}t^{2\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt \\ &\leq 2\mathscr{B}\int_0^\infty\sum_{j=0}^\infty\frac{1}{(4\ell)!}\left(\frac{2\pi^2 t}{h}\right)^{2\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt \\ &\leq 2\mathscr{B}\int_0^\infty\sum_{j=0}^\infty\frac{1}{\ell!}\left(\pi\sqrt{\frac{2t}{h}}\right)^{\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt \\ &< 2\mathscr{B}\int_0^\infty\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\exp\left(\pi\sqrt{\frac{2t}{h}}+\left(\frac{\pi}{2}-2u\right)t\right)\,dt \\ &\ll_{k,z,\Lambda}\int_0^\infty|t|^{B(k)}\,\exp\left(-2ut+\left(\pi\sqrt{\frac{2}{h}}+|z|\right)\sqrt{t}\right)\,dt<\infty, \end{aligned}$$

where in the last step we have used the estimate (4.1). Having assured that we can perform the operation (5.8), it now follows from Lemma 4.4 above that

$$Q_f(u) = \sum_{j=0}^{\infty} (-1)^{\ell} a_{2\ell} \int_0^{\infty} i^{-k/2} \tilde{G}_{z,f}\left(\frac{k}{2} + it\right) t^{2\ell} \cosh\left(\left(\frac{\pi}{2} - 2u\right) t\right) dt$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} a_{2\ell} \frac{2\pi}{2^{2\ell}} e^{\frac{z^2}{4}} \operatorname{Re}\left(i^{-\frac{k}{2}} \frac{d^{2\ell}}{du^{2\ell}} \left\{ \sum_{j\neq 0} c_j \, e^{i\left(\frac{\pi}{4} - u\right)k - \left(\frac{\pi}{2} - 2u\right)\lambda_j} \psi_f\left(ie^{-2iu}, z\right) \right\} \right). \tag{5.9}$$

Using Lemma 4.3 together with the estimate for $a_{2\ell}$ (5.7), we can bound uniformly the previous series with respect to u. Indeed, since the function

$$g(u) = e^{i\left(\frac{\pi}{4} - u\right)k} \sum_{j\neq 0} c_j \, e^{-\left(\frac{\pi}{2} - 2u\right)\lambda_j}$$

is analytic and $\|g(u)\|_{L^{\infty}(C_0(1))} \leq \mathcal{M}$, with $\mathcal{M}$ only depending on the sequence $\sum_{j\neq 0} |c_j|$ (see (3.7) above), it follows from (4.9) that

$$\left| \frac{d^{2\ell}}{du^{2\ell}} \left\{ g(u) \, \psi_f\left(ie^{-2iu}, z\right) \right\} \right| < \mathcal{D} \frac{2^{14\ell} (2\ell)! e^{-\frac{A}{u}}}{u^{k+2\ell}}, \tag{5.10}$$

for some constant $\mathcal{D}$ depending on the weight of the cusp form and on the sequences $(c_j)_{j\in\mathbb{N}}$ and on an upper bound for the sequence $(\lambda_j)_{j\in\mathbb{N}}$. Thus, returning to (5.9) and invoking (5.7)

$$|Q_f(u)| < 2\pi \mathcal{D} e^{\frac{z^2}{4}} \frac{e^{-A/u}}{u^k} \sum_{\ell=0}^{\infty} |a_{2\ell}| \frac{2^{12\ell} (2\ell)!}{u^{2\ell}} \leq 2\pi \mathscr{C} e^{\frac{z^2}{4}} \frac{e^{-A/u}}{u^k} \sum_{\ell=0}^{\infty} \frac{2^{14\ell+1} \pi^{4\ell} (2\ell)!}{h^{2\ell} (4\ell + 2)!} \cdot \frac{1}{u^{2\ell}}$$

$$= \mathscr{C} \pi^{\frac{3}{2}} e^{\frac{z^2}{4}} \frac{e^{-A/u}}{u^k} \sum_{\ell=0}^{\infty} \frac{2^{10\ell} \pi^{4\ell}}{\Gamma\left(2\ell + \frac{3}{2}\right) (2\ell + 1) (hu)^{2\ell}} < \mathscr{C} \pi^{\frac{3}{2}} e^{\frac{z^2}{4}} \frac{e^{-A/u}}{u^k} \cosh\left(\frac{32\pi^2}{hu}\right)$$

$$< \mathscr{C} \frac{\pi^{\frac{3}{2}} e^{\pi/36}}{u^k} \exp\left(-\left(A - \frac{32\pi^2}{h}\right) \frac{1}{u}\right), \tag{5.11}$$

where we have used the fact that $|z| < \frac{1}{3}\sqrt{\pi}$, (1.25). From (5.11), if we choose the constant h in such a way that $h > \frac{32\pi^2}{A}$, then

$$\lim_{u\to 0^+} |Q_f(u)| < \mathscr{C} \pi^{\frac{3}{2}} e^{\pi/36} \lim_{u\to 0^+} u^{-k} \exp\left(-\left(A - \frac{32\pi^2}{h}\right) \frac{1}{u}\right) = 0$$

which contradicts the fact that $|Q_f(u)|$ is positive decreasing. Consequently, we have found $h > \frac{32\pi^2}{A}$ such that $\tau_n < h\, n^2$ for infinitely many values of n. This shows (5.1) and so we complete the proof of our Theorem. Finally, we note that we can replace the value of d in (1.27) by an explicit constant. Using the value $A = 0.03$ (calculated in the proof of Lemma 4.2) and taking $h = \frac{36\pi^2}{0.03}$, we find from (5.1) that $d = \frac{1}{36\pi}$ works.

In order to consider the case where $k \equiv 2 \mod 4$, the argument is the same with a small modification. Instead of considering $Q_f(u)$ we will study the similar function,

$$P_f(u) := \int_0^{\infty} i^{-\frac{k}{2}} \, t \, \tilde{G}_{z,f}\left(\frac{k}{2} + it\right) \varphi_{f,z}(t) \cosh\left(\left(\frac{\pi}{2} - 2u\right) t\right) dt, \quad 0 < u < \frac{\pi}{4},$$

where $\varphi_{f,z}(t)$ is given by (5.2). Invoking the second integral representation (4.11) and using the bounds for a_{2j} found in (5.7), one can prove that $|P_f(u)| \to 0$ as $u \to 0^+$, which assures the result. ■

6 Concluding Remarks

In this paper we have presented estimates for the number of critical zeros of the arbitrary shifted combinations

$$\tilde{F}_{z,\alpha}(s):=\sum_{j\neq0}c_j\,\eta_{\alpha}\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\}, \tag{6.1}$$

$$\tilde{G}_{z,f}(s):=\sum_{j\neq0}c_j\,\eta_{f}\left(s+i\lambda_j\right)\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{z^2}{4}\right)\right\}. \tag{6.2}$$

The proofs of both estimates relied on the summation formulas (2.19) and (4.2). One may wonder how to extend our results in different directions. For example, one may ask if we can relax the class of hypergeometric functions appearing in the combinations (6.1) and (6.2). Note that the second parameter of the hypergeometric functions above depends on the Dirichlet series that preceeds them: in the first case, it is equal to $\alpha/2$ and in the second is k. Therefore, one may consider the study of the critical zeros of the functions,

$$\tilde{F}_{z,\alpha}(s;\nu):=\sum_{j\neq0}c_j\,\eta_{\alpha}\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)\right\}, \tag{6.3}$$

$$\tilde{G}_{z,f}(s;\nu):=\sum_{j\neq0}c_j\,\eta_{f}\left(s+i\lambda_j\right)\left\{{}_1F_1\left(k-s-i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(k-\overline{s}+i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)\right\}, \tag{6.4}$$

where ν is now a real parameter independent of α (in the first case) and of k in the second. We remark that it is indeed possible to get analogues of Theorems 1.1 and 1.2 to the shifted combinations (6.3) and (6.4) provided $\nu\geq\frac{\alpha}{2}-1$ in the first case and $\nu\geq k-1$ in the second. However, instead of using the summation formulas (2.19) and (4.2), we will need more general transformations. Indeed, it is not difficult to show[9] that, when $\text{Re}(\nu)>-1$ and $\text{Re}(x)>0$, $y\in\mathbb{C}$, the following summation formula takes place

$$\sum_{n=1}^{\infty}(-1)^n r_{\alpha}(n)\,n^{-\nu/2}\,e^{-\pi nx}\,J_{\nu}(\sqrt{\pi n}y)=-\frac{y^{\nu}\pi^{\nu/2}}{2^{\nu}\Gamma(\nu+1)}$$
$$+\frac{\pi^{\nu/2}y^{\nu}e^{-\frac{y^2}{4x}}}{2^{\nu}\Gamma(\nu+1)x^{\alpha/2}}\sum_{n=1}^{\infty}\tilde{r}_{\alpha}(n)e^{-\frac{\pi}{x}\left(n+\frac{\alpha}{4}\right)}\,\Phi_3\left(1-\frac{\alpha}{2}+\nu;\nu+1;\frac{y^2}{4x},\frac{\pi y^2}{4x^2}\left(n+\frac{\alpha}{4}\right)\right), \tag{6.5}$$

where $\Phi_3(b;c;w,z)$ is the usual Humbert function,

$$\Phi_3(b;c;\,w,z)=\sum_{k,m=0}^{\infty}\frac{(b)_k}{(c)_{k+m}}\frac{w^k z^m}{k!\,m!}, \tag{6.6}$$

whose series converges absolutely for any $w,z\in\mathbb{C}$. By using (6.5) instead of (2.19), we can establish a lower bound for the number of critical zeros of the function (6.3). Analogously, Theorem 1.2 can be extended to shifted combinations of the form (6.4) by using the transformation formula

$$\sum_{n=1}^{\infty}a_f(n)\,n^{-\frac{\nu}{2}}\,e^{-2\pi nx}\,J_{\nu}\left(\sqrt{2\pi n}\,y\right)=\frac{(-1)^{k/2}\pi^{\nu/2}y^{\nu}x^{-k}}{2^{\nu/2}\Gamma(\nu+1)}e^{-\frac{y^2}{4x}}\sum_{n=1}^{\infty}a_f(n)e^{-\frac{2\pi n}{x}}\,\Phi_3\left(1-k+\nu;\nu+1;\frac{y^2}{4x},\frac{\pi y^2 n}{2x^2}\right). \tag{6.7}$$

[9] The proof of this summation formula is quite standard and it follows the same steps as the ones given in [[24], pp. 18-19]. Although our formulas (6.5) and (6.7) do not have an intrinsic interest themselves, they can be used to establish new summation formulas for $\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu-\mu}{2}}\,I_{\mu}\left(Y\sqrt{n}\right)\,K_{\nu}\left(X\sqrt{n}\right)$, where $X>Y$ and $\text{Re}(\mu),\text{Re}(\nu)>0$. Such a formula will be given in a forthcoming investigation and it constitutes a generalization of a formula of Berndt, Dixit, Kim and Zaharescu [4].

Note that (6.5) and (6.7) reduce to (2.19) and (4.2) when $\nu = \frac{\alpha}{2} - 1$ and $\nu = k-1$ respectively. This is the case due to the well-known reduction formula for Humbert's function

$$\Phi_3\left(0; c; w, z\right) = \Gamma(c)\, z^{-\frac{c-1}{2}} I_{c-1}\left(2\sqrt{z}\right),$$

which can be immediately established from the power series (6.6). Analogous theorems can be established when the combination of confluent hypergeometric functions ${}_1F_1\left(a; c; z\right)$ is replaced by ${}_2F_2\left(a, b; c, d; z\right)$ and Gauss' hypergeometric function ${}_2F_1(a, b; c; z)$.

Another direction that this work can take concerns even more general shifted combinations of completed Dirichlet series. To continue, recall again that $\eta(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$. In the paper [7], the authors have posed the following interesting problem: if one considers a combination of shifted products of the form

$$H_z(s) := \sum_{m,n=1}^{\infty} c_{m,n}\, \eta\left(s + i\lambda_m\right)\, \eta\left(s+i\lambda_n\right)\, \text{Re}\left({}_1F_1\left(\frac{1-s-i\lambda_m}{2}; \frac{1}{2}; \frac{z^2}{4}\right)\right)\, \text{Re}\left({}_1F_1\left(\frac{1-s-i\lambda_n}{2}; \frac{1}{2}; \frac{z^2}{4}\right)\right), \tag{6.8}$$

under what circumstances will $H_z(s)$ have infinitely many zeros on the critical line? Even when restrict this problem to $z = 0$, it seems complicated to formulate a result covering this case. Indeed, consider the case where $c_{m,n} := d_m\, \delta_{m,n}$, $d_m \geq 0$ and $z = 0$: then we are dealing with the study of the zeros of the function

$$\mathcal{H}(s) = \sum_{m=1}^{\infty} d_m\, \eta^2\left(s + i\lambda_m\right), \quad d_m \geq 0. \tag{6.9}$$

It is simple to give an example where $\mathcal{H}(s)$ does not have a single zero at the critical line $\text{Re}(s) = \frac{1}{2}$. For some sufficiently large M, consider the finite sequence $\lambda_m = m, \;\; 1 \leq m \leq M$ and assume that $d_j = 0$ when $m \geq M+1$. If $\mathcal{H}(s) = \sum_{1\leq m\leq M} d_m\, \eta^2\left(s + i\, m\right)$ has a zero on the critical line, say $s = \frac{1}{2} + i\tau$, then due to the non-negativity of d_m, we have that

$$\eta\left(\frac{1}{2} + i(\tau+m)\right) = 0, \quad \text{for } 1 \leq m \leq M. \tag{6.10}$$

But this is impossible for N large enough because every arithmetical sequence contains infinitely many elements which are not zeros of $\zeta(z)$.[10] Therefore, it seems difficult to assure a theorem of the form given here for combinations of shifted products. However, one may conjecture that a combination of the form (6.9) has infinitely critical zeros for "infinitely many" pairs of bounded shifts.

Conjecture 6.1. *Let $(c_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence of non-zero real numbers such that $\sum_{j,k=1}^{\infty} |c_{j,k}| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$, $(\lambda_k^{\prime})_{k\in\mathbb{N}}$ be two bounded sequences of distinct real numbers that attain their bounds. Then there are infinitely many $\tau \neq 0$ such that the function*

$$G(s;\, \tau) := \sum_{j,k=1}^{\infty} c_{j,k}\, \eta_{\alpha}\left(s + i\lambda_j\right)\, \eta_{\alpha}\left(s + i\tau + i\lambda_k^{\prime}\right)$$

has infinitely many zeros at the critical line $Re(s) = \frac{\alpha}{4}$.

[10]This is a beautiful observation (with a simple proof) made by Putnam [20, 21]. Note that, if we pick $\lambda_m := \frac{2\pi m}{\log(2)}$, then our reasoning would also contradict a very interesting result due to Steuding and Wegert [29], which states that

$$\frac{1}{M}\sum_{0\leq m<M} \zeta\left(\frac{1}{2} + i\tau + i\frac{2\pi m}{\log(2)}\right) = \frac{1}{1-2^{-\frac{1}{2}-i\tau}} + O\left(\frac{\log(M)}{\sqrt{M}}\right), \quad M \to \infty.$$

Finally, we should also mention a case where a zero counting problem of the form considered in this paper has a better estimate than those given by our Theorems. Under the conditions of Theorem 1.1, one may consider the problem of counting the zeros of the function

$$\mathcal{Z}\left(t\right) := \sum_{j=1}^{\infty} c_j \, Z\left(t+\lambda_j\right), \tag{6.11}$$

where $Z(t)$ represents Hardy's $Z-$function [16],

$$Z(t) := \zeta\left(\frac{1}{2}+it\right)\left(\chi\left(\frac{1}{2}+it\right)\right)^{-1/2}, \quad \chi(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{6.12}$$

Building on Ingham's work [15] and following an approach very similar to Atkinson's [2], Hall [[12], p. 103, Theorem 5] deduced a curious integral formula involving the product of $\mathcal{Z}(t)$ with a shifted version of itself. He proved the asymptotic formula

$$\int_0^T \mathcal{Z}\left(t\right)\mathcal{Z}\left(t+\frac{\beta}{\log(T)}\right) dt = \sum_{j=1}^{\infty} c_j^2 \cdot \frac{\sin(\beta/2)}{\beta/2} T\log(T) + O(T), \tag{6.13}$$

where $0<\beta<1$. A formula like (6.13) was derived for the first time by Atkinson [2] [11]and it was used by him to give a very short proof that the Riemann zeta function has $\gg T/\log(T)$ critical zeros. Under the assumption that $\sum c_j^2 < \infty$ (which is actually implied by our condition $\sum |c_j| < \infty$), Hall proved the analogous result to a shifted combination of the form (6.11), establishing that

$$\mathcal{N}_0(T) \gg \frac{T}{\log(T)}, \tag{6.14}$$

where $\mathcal{N}_0(T)$ denotes the number of the zeros of the form $s=\frac{1}{2}+it,\ 0<t<T$ of the function $\mathcal{Z}\left(t\right)$. Although the function (6.11) is very different from the completed Dirichlet series $\eta(s)$, it is not unreasonable to conjecture that lower bounds like (6.14) hold true in the setting of shifted combinations of our class of completed Dirichlet series.

We state a conjecture which, if proven, would drastically improve on our Theorem 1.1 when $z=0$.

Conjecture 6.2. *Assume the conditions of Theorem 1.1 (with $z:=0$) and let $\mathcal{N}_{\alpha}(T)$ denote the number of zeros written in the form $s=\frac{\alpha}{4}+it$, $0\leq t\leq T$, of the function*

$$\tilde{F}_{\alpha}(s) := \sum_{j\neq 0} c_j \, \pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\, \zeta_{\alpha}\left(s+i\lambda_j\right).$$

Then there exists some $c>0$ such that

$$\liminf_{T\to\infty} \frac{\mathcal{N}_{\alpha}(T)}{T} \geq c. \tag{6.15}$$

As it can be seen from the 4-square Theorem,

$$\zeta_4(s) = 8(1-2^{2-2s})\,\zeta(s)\,\zeta(s-1),$$

[11]The reader may check [[[16]], p. 121] to see the differences between the results of Hall and Atkinson.

the estimate (6.15) cannot be in general improved to $\mathcal{N}_\alpha(T) \gg T\,\log(T)$. By Siegel's result [28], whenever $\alpha \in \mathbb{N}_{\geq 4}$, $\zeta_\alpha(s)$ has $\asymp T$ zeros of the form $s = \frac{\alpha}{4} + it$, $0 \leq t \leq T$ so, again, (6.15) seems to be the best estimate that one may be able to prove. In any case, it can be conjectured that, in the situation of $\zeta(s)$, one has the estimate

$$\liminf_{T\to\infty} \frac{\mathcal{N}_1(T)}{T\log(T)} \geq c, \tag{6.16}$$

which could act as an extension of Selberg's famous result [27] in a different direction than [26].

Analogously, we may formulate the following conjecture for holomorphic cusp forms which, if true, would constitute a massive generalization of Hafner's result [13] in a different setting than Selberg's result adapted to combinations of degree 2 $L-$ functions [22].

Conjecture 6.3. *Assume the conditions of Theorem 1.2 (with $z := 0$) and let $\mathcal{N}_f(T)$ denote the number of zeros written in the form $s = \frac{k}{2} + it,\ \ 0 \leq t \leq T$, of the function*

$$\tilde{G}_f(s) := \sum_{j\neq 0} c_j\ (2\pi)^{-s-i\lambda_j}\,\Gamma(s+i\lambda_j)\,L\left(s+i\lambda_j, f\right).$$

Then there exists some $d > 0$ such that

$$\liminf_{T\to\infty} \frac{N_{f,z}(T)}{T\log(T)} \geq d.$$

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